1A

SIMULTMEOUS LINEAR EQUATION

UNIT STRUCTURE

1A.1. INTRODUCTION

Equation of a line in a plane is of first degree in x & y and Conversely every equation of first degree in x & y represents a line.

- **Linear equation:**

The general equation of the set of equation 'x' & 'y' variable of the type $ax+by+c=o$ with a, b, c real number and at least one of a and b is not zero, is closely associated with the set of lines in a plane is called linear equation. Linear equation is obtained by equating to zero a linear expression.

- **Linear Expression:**

Any expression of the type $ax+by+c$ a, b, c in R and at least one of a and b is non zero, is called linear expression.

1A.2. METHODS FOR SOLVING SIMULTANEOUS LINEAR EQUATIONS:

<u>*Simultaneous Linear Equation in Two variables:</u>

Let $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$. A value for each variable which satisfies simultaneously both the equations will give the roots of the equation. There are two methods to solve the given simultaneous equation

1) Elimination method
2) Cross Multiplication Method

(1) Elimination Method:-

In this method, two given equations are reduced to a linear equation in one unknown by eliminating one of the unknowns and then solving for the other unknown.

Eg. (i) Solve $2x + 5y = 9$ and $3x - y = 5$

$2x + 5y = 9$...... (1)

$3x - y = 5$........ (2)

Multiply eq^n (2) by 5,

We get, $15x - 5y = 25$......(3)

Adding (1) & (3),

$$\begin{array}{r} 2x + 5y = 9 \\ 15x - 5y = 25 \\ \hline 17x \qquad = 34 \end{array}$$

$$\boxed{\therefore x = 2}$$

Substituting $x = 2$ in eq^n (1),

$\therefore 2x + 5y = 9$

$\therefore 2(2) + 5y = 9$

$\boxed{\therefore y = 1}$

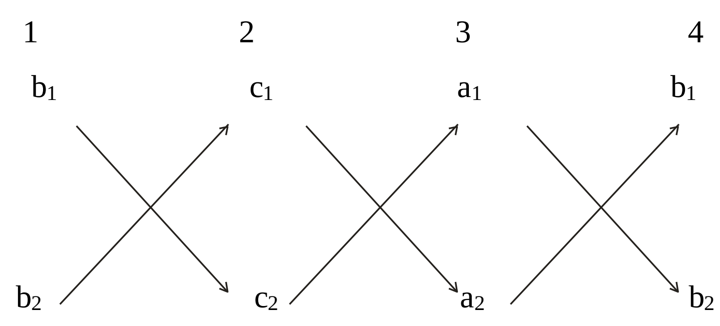
Thus $x = 2$, $y = 1$ is the required solution.

(2) Cross-Multiplication Method:-

$a_1x + b_1y + c_1 = 0$

$a_2x + b_2y + c_2 = 0$

We write the coefficients of x, y, and constant terms and two more columns by repeating the coefficients of x and y as follows:



and the result is given by:

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$$\frac{x}{(b_1c_2 - b_2c_1)} = \frac{y}{(c_1a_2 - c_2a_1)} = \frac{1}{(a_1b_2 - a_2b_1)}$$

So, the solution is:

$$x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1} \qquad y = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$$

E.g.

1) Solve $3x + 2y + 17 = 0$

$5x - 6y - 9 = 0$

Solution:

By comparing with $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ then using the result $\frac{x}{(b_1c_2 - b_2c_1)} = \frac{y}{(c_1a_2 - c_2a_1)} = \frac{1}{(a_1b_2 - a_2b_1)}$

$$\therefore \frac{x}{2(-9) - 17(-6)} = \frac{y}{(17)(5) - (3)(-9)} = \frac{1}{3(-6) - 5(2)}$$

$$\therefore \frac{x}{84} = \frac{y}{112} = \frac{1}{-28}$$

$$i.e\ \frac{x}{3} = \frac{y}{4} = \frac{1}{-1}$$

$$\therefore \boxed{x = -3} \quad \boxed{y = -4}$$

Thus $x = -3$, $y = -4$ is the required solution.

- **<u>Simultaneous Linear Equations with Three variables Methods</u>:**

Methods: (1) Elimination method

(2) Cross Multiplication method

E.g 1) Solve for x, y and z.

$3x - 2y + 4z = 1$

$2x - y + z = 3$

$x + 3y - 2z = 11$

(a) **Method of Elimination:** Any two of three equations can be chosen for limitation of one of the variable.

$2x - y + z = 3$ .... (i)

$x + 3y - 2z = 11$ .... (ii)

$3x - 2y + 4z = 1$ .... (iii)

Multiplying eq^n (i) by 2

$4x - 2y + 2z = 6$...... (iv)

Eliminating variable 'z' by adding (ii) and (iv),

$\therefore\ 5x + y = 17$ ...... (v)

Multiplying eq^n (ii) by 2

$2x + 6y - 4z = 22$...... (vi)

By Adding (iii) and (vi), $5x + 4y = 23$......... (vii)

Subtracting eq^n (vii) from eq^n (v),

$$-3y = -6$$
$$\boxed{\therefore y = 2}$$

Substituting $y = 2$ in eq^n (v)

$$\therefore\ 5x + 2 = 17$$
$$\boxed{\therefore x = 3}$$

Substituting $\boldsymbol{x} = 3$ and $y = 2$ in eq^n (i)

$$2(3) - 2 + z = 3$$
$$\boxed{\therefore z = -1}$$

$\therefore\ x = 3, y = 2, z = -1$ is the required solution.

(b) Method of Cross Multiplication:

$2x - y + z = 3$ .... (i)

$x + 3y - 2z = 11$ .... (ii)

$3x - 2y + 4z = 1$ .... (iii)

The equations (i) & (ii) can be written as follows:-

$$2x - y + (z-3) = 0$$
$$x + 3y + (-2z-11) = 0$$

By cross multiplication,

$$\frac{x}{-1(-2z-11)-3(z-3)}=\frac{y}{(z-3)-2(-2z-11)}=\frac{1}{2(3)-1(-1)}$$

$$\therefore \frac{x}{20-z}=\frac{y}{5z+19}=\frac{1}{7}$$

$$\therefore x=\frac{20-z}{7} \quad y=\frac{5z+19}{7}$$

Substituting above values for x & y in eq^n (iii) i.e $3x - 2y + 4z = 1$

$$\therefore 3\left(\frac{20-z}{7}\right)-2\left(\frac{5z+19}{7}\right)+4z=1$$

$$\therefore 60 - 3z - 10z - 38 + 28z = 7$$

$$\therefore 15z = 7- 22 \quad \therefore 15z = -15$$

$$\boxed{\therefore z=-1}$$

Now, $x=\frac{20-(-1)}{7}=\frac{21}{7} \quad \therefore \boxed{x=3}$

$$y=\frac{5(-1)+19}{7}=\frac{14}{7} \quad \therefore \boxed{y=2}$$

$\therefore$ $x = 3$, $y = 2$, $z = -1$ is the required solution.

3) It the numerator of a fraction is increased by 2 and the denominator by 1, it becomes 1. Again, if the numerator is decreased by 4 and the denominator by 2, it becomes $1/2$. Find the fraction.

Let x/y be the required fraction (given)

$$\therefore \left.\begin{array}{l}\frac{x+2}{y+1}=1\\ \frac{x-4}{y-2}=\frac{1}{2}\end{array}\right\}$$

$\therefore$ $x + 2 = y + 1$ i.e. $x - y = -1$ ..... (i)

$2x - 8 = y - 2$ $\therefore$ $2x - y = 6$ .... (ii)

Subtracting (i) from (ii)

$$
\begin{array}{rcl}
2x - y &=& 6 \quad \text{(i)} \\
x - y &=& -1 \quad \text{(ii)} \\
- \quad + && + \\
\hline
x &=& 7
\end{array}
$$

$\therefore \boxed{x = 7}$ and

Substituting x=7 in eq^n (i) we get, 2(7)-y=6

$\therefore$ 14-y=6

$\therefore$ y= 14 - 6 = 8

$\therefore \boxed{y = 8}$

$\therefore$ The fraction = $\dfrac{x}{y} = \boxed{\dfrac{7}{8}}$

4) The age of a man is three times the sum of the ages of his two sons and 5 years hence his age will be double the sum of their ages. Find the present age of man?

Let 'x' years be the present age of man and sum of present ages of the two sons be 'y' years.

Given condition $x = 3y$ .... (i)

$x + 5 = 2\ (y + 5 + 5)$ .... (ii)

Substituting eq^n (i) in eq^n (ii),

$3y + 5 = 2(y + 10)$

$\therefore \boxed{y = 15}$

by substituting y = 15 in (i) $\therefore$ $x = 3y = 3(15) = 45$

$\boxed{x = 45}$

Hence, the present age of man is 45 years

1A.3. UNIT END EXERCISE:

➢ **Multiple Choice Questions**

1) The solution of the set of equations $3x + 4y = 7$, $4x - y = 3$ is
[A] (1, -1) [B] (1, 1) [C] (2, 1) [D] (1,-2)

2) The values of x and y satisfying the equations
$\dfrac{x}{y} + \dfrac{y}{3} = 2,\quad x + 2y = 8$ are given by the pair
[A] (3, 2) [B] (-2, -3) [C] (2, 3) [D] None of these

3) Solve for x and y: $x - 3y = 0,\quad x + 2y = 20$

[A] $x = 4, y = 12$ [B] $x = 12, y = 4$ [C] $x = 5, y = 4$ [D] None of these

4) Solve: $\frac{x}{4} = \frac{y}{3} = \frac{z}{2}$ $\quad 7x + 8y + 5z = 62$

[A] (4, 3, 2) [B] (2, 3, 4) [C] (3, 4, 2) [D] (4, 2, 3)

5) Solve: $3x - 4y + 70z = 0$, $2x + 3y - 10z = 0$, $x + 2y + 3z = 13$

[A] (1, 3, 7) [B] (1, 7, 3) [C] (2, 4, 3) [D] (-10, 10, 1)

6) Monthly incomes of two persons are in the ratio 4:5 and their monthly expenses are in the ratio 7:9. If each saves Rs 50 per month find their monthly incomes.

[A] (500,400) [B] (400,500) [C] (300,600) [D] (350,550)

7) The age of a person is twice the sum of the ages of his two sons and five years ago his age was thrice the sum of their ages. Find his present age.

[A] 60 years [B] 52 years [C] 51 years [D] 50 years

8) The sum of the digits in a three digit no. is 12. If the digits are reversed, the no. is increased by 495 but reversing only of tens & units digits increases the number by 36. The number is

[A] 327 [B] 372 [C] 237 [D] 273

Exercises

1) Solve $x + y + z = 5$, $2x - 3y - 4z = -11$ and $3x + 2y - z = -6$

2) Find the values of x and y for equations $x + 5y = 36$ and $\frac{x+y}{x-y} = \frac{5}{3}$

3) The wages of 8 men and 6 boys amount to Rs.33. If 4 men earn Rs.4.50 more than 5 boys. Determine the wages each man and boy.

4) The demand and supply equations for a certain commodity are

$4q + 7p = 17$ and $p = \frac{q}{3} + \frac{7}{4}$ respectively.

Where p market price; q: quantity. Find the equilibrium price and quantity.

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1B

QUADRATIC EQUATIONS

UNIT STRUCTURE

1B.1. INTRODUCTION AND DEFINITIONS

***Polynomial:**

An expression of the form $a_0x^n + a_1x^{n-1}$ ……… +……. + $a_{n-1}\, x + a_n$, where $a_0 \neq 0$

$a_1, a_2 \ldots\ldots a_n$ are constants

n : Positive integer

x : Variable (un known)

is called a polynomial in 'x' of degree 'n' and it is denoted by f(x).

Eg. $x^2 - 3x + 1$, $2x^3 - 3x^2 + 5$, $\sqrt{2}x^4 - 2x + 3$ polynomials in 'x'.

- **Quadratic Expression :**

An expression of the form $ax^2 + bx + c$, $a \neq 0$ and a, b, c$\in$R, is called a quadratic expression, where R is the set of real numbers.

Eg. (i) $2x^2 + 3x + 4$

(ii) $x^2 - 4x + 5$ are examples of quadratic expressions.

- **Quadratic Equation:**

The equation of the form $ax^2 + bx + c = 0, a \neq 0$, a, b, c, $\in$ R is called a quadratic equation where R is the set of real numbers. It is an equation of degree 2. Generally, coefficients of quadratic equations encountered are rational numbers.

Eg (i) $3x^2 + 7x + 12 = 0$

(ii) $x^2 + 4 = 0$ are examples of quadratic equations.

- **Root of a quadratic equation :**

The value of x which satisfies the quadratic equation $ax^2 + bx + c = 0$ is called the root of given quadratic equation.

Thus, if $x = \alpha$, is the root of quadratic equation

$$ax^2 + bx + c = 0, \text{ then } a\alpha^2 + b\alpha + c = 0$$

- **Roots of a Quadratic equation:**

The roots of the quadratic equation $ax^2 + bx + c = 0$ are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

i.e. $\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$, $\beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$

- **Sum and Product of the Roots of a Quadratic Equation:**

Let α, β be the roots of quadratic eq^n $ax^2 + bx + c = 0$. If S is the sum of roots and P the product of the roots of quadratic equation, then the quadratic equation is $x^2 - Sx + P = 0$

i.e. x^2 – (Sum of roots) x + (Product of roots) = 0

Where, $S = \alpha + \beta = \frac{-b}{a}$ and $P = \alpha\beta = \frac{c}{a}$

There are two types of quadratic equations

(1) Pure
(2) Affected

(1) Pure Quadratic Equation:

A quadratic equation is said to be Pure if coefficient of x is zero. Thus, a pure quadratic equation is of the type $ax^2 + b = 0$; $a \neq 0$

(2) Affected Quadratic Equation:

A quadratic equation which is not pure is called an affected quadratic equation. The most general form of an affected quadratic equation is

$$ax^2 + bx + c = 0 \;;\; a \neq 0 \;\&\; b \neq 0$$

- **Methods of Solving Pure Quadratic Equations:**

Let $ax^2 - b = 0$ be a pure quadratic equation. This implies

$$ax^2 = +b \;\Rightarrow\; x^2 = \frac{+b}{a} \;\Rightarrow\; x = \pm\sqrt{\frac{+b}{a}}$$

The roots of $ax^2 - b$ are real if 'a' and 'b' are of opposite sign.

Eg. Solve $16x^2 - 9 = 0 \quad \Rightarrow \quad x = \pm\frac{3}{4}$

- **Methods of Solving Affected Quadratic Equations:**

(1) Method of factorization
(2) Method of perfect square

(1) Method of factorization:

If the expression $ax^2 + bx + c$ can be factorized into linear factors then each of the factors, put to zero, we get two roots α & β for the given quadratic equation

Thus if $ax^2 + bx + c = a(x-\alpha)(x-\beta)$, then the roots of $ax^2 + bx + c = 0$ are α & β

Eg (i) **Solve:** $x^2 - 8x + 12 = 0$

12 → -6, -2

$\therefore (x-6)(x-2) = 0$

$\therefore x - 6 = 0$ or $x - 2 = 0$

$\therefore x = 6$ or $x = 2$

Hence, the roots of given equation are 6 or 2

(ii) **Solve:** $x^2 - 8x - 48 = 0$

48 → -12, 4

$\therefore x^2 - 12x + 4x - 48 = 0 = 0$

$\therefore x(x-12) + 4(x-12) = 0$

$\therefore (x+4)(x-12) = 0$

$\therefore x + 4 = 0 \quad or \quad x - 12 = 0$

$\therefore x = -4 \quad or \quad x = 12$

Hence, the roots of given equation are -4 or 12

(iii) **Solve:** $3x^2 + 7x + 2 = 0$

$\therefore 3x^2 + 6x + x + 2 = 0$

$\therefore 3x(x+2) + 1(x+2) = 0$

$\therefore (3x+1)(x+2) = 0$

$\therefore 3x + 1 = 0 \quad or \quad x + 2 = 0$

$x = -1/3 \quad or \quad x = -2$

Hence, the roots of given equation are -1/3 or -2

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(2) Method of Perfect Square:

When $ax^2 + bx + c$ cannot be factorized easily into its linear factors. If can be solved by the method of perfect square.

Step 1: Let $ax^2+bx+c=0$ be the given equation. Divide both sides of equation by a, we get $x^2+\frac{b}{a}x+\frac{c}{a}=0 \quad (\therefore a\neq 0)$

Step 2: Transpose the constant term (i.e. the term independent of x) on R.H.S.

$$x^2+\frac{b}{a}x=\frac{-c}{a}$$

Step 3: Add $\frac{b^2}{4a^2}$ on both sides to make R.H.S. a perfect square

Thus, $x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=\frac{-c}{a}+\frac{b^2}{4a^2}$

$$\therefore \left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$$

Thus, a pure equation in the variable $x+\frac{b}{2a}$ is

$$x+\frac{b}{2a}=\frac{\pm\sqrt{b^2-4ac}}{2a}$$

$$\therefore x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$\therefore x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Solve:

(i) $2x^2-5x-1=0$ cannot be easily factored into linear factors

Comparing with $ax^2+bx+c=0$

$a = 2$, b = -5, $c = -1$

Roots are given by

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

i.e. $x=\frac{-(-5)\pm\sqrt{(-5)^2-4(2)(-1)}}{2(2)}$

$$= \frac{5}{4} \pm \frac{\sqrt{33}}{4}$$

Thus, the roots are $\left(\frac{5}{4} + \frac{\sqrt{33}}{4} \right)$ and $\left(\frac{5}{4} - \frac{\sqrt{33}}{4} \right)$

- **Nature of the Roots of a Quadratic Equation:**

The roots of the quadratic equation $ax^2 + bx + c = 0$

are given by $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

The quantity, $b^2 - 4ac$ is called Discriminant of the quadratic equation

and is generally denoted by 'Δ'.

Thus, $\Delta = b^2 - 4ac$

Nature of the roots of a quadratic equation depends upon its discriminant Δ for a, b, c $\in R$

(1) If $\Delta = b^2 - 4ac > 0$, roots are real and different.

(2) If $\Delta = b^2 - 4ac > 0$ and is a perfect square, then the roots are real. different and rational.

(3) If $\Delta = b^2 - 4ac > 0$ and is not a perfect square, then the roots are real different and irrational.

(4) If $\Delta = b^2 - 4ac = 0$, then the roots are real & equal.

(5) If $\Delta = b^2 - 4ac < 0$ then the roots are complex and different.

Note:

(i) If a, b, c are rational, then the irrational roots occur in pairs If one of the root is $p + \sqrt{q}$, the other must be p -iq

(ii) If a, b, c are real, then the complex roots, if any occur in the other must be p – iq

(iii)If α, β are roots of quadratic eq^n $ax^2 + bx + c = 0$, then

$$a(x - \alpha)(x - \beta) = 0$$

- **Important Results:**

(1) If in the quadratic equation $ax^2 + bx + c = 0$ $a + b + c = 0$, then one root of quadratic eq^n is unity (1) and the other root is c/a.

(2) If the quadratic eq^n $ax^2 + bx + c = 0$ is satisfied by more than two values of x i.e it has more than two roots, then it must be an identity for which *a = b = c = 0*

(3) If the roots of quadratic equation $ax^2+bx+c=0$ are real and distinct, then one root must be greater than $-b/2a$ and the other less than $-b/2a$

- **Symmetric Function:**

Any expression involving α and β as its roots is called a symmetric function of α and β, if it remains unchanged when α and β are interchanged

Eg. $\alpha\beta,\ \ \alpha^2\beta+\alpha\beta^2,\ \ \alpha^2+\beta^2$ are all symmetric functions of α and β, whereas $\alpha^3-\beta$ is not a symmetric function since in general $\alpha^3-\beta$ need not be equal to $\beta^3-\alpha$

A symmetric functions of α and β can be solved with help of

Sum of roots $S=\alpha+\beta=\dfrac{-b}{a}$

Product of roots $P=\alpha\beta=\dfrac{c}{a}$

Examples:

1) Examine the nature of roots of following equations:

(i) $x^2-8x+16=0$

Here, a = 1, b = -8, c = 16

$b^2-4ac=(-8)^2-4(16)(1)$

$=64-64=0$

$\because$ Discriminate $\Delta=b^2-4ac=0,$ the roots are real and equal

(ii) $3x^2-8x+4=0$

Here, a = 3, b = -8, c = 4

$b^2-4ac=(-8)^2-4(3)(4)$

$=64-48$

$=16$

$\because$ Discriminate $\Delta=b^2-4ac>0$ and a perfect square, the roots are real, rational and unequal.

2) Solve :

$$\left(x-\frac{1}{x}\right)^2+2\left(x+\frac{1}{x}\right)=7\frac{1}{4}$$

$$\therefore \left(x-\frac{1}{x}\right)^2+2\left(x+\frac{1}{x}\right)=\frac{29}{4}$$

$$\therefore \left(x+\frac{1}{x}\right)^2-4+2\left(x+\frac{1}{x}\right)=\frac{29}{4} \quad [\because (a-b)^2=(a+b)^2-4ab\,]$$

Let $x+\frac{1}{x}=t,$

$$\therefore t^2-4+2t=\frac{29}{4}$$

$$\therefore 4t^2+8t-45=0$$

$$\therefore \; 4t^2+18t-10t-45=0$$

$$\therefore \; 2t(2t+9)-5(2t+9)=0$$

$$\therefore \; 2t-5=0 \quad or \quad 2t+9=0$$

$$\therefore \; t=5/2 \quad or \quad t=-9/2$$

Either $x+\frac{1}{x}=\frac{5}{2}$ *or* $x+\frac{1}{x}=-9/2$

$$\therefore 2x^2-5x+2=0 \quad or \quad 2x^2-5x+2=0$$

Either $x=\frac{-(-5)\pm\sqrt{25-16}}{4}$ *or* $x=\frac{-9\pm\sqrt{81-16}}{4}$

$$\therefore x=\frac{5\pm 3}{4} \; or \; x=\frac{-9\pm\sqrt{65}}{4}$$

$$\therefore \boldsymbol{x=\frac{8}{4}=2 \; or \; x=\frac{2}{4}=\frac{1}{2} \; or \; x=\frac{-9+\sqrt{65}}{4}, x=\frac{-9-\sqrt{65}}{4}}$$

$$\therefore x=2,\frac{1}{2},\frac{-9+\sqrt{65}}{4},\frac{-9-\sqrt{65}}{4}$$

Hence, the roots of given equation are 2, ½ , $\frac{-9+\sqrt{65}}{4}$, $\frac{-9-\sqrt{65}}{4}$

3) **Solve the equation:**

$$x^2-6x+9=4\sqrt{x^2-6x+6}$$

Let $x^2-6x+6=t$

$$\therefore t+3=4\sqrt{t}$$

$$\therefore \; t^2+6t+9=16t$$

$$\therefore t^2-10t+9=0$$

$$\therefore \; (t-1)(t-9)=0$$

∴ t = 1 or t = 9

$$\therefore \; x^2-6x+6=1 \quad or \quad x^2-6x+6=9$$

$$\therefore \; x^2-6x+5=1 \quad or \quad x^2-6x-3=0$$

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$$\therefore (x-1)(x-5)=0 \quad or \quad x=\frac{6\pm\sqrt{36-4(1)(-3)}}{2}$$

$$\therefore\ x=1,5 \qquad or \qquad x=\frac{6\pm 4\sqrt{3}}{2}$$

$$\therefore\ x=1,5 \qquad or \qquad x=3\pm 2\sqrt{3}$$

Hence, the roots of given equation are 1, 5, $3+2\sqrt{3},\ 3-2\sqrt{3}$

4) Find value of $\sqrt{6+\sqrt{6+\sqrt{6+....\infty}}}$

Let $\quad x=\sqrt{6+\sqrt{6+\sqrt{6+....\infty}}}$

$x=\sqrt{6+x}$

$\therefore\quad x^2=6+x$ [$\because$ taking square on both the sides]

$\therefore\quad x^2-x-6=0$

$\therefore\quad (x-3)\ (x+2)=0$

$\therefore x=3 \quad or \quad x=-2$

$\therefore$ The value of $\sqrt{6+\sqrt{6+\sqrt{6+....\infty}}}$ is 3 or -2

5) It the roots of eq. $x^2-px+q=0$ 'α' & 'β' and form the equation whose roots are $\frac{\alpha}{\beta}$ & $\frac{\beta}{\alpha}$

$\because \alpha,\beta$ are roots of eq^n $x^2-px+q=0$.

Comparing with $ax^2+bx+c=0$. $a=1$, $b=-p$, $c=q$

$\therefore \alpha+\beta=-(-p)=p$ and $\alpha\beta=q$

$\left[\because \alpha+\beta=\frac{-b}{a}, \alpha\beta=\frac{c}{a}\right]$

Now, $\quad \frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{\alpha^2+\beta^2}{\alpha\beta}=\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}=\frac{p^2-2q}{q}$

$\frac{\alpha}{\beta}\ \frac{\beta}{\alpha}=1$

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$\therefore$ Required eq^n *is* $x^2 - \left(\frac{p^2 - 2q}{q}\right)x + 1 = 0$

$[\because x^2 - (Sum\, of\, roots)x + (Product\, of\, root)) = 0]$

$$\therefore qx^2 - \left(p^2 - 2q\right)x + q = 0$$

6) **Solve :** $x^4 - 5x^3 + 15x + 9 = 0$

$$\therefore x^4 - 5x\left(x^2 - 3\right) + 9 = 0$$

$$\therefore \left(x^4 - 6x^2 + 9\right) - 5x\left(x^2 - 3\right) + 6x^2 = 0$$

$$\therefore (x^2 - 3)^2 - 5x(x^2 - 3) + 6x^2 = 0$$

Substitute $x^2 - 3 = t$

$$\therefore t^2 - 5xt + 6x^2 = 0$$

Roots of above eq^n t = 2x or t = 3x

$$\therefore x^2 - 3 = 2x \; or \; x^2 - 3 = 3x$$

$$\therefore x^2 - 2x - 3 = 0 \; or \; x^2 - 3x - 3 = 0$$

$$\therefore (x+1)(x-3) = 0 \; or \; x = \frac{3 \pm \sqrt{9 - 4(-3)(1)}}{2}$$

$$\therefore x = -1, 3 \; 0r \; x = \frac{3 \pm \sqrt{21}}{2}$$

Hence, the roots of given equation are -1, 3, $\frac{3+\sqrt{21}}{2}$, $\frac{3-\sqrt{21}}{2}$

7) *Solve* : $4^x - 3.2^{x+2} + 2^5 = 0$

$$4^x - 3.2^{x+2} + 2^5 = 0$$

$$\therefore \left(2^x\right)^2 - 3.2^x.2^2 + 32 + 0$$

$$\therefore \left(2^x\right)^2 - 12.2^x + 32 + 0$$

Let $2^x = y$

$$\therefore y^2 - 12y + 32 = 0$$

$$\therefore y^2 - 8y - 4y + 32 = 0$$

$$\therefore y(y-8) - 4(y-8) = 0$$

$$\therefore\ y = 4 \quad or \quad y = 8$$

$$\therefore 2^x = 4 \quad or \quad 2^x = 8$$

$$\therefore 2^x = 2^2 \quad \text{or} \quad 2^x = 2^3$$

$$\therefore x = 2 \quad or \quad x = 3$$

8) If α, β be the roots of $2x^2 - 4x - 1 = 0$ the value of $\dfrac{\alpha^2}{\beta} + \dfrac{\beta^2}{\alpha}$

$$\therefore \alpha + \beta = \frac{-1(-4)}{2} = 2 \qquad [\because \alpha+\beta = -\frac{b}{a} \ and \ \alpha\beta = \frac{c}{a}]$$

and $\alpha\beta = {}^{-1}\!/_{2}$

$$\therefore \frac{\alpha^2}{\beta} + \frac{\beta^2}{\beta} = \frac{\alpha^3 + \beta^3}{\alpha\beta} = \frac{\left[(\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)\right]}{\alpha\beta}$$

$$= \frac{(2)^3 - 3\left({}^{-1}\!/_{2}\right)(2)}{-{}^{1}\!/_{2}}$$

$$= \frac{8+3}{-1/2}$$

$$= \frac{11}{-1/2}$$

$$= -22$$

$\therefore$ The value of $\dfrac{\alpha^2}{\beta} + \dfrac{\beta^2}{\alpha}$ = -22

9) Divide 25 into two parts so that sum of their reciprocals is ${}^{1}\!/_{6}$.

Let the parts be x and 25 – x

Given condition:

$$\frac{1}{x} + \frac{1}{25-x} = \frac{1}{6}$$

$$\therefore \frac{25 - x + x}{x(25-x)} = \frac{1}{6}$$

$$\therefore 150 = 25x - x^2$$

$$\therefore x^2 - 25x + 150 = 0$$

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$$\therefore\ x(x-15)-10(x-15)=0$$

$$\therefore\ (x-15)(x-10)=0$$

$$\therefore x = 10,\ 15$$

$\therefore$ Part of 25 are 10 and 15.

10) A piece of iron rod casts ` 69. If the rod was 2 meter shorter and each meter unchanged, what is the length of rod?

Let the length of rod be 'x' meters

The rate per meter $= \dfrac{60}{x}$

New length = $(x - 2)$

As cost is same, new rate per metre $= \dfrac{60}{x-2}$

Given condition:

$$\frac{60}{x-2}=\frac{60}{x}+1$$

$$\therefore\ \frac{60}{x-2}-\frac{60}{x}=1$$

$$\therefore\ \frac{120}{x(x-2)}=1$$

$$\therefore 120=x(x-2)$$

$$\therefore x^2-2x-120=0$$

$$\therefore\ (x-12)(x+10)=0$$

$\therefore$ Either x = 12 or x = -10

The length cannot be negative. Therefore, the length of the rod is 12m.

- **Simultaneous Equations in two unknowns**

(1) Linear simultaneous equations
(2) Non-linear simultaneous equations

Eg. (1) Solve: $4x-3y=1,\ 12xy+13x^2=25$

From $4x-3y=1, \quad y=\frac{4x-1}{3}$

Substituting the above value of y in $12xy+13x^2=25$

$$\therefore 12x\left(\frac{4x-1}{3}\right)+13x^2=25$$

$$\therefore 16x^2-4x+13x^2=25$$

$$\therefore 29x^2-4x-25=0$$

$$\therefore (29x+25)(x-1)=0$$

$$\therefore x=1 \quad or \quad x=\frac{-25}{29}$$

In terms of y $x=\frac{3y+1}{4}$

$$\therefore \frac{3y+1}{4}=1 \quad or \quad \frac{3y+1}{4}=\frac{-25}{29}$$

$$\therefore y=1 \quad or \quad y=\frac{-43}{29}$$

$\therefore$ Required solution is $x=1, y=1 \quad or \quad x=\frac{-25}{29}, y=\frac{-43}{29}$

(2) Solve: $2x+3y=5, \; xy=1$

$xy=1 \qquad \therefore (2x)(3y)=6xy=6(1)=6$

$$\therefore (2x)(3y)=6$$

$$2x+3y=5$$

As $(2x-3y)^2=(2x+3y)^2-4(2x)(3y)$

$=25-24 \qquad [\because 2x+3y=5 \;\&\; (2x)(3y)=6]$

$=1$

$$\therefore 2x-3y=\pm 1$$

$\therefore \quad 2x-3y=1 \quad or \quad 2x-3y=-1$

Now $\quad 2x+3y=5, 2x-3y=1 \quad or \quad 2x+3y=5, 2x-3y=-1$

$\therefore \quad x={}^{3}/_{2},\; y={}^{2}/_{3} \quad or \quad x=1,\; y=1$

Hence, the required solution is $x=1, \quad y=1$

or $x=3/2, \quad y=2/3$

(3) Solve $x+y-\sqrt{xy}=7, \quad x^2+y^2+xy=133$

$$x^2+y^2+xy=(x+y)^2-xy$$

$$=\left(x+y+\sqrt{xy}\right)\left(x+y-\sqrt{xy}\right)$$

$$\therefore 133=7\left(x+y+\sqrt{xy}\right)$$

$$\left[\because x+y-\sqrt{xy}=7\right]$$

$$\therefore x+y+\sqrt{xy}=19 \quad \ldots. \quad (1)$$

$$x+y-\sqrt{xy}=7 \quad \ldots. \quad (2)$$

Adding (1) & (2) $\quad 2(x+y)=26$

$$\therefore \quad x+y=13 \quad \ldots. \quad (3)$$

Subtracting (2) from (1) $\quad 2\sqrt{xy}=12$

$$\therefore \sqrt{xy}=6 \therefore xy=36 \quad \ldots. \quad (4)$$

Substituting $\quad y=36/x \quad$ in eq^n (3)

$$\therefore x+\frac{36}{x}=13$$

$$\therefore x^2-13x+36=0$$

$$\therefore x^2-9x-4x+36=0$$

$$\therefore x(x-9)-4(x-9)=0$$

$$\therefore x=9 \quad or \quad x=4$$

$$\therefore y=4 \quad or \quad y=9$$

$\therefore$ Required solution is $\quad x=9, \quad y=4$

$or \quad x=4, \quad y=9$

Simultaneous Equations in three unknowns

Eg: (1) $5x-4y+z=0$ ……(i)

$2x+5y-4z=0$ …… (ii)

$x^2-2y^2+z^2=0$ …… (iii)

From (i) & (ii)

By cross multiplication,

$$\frac{x}{16-5}=\frac{y}{2-(-20)}=\frac{z}{25-(-8)}$$

$$\therefore \frac{x}{11}=\frac{y}{22}=\frac{z}{33}$$

$$\therefore x=\frac{y}{2}=\frac{z}{3}=k$$

$\therefore\ x=k,\ \ y=2k,\ z=3k$

Substituting the above values in eq^n (iii)

$$k^2-8k^2+9k^2=0$$

$$\therefore\ 2k^2=0$$

$$\therefore\ k=0$$

$$\therefore\ x=0, y=0,\ z=0$$

(2) $x(y+z)=5,$ …… (i)

$y(z+x)=8$ …… (ii)

$z(x+y)=9$ …… (iii)

Adding (i) & (ii),

$xy+xz+yz=13$ ………. (iv)

Subtracting (iii) from (iv),

$$2xy=4$$

$\therefore\ \boxed{xy=2}$ ………………………… (v)

Similarly adding (2) & (3) & subtracting eq^n (1) from the sum,

$$\begin{array}{r} 2yz + xz + xy = 17 \\ xz + xy = 5 \\ \hline 2yz = 12 \end{array}$$

$\therefore$ $\boxed{yz = 6}$ ……………………………….. (vi)

Similarly adding (1) & (3), and subtracting eq^n (2) from sum

$$\begin{array}{r} 2xz + xy + yz = 14 \\ xy \pm yz = 8 \\ \hline 2xz = 6 \end{array}$$

$\boxed{\therefore xz = 3}$ ………………………………… (vii)

$\therefore \frac{x}{z} = \frac{1}{3} \qquad \frac{y}{x} = 2 \qquad \frac{y}{z} = \frac{2}{3}$ [$\because$ From (v), (vi) and (vii)]

$$x = \frac{y}{2} = \frac{z}{3} = k$$

$$\therefore x = k,\ \ y = 2k,\ \ z = 3k$$

Substituting the above values in eq^n (i)

$$x(y+z) = 5$$

$$\therefore k(2k+3k) = 5$$

$$\therefore 2k^2 + 3k^2 = 5$$

$$\therefore 5k^2 = 5$$

$$\therefore k^2 = 1$$

$$\therefore k = \pm 1$$

$$\therefore x = \pm 1 \quad y = \pm 2,\ \ z = \pm 3$$

Required solution: $x = 1,\ y = 2,\ z = 3$

OR $x = -1,\ y = -2,\ z = -3$

(3) $x^2 + xy + xz = 45$

$yx + y^2 + yz = 75$

$zx + zy + z^2 = 105$

$\therefore\ x(x + y + z) = 45$ .... (1)

$\therefore\ y(x + y + z) = 75$ .... (2)

$\therefore\ z(x + y + z) = 105$ .... (3)

Adding we get,

$(x + y + z)^2 = 225$

$\therefore\ x + y + z = \pm 15$

$\therefore x = \pm 3,\quad y = \pm 5,\quad z = \pm 7$ [$\because$ From (1), (2) and (3)]

Required solution:

$x = 3,\ y = 5,\ z = 7$

or $x = -3,\ y = -5,\ z = -7$

1B.2. UNIT END EXERCISE

***Multiple Choice Questions:**

(1) If $2^{2x+3} - 3^2.2^x + 1 = 0,$ then values of x are

[A] 0, 1 [B] 1, 2 [C] 0, 3 [D] 0, -3

(2) If $\alpha\beta$ be the roots of eq^n $2x^2 - 4x - 3 = 0,$ the values of $\alpha^2 + \beta^2$ is

[A] 5 [B] 7 [C] 3 [D] -4

(3) The equation $x^2 - (p + 4)x + 2p + 5 = 0$ has equal roots, the value of p will be

[A] ± 1 [B] 2 [C] ± 2 [D] – 2

(4) If the roots of eq^n $x^2 + (2p - 1)x + p^2 = 0$ are real, then

[A] $p \geq 1$ [B] $p \leq 4$ [C] $p \geq 1/4$ [D] $p \leq 1/4$

(5) If the roots of the eq^n exceeds the other by 4, then the value of m is

[A] $m=10$ [B] $m=11$ [C] m = 9 [D] m = 12

(6) The area of a rectangular field is 2000 m^2 and its perimeter is 180 m. Form a quadratic eq^n by taking the length of field as x and solve it to find length of breadth of field. The length and breadth are

[A] $(205m, 80m)$ [B] $(50m, 40m)$ [C] (60m, 50m) [D] None

➢ **Exercise:**

1) $\left(x^2+\frac{1}{x^2}\right)+\left(x+\frac{1}{x}\right)-4=0$ Solve for x

2) Solve $\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1=x^2}}=3$

3) Solve $x^{2/3}+x^{1/3}-2=0$

4) Solve $3x^2-18+\sqrt{3x^2-4x+6}=4x$

5) If α and β be the roots of $x^2+7x+12=0$. Find the equation whose roots are $(\alpha+\beta)^2$ and $(\alpha-\beta)^2$

6) If are roots of $2x^2+3x+7=0$. the valves of $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}$ and $\alpha^3+\beta^3$

7) Solve $x^2+y^2=29,\quad x-y=3$

8) Solve $x+y+\sqrt{xy}=14$
$x^2+y^2+\sqrt{xy}=84$

9) Solve $3x + y - 2z = 0,\ 4x - y - 3z = 0$

$$x^3 + y^3 + z^3 = 467$$

10) Solve $xy + x + y = 23$

$$xz + x + z = 41$$

$$yz + y + z = 27$$

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2A

Determinants

UNIT STRUCTURE

2A.1. OBJECTIVES

In this chapter a student has to learn the

- Concept of Determinant.
- Minors and co-Factors
- Applications of Determinant in solving Linear equations

2A.2. INTRODUCTION

* Determinants:

The determinant has definite value. In determinant number of rows and columns are always equal.

e.g.

$$D_1 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}_{2x2} \quad \textit{is determinant of } 2^{nd} \textit{ order and } a_{11}, a_{12}, a_{21}, a_{22} \textit{ are called its elements.}$$

.

$$D_2 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}_{3x3} \quad \textit{is determinant of } 2^{nd} \textit{ order; and}$$

$$D_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}_{4x4} \quad \textit{is determinant of } 4^{th} \textit{ order.}$$

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2A.3. EVALUATION OF DETERMINANT

(a)<u>Second order determinant:</u>

The value of determinant of order 2, $A=\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is

$$|A|=\det(A)=\begin{vmatrix} a & b \\ c & d \end{vmatrix}=ad-bc$$

e.g.

The value of $A=\begin{vmatrix} 2 & -3 \\ -1 & 3 \end{vmatrix}$ is

$$|A|=\det(A)=\begin{vmatrix} 2 & -3 \\ -1 & 3 \end{vmatrix}=(2)(3)-(-3)(-1)=6-3=3$$

$\therefore$ The value of determinant is 3.

(b)<u>Third order determinant:</u>

The value of determinant of order 3

$$A=\begin{vmatrix} a_1 & a_2 & a_2 \\ b_1 & b_2 & b_2 \\ c_1 & c_2 & c_2 \end{vmatrix}$$

is given by

$$|A|=\det(A)=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}=a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}-a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}+a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

E.g.

The value of A= $\begin{vmatrix} 3 & 2 & 1 \\ -5 & 1 & 0 \\ 3 & -1 & 4 \end{vmatrix}$ is given by

$$|A|=\det(A)=\begin{vmatrix} 3 & 2 & 1 \\ -5 & 1 & 0 \\ 3 & -1 & 4 \end{vmatrix}$$

$$=3\begin{vmatrix} 1 & 0 \\ -1 & 4 \end{vmatrix}-2\begin{vmatrix} -5 & 0 \\ 3 & 4 \end{vmatrix}+1\begin{vmatrix} -5 & 1 \\ 3 & -1 \end{vmatrix}$$

$\therefore$ det (A)=3(4-0)-2(-20-0)+1(5-3)=3(4)-2(-20)+1(2)=12+40+2=54

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$\therefore$ The value of determinant is 54.

2A.4. PROPERTIES OF DETERMINANT

(1) The value of determinant is not attend by changing the rows into the corresponding columns and the columns into the corresponding rows.

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

(2) If two rows or two columns of a determinant are identical, the determinant has the value zero.

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0 \text{ (Since } R_2=R_3\text{)} \quad \text{OR} \quad \begin{vmatrix} a_1 & a_2 & a_2 \\ b_1 & b_2 & b_2 \\ c_1 & c_2 & c_2 \end{vmatrix} = 0 \text{ (Since } C_2=C_3\text{)}$$

(3) If two adjacent rows or columns of the determinant are interchanged, the value of the determinant so obtained is the negative of the value of the original determinant.

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -\begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

(4) If the elements of any row or column are multiplied by the same factor, the value of the

determinant so obtained is equal to the value of the original determinant multiplied by that factor.

$$\begin{vmatrix} ma_1 & ma_2 & ma_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = m\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

(5) **Sum of determinants:**

If any element in any row (or columns) consists of the sum of two terms, the determinant can be expressed as the sum of two other determinants whose other rows (or columns) remain the same, while the remaining row (or column) consists of these terms respectively.

Thus,

$$\begin{vmatrix} a_1+\alpha_1 & a_2 & a_3 \\ b_1+\beta_1 & b_2 & b_3 \\ c_1+\lambda_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & a_2 & a_3 \\ \beta_1 & b_2 & b_3 \\ \lambda_1 & c_2 & c_3 \end{vmatrix}$$

***PRODUCT OF DETERMINANT**:

The product of two determinants is possible only if both the determinants are of same order.

Let A and B two determinants of the order 3.

$$A=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad \text{and} \quad B=\begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix}$$

Then,

$$AXB=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} X \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix}$$

$$=\begin{vmatrix} a_1l_1+a_2l_2+a_3l_3 & a_1m_1+a_2m_2+a_3m_3 & a_1n_1+a_2n_2+a_3n_3 \\ b_1l_1+b_2l_2+b_3l_3 & b_1m_1+b_2m_2+b_3m_3 & b_1n_1+b_2n_2+b_3n_3 \\ c_1l_1+c_2l_2+c_3l_3 & c_1m_1+c_2m_2+c_3m_3 & c_1n_1+c_2n_2+c_3n_3 \end{vmatrix}$$

$$=\begin{vmatrix} R_1R_1^{'} & R_1R_2^{'} & R_1R_3^{'} \\ R_2R_1^{'} & R_2R_2^{'} & R_2R_3^{'} \\ R_3R_1^{'} & R_3R_2^{'} & R_3R_3^{'} \end{vmatrix};$$

where, R_1, R_2, R_3 are rows of A and $R_1^{'}, R_2^{'}, R_3^{'}$ are rows of B.

2A.5. MINORS AND CO-FACTORS

(a) Minor of an element:

Consider the determinant A of order n written as

$$A=\begin{vmatrix} a_{11} & a_{12} & a_{13} & - & - & - & a_{1n} \\ a_{21} & a_{22} & a_{23} & - & - & - & a_{2n} \\ - & - & - & - & - & - & - \\ - & - & - & - & - & - & - \\ a_{n1} & a_{n2} & a_{n3} & - & - & - & a_{nn} \end{vmatrix}_{n\times n}$$

Then $|M_{ij}|$ is called the minor of the element a_{ij} of determinant A, where M_{ij} is obtained by deleting i^{th} row and j^{th} column of A of the order (n-1)X(n-1).

E.g. Consider the determinant of order 3.

$$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}_{3x3}$$

M_{11} = Minor of an element a_{11}

$$= \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

Similarly,

M_{12} = Minor of an element a_{12}

$$M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

E.g. Let,

$$A = \begin{vmatrix} 2 & 5 & 8 \\ 1 & 3 & 2 \\ 0 & 4 & 6 \end{vmatrix}$$

$$M_{11} = \begin{vmatrix} 3 & 2 \\ 4 & 6 \end{vmatrix}, M_{12} = \begin{vmatrix} 1 & 2 \\ 0 & 6 \end{vmatrix}, M_{13} = \begin{vmatrix} 1 & 3 \\ 0 & 4 \end{vmatrix}$$

$$M_{21} = \begin{vmatrix} 5 & 8 \\ 4 & 6 \end{vmatrix}, M_{22} = \begin{vmatrix} 2 & 8 \\ 0 & 6 \end{vmatrix}, M_{23} = \begin{vmatrix} 2 & 5 \\ 0 & 4 \end{vmatrix}$$

Similarly we can find M_{31}, M_{32}, M_{33}.

(b) <u>Co-factor of an element:</u>

If A is the determinant of order n and C_{ij} denotes Co-factor of the element a_{ij} and is obtained by multiplying the minor M_{ij} multiplies by by $(-1)^{i+j}$.

$C_{ij} = (-1)^{i+j} M_{ij}$ Where M_{ij} is minor of a_{ij}.

$$\text{If } A = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

C_{11} = The co-factor of a_1 =$(-1)^{1+1} \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$

C_{12} = The co-factor of b_1 $=(-1)^{1+2}\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$

C_{13} = The co-factor of c_1 $=(-1)^{1+3}\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$

E.g. Consider,

$$A=\begin{vmatrix} 1 & 3 & 4 \\ 0 & 2 & 1 \\ 3 & 7 & 6 \end{vmatrix}$$

$$C_{12} = (-1)^{1+2} M_{12}$$

$$\therefore C_{12} = (-1)^{1+2}\begin{vmatrix} 0 & 1 \\ 3 & 6 \end{vmatrix}$$

$$=(-1)^3(0-3)$$

$$=(-1)(-3)=3$$

$$C_{11}=(-1)^{1+1} M_{11}$$

$$\therefore C_{11}= (-1)^{1+1}. \begin{vmatrix} 2 & 1 \\ 7 & 6 \end{vmatrix}$$

$$= (1) \times (12-7) =5$$

2A.6. CRAMER'S RULE FOR SOLUTION OF LINEAR EQUATIONS:

(a) <u>Cramer's Rule for solution of Linear equation in two variables</u>:

The solution system:

a_1x+b_1y = c_1 ……. (1)

a_2x+b_2y= c_2 ……. (2)

is given by $x=\dfrac{D_x}{D}, y=\dfrac{D_y}{D}$;

Where $D=\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, $D_x=\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$, $D_y=\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$

$$\therefore x=\frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \quad y=\frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

E.g. Solve:

3x+2y=5 …….(1)
4x+ y =3 …….(2)

Solution:

By using Cramer's Rule,

$$x=\frac{D_x}{D}, y=\frac{D_y}{D}$$

Where D= $\begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix}$ = 3 - 8 = -5, $D_x=\begin{vmatrix} 2 & 5 \\ 1 & 3 \end{vmatrix}=6-5=1$, $D_y=\begin{vmatrix} 3 & 5 \\ 4 & 3 \end{vmatrix}=9-20=-11$

Thus we get, D=-5 , D_x = 1, D_y = -11

Substituting in $x=\frac{D_x}{D}, y=\frac{D_y}{D}$ we get

$$x=\frac{1}{-5}=-\frac{1}{5}, y=\frac{-11}{-5}=\frac{11}{5}$$

Therefore, solution is x = $-\frac{1}{5}$, y = $\frac{11}{5}$

(b)<u>Cramer's Rule for solution of Linear equation in three variables</u>:

The solution system:

a_1x+ b_1y+ c_1z = d_1 ……. (1)
a_2x+ b_2y+ c_2z = d_2 ……. (2)
a_3x+ b_3y+ c_3z = d_3 ……. (3)

is given by $x=\frac{D_x}{D}, y=\frac{D_y}{D}, z=\frac{D_z}{D}$, ,

Where $D=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, D_x=\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}, D_y=\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, D_z=\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

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$$\therefore \; x=\frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \quad y=\frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \quad z=\frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

e.g. Solve the following:

3x+ 2y+ z =10
5x+ 3y+2z=17
7x+ 8y+ z = 26

is given by $x=\frac{D_x}{D}, y=\frac{D_y}{D}, z=\frac{D_z}{D}$

Where

$$D=\begin{vmatrix} 3 & 2 & 1 \\ 5 & 3 & 2 \\ 7 & 8 & 1 \end{vmatrix}=3(3-16)-2(5-14)+1(40-21)$$

=3(-13)-2(-9)+1(19)

=-39+18+19 =-2

$$D_x=\begin{vmatrix} 10 & 2 & 1 \\ 17 & 3 & 2 \\ 26 & 8 & 1 \end{vmatrix}=10(3-16)-2(17-52)+1(136-78)$$

=10(-13)-2(-35) + 1(58)

= -130+70+58 = -2

$$D_y=\begin{vmatrix} 3 & 10 & 1 \\ 5 & 17 & 2 \\ 7 & 26 & 1 \end{vmatrix}=3(17-52)-10(5-14)+1(130-119)$$

= 3(-35)-10(-9) + 1(11)

$=-105+90+11= - 4$

$$D_z = \begin{vmatrix} 3 & 2 & 10 \\ 5 & 3 & 17 \\ 7 & 8 & 26 \end{vmatrix} = 3(78-136) - 2(130-119) + 10(40-21)$$

$=3(-58) -2(11) + 10(19)$

$= -174-22+190$

[35]

$= - 196+190$

$= -6$

Thus we get,

D=-2 , D_x = -2, D_y = -4, D_z = -6

Substituting in $x = \frac{D_x}{D}, y = \frac{D_y}{D}, z = \frac{D_z}{D}$ we get

$$x = \frac{-2}{-2} = 1, y = \frac{-4}{-2} = 2, z = \frac{-6}{-2} = 3$$

Therefore, solution is x=1, y=2 and z=3.

2A.7. UNIT END EXERCISE

i) Find the value of determinant

$$A = \begin{vmatrix} 2 & 1 & 1 \\ 4 & -3 & 1 \\ -2 & 5 & 1 \end{vmatrix}$$

ii) Write the minors and co-factors of the elements of the determinant

$$A = \begin{vmatrix} 2 & 3 & 2 \\ 1 & 4 & -1 \\ 5 & 6 & 8 \end{vmatrix}$$

iii) Solve the following system of equations by using Cramer's rule:

3x+y= 19

3x-y = 23

iv) Solve the following system of equations by using Cramer's rule:

x+ 2y+ 3z=6
2x+ 4y+z=17
3x+ 2y+9z=2

v) Find product of the following two determinants:

$$A = \begin{vmatrix} 3 & 2 & 1 \\ 5 & 3 & 2 \\ 7 & 8 & 1 \end{vmatrix} \quad \text{and} \quad B = \begin{vmatrix} 2 & 5 & 8 \\ 1 & 3 & 2 \\ 0 & 4 & 6 \end{vmatrix}$$

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2B

MATRICES

UNIT STRUCTURE

2B.1. OBJECTIVES

In this chapter a student has to learn the

- Concept of adjoint of a matrix.
- Inverse of a matrix.
- Rank of a matrix and methods finding these.

2B.2. INTRODUCTION

At higher secondary level, we have studied the definition of a matrix, operations on the matrices, types of matrices inverse of a matrix etc.

In this chapter, we are studying adjoint method of finding the inverse of a square matrix and also the rank of a matrix.

2B.3. DEFINITIONS

<u>*Matrix:</u>

A matrix is a rectangular grid of numbers, symbols or expressions that is arranged in a row or column format enclosed in Square or curved brackets.

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A system of $m \times n$ numbers arranged in the form of an ordered set of m horizontal lines called rows & n vertical lines called columns is called a matrix of order $m \times n$.

A matrix with m rows and n columns is a matrix of order m x n is written as

$$\begin{bmatrix} a_{11} & a_{12} & \ldots\ldots\ldots & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots\ldots & a_{2n} \\ \vdots & \vdots & \ldots\ldots\ldots & \vdots \\ a_{m1} & a_{m2} & \ldots\ldots\ldots & a_{mn} \end{bmatrix}_{mxn} \quad \text{OR} \quad \begin{pmatrix} a_{11} & a_{12} & \ldots\ldots\ldots & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots\ldots & a_{2n} \\ \vdots & \vdots & \ldots\ldots\ldots & \vdots \\ a_{m1} & a_{m2} & \ldots\ldots\ldots & a_{mn} \end{pmatrix}_{mxn}$$

* **Note:**

i) Matrices are generally denoted by capital letters.

ii) The elements are generally denoted by corresponding small letters.

iii) A matrix has no numerical value.

Types of Matrices:

1) Rectangular matrix:-

Any m x n Matrix where $m \neq n$ is called rectangular matrix.

For e.g.

$$A=\begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \end{bmatrix}_{2\times 3}$$

2) Column Matrix:

It is a matrix in which there is only one column.

$$A=\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}_{3\times 1}$$

3) Row Matrix:

It is a matrix in which there is only one row.

$$A=\begin{bmatrix} 5 & 7 & 9 \end{bmatrix}_{1\times 3}$$

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4) Square Matrix:

It is a matrix in which number of rows equals the number of columns. i.e. it is n x n matrix of order n.
e.g.

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}_{2\times 2}$$

Matrix A is a square matrix of order 2.

5) Diagonal Matrix:

It is a square matrix in which all non-diagonal elements are zero.

e.g.

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times 3}$$

6) Scalar Matrix:

It is a square diagonal matrix in which all diagonal elements are equal.

e.g.

$$A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}_{3\times 3}$$

7) Unit Matrix:

It is a scalar matrix with diagonal elements as unity.

e.g.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$$

8) Null Matrix:

A matrix whose all elements are zero is said to be Null matrix.
e.g.

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times3}$$

9) Upper Triangular Matrix:

It is a square matrix in which all the elements below the principle diagonal are zero.

e.g.

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 5 \end{bmatrix}_{3\times3}$$

10) Lower Triangular Matrix:

It is a square matrix in which all the elements above the principle diagonal are zero.

e.g.

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 4 & 0 \\ -1 & 3 & 2 \end{bmatrix}_{3\times3}$$

11) Symmetric Matrix:

If for a square matrix A, $A = A^T$ then A is symmetric
e.g.

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 4 & 1 \\ 5 & 1 & 9 \end{bmatrix}$$ is symmetric matrix.

12) Skew -Symmetric Matrix:

If for a square matrix A, $A = -A^T$ then it is skew -symmetric matrix.
e.g.

$$A = \begin{bmatrix} 0 & 5 & 7 \\ -5 & 0 & 3 \\ -7 & -3 & 0 \end{bmatrix}$$ is skew -symmetric matrix.

Note: For a skew -Symmetric matrix, diagonal elements are always zero.

***Some more types of Matrices:**

(a)Transpose of Matrix:

It is a matrix obtained by interchanging rows into columns or columns into rows.
e.g.

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 7 & 9 \end{bmatrix}_{2\times 3}$$

$$A^T = Transpose\ of\ A = \begin{bmatrix} 1 & 3 \\ 3 & 7 \\ 5 & 9 \end{bmatrix}_{3\times 2}$$

(b)Determinant of a Square Matrix:

Let A be a square matrix then

$|A| = determinant\ of\ A\ i.e\ det\ A = |A|$

If (i) then $|A| \neq 0$ matrix A is called as non-singular and
If (i) then $|A| = 0,$ matrix A is singular.

Note: For non-singular matrix A^{-1} exists.

(c) Co-factor Matrix:-

A matrix C = $\left[C_{ij}\right]$ where C_{ij} denotes co-factor of the element a_{ij}.

of a matrix A of order n x n, is called a co-factor matrix.

In above matrix $A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 1 \\ 3 & 7 & 6 \end{bmatrix}$

Co-factor matrix is

$$C = \begin{bmatrix} 5 & 3 & -6 \\ 10 & -6 & 9 \\ -3 & -1 & 2 \end{bmatrix}$$

$$\therefore C = \begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix}$$

Similarly for a matrix, $A = \begin{bmatrix} 1 & 2 \\ 3 & 9 \end{bmatrix}$ the co-factor matrix is

$$C = \begin{bmatrix} 9 & -3 \\ -2 & 1 \end{bmatrix}$$

(d) Adjoint of Matrix:-

If A is any square matrix then transpose of its co-factor matrix is called Adjoint of A.

Thus in the notations used,

Adjoint of $A = C^T$ = transpose of its co-factor matrix

$$\Rightarrow Adj\ A = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$$

Adjoint of a matrix A is denoted as Adj. A

Thus if,

$A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 1 \\ 3 & 7 & 6 \end{bmatrix}$ then Co-factor of matrix $A = \begin{bmatrix} 5 & 3 & -6 \\ 10 & -6 & 9 \\ -3 & -1 & 2 \end{bmatrix}$

$$Adj.\ A = \begin{bmatrix} 5 & 10 & -3 \\ 3 & -6 & -1 \\ -6 & 9 & 2 \end{bmatrix}$$

Note:

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times 2}$ then Co-factor of matrix $A = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$

$$\text{Adj. } A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

(e) Inverse of a square Matrix:-

Two non-singular square matrices of order n, A and B are said to be inverse of each other if,AB=BA=I, where I is an identity matrix of order n.Inverse of A is denoted as A^{-1} and read as A inverse.

Thus, $AA^{-1}=A^{-1}A=I$

Inverse of a matrix can also be calculated by the Formula.

$A^{-1} = \dfrac{1}{|A|}$ Adj. A where $|A|$ denotes determinant of A.

Note:

(i)From this relation it is clear that A^{-1} exist if and only if $|A| \neq 0$

i.e. A^{-1} exist if and only if A is non-singular matrix.

(ii)An easy method to find the inverse of the second order matrix:

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times 2}$ then, $A^{-1} = \dfrac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}_{2\times 2}$

i.e. interchange the position of a and d and change signs of b and c and divide by $|A|$.

<u>Properties of Matrix:</u>

[1] Addition of matrices:

Addition of two matrices is possible if the number of row and columns of two matrices are equal**.**

e.g.

$$A=\begin{bmatrix}3 & 2 & 1\\ -5 & 1 & 0\\ 3 & -1 & 4\end{bmatrix} \text{ and } B=\begin{bmatrix}1 & 2 & 0\\ 5 & 1 & 3\\ 3 & 2 & 4\end{bmatrix} \text{ Then } A+B=\begin{bmatrix}4 & 4 & 1\\ 0 & 2 & 3\\ 6 & 1 & 8\end{bmatrix}$$

[2] Multiplication of matrices:

The Multiplication of two matrices A and B is defined as AB. AB exists if number of columns of matrix A is equal to number of rows of matrix B. It is not necessary that AB=BA.

Sometimes BA may not exists.

eg.

$$A=\begin{bmatrix}3 & 2 & 1\\ -5 & 1 & 0\\ 3 & -1 & 4\end{bmatrix} \quad \text{and} \quad B=\begin{bmatrix}1 & 2 & 0\\ 5 & 1 & 3\\ 3 & 2 & 4\end{bmatrix}$$

$$\text{Then } A.B=\begin{bmatrix}3+10+3 & 6+2+2 & 0+6+4\\ -5+5+0 & -10+1+0 & 0+3+0\\ 3-5+12 & 6-1+8 & 0-3+16\end{bmatrix}=\begin{bmatrix}6 & 10 & 10\\ 0 & -9 & 3\\ 10 & 13 & 13\end{bmatrix}$$

2B.4. ILLUSTRATIVE EXAMPLES

Example 1: Find the inverse of the matrix by finding its adjoint.

$$A=\begin{bmatrix}2 & 1 & 3\\ 3 & 1 & 2\\ 1 & 2 & 3\end{bmatrix}$$

Solution: We have,

$$|A|=2\ (3-4)-1\ (9-2)+3\ (6-1)$$

$$=-2-7+15$$

$$\therefore |A|=6$$

Here, $|A|\neq 0$

$\therefore\ A^{-1}$ Exists

Transpose of matrix A = A^T

$$\therefore \quad A^T = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 1 & 2 \\ 3 & 2 & 3 \end{bmatrix}$$

We find co-factors of the elements of A^T (Row-wise)

$$\begin{aligned} & C.F.(2) = -1, \quad C.F.(3) = 3, \quad C.F.(1) = -1 \\ \therefore \quad & C.F.(1) = -7, \quad C.F.(1) = 3, \quad C.F.(2) = -5 \\ & C.F.(3) = 5, \quad C.F.(2) = -3, \quad C.F.(3) = -1 \end{aligned}$$

$$\therefore \text{Adj A} = \begin{bmatrix} -1 & 3 & -1 \\ -7 & 3 & -5 \\ 5 & -3 & -1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|}, \ Adj\ A = \frac{1}{6}\begin{bmatrix} -1 & 3 & -1 \\ -7 & 3 & -5 \\ 5 & -3 & -1 \end{bmatrix}$$

Example 2: Find the inverse of matrix A by Adjoint method, if

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

Solution: Consider

$$\begin{aligned} |A| &= \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} \\ &= 0(-1) - 1(-8) + 2(-5) \\ &= 0 + 8 - 10 \\ &= -2 \end{aligned}$$

Co- factor of the elements of A are as follows

$$C_{11} = (-1)^{1+1} . \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = -1$$

$$C_{12} = (-1)^{1+2}.\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} = 8$$

$$C_{13} = (-1)^{1+3}.\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = -5$$

$$C_{21} = (-1)^{2+1}.\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1$$

$$C_{22} = (-1)^{2+2}.\begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix} = -6$$

$$C_{23} = (-1)^{2+3}.\begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix} = 3$$

$$C_{31} = (-1)^{3+1}.\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -1$$

$$C_{32} = (-1)^{3+2}.\begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} = 2$$

$$C_{33} = (-1)^{3+3}.\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} = -1$$

Thus,

Co-factor of matrix $C = \begin{bmatrix} -1 & 8 & -5 \\ 1 & -6 & 3 \\ -1 & 2 & 1 \end{bmatrix}$

And Adjoint of A= C^T

$$= \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & 1 \end{bmatrix} \Rightarrow A^{-1} = -\frac{1}{2}\begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & 1 \end{bmatrix}$$

Note: - A Rectangular matrix does not possess inverse.

Properties of Inverse of Matrix:-

i) The inverse of a matrix is unique

ii) The inverse of the transpose of a matrix is the transpose of inverse i.e. $(A^T)^{-1} = (A^{-1})^T$

iii) If A & B are two non-singular matrices of the same order $(AB)^{-1} = B^{-1}A^{-1}$

This property is called reversal law.

Definition:-Orthogonal matrix:-

If a square matrix satisfies the relation $AA^T = I$, then the matrix A is called an orthogonal matrix and also

$$A^T = A^{-1}$$

Example 3:

Show that $A = \begin{bmatrix} Cos\theta & Cos\theta \\ Sin\theta & Cos\theta \end{bmatrix}$ is orthogonal matrix.

Solution:

To show that A is orthogonal i.e. To show that $AA^T = I$

$$A = \begin{bmatrix} \text{Cos}\theta & \text{Sin}\theta \\ -\text{Sin}\theta & \text{Cos}\theta \end{bmatrix}$$

$$A^T = \begin{bmatrix} \text{Cos}\theta & -\text{Sin}\theta \\ \text{Sin}\theta & \text{Cos}\theta \end{bmatrix}$$

$$AA^T = \begin{bmatrix} \text{Cos}\theta & \text{Sin}\theta \\ -\text{Sin}\theta & \text{Cos}\theta \end{bmatrix}\begin{bmatrix} \text{Cos}\theta & -\text{Sin}\theta \\ \text{Sin}\theta & \text{Cos}\theta \end{bmatrix}$$

$$= \begin{bmatrix} \text{Cos}^2\theta + \text{Sin}^2\theta & -\text{Cos}\theta\text{Sin}\theta + \text{Sin}\theta\text{Cos}\theta \\ -\text{Sin}\,\theta\text{Cos}\theta + \text{Cos}\theta\text{Sin}\theta & \text{Sin}^2\theta + \text{Cos}^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$\therefore$ A is an orthogonal matrix.

Check Your Progress:

Q. 1) Find the inverse of the following matrices using Adjoint method, if they exist.

i) $\begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$, ii) $\begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}$, iii) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$,

iv) $\begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -5 \\ 2 & 5 & 0 \end{bmatrix}$, v) $\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

vi) $\begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$ vii) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$

Q.2) **I**f A = $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, B = $\begin{bmatrix} 1 & -\tan\frac{\theta}{2} \\ \tan\frac{\theta}{2} & 1 \end{bmatrix}$,

C = $\begin{bmatrix} 1 & \tan\frac{\theta}{2} \\ -\tan\frac{\theta}{2} & 1 \end{bmatrix}$, prove that A= B.$C^{-1}$

Q. 3) **I**f A = $\begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$, prove that Adj. A= A

Q. 4) **I**f A = $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$, verify if $(\text{Adj}A)' = (\text{Adj}A')$

Q.5) **Find the inverse of** A = $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 2 & 2 & 3 \end{bmatrix}$, hence find inverse of

A = $\begin{bmatrix} 3 & 6 & -3 \\ 0 & 3 & -3 \\ 6 & 6 & 9 \end{bmatrix}$

2B.5. RANK OF A MATRIX

a) Minor of a matrix

Let A be any given matrix of order mxn. The determinant of any sub-matrix of a square order is called minor of the matrix A.

We observe that, if 'r' denotes the order of a minor of a matrix of order m x n then $1 \le r \le m$, if m<n and $1 \le r \le n$, if n<m.

e.g. Let

$$A = \begin{bmatrix} 1 & 3 & -1 & 4 \\ 4 & 0 & 1 & 7 \\ 8 & 5 & 4 & -3 \end{bmatrix}$$

From matrix A we get four 3rd Order determinants.

$$\begin{vmatrix} 1 & 3 & -1 \\ 4 & 0 & 1 \\ 8 & 5 & 4 \end{vmatrix}, \begin{vmatrix} 3 & -1 & 4 \\ 0 & 1 & 7 \\ 5 & 4 & -3 \end{vmatrix}, \begin{vmatrix} 1 & -1 & 4 \\ 4 & 1 & 7 \\ 8 & 4 & -3 \end{vmatrix}, \begin{vmatrix} 1 & 3 & 4 \\ 4 & 0 & 7 \\ 8 & 5 & -3 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 3 \\ 4 & 0 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 5 & 4 \end{vmatrix}, \begin{vmatrix} 3 & 4 \\ 0 & 7 \end{vmatrix}, |1|, |0|, |-3|,$$

Are some examples of minors of A.

b) Definition – Rank of a matrix:

A number 'r' is called rank of a matrix of order mxn if there is almost one minor of the matrix which is of order r whose value is non-zero and all the minors of order greater than 'r' will be zero.

e.g.(i) Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 4 & 1 \\ 3 & 5 & 7 \end{bmatrix}$

$$|A_1| = \begin{vmatrix} 1 & 0 \\ 2 & 4 \end{vmatrix} = 4, \quad |A_2| = \begin{vmatrix} 0 & 2 \\ 4 & 1 \end{vmatrix} = -8 \text{ etc.}$$

$$|A| = \begin{vmatrix} 1 & 0 & 2 \\ 2 & 4 & 1 \\ 3 & 5 & 7 \end{vmatrix} = 1(23) + 2(-2) = 19 \neq 0$$

$\therefore$ Rank of A= 3

(ii) $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$

Here,

$$|A| = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{vmatrix} = 1(1) - 1(-1) + 2(-1) = 0$$

Third order determinant is zero.

$\therefore$ rank of A ≤ 2

$$A_1 = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1 \neq 0$$

Thus minor of order 3 is zero and at least one minor of order 2 is non-zero
$\therefore$ Rank of A = 2.

Some results:

(i)Rank of null matrix is always zero.

(ii)Rank of any non-zero matrix is always greater than or equal to 1.

(iii)If A is a non-singular n x n matrix then Rank of A is equal to n and if A is n x n unit matrix then rank of A is equal to n.

(iv)Rank of transpose of matrix A is always equal to rank of A.

(v)Rank of product of two matrices cannot exceed the rank of both of the matrices.

(vi)Rank of a matrix remains unaffected by **elementary transformations**.

Elementary Transformations:

Following changes made in the elements of any matrix are called elementary transactions.

(i) Interchanging any two rows (or columns).

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(ii) Multiplying all the elements of any row (or column) by a non-zero real number.

(iii) Adding non-zero scalar multiples of all the elements of any row (or columns) into the corresponding elements of any another row (or column).

Definition:- Equivalent Matrix:

Two matrices A and B are said to be equivalent if one can be obtained from the other by a sequence of elementary transformations. Two equivalent matrices have the same order & the same rank. It can be denoted by $A \sim B$

[It can be read as A equivalent to B]

Example 4: Determine the rank of the matrix.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$$

Solution:

Given $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$

$R_2 \to R_2 - R_1 \quad \& \quad R_3 \to R_3 - 2R_1$

We get,

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix}$$

Here two columns are identical. Hence 3rd order minor of A vanished

$\therefore \rho(A) \neq 3$

Here, 2nd order minor $\begin{vmatrix} 1 & 3 \\ 0 & -1 \end{vmatrix} = -1 \neq 0$

$\therefore \rho(A) = 2$

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Thus, 3rd order minor is zero and at least one minor of order 2 is non-zero.

Hence the rank of the given matrix is 2.

2B.6. CANONICAL FORM OR NORMAL FORM

If a matrix A of order m x n is reduced to the form $\begin{bmatrix} I_r & o \\ o & o \end{bmatrix}$ using a sequence of elementary transformations then it called canonical or normal form. I_r denotes identity matrix of order 'r'.

Note:- If any given matrix of order m x n can be reduced to the canonical form which includes an identity matrix of order 'r' then the matrix is of rank 'r'.

Example 5: Determine rank of the matrix A if

$$A = \begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 3 & -3 & 1 & 2 \\ 2 & 1 & -3 & 6 \end{bmatrix}$$

$R_2 - 3R_1,\ R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 3 & -6 & -2 & -4 \\ 0 & -1 & -5 & -10 \end{bmatrix}$$

$R_2 - 7R_3$

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$$\sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 33 & 66 \\ 0 & -1 & -5 & -10 \end{bmatrix}$$

$R_1 - R_2,\ R_3 + R_2$

$$\sim \begin{bmatrix} 1 & 0 & -32 & -64 \\ 0 & 1 & 33 & 66 \\ 0 & 0 & 28 & -56 \end{bmatrix}$$

$R_3 \times \frac{1}{28}$

$$\sim \begin{bmatrix} 1 & 0 & -32 & -64 \\ 0 & 1 & 33 & 66 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$R_1 + 32\, R_3,\ \ R_2 - 33\, R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} I_3 & o \end{bmatrix}$$

$\therefore$ Rank of A=3

Example 6: Determine the rank of matrix

$$A = \begin{bmatrix} 1 & 2 & 7 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{bmatrix}$$

$R_2 - 2R_1,\ R_3 - 3R_1$

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$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{bmatrix}$$

$R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$R_1 - 3R_2$

$$\sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$C_2 - 2C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$C_2 \leftrightarrow C_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} I_2 & 0 \end{bmatrix}$$

$\therefore$ Rank of A= 2

Example 7: Determine the rank of matrix A if

$$A = \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

Solution:

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$$A = \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$R_2 - 2R_1,\ R_3 - 3R_1,\ R_4 - 6R_1,$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

$R_2 - R_3$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & -6 & -3 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

$R_1 + R_2,\ R_3 - 4R_2,\ R_4 - 9R_2$

$$\sim \begin{bmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 66 & 44 \end{bmatrix}$$

$R_4 - 2R_3$

$$\sim \begin{bmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 \times \dfrac{1}{11}$

$$\sim \begin{bmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & -6 & -2 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 - C_4$

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$$\square \begin{bmatrix} 1 & 0 & -1 & -7 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 + R_3,\ R_2 + 3R_3$

$$\square \begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_4 - \left(5C_1 + 3C_2 + 2C_2\right)$

$$\square \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \square \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore$ Rank of A= 3

Check Your Progress:-

Reduce the following to normal form and hence find the ranks of the matrices.

i) $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$ ii) $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$ iii) $\begin{bmatrix} -3 & 4 & 6 \\ 5 & -5 & 7 \\ 3 & 1 & -4 \end{bmatrix}$

iv) $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$ v) $\begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$ vi) $\begin{bmatrix} 1 & 2 & 1 & 0 \\ 3 & 2 & 1 & 2 \\ 2 & -1 & 2 & 5 \\ 5 & 6 & 3 & 2 \\ 1 & 3 & -1 & -3 \end{bmatrix}$

2B.7. NORMAL FORM PAQ

If A is any m x n matrix of rank ‘r’ then there exist non-singular matrices P and Q such that, PAQ is in normal form.

$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = PAQ$, where I_r is the unit matrix of order r, hence $\rho(A) = r$

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We observe that, the matrix A can be expressed as

$A = I_m I_n$ ………… (i)

Where I_m and I_n are the identity matrices of order m and n respectively. Applying the elementary transformations on this equation. A in L.H.S. can be reduced to normal form. The equation can be transformable into the equations.

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = PAQ \text{ ………… (ii)}$$

Note that, the row operations can be performed simultaneously on L.H.S. and pre-factor in R.H.S. [i.e. I_m in equation (i)] and column operations can be performed simultaneously on L.H.S. and post factor in R.H.S. [i.e. I_n in equation (i)]

Examples 8: Find the non-singular matrices P and Q such that PAQ is in normal and hence find the rank of A.

i) $$A = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 4 & -1 \\ 1 & 5 & -4 \end{bmatrix}$$

Solution: Consider

$A = I_3 A I_3$

$$\begin{bmatrix} 2 & -1 & 3 \\ 3 & 4 & -1 \\ 1 & 5 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 5 & -4 \\ 3 & 4 & -1 \\ 2 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 - 5C_1, \; C_3 + 4C_1$

$$\begin{bmatrix}1 & 0 & 0\\3 & -11 & -11\\2 & -11 & -11\end{bmatrix} = \begin{bmatrix}0 & 0 & 1\\0 & 1 & 0\\1 & 0 & 0\end{bmatrix} A \begin{bmatrix}1 & 5 & -4\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

$R_2 - R_3$

$$\begin{bmatrix}1 & 0 & 0\\1 & 0 & 0\\2 & -11 & -11\end{bmatrix} = \begin{bmatrix}0 & 0 & 1\\-1 & 1 & 0\\1 & 0 & 0\end{bmatrix} A \begin{bmatrix}1 & -5 & 4\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

$R_2 - R_1,\ R_3 - 2R_1$

$$\begin{bmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & -11 & 11\end{bmatrix} = \begin{bmatrix}0 & 0 & 1\\-1 & 1 & -1\\1 & 0 & -2\end{bmatrix} A \begin{bmatrix}1 & 5 & -4\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

$C_3 + C_2$

$$\begin{bmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & -11 & 0\end{bmatrix} = \begin{bmatrix}0 & 0 & 1\\-1 & 1 & -1\\1 & 0 & -2\end{bmatrix} A \begin{bmatrix}1 & -5 & -1\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix}$$

$R_3 \times \frac{1}{11}$

$$\begin{bmatrix}1 & 0 & 0\\0 & 0 & 0\\0 & 1 & 0\end{bmatrix} = \begin{bmatrix}0 & 0 & 1\\-1 & 1 & -1\\1/11 & 0 & -2/11\end{bmatrix} A \begin{bmatrix}1 & -5 & -1\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix} = \begin{bmatrix}0 & 0 & 1\\-1/11 & 0 & 2/11\\-1 & 1 & -1\end{bmatrix} A \begin{bmatrix}1 & -5 & -1\\0 & 1 & 1\\0 & 0 & 1\end{bmatrix}$$

Thus,

$$P = \begin{bmatrix}0 & 0 & 1\\-1/11 & 0 & 2/11\\-1 & 1 & -1\end{bmatrix} \text{ and } |P| = \begin{vmatrix}0 & 0 & 1\\-1/11 & 0 & 2/11\\-1 & 1 & -1\end{vmatrix} = \frac{-1}{11}$$

$$Q = \begin{bmatrix} 1 & -5 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad |Q| = \begin{vmatrix} 1 & -5 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

P and Q are non-singular matrices. Also Rank of A = 2

ii) $$A = \begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

Solutions:
Consider:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 3 & -6 & -2 & -4 \\ 2 & -1 & -5 & -10 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_2 - C_1, \ C_3 - C_1, \ C_4 - 2C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & -6 & -2 & -4 \\ 2 & -1 & -5 & -10 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 - 3R_1, \ R_3 - 2R_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -6 & -2 & -4 \\ 0 & -1 & -5 & -10 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 - 6R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 28 & 56 \\ 0 & -1 & -5 & -10 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 6 & 1 & 9 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_4 - 2C_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 28 & 0 \\ 0 & -1 & -5 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -6 & 1 & 9 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 - 5C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 28 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -6 & 1 & 9 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \times \frac{1}{28}, \; R_3 \times (-1)$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ \frac{3}{14} & \frac{1}{28} & \frac{9}{28} \\ -1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 2 \\ \frac{3}{14} & \frac{1}{28} & \frac{9}{28} \end{bmatrix} A \begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore [I_3 \quad 0] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ \frac{3}{14} & \frac{1}{28} & \frac{9}{28} \end{bmatrix} A \begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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$$\therefore P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ \frac{3}{14} & \frac{1}{28} & \frac{9}{28} \end{bmatrix}, \ |P| = \frac{1}{28}$$

$$Q = \begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, |Q| = 1$$

$\therefore$ P & Q are non- singular. Also, Rank of A = 3.

Check Your Progress:

A) Find the non-singular matrices P and Q such that PAQ is in normal form and hence find rank of matrix A.

i) $\begin{bmatrix} 1 & 0 & -2 \\ 2 & 3 & -4 \\ 3 & 3 & -6 \end{bmatrix}$ ii) $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$ iii) $\begin{bmatrix} 3 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$

iv) $\begin{bmatrix} 2 & 3 & 4 & 7 \\ -3 & 4 & 7 & -9 \\ 5 & 4 & 6 & -5 \end{bmatrix}$ v) $\begin{bmatrix} 1 & 3 & 5 & 7 \\ 4 & 6 & 8 & 10 \\ 15 & 27 & 39 & 51 \\ 6 & 12 & 18 & 24 \end{bmatrix}$

2B.8. LET US SUM UP

- Definition of matrix & its types of matrices.
- Using Adjoint method to find the A^{-1} by using formula $A^{-1} = \frac{1}{|A|} adjA$
- Rank of the matrix using row & column transformation
- Using canonical & normal form to find Rank of matrix.

2B.9. UNIT END EXERCISE

i)Find the inverse of matrix $A=\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix}$ if exists.

ii) Find Adjoint of Matrix $A=\begin{bmatrix}1 & 1 & -1\\0 & 2 & 1\\2 & -1 & 1\end{bmatrix}$

iii)Find the inverse of A by adjoint method if $A=\begin{bmatrix}1 & 0 & 2 & 1\\1 & 1 & 0 & 1\\-1 & 0 & 1 & 2\\2 & 3 & 1 & 0\end{bmatrix}$

iv) Find Rank of matrix $A=\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\\7 & 8 & 9\end{bmatrix}$

v) Prove that the matrix $A=\begin{bmatrix}Cos\theta & -Sin\theta & 0\\Sin\theta & Cos\theta & 0\\0 & 0 & 1\end{bmatrix}$ is orthogonal.

Also find A^{-1}.

vi) Reduce the matrix $A=\begin{bmatrix}0 & 1 & -3 & -1\\1 & 0 & 1 & 1\\3 & 1 & 0 & 2\\1 & 1 & -2 & 0\end{bmatrix}$ to the normal form θ and find its rank.

vii) Find the non- singular matrix ρ *and* α such that $\rho\ A\ \alpha$ is the normal form when $A=\begin{bmatrix}1 & 1 & 1\\1 & -1 & -1\\3 & 1 & 1\end{bmatrix}$ .Also find the rank of matrix A.

viii) Under what condition the rank of the matrix will be 3?

$$A = \begin{bmatrix} 2 & 4 & 2 \\ 2 & 1 & 2 \\ 1 & 0 & \lambda \end{bmatrix}$$

ix)If $X = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$ & $Y = \begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix}$

Then show that $\rho(\text{xy}) \neq \rho(\text{yx})$ where ρ denotes Rank.

x) Find the rank of matrix $A = \begin{bmatrix} 8 & 3 & 6 & 1 \\ -1 & 6 & 4 & 2 \\ 7 & 9 & 10 & 3 \\ 15 & 12 & 16 & 4 \end{bmatrix}$

♦♦♦♦

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3A

SIMPLE INTEREST
&
COMPOUND INTEREST

UNIT STRUCTURE

3A.1. INTRODUCTION

Suppose we invest money in the bank for a specific period. At the end of the period, the bank not only returns our money, invested, but in addition will give some 'extra money' for using our money which was kept with them. That extra money we earn is called interest.

Many times we borrow some money from our friends or relatives for personal needs. A large amount may be needed which can be taken from banks, financial institutions etc. as a loan. In this case we have to pay extra money while repaying the loan. That extra money paid for making use of their money is called Interest.

Some useful terms:

(i) **Principal (P):** The amount or the sum of money which is borrowed, invested in a bank or landed is called the principal (P).

(ii) **Interest (I):** The 'extra money' paid in addition to the principal, is called Interest (I).

(iii) **Amount (A):** The total money, including Principal and interest is called Amount (A).

(iv) **Period ('n' or 't'):** The specified pre-decided period is called the period of investment, it is expressed in 'years' and it is denoted by 'n' or 't'.

(v) **Rate of Interest:** The interest to be paid by the concerned party is calculated as a 'Percentage' of the Principal for specific time period, at the pre-decided yearly rate. This rate is called the Rate of Interest per year. It is usually expressed as a percentage of the principal.

- The Interest is calculated in two ways as Simple Interest and Compound Interest.

***Simple Interest: (I)**

If the interest is charged or calculated on the principal, then it is called simple interest.

The simple interest I on the principal P at the rate of r %, for the period n (years) is given by

$I = \frac{P \times n \times r}{100}$ OR $I = P \times n \times i$ Where $i = \frac{r}{100}$

***Amount (A)** at the end of n (years) is given by

$$A = P + I$$
$$= P + \frac{pnr}{100}$$
$$= P\left(1 + \frac{nr}{100}\right)$$
$$= P(1 + in)$$
$$\therefore \boxed{A = P(1 + i\,n)}$$

Examples:

(1) If Rs.5, 000 is invested at 5% per annum. Find the amount after (1) One year (ii) Five years (iii) 6 months (half yearly) (iv) 4 months.

Sol^n:

If A = Amount, P = Principal, r = rate of interest, $i = \frac{r}{100}$,

n = time period in years. Then $A = P(1 + in)$

Here P = 5000, r = 5% $\therefore$ i = 0.05

(i) After one year, n = 1

A = P (1 + in) = 5000(1 + (0.05) × 1)
= 5000(1 + 0.05)
= 5000(1.05)
= 5250

(ii) After five years, n = 5

A = P (1 + in) = 5000(1 + (0.05) × 5)

$= 5000(1 + 0.25)$
$= 5000(1.25)$
$= 6250$

(iii) After six months, $n = \frac{6}{12} = \frac{1}{2}$ years

A = P (1 + in) $= 5000(1 + (0.05)(1/2))$
$= 5000(1 + 0.025)$
$= 5125$

(iv) After 4 months, $n = \frac{4}{12} = \frac{1}{3}$ years

A = P (1 + i n) $= 5000(1 + (0.05)(1/3))$
$= 5000\left(1 + \frac{0.05}{3}\right)$
$= 5000(1.0167)$
$= 5835$

(2) Find the simple interest of Rs.2000 for 5 years at 6% per annum. Also find amount after 5 years.

Sol^n:

Given: P = Rs.2000, n = 5 years, r = 6% p.a. i.e. i = 0.06

$\therefore$ Simple Interest $I = P \times n \times i$

$= 2000 \times 5 \times 0.06$

$= Rs.600$

$\therefore$ Amount after 5 years = A = P + I

= 2000 + 600

= Rs.2600

Hence the simple Interest is Rs.600 and the amount after 5 years is Rs.2600.

(3) At what simple interest rate will Rs.6, 000 get Rs.1, 080 as simple interest in 3 years.

Sol^n:

Given P = Rs.6000, n = 3 years, I = Rs.1080

I = $P \times n \times$ i

$\therefore$ 1080 = 6000 × 3 × i

$\therefore$ 1080 = 18000 i

$\therefore i = \frac{1080}{18000} = \frac{6}{100} = 0.06$

$\therefore$ The simple interest rate is 6% per annum.

(4) What sum of money will amount to Rs.6050 in 3 years at 7% p.a. simple interest?

Sol^n:

Given A = Rs.6050, n = 3, r = 7%

$\therefore\ i = \frac{7}{100} = 0.07$, P = (?)

$A = P\,(1 + n\,i)$

$\therefore\ 6050 = P\,(1 + 3 \times 0.07)$

$\therefore\ 6050 = P\,(1 + 0.21)$

$\therefore\ 6050 = P\,(1.21)$

$\therefore\ P = \frac{6050}{1.21} = 5000$

$\therefore$ Rs.5000 is the required sum of money.

(5) In how many years will Rs.3, 500 amount to Rs.4, 200 at 5% p.a. simple Interest?

Sol^n:

Given: P = 3,500, A = Rs.4, 200, r = 5%p.a. $\Rightarrow i = \frac{5}{100} = 0.05$, n = (?)

$A = P\,(1 + ni)$

$\therefore 4200 = 3500(1 + 0.05 \times n)$

$\therefore\ \frac{4200}{3500} = (1 + 0.05n)$

$\therefore\ 1.2 = 1 + 0.05n$

$\therefore\ 0.05n = 1.2 - 1 = 0.2$

$\therefore 0.05n\ =\ 0.2$

$\therefore n = \frac{0.2}{0.05} = 4$

$\therefore\ \boxed{n = 4}$

$\therefore$ The time required is 4 years.

(6) A sum of money amounts to Rs.6, 600 in 2 years and Rs.7, 200 in 4 years. Find the sum and the rate of simple interest.

Sol^n:

Let P = Principal, r = Rate of S.I. Per annum

We have $A = P + I$

$\therefore A = P + \frac{pnr}{100}$

$$\therefore 6600 = P + \frac{p \times 2 \times r}{100}$$

$$\therefore 6600 = P + \frac{2pr}{100} \quad \text{....(1)}$$

$$\therefore 7200 = P + \frac{p \times 4 \times r}{100}$$

$$\therefore 7200 = P + \frac{4pr}{100} \quad \text{.... (2)}$$

Subtracting (1) from (2)

$$\therefore 7200 - 6600 = p - p + \frac{4pr}{100} - \frac{2pr}{100}$$

$$\therefore \frac{2pr}{100} = 600$$

Substituting $\frac{2pr}{100}$ in eq^n. (1)

$$\therefore 6600 = P + 600$$

$$\therefore \boxed{P = Rs.6000}$$

Also $\frac{2pr}{100} = 600$

$$\therefore 2pr = 60000$$

$$\therefore \text{Pr} = 30000$$

Substitute P = 6000,

We get 6000 r = 30,000

$$\therefore r = \frac{30000}{6000} = 5$$

$\therefore$ The rate of simple interest is 5% p.a.

(7) Mr. Amit lent Rs.17, 000 for 3 years and Rs.12, 000 for 4 years at the same rate of simple interest. Find the rate if the total interest received was Rs.9900.

Sol^n:

Let r be the common rate of simple interest percent per annum.

For the first loan, P = Rs.17000, n = 3 year

$$I = p \times n \times i$$
$$= 17000 \times 3 \times i$$
$$= 51000i$$

For the second loan, P = Rs.12000, n = 4 years

$$I = p \times n \times i$$
$$= 12000 \times 4 \times i$$
$$= 48000i$$

$\therefore$ Total interest = 51000 i + 48000 i = 99000i

But this is given to be Rs.9900

$\therefore$ 99000 i = 9900

$$\therefore i = \frac{9900}{99000} = 0.1$$

$\therefore$ r = 10%

$\therefore$ The common rate of S.I. is 10% p.a.

(8) Hina and Mita borrowed Rs.8000 and Rs.15, 000 respectively at the same rate Simple Interest. After 3 years Hina repayed the loan by giving Rs.10160. How much amount should Mita pay after $4\frac{1}{2}$ years, to pay off the loan, including simple interest?

Sol^n:

Let r be the rate of interest p.a.

For Hina:

$$S.I._1 = \frac{p \times n \times r}{100}$$

$$= \frac{8000 \times 3 \times r}{100}$$

$$= 240r$$

Now the S.I., she paid = 10160 – 8000 = 2160

$\therefore$ 2160 = 240r $\therefore$ r = 9

For Mita:

$$S.I._2 = \frac{p \times n \times r}{100}$$

$$= \frac{15000 \times 4.5 \times 9}{100}$$

$$= 6075$$

The total amount = 15, 000 + 6075

= Rs.21, 075

$\therefore$ Mita should pay Rs.21, 075 after $4\frac{1}{2}$ years to pay off the loan, including Simple Interest.

*<u>Compound Interest (C. I.):</u>

If periodically the interest due is added to the principal and the interest for the next period is calculated on this addition, then it is called as compound interest. Since the compound interest is the interest on interest over a period of time, it depends on the frequency of the interest redeemed.

e.g. If Rs.50,000 you deposit in a bank for 2 years at 7% p.a. compounded annually. The interest will be calculated in the following way.

Interest for the First Year:

$I = P \times i \times n = 50,000 \times 0.07 \times 1 = \text{Rs.}3,500$

Interest for the Second year.

Here for calculating interest for the second year principal would not be the initial deposit, but Principal for calculating second year will be initial deposit plus interest for the first year. Therefore, principal for calculating second year interest would be

Rs.50000 + Rs.3500 = Rs. 53, 500

∴ Interest for the second year = 53500 × 0.07 × 1

= Rs. 3, 745

∴ Total interest = Interest for first year + Interest for 2^{nd} year.

= Rs.3500 + Rs.3745

= Rs.7, 245

At the same time the simple interest for 2 years of Rs.5000 at 7% p.a. is

S.I. = P × n × i

= 50000 × 0.07 × 2

= Rs.7, 000

So compound interest for a principal is more than simple interest on the same amount for the same period.

So we can summarize the main difference between simple interest and compound interest is that in simple interest the principal remains constant throughout whereas in the case of compound interest principal goes on changing at the end of specified period.

The Formula for amount A is

$A = P\left(1 + \frac{r}{100}\right)^n = P(1+i)^n$, Where $\frac{r}{100} = i$, A = Amount.

P = Principal, r = rate of interest per period, n = period of time

The formula for compound interest is

$$\begin{aligned} CI &= A - P \\ &= P(1+i)^n - P \\ &= P\left[(1+i)^n - 1\right] \end{aligned}$$

Note: In case of compound interest calculations, it is easier to first calculate the amount A.

- The compound interest for k^{th} period is calculated as follows:

Interest for k^{th} period = $P(1+i)^{k-1}(i)$

The interest can be compounded yearly, half-yearly quarterly or, monthly, then amount A at the end of n years is given by

$$A = \left(1 + \frac{i}{m}\right)^{mn}$$

Where $i = \frac{r}{100}$ rate of interest p.a.

n = number of years

m = number of times the interest is compounded per year.

e.g. m = 2 if interest is compound half yearly

m = 4 if interest is compound quarterly

m = 12 if interest is compound monthly.

Examples:

(1) Find the compound interest and the amount after 3 years on a Principal of Rs.15000 at 10% p.a.

Sol^n: Given: P=Rs.15,000,n=3 years, r = 10% $\Rightarrow i = \frac{10}{100} = 0.10$

A= P (1 + i)^n

= 15000 (1 + 0.10)^3

= 15000 (1.10)^3

= 15000 (1.331)

= 19965

C.I. = A – P

= 19965 – 15000

= Rs.4965

Thus the compound interest is Rs.4965 and the amount after 3 years is Rs.19, 965.

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(2) What sum of money will amount to Rs.40, 31, 078.40 in 3 years at 8% p.a. compound interest?

Sol^n:

Given: n = 3 years, r = 8% = i = 0.08, A = 40, 31, 078.40, P = (?)

$$A = P\,(1 + i)^n$$

$\therefore$ 40, 31,078.40 = P (1+0.08)^3

$\therefore$ 40, 31, 078.40 = P (1.08)^3

$\therefore$ 40, 31, 078.40 = P (1.259712)

$$\therefore\ P = \frac{40,31,078.40}{1.259712} = 32,00,000$$

The required sum is Rs.32, 00, 000

(3) At what rate of compound interest would an amount double itself in 4 years? (Given: $2^{\frac{1}{3}}$=1.2611, $2^{\frac{1}{3}}$=1.1892)

Sol^n:

Given that A = 2P & n = 4 years

$$A = P\left(1+i\right)^n$$

$$\therefore 2p = p\left(1+i\right)^4$$

$$\therefore 2 = (1+i)^4$$

$$\therefore 2^{\frac{1}{4}} = 1+i$$

$$\therefore 1.892 - 1 = i$$

$$\therefore i = 0.1892$$

$$\therefore\ r = 18.92\%$$

$\therefore$ The required rate of compound interest is 18.92%

(4) The bank offer fixed deposits for 4 years. Under the following schemes.

(i) At 12%, if the interest compounded annually.

(ii) At 11% if the interest compounded half-yearly?

State which scheme is more beneficial to the public?

n = 4 years for both schemes.

Sol^n:

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Suppose P = 100 then

(i) For the interest compounded annually,

P = 100, n = 4, r = 12% i.e. i = 0.12

$\therefore$ A = P $(1 + i)^n$

$= 100\,(1 + 0.12)^4$

$= 100(1.12)^4$

= 100 (1.573519)

= 157.3519

$\therefore$ Interest on first scheme = A – P

= 157.3519 – 100 = 57.3519

(ii) For the interest compounded half yearly i.e. m = 2

i = 0.11, n = 4 years

$$\therefore A = P\left(1+\frac{i}{2}\right)^{2n}$$

$$= 100\left(1+\frac{0.11}{2}\right)^{2\times 4}$$

$$= 100(1+0.55)^{8}$$

$$= 100(1.055)^{8}$$

$$= 100(1.5346865)$$

$$= 153.46865$$

$\therefore$ Interest on second scheme = A – P

= 153.46865 – 100

= 53.46865

$\therefore$ Since interest on first scheme is more than the interest on second scheme, therefore the first scheme is more beneficial to the public.

(5) A man borrowed a certain amount for 2 years from his friend at 3% and had to pay a simple interest of Rs.120. He once again took a loan of the same amount for 4 years from a bank at 15% interest, compounded quarterly. Find the interest he will have to pay to the bank.

Sol^n:

The problem is in two parts.

(i) First find the <u>principal P</u> by using simple interest formula for given S.I = 120, n = 2 & r = 3% i.e. i = 0.03

S.I. = P× n × i, Where S.I. = 120, n = 2 & i = 0.03

120 = P × 2 × 0.03

$\therefore$ 120 = 0.06P

$$\therefore P = \frac{120}{0.06} = \frac{12000}{6} = 2000$$

$$\therefore \boxed{P = 2000}$$

(ii)The loan of same amount P=2000 (P we found in (i)),

n=4years, r=15% $\therefore$ i = 0.15 and compounded quarterly.

By using formula of A & C.I. find the final answer.

P = 2000, i = 0.15, n = 4, m = 4

$$A = p\left(1+\frac{i}{m}\right)^{mn}$$

$$= 2000\left(1+\frac{0.15}{4}\right)^{4\times 4}$$

= 2000(1 + 0.0375)16

= 2000(1.0375)16

= 2000(1.8022278)

= 3604.4556

& C. I = A – P

= 3604.4556 – 2000

= 1604.4556

- **Nominal and Effective rate of Interest:**

Suppose the interest is compounded in times a year

Let i = nominal rate (stated rate) of interest per Re.1 per year

i_e = effective rate of interest per Re.1 per year

P = Re. 1

N = 1 year

Then Accumulated value A after 1 year

(i) Using the nominal rate of interest $A = \left(1+\frac{p}{m}\right)^{m}$

(ii) Using the effective rate of interest $A = (1+i_e)^1 = 1+i_e$

Equating (i) & (ii)

$$1+i_e = \left(1+\frac{i}{m}\right)^m$$

i.e. $\boxed{i_e = \left(1+\frac{i}{m}\right)^m - 1}$

Ex.: Find the effective rate equivalent to the nominal rate 16% p.a. when compounded (i) half yearly (ii) quarterly

Sol^n: $i = \frac{16}{100} = 0.16$

(i) Interest compounded half years, m = 2

$$\therefore i_e = \left(1+\frac{i}{m}\right)^m - 1$$

$$= \left(1+\frac{0.16}{2}\right)^2 - 1$$

$$= (1+0.08)^2 - 1$$

$$= 1.1664 - 1$$

$$= 0.1664$$

∴ Effective rate of interest per Re.1 per year is 0.1664

∴ Effective rate of interest percent per year

= 100 × 0.1664 = 16.64%

(ii) Interest is compounded quarterly, m = 4

$$\therefore i_e = \left(1+\frac{i}{m}\right)^m - 1$$

$$= \left(1+\frac{0.16}{4}\right)^4 - 1$$

$$= (1+0.04)^4 - 1$$

$$= (1.04)^4 - 1$$

$$= 1.16985856 - 1$$

$$= 0.16985856$$

∴ Effective rate of interest percent per year is

= 100 × 0.16985856 = 16.98585690

Ex: Which rate yields more interest?

5.8% compounded half-yearly or 6% compounded quarterly.

Sol^n:

(i) The effective rate of 5.8% compound half-yearly

i = 0.058 and m = 2

$$i_e = \left(1+\frac{i}{m}\right)^m - 1$$

$$= \left(1+\frac{0.0.058}{2}\right)^2 - 1$$

$$= (1+0.029)^2 - 1$$

= 1.058841 – 1 = 0.058841

Effective rate 5.8841% p.a.

(ii) The effective rate of 6% compounded quarterly i = 0.06 and m = 4

$$i_e = \left(1+\frac{i}{m}\right)^m - 1$$

$$= \left(1+\frac{0.06}{4}\right)^4 - 1$$

$$= (1+0.015)^4 - 1$$

$$= 1.06136355 - 1$$

$\therefore$ i_e = 0.06136355

$\therefore$ Effective rate 6.13 6355% p.a.

6.136355 > 5.8841

Hence 6% compounded quarterly yields more interest than 5.8% compounded half-yearly.

* **Future Value**

An amount (Accumulated Amount) of a sum of money including the interest amount after specified period at a given rate of interest is called Future Value.

If principal P is kept in a fixed deposit for n years 1% rate of interest, compounded annually then Future Value is calculated by using the formula.

$$F.V = A = P(1+i)^n$$

i.e. sum due = Principal $(1+i)^n$

Ex: Find the future value of Rs.24, 500 kept as a fixed deposit, after years at 7% p.a. compounded annually.

Sol^n: P = Rs.24, 500, n = 7 years, r = 7% i = 0.07, A = (I)

A = P $(1+i)^n$

= $24500(1+0.07)^7$

= $24500(1.07)^7$

= 24,500(1.60578)

= Rs.39341.65

Ex: Mr. Mehta was approached by a person with two schemes, as he wanted to invest Rs.1, 20, 000. In Schemes A, the period was 8 years with 9% rate p.a. compounded annually. In schemes, the period was 10 years with 8% compounded interest p.a. Advise him about the choice of scheme w.r.t. the amount to be received.

Sol^n:

Scheme A:

P = 1, 20, 000, n = 8 years, r = 9% $\Rightarrow$ i = 0.09, A =?

A = P $(1 + i)^n$

= $(1, 20, 000)\ (1+0.09)^8$

= $1, 20, 000(1.09)^8$

= 1, 20, 000(1.9925626.4168)

= 239107.52

Scheme B:

P = 1, 20, 000,n = 10 years, r = 8% $\Rightarrow$ i = 0.08

A = P $(1 + i)^n$

=$1, 20, 000(1 + 0.08)^{10}$

=$1, 20, 000\ (1.08)^{10}$

=1, 20, 000(2.15892499725)

=259070.9 = 259071

In scheme B Mr. Mehta received more amount, so choice is scheme B.

* **Present Value:**

The present value concept is useful when we wish to target for an amount A after n years and wish to know what amount p should be invested presently to achieve the target. Then A, the amount is called sum due and P is called its present value (Present worth) or discounted value.

* **Discounting:**

The process of finding the present value of a sum due is called discounting.

$$\text{Present Value} = \frac{\text{Sum due}}{(1+i)^n}$$

$$\text{i.e. } P.V. = \frac{A}{(1+i)^n}$$

Ex: Find the Present Value of Rs.14, 641 at 10% rate of interest, payable 4 years from now.

Solu^n:

Here A = Rs.14641, r = 10% $\Rightarrow$ i = 0.10, n = 4 year

$$P.V. = \frac{A}{(1+i)^n}$$

$$= \frac{14641}{(1+0.10)^4}$$

$$= \frac{14641}{(1.10)^4}$$

$$= \frac{14641}{1.4641}$$

= 10,000

$\therefore$ Present value is Rs.10, 000

Ex: Sohail promised to pay Amir Rs.15, 000 after 3 years with compound rate of interest 8% p.a. He also promised to pay Aakash Rs.20, 000 after 4 years with compound rate of interest 9% p.a. Find the present worths of these payments. Also find the total present worth of the money Sohail has to pay.

Solu^n:

P.V. of Payment of Amir:

$A = 15000, n = 3, r = 8\% \therefore i = 0.08$

$$P.V. = \frac{A}{(1+i)^n}$$

$$= \frac{15000}{(1+0.08)^3}$$

$$= \frac{15000}{(1.08)^3}$$

$$= \frac{15000}{1.259712}$$

$\therefore$ P.V. = $Rs.10,907.484$

P.V. of Payment of Aakash:

$A = 20000, n = 4, r = 9\% \therefore i = 0.09$

$$P.V. = \frac{A}{(1+i)^n}$$

$$= \frac{20000}{(1+0.09)^4}$$

$$= \frac{20000}{1.41158161}$$

$\therefore$ P.V. = $Rs.14,168.504$

Total present worth of the money Sohil has to pay

= 11,907.484 + 14,168.504=Rs.26, 075.988

Ex: Mr. XYZ has to pay an institution Rs.16, 800 after 3 years. He offers to pay the institution now at the present value at interest compounded 8%p.a. What amount should he pay now?

Sol^n:

Here A = 16800, i = 0.08, n = 3

$$P.V. = \frac{A}{(1+1)^n}$$

$$= \frac{16800}{(1.08)^3}$$

$$= \frac{16800}{1.259712}$$

$$= 13336.3816$$

∴ Mr. XYZ should pay Rs. 13336.3816 now.

Ex: Mr. PQR has to pay an institution Rs.13336.38 now.

If he agrees to pay a lump sum after 3 years with interest compounded at 8% p.a. What is the amount that he will have to pay?

Sol^n:

P = 13336.38, n = 3, i = 0.08

$\therefore$ F.V. $= P(1+i)^n$

$= 13336.38(1+0.08)^3$

$= 13336.38(1.08)^3$

$= 13336.38(1.259712)$

$= 16799.9979 = \text{Rs.}16,800$

Mr. PQR will have to pay Rs.16, 800.

Ex.: Mr. Das has to pay an institution Rs.10, 000 at the end of 2 years and Rs.6, 000 at the end of 3 years from now. If he opts for paying a lump sum at the end of 3 years, what will be the future value at that time at interest compounded 8% p.a.?

Sol^n:

The amount of the payment of Rs.6000 paid at the end of 3 years is Rs.6000 itself.

The amount of the payment of Rs.10, 000 at the end of 2 years. Its amount at the end of 3 years is

$A = P(1+i)^n$

$\therefore A = 10000(1.08)^1$

$= 10000 \times 1.08$

$= 10,800$

∴ The total future value of all payments at the end of 3 years

$= 6000 + 10,800$

$= \text{Rs.}16,800$

Mr. Das has to pay Rs.16, 800 at the end of 3 years.

Ex.: Mr. Patel has to pay an institution Rs.10, 000 at the end of 2 years and Rs.6000 at the end of 3 years from now. If he decides to settle the payments now, what is the present value at interest compounded 8% p.a.?

Sol^n:

1st Payment:

Here A = 10000, n = 2 & i = 0.08

$$P.V. = \frac{A}{(1+i)^n}$$

$$= \frac{10000}{(1.08)^2}$$

$$= \frac{10000}{(1.08)^2}$$

$$= \frac{10000}{1.1664}$$

= 8573.3882

2nd Payment:

A=6000, n=3, i=0.08

$$P.V. = \frac{A}{(1+1)^n}$$

$$= \frac{6000}{(1.08)^3}$$

$$= \frac{6000}{(1.08)^3}$$

$$= \frac{6000}{1.259712}$$

= 4762.99344

∴ Total present value of all payments taken together

= 8573.3882 + 4762.99344

= 13336.38164

∴ Present value is Rs.13336.38164 at interest compounded 8% p.a.

3A.2. UNIT END EXERCISE:

(1) At what rate will the simple interest on Rs. 15,000 for 4 years are equal to the simple interest on Rs. 16,000 for 3 years at 10% p.a.?

(2) A principal amounts to Rs.9, 680 after 3 years and to Rs.10, 800 after 5 years. Find the principal and rate of interest.

(3) Anita and Amisha borrowed Rs.8, 000 and Rs.15, 000 respectively at the same rate of simple interest. After 3 years Anita repaid the loan by giving Rs.10, 160. How much amount should Amisha pay after four and half years, to pay off the loan, including simple interest?

(4) **T**he simple interest at 20% p.a. on a certain sum of money for 4 years is Rs. 25,600. Find compound interest on the sum at the same rate for the same period?

(5) Mr. XYZ wants to purchase smart phone after 4 years which will cost him Rs. 25,000. How much money he should invest in bank at present so as to receive Rs. 25,000? If the bank is giving 12% per year rate of compound interest.

(6) A particular sum of money amounts to Rs.7, 69,824 in 2 years and Rs. 8, 31,409.92 in 3 years. Find the sum and compound interest rate.

(7) On what sum of money will be the difference between the simple interest and compound interest for 2 years at 4% p.a. be Rs. 56?

(**8)**The simple interest and the compound interest on a sum of money at a certain rate for 2 years is Rs. 1260 and Rs.1323 respectively. Find the sum and the rate.

(9)A sum of Rs. 6,55,000 is invested in a fixed deposit giving 10% p.a. compound interest. Find the interest in 4^{th} year.

(10) Find the maturity amount of a 2 year fixed deposit of Rs. 10,000 at 10% p.a. if the interest is compounded semi-annually.

(**11**)Find the effective rate equivalent to the nominal rate 16%p.a. when Compounded(i)half yearly(ii)quarterly

(**12)**Which rate yields more interest :5.8% compounded half-yearly or 6% compounded quarterly?

(13) Find the future value of Rs. 20,000 after 4 years if the compound interest rate is 10%.

(**14)**Find the present value of Rs. 35,730.48 to be paid three years from now with the rate of compounding at 6% p.a.

(15) Mr. ABC estimates that after 3 years he would requires 50, 00,000 for his new business. He wishes to put aside money now, invested in an instrument giving interest 7% p.a. compounded half yearly to meet his requirement then. How much money should he invest presently?

(**16)**A person is supposed to pay a bank Rs. 5000, Rs. 6000 and Rs. 7000 at the end of 1,2and 3 years respectively. He offers to settle the payment now itself. How much will he have to pay now, with rate of compounding at 12% p.a.?

(17) A person is supposed to pay a company Rs. 5000, Rs. 6000 and Rs. 7000 at the end of 1,2and 3 years respectively. He asks the company if he can settle the payment by directly by paying a lump sum at the end of 3 years. The company puts a condition that he should pay a compound interest at 12% p.a. What amount will he have to pay at the end of 3 years?

♦♦♦♦

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3B
Annuity

UNIT STRUCTURE

3B.1. THE CONCEPT OF ANNUITY:

In real life, we all must have gone through with different situations where we do not have enough amount of money but still we want to buy things for our use. At that time we borrow the amount and to repay back we fix some time in which we return fixed amount at regular intervals.
These equal amounts being returned at specific periods and introduces the concept of annuity.

Pension honoured by the employer to the retired personnel, instalment to the loans, insurance premiums, salary of the employees distributed by the company, etc. are few examples of annuity.

*** <u>Characteristics of Annuity:</u>**

- Annuity is a series of payments.
- Annuity paid is of equal amounts and is fixed.
- Annuity is paid at equal interval of time. It could be either annually, semi-annually, quarterly, monthly etc.
- Annuity can be paid either at the beginning or at the end of each payment period.
- Annuity payments are periodic in nature.

***<u>Annuity</u>**: The series of payments made at successive interval of time is called an Annuity.

3B.2. TYPES OF ANNUITY:

***<u>Uniform Annuity:</u>**

If the payments of an annuity are of equal amounts for equal interval of time and continue for entire term period, it is called uniform annuity. E.g. Loan instalments.

***Varying Annuity:**

If the amount of payments of an annuity are unequal and non-uniform. Though the payments are made at regular intervals but their amounts are not same is also known as non-uniform or varying annuity. E.g. Dividends on bonds, mutual funds.

***Annuity Certain**:

If the payments are made or received after a fixed period of time, it is known as Annuity Certain. Any form of loan instalments or advancements, bank recurring are examples of annuity certain.

***Contingent Annuity**:

An annuity with number of payments depending on happening an event is Contingency Annuity i.e. Annuity that does not begin making payments to the annuitant or the beneficiary until a certain stated event occurs.E.g. Annuities that do not begin payments until an individual's **retirement** or death.

***Perpetual Annuity (perpetuity):** An annuity supposed to go on perpetually or endlessly is a Perpetual Annuity or a Perpetuity. i.e. The payments continues forever. E.g. Pension, Cap rate in real estate and dividend stream on shares.

***Immediate Annuity (Ordinary annuity)**: If the annuities are paid at the end of each period, it is known as an Immediate Annuity. It is also known as an Ordinary Annuity. E.g. repayment of loan instalments.

***Annuity Due**: If Payments are paid at the beginning of each period, it is called an Annuity Due. E.g. payment of life insurance policies.

***Deferred Annuity**:

If the periodic payments are not made in the beginning of some time periods and thereafter continues periodically is called Deferred Annuity. The period during which payments are not made are known as period of deferment. E.g. In general repayment of home loan instalments begins after the loan has been disbursed.

***Accumulated Values (or Future value)**(Immediate Annuity):

Let each annuity be of Rs. C, rate of interest per unit per annum be i. Payments are made at the end of each period, then amount or accumulated value A

$$\text{Accumulated Value} = F.V. = \frac{C}{i}\left[(1+i)^n - 1\right]$$

*__Present Value__ (Immediate Annuity):

Let C-an annuity, n-no. of time periods, i-rate of interest per unit p.a.

$$P.V. = \frac{C}{i}\left[1-(1+i)^{-n}\right]$$

*__Accumulated Values (or Future value)__ (Annuity due):

Let each annuity be of Rs. C, rate of interest per unit per annum be i. Payments are made at the beginning of each period, then amount or accumulated value A

$$\text{Accumulated Value} = \frac{C(1+i)}{i}\left[(1+i)^n - 1\right]$$

*__Present Value__ (Annuity due):

Let C-an annuity, n-no. of time periods, i-rate of interest per unit p.a.

$$P.V. = \frac{C(1+i)}{i}\left[1-(1+i)^{-n}\right]$$

***Present value of Deferred Annuity:**

A deferred annuity is characterized by a payment which is made at some later date, rather than the beginning or end of the time period.

Let us consider, 'm' to be the deferment period and for 'n' periods payments are due to be made at the end of each periods after deferment period. The following formula calculates the present value of deferred annuity.

$$P.V. = \frac{C(1+i)^{-m}}{i}\left[1-(1+i)^{-n}\right]$$

***Amortization of the Loan**:

The process of gradual elimination of a loan through regular instalments that are sufficient to cover both the principal and the interest, is called as amortization of the loan.

* Each EMI consists of two parts, one representing interest on the outstanding balance loan and the other representing part of the principal to be repaid.

***Amortization Table:**

The calculations of an amortized loan may be displayed in an amortization table. The table lists relevant balances and dollar amounts for each period. Each period is a row in the table, while the columns are typically current loan balance, total monthly payment, interest portion of payment, principal portion of payment and ending outstanding balance. The ending outstanding loan balance of one period becomes the current loan balance for the next.

***Equated Monthly Instalments (EMI):**

The loan taken from banks or any financial institutions repaid as an immediate annuity in equal instalments with the unit time period of loan. This instalment is known as EMI.

***Interest on reducing balance method**:

The method using the present value of annuity using compound interest to calculate the EMI is called the method of reducing balance.
is called the method of reducing balance.

3B.3. EXAMPLES:

[1] Himanshi opened a recurring deposit in a bank for 3 years with payments of Rs.
4000 paid at the end of each year. Find the money obtained at the end of period with
6% p.a.

Solution:

Annuity Immediate-Payment at the end of the year

Here C=Rs.4000, i=0.06, n=3 years

$$F.V. = \frac{C}{i}\left[(1+i)^n - 1\right]$$

$$= \frac{4000}{0.06}\left[(1+0.06)^3 - 1\right]$$

$$= 66666.6667\left[(1.06)^3 - 1\right]$$

$$= 66666.6667\left[1.191016 - 1\right]$$

$$= 66666.6667\left[(0.191016)\right]$$

$$= Rs.12734.40$$

Hence, Himanshi obtained Rs. 12734.40 at the end of 3 years at 6% p.a.

[2] Rashmi deposits Rs.6000 at the end of every month for 4 years with 9% compound
interest p.a. What is the total amount she will receive at the end of the period?

Solution:

Annuity Immediate-Payment at the end of the month

Here C=Rs.6000, i=0.09/12=0.0075, n=4x12=48month

$$F.V. \quad = \frac{C}{i}\left[(1+i)^n - 1\right]$$

$$= \frac{6000}{0.0075}\left[(1+0.0075)^{48} - 1\right]$$

$$= 8,00,000\left[(1.0075)^{48} - 1\right]$$

$$= 8,00,000\left[(1.43140533 - 1\right]$$

$$= 8,00,000\left[(0.43140533\right]$$

$$= 345124.264$$

Thus, Rashmi will receive the final amount Rs. 345124.264 after 4 years.

[3] Find the present value of immediate annuity of Rs.16, 000 per year for 3 years at 10% p.a.

Solution:

Here C=Rs.16, 000, n=3 years, i=0.10, P=?

$$P.V. \quad = \frac{C}{i}\left[1 - (1+i)^{-n}\right]$$

$$P.V. \quad = \frac{16000}{0.10}\left[1 - (1+0.10)^{-3}\right]$$

$$= 160000[1 - (1.10)^{-3}]$$

$$=160000\left[1-\frac{1}{(1.10)^3}\right]$$

$$=160000\left[1-\frac{1}{1.331}\right]$$

$$=160000[1-07513]$$

$$=160000[0.2487]$$

$$=39792$$

$\therefore$ P.V. is Rs.39792

[4] A man purchases a house and takes a mortgage on it for Rs. 10, 00,000 to be paid off in 4 years by equal annual payments payable at the end of each year. If the interest rate is 6% p.a., find the sum of money that he will pay each year.

Solution:

Here P=Rs.10, 00, 000, n=4 years, i=0.06, C=?

$$10,00,000=\frac{C}{i}\left[1-(1+i)^{-n}\right]$$

$$\therefore 10,00,000=\frac{C}{0.06}\left[1-(1+0.06)^{-4}\right]$$

$$\therefore 10,00,000X\,0.06=C[1-(1.06)^{-4}]$$

$$\therefore 60,000=C\left[1-\left(\frac{1}{(1.06)^4}\right)\right]$$

$$\therefore 60,000 = C\left[1-\left(\frac{1}{1.262477}\right)\right]$$

$$\therefore 60,000 = C[1-0.79209]$$

$$\therefore 60,000 = C[0.20791]$$

$$\therefore 60,000 = 0.20791C$$

$$\therefore C = \frac{60,000}{0.20791} = 288586.40$$

Hence, each instalment would be of Rs. 288586.40

[5] Miss. MNO purchased a home-theatre on instalment basis such that Rs. 6000 and the remaining amount to be paid in 4 equal quarterly instalments of Rs.3000 each payable at the end of each quarter. Find the cash price of the system if the rate of compound interest is 7% p.a.

Solution:

Here C=Rs.6, 000, n=1x4=4 years, i=0.07/4 =, P=?

$$P.V. = \frac{C}{i}\left[1-(1+i)^{-n}\right]$$

$$= \frac{3000}{0.0175}\left[1-(1+0.0175)^{-4}\right]$$

$$= 171428.57[1-(1.0175)^{-4}]$$

$$= 171428.57\left[1-\left(\frac{1}{(1.0175)^{4}}\right)\right]$$

$$= 171428.57\left[1-\left(\frac{1}{(1.07185903}\right)\right]$$

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$$= 171428.57[1 - 0.9329585]$$

$$= 171428.57[0.0670415]$$

$$= Rs.11492.82847$$

Hence, Cash price of the system=initial payment + P.V.

=Rs.6000+Rs.11492.82847

=Rs.17492.83

[6] Find the accumulated value of an annuity due of Rs.1000 per annum for 3 years at 10% p.a.

Solution:

Annuity due-Payment at the end of the year

Here C=Rs.1000, i=0.10, n=3 years

$$F.V. = \frac{C(1+i)}{i}\left[(1+i)^n - 1\right]$$

$$= \frac{1000(1+0.10)}{0.10}\left[(1+0.10)^3 - 1\right]$$

$$= \frac{1000(1.10)}{0.10}\left[(1.10)^3 - 1\right]$$

$$= \frac{1100}{0.10}\left[1.331 - 1\right]$$

$$= 11,000[0.331]$$

$$= Rs.3641$$

The accumulated value of an annuity due is Rs.3641

[7] A person plans to put Rs.200 at the beginning of each year in a deposit giving 2% p.a. compounded annually. What will be the accumulated amount after 2 years?

Solution: Annuity due-Payment at the end of the year

Here,C=Rs.200, i=0.02, n=2 years

$$F.V. = \frac{C(1+i)}{i}\left[(1+i)^n - 1\right]$$

$$= \frac{200(1+0.02)}{0.02}\left[(1+0.02)^2 - 1\right]$$

$$= \frac{200(1.02)}{0.02}\left[(1.02)^2 - 1\right]$$

$$= \frac{204}{0.02}\left[1.0404 - 1\right]$$

$$= 10200[0.0404]$$

$$= Rs.412.08$$

[8] Find the present value of annuity due of Rs. 100 p.a. for a period of 4 years if interest is charged at 8% p.a. effective rates.
Solution: Annuity due: yearly payment, C=100, n=4 years, i=0.08,P=?

$$P.V. = \frac{C(1+i)}{i}\left[1-(1+i)^{-n}\right]$$

$$= \frac{100(1+0.08)}{0.08}\left[1-(1+0.08)^{-4}\right]$$

$$= \frac{100(1.08)}{0.08}\left[1-\frac{1}{(1.08)^4}\right]$$

$$= \frac{108}{0.08}\left[1-\frac{1}{1.36049}\right]$$

$$= 1350(1-0.735029)$$

$$= 1350(0.2650)$$

$$= 357.75$$

Therefore, P.V. of annuity due is Rs.357.75

[9] A deferred annuity is purchased that will pay Rs. 10,000 for 15 years after being deferred for 5 years. If money is worth 6% compounded quarterly, what is the present value of this annuity?

Solution:

Deferred Annuity: Quarterly payment

C=10,000, n=15(4) =60, m=5(4) = 20,i=0.06/4=0.015

$$P.V. = \frac{C(1+i)^{-m}}{i}\left[1-(1+i)^{-n}\right]$$

$$= \frac{10,000(1+0.015)^{-20}}{0.015}\left[1-(1+0.015)^{-60}\right]$$

$$= \frac{10,000(1.015)^{-20}}{0.015}\left[1-(1.015)^{-60}\right]$$

$$= \frac{10,000}{0.015(1.015)^{20}}\left[1-\frac{1}{(1.015)^{60}}\right]$$

$$= \frac{10,000}{0.015(1.346855)}\left[1-\frac{1}{2.4432198}\right]$$

$$= \frac{10,000}{0.015(1.346855)}\left[1-\frac{1}{2.4432198}\right]$$

$$= \frac{10,000}{0.015(1.346855)}\left[1-\frac{1}{2.4432198}\right]$$

$$= \frac{10,000}{0.0202028}\left[1-\frac{1}{2.4432198}\right]$$

$$= 494980.8937\left[1-0.409296\right]$$

$$= 494980.893(0.590704)$$

$$= Rs.292387.19$$

Present value of this annuity is $Rs.292387.19$

[10] A deferred annuity is purchased that will pay Rs. 500 for 10 years after being deferred for 6 years. If money is worth 3% compounded annually, what is the present value of this annuity?

Solution:

Deferred Annuity: C=500, m=6, n=4,i=0.03

$$P.V. = \frac{C(1+i)^{-m}}{i}\left[1-(1+i)^{-n}\right]$$

$$= \frac{500(1+0.03)^{-6}}{0.03}\left[1-(1+0.03)^{-10}\right]$$

$$= \frac{500(0.03)^{-6}}{0.03}\left[1-(1.03)^{-10}\right]$$

$$= \frac{500}{0.03(1.03)^{6}}\left[1-\frac{1}{(1.03)^{10}}\right]$$

$$= \frac{500}{0.03(1.19405)}\left[1-\frac{1}{1.343916}\right]$$

$$= \frac{500}{0.03(1.1945)}\left[1-0.744094\right]$$

$$= \frac{500}{(0.035821)}\left[0.255591\right]$$

$$= 13958.2926(0.255591)$$

$$= Rs.3567.6084$$

Present value of this annuity is $Rs.3567.6084$

[11] A deferred annuity is purchased that will pay Rs. 5000 for 4 years after being deferred for 2 years. If money is worth 4% compounded annually, what is the present value of this annuity?

Solution:

Deferred Annuity: C=5000, m=2, n=4,i=0.04

$$P.V. = \frac{C(1+i)^{-m}}{i}\left[1-(1+i)^{-n}\right]$$

$$= \frac{5000(1+0.04)^{-2}}{0.04}\left[1-(1+0.04)^{-4}\right]$$

$$= \frac{5000(1.04)^{-2}}{0.04}\left[1-(1.04)^{-4}\right]$$

$$= \frac{5000}{0.04(1.04)^{2}}\left[1-\frac{1}{(1.04)^{4}}\right]$$

$$= \frac{5000}{0.04(1.0816)}\left[1-\frac{1}{1.16986}\right]$$

$$= \frac{5000}{0.043264}\left[0.854803\right]$$

$$= 115569.527(0.854803)$$

$$= Rs.98789.19$$

Present value of this annuity is $Rs.98789.19$

3B.4. UNIT END EXERCISE:

(1) Find the accumulated value after 4 years of an immediate annuity of Rs. 20,000 p.a. with interest Compounded at 6% p.a.

(2) Divya deposited Rs.2000 at the end of each year, for 2 years in a company and received Rs.4200 as the accumulated value. Find rate of compound interest.

(3) Manini deposits Rs. 500 with 12%compound interest for 3 years. Find the final amount if payment is at the end of each quarter.

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(4) Sarita invested Rs.1000 at the end of every month for 4 years at 12 % p.a. compound interest. Find the amount she will receive at the end of the period.

(5) Ms. Rukmini plans to save for her daughter's higher studies. She wants to accumulate an amount of Rs.1,00,000 at the end of 4 years. How much should she invest at the end of each year from now, if she can get interest compounded at 7% p.a.?

(6) Ms.Sima deposited Rs.20,000 at the end of every year for two years. The rate of interest is 10% p.a., compounded half-yearly. What is the amount accumulated at the end of 2 years?

(7) Find the accumulated value at the end of 4 years and the present value of an immediate annuity of Rs.50,000 p.a. for 4 years at 4% p.a.

(8) Kartik purchased a TV set and paid Rs. 5,000 immediately, anther Rs. 5,000 after a year and Rs.5,000 after 2 years and thus became debt free. Find the price of TV set if compound interest charged was 3.5%p.a.

(9) How much money should a person invest at 7% p.a. compound interest so that he would get an annuity of Rs. 1, 00,000 at the end of each year for the next four years after which his principal money will be over?

(10) A TV is purchased for Rs.5,000 cash down and Rs. 10,000 at end of each month, for 4 months. Find the cash price of the TV if the payments include interest payment at 12% p.a. compounded monthly.

(11) A man purchases a house and takes a mortgage on it for Rs. 10, 00,000 to be paid of in 4 years by equal annual payments payable at the end of each year. If the interest rate is 6% p.a., find the sum of money that he pays each year.

(12) Miss. MNO purchased a refrigerator with a down payment of Rs. 2500 and the remaining amount to be paid in 6 equal monthly instalments of Rs.1000 each. Find the price of the fridge if the company wants to earn 12% p.a.

(13) A company decide to set aside a certain sum at the end of each year to create a sinking fund, which should amount to Rs.5 lakhs in 4 years at 12% p.a. Find the amount to be kept aside each year.

(14) Find the present value of an immediate annuity of Rs. 30,000 p.a. for 3 years with interest compounded at 8% p.a.

(15) Raju took a loan of Rs.1,20,000 from a friend for a period of 9 months. Compute the EMI at 10%p.a.using Reducing balance method.

(16) Mr. Bhatt wants to take a loan of Rs.4 lakhs, which he intends to return after 4 years, with interest. Bank A offers him the loan of 4 lakhs at 6% p.a., flat interest rate and bank B offers him at 8% p.a., on monthly reducing balance. Comparing the EMI's decide about the choice of bank he should make.

(17) A loan of Rs. 1,00,000 is to be repaid in 4 years in 4 equal instalments , with the first instalment at the end of the first year. The rate of interest is 10% p.a. (a) Find the yearly instalment using interest on reducing balance. (b)Find the interest and principal repayment for each month. [Make the amortization table.]

♦♦♦♦

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4

LIMITS AND FUNCTIONS

UNIT STRUCTURE

4.1 INTRODUCTION

In this chapter we learn the concepts of functions and their limit

Concept of a functions

Dependent and Independent Variables

Consider an equation $y = 3x + 4$

As we assign different value to x, we obtain the corresponding values of y in such a relation, x is called as the independent variable and y is called the dependent variable. For each value of x we get a unique value of y and we say that y is a function of x and denote it as $y = f(x)$

Let f be a function from set A to Set B. This is denoted as $f : A \rightarrow B$.The set A is called the domain of the function and set B is called the co-domain of the function. Further set $R = \{f(x) \mid x \in A\}$ is called the range of the function.

Examples:

1. Write the domain and range of the function y = $f(x) = x^2 + 5$. Where $0 \le x \le 5$, $x \in I$

$y = f(x) = x^2 + 5$, Where $0 \le x \le 5$, $x \in I$

Domain = {0, 1, 2, 3, 4, 5,}

Here, f(0)=5, f(1)=6,f(2)=9,f(3)=14,f(4)=21 and f(5)=30

$\therefore$ Range = {5, 6, 9, 14, 21, 30}

2. If $f(x) = x^2 + 3x - 1$. Find $f(0)$, $f(1)$, $f(x+1)$

$f(0) = (0)^2 + 3(0) - 1 = -1$

$f(1) = 1+3-1 = 3$

$f(x+1) = (x+1)^2 + 3(x+1) - 1$

$= x^2 + 2x + 1 + 3x + 3 - 1$

$= x^2 + 5x + 3$

<u>Examples for Practice:</u>

(1) Find the domain and range for the following functions

(i) $f(x) = \frac{x+6}{x-1} \quad 3 \le x \le 5$

(ii) $f(x) = |x| \quad -2 \le x \le 2$

(2) For the function $f(x) = 3x^2 + 2x - 1$ find $f(2)$, $f(-3)$ $f(0)$ and $f(x+4)$

4.2 TYPES OF FUNCTIONS

I Constant Function:-

For every value of x

$f(x)$ takes the same value. e.g. $y = f(x) = 8$

II Linear Function:-

Consider the function $y = f(x) = ax + b$ where a and b are real numbers and x is a variable. Such a function in called as a linear function the graph of this function will be a straight line and the power of x is 1.

III Quadratic Function:-

Consider the function $y = f(x) = ax^2 + bx + c$ where a, b, c are real numbers $(a \ne 0)$ and x is a variable. The highest degree of x is 2 Hence such a function is called a quadratic function the graph of this function is a parabola.

IV Polynomial Function:-

A function of the type

$y = f(x) = a_0 + a_1x + a_2x^2 + \dots\dots\dots + a_nx^n$ is called a polynomial function where $a_0, a_1, a_2, \dots\dots a_n$ are real numbers and x is a variable.

The constant function, linear function and quadratic function are special cases of polynomial function.

V Exponential function:-

A function of the type f(**x**) = a^x where **a** is a positive real number $(a \neq 0)$ and x is a rational number, is called exponential function.

If a = e where e is the natural logarithmic base whose value is approximately 2.71828183, then we get the exponential function as $y = e^x$

VI Logarithmic Function:-

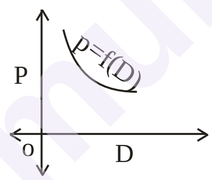
A function of the type $f(x) = \log_a x$ where a is positive real number $(a \neq 1)$ is called Logarithmic function

Functions in Economics:-

In business activity we use terms like price, demand, supply, revenue, cost, profit price, demand and supply are related to each other and one can be expressed as a function of the other. Similarly revenue and cost are related to the number of units produced and sold. The various function in Economics are as follows

I) Demand Function :-

Let p denote the price of a commodity whose demand is D. Then the two variable p and D are related to each other and we can write the relation as p = *f* (D) & D = g(p). It is a convention to write the demand function s p = *f* (D). If we plot Demand D on X axis and price P on the Y axis, the demand curve appears as shown below:

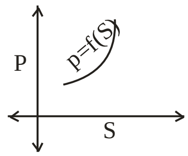


The curve indicates that if price decreases the demand increases and as price increases demand decreases.

II) Supply Functions :-

Let p denote the price of a commodity whose supply is S. Then the two variables are related to each other and we can write the relation as p = *f* (*s*) or s = g(p). It is a convention to write the supply function as p = f(s). If we plot supply S on X axis and price p on Y axis, then the supply curve appears as shown below.

Generally supply and price increases or decreases together.



* **<u>Point of Equilibrium price (Demand = Supply):-</u>**

If both the graphs are plotted on the same co-ordinate axes, the point of intersection of the two curves is the point of equilibrium price.

III) Cost Function:- Let *x* denote the quantity of goods produced at price p. The total cost of producing the goods consists of the fixed cost and variable cost.

Thus Total cast = Fixed cost + Variable Cost

i.e. Total cost = Fixed Coast + *x. P*

Fixed cost will be the cost when no goods are produced i.e. when $x = 0$

IV) Revenue Function:- Let *x* denote the quantity of goods sold at price *p* then total revenue generated can be written as R= p.*x*.

V) Profit Function:- Let C denote the total cost of *x* units of goods produced and sold and let R denote the total revenue generated.

Then Profit = R – C.

4.3 CONCEPT OF LIMIT OF A FUNCTION

Let *x* be a variable and *a* be *a* constant, then $x \to a$ means x tends to a i.e. *x* approaches a, but $x \neq a$

$x \to a+$ means x approaches a from the right i.e. $x \neq a$, x > a

$x \to a\_$ means x approaches a from the left i.e $x \neq a$, x < a

*** <u>Limit of a function</u> :**

Consider a function *y* = f(*x*) then as $x \to a$ if $f(x) \to a$ then we say that $\lim_{x \to a} f(x) = b$, where a and b are constants.

* **<u>Left hand limit and right hand Limit</u>**:

$\lim_{x \to a} f(x) = b_1$ $\quad \lim_{x \to a} f(x) = b_2$

Further if $b_1 = b_2 = b$ we say that

$\lim_{x \to a} f(x) = b$

Example: *y* = f(*x*) = *x* +1

X	1	1.5	1.9	1.99	2.0001	2.01	2.1	2.5
f(x)	2	2.5	2.9	2.99	3.001	3.01	3.1	3.5

In the above example we observe that as $x \to 2 \Rightarrow f(x) \to 3$

*** <u>Rules for finding Limits:</u>**

If f(x) and g(x) are two functions then

(i) $\lim_{x\to a}\left(f(x)+g(x)\right)=\lim_{x\to a} f(x)+\lim_{x\to a} g(x)$

(ii) $\lim_{x\to a}\left(f(x)-g(x)\right)=\lim_{x\to a} f(x)-\lim_{x\to a} g(x)$

(iii) $\lim_{x\to a}\left(f(x).g(x)\right)=\lim_{x\to a}\text{f(x)}.\lim_{x\to a} g(x)$

(iv) $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ Where $g(x)$ = 0

*** <u>Methods for finding Limits:-</u>**

1. **<u>Substitution Method</u>**:- In this method, we substitute the limiting value of x, in the given function to obtain the limit.
   Example:

$$\lim_{x\to 2}\frac{x^2+5x+1}{x+2}$$

$$=\frac{(2)^2+5(2)+1}{2+2} \quad \left[x\to 2, x\neq 2(x-2)\neq 0\right]$$

$$=\frac{4+10+1}{4}$$

$$=\frac{15}{4}$$

2. **<u>Factorization Method</u>**:- In this method, we factories the numerator and denominator, cancel the common factor and then substitute the limiting value of x to obtain the limit of the function

   Ex $\lim_{x\to 3}\frac{x^3-27}{x-3}$

In this example if we substitute x = 3, we will obtain the value of the function as $\frac{0}{0}$ form which is called as indeterminate form the other indeterminate forms are $\frac{\infty}{\infty}, 1^{\infty}, 0^0, \infty^0, 0\times\infty, \infty-\infty$

In the above example (x- 3) is a common factor, so we proceed as follows.

$$\lim_{x\to 3}\frac{x^3-27}{x-3}$$

$$=\lim_{x\to 3}\frac{(x-3)(x^2+3x+9)}{(x-3)}\left[x\to 3, x\neq 3 (x-3)\neq 0\right]$$

Now substituting the limiting value of x we obtain the limit of the function as

$$=\lim_{x\to 3}x^2+3x+9=(3)^2+3(3)+9=9+9+9=27$$

3. **<u>Simplification Method</u>:-** In this method we simplify the function using the common denominator cancel the common factor and then substitute the limiting value of *x* to find the limit of the function

Ex. $$\lim_{x\to 2}\left[\frac{1}{x-2}-\frac{4}{x^3-2x^2}\right]$$

$$=\lim_{x\to 2}\left[\frac{1}{x-2}-\frac{4}{x^2(x-2)}\right]$$

$$=\lim_{x\to 2}\frac{x^2-4}{x^2(x-2)}$$

$$=\lim_{x\to 2}\frac{\cancel{(x-2)}(x+2)}{x^2\cancel{(x-2)}}\quad [x\to 2,\ x\neq 2 (x-2)\neq 0]$$

$$=\frac{2+2}{(2)^2}=\frac{4}{4}=1$$

4. **<u>Rationalization method</u>**:- In this method we rationalize the numerator or denominator by multiplying and dividing by the rationalizing factor, then simplifying and canceling the common factor and substituting the limiting value of x to obtain the limit of the function.

Ex. $$=\lim_{x\to 3}\frac{\sqrt{x^2+1}-\sqrt{10}}{x-3}$$

$$=\lim_{x\to 3}\frac{(\sqrt{x^2+1}-\sqrt{10})(\sqrt{x^2+1}+\sqrt{10})}{(x-3)(\sqrt{x^2+1}+\sqrt{10})}$$

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$$= \lim_{x \to 3} \frac{(\sqrt{x^2+1} - \sqrt{10})(\sqrt{x^2+1} + \sqrt{10})}{(x-3)(\sqrt{x^2+1} + \sqrt{10})}$$

$$= \lim_{x \to 3} \frac{(x^2+1-10)}{(x-3)(\sqrt{x^2+1} + \sqrt{10})}$$

$$= \lim_{x \to 3} \frac{(\sqrt{x^2+1} - \sqrt{10})(\sqrt{x^2+1} + \sqrt{10})}{(x-3)(\sqrt{x^2+1} + \sqrt{10})}$$

$$= \lim_{x \to 3} \frac{x^2 - 9}{(x-3)(\sqrt{x^2+1} + \sqrt{10})}$$

$$= \lim_{x \to 3} \frac{\cancel{(x-3)}(x+3)}{\cancel{(x-3)}(\sqrt{x^2+1} + \sqrt{10})} \quad [x \to 3, x \neq 3, \ x-3 \neq 0]$$

$$= \frac{3+3}{\sqrt{10}+\sqrt{10}} = \frac{6}{2\sqrt{10}} = \frac{3}{\sqrt{10}} = \frac{3\sqrt{10}}{10}$$

* **Limit to infinity:-**

So far we have discussed problems where the value of a limit of a function was finite. However sometimes the variable x may take values which go on increasing indefinitely and we say that $x \to \infty$ and then find the limit of the function

Ex. $\lim_{x \to \infty} \frac{(x^2+1)(x+4)}{3x^3+5x-1}$

$$= \lim_{x \to \infty} \frac{x^3 + 4x^2 + x + 4}{3x^3 + 5x - 1}$$

Dividing the numerator and denominator by x^3 we get

$$= \lim_{x \to \infty} \frac{\left(1 + \frac{4}{x} + \frac{1}{x^2} + \frac{4}{x^3}\right)}{3 + \frac{5}{x^2} - \frac{1}{x^3}} \qquad [\because \ x \to \infty, \frac{1}{x} \to 0, \frac{1}{x} \neq 0]$$

$$= \frac{1+0+0+0}{3+0-0} = \frac{1}{3}$$

*** Limit of exponential series:-**

1. $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$

Using Binomial theorem

$$\left(1+\frac{1}{n}\right)^n = 1+n.\frac{1}{n}+\frac{n(n-1)}{2!}\frac{1}{n^2}+\frac{n(n-1)(n-2)}{3!}\frac{1}{n^3}+.......$$

$$=1+1+\frac{\left(1-\frac{1}{n}\right)}{2!}+\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{3!}+........$$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 1+1+\frac{1}{2!}+\frac{1}{3!}+............ = \text{e} \quad As \quad n\to\infty, \frac{1}{n},\frac{2}{n}.......\to 0$$

Thus $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$ and $\lim_{n\to 0}(1+n)^{1/n} = e$

2. $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \lim_{n\to\infty}\left\{\left(1+\frac{x}{n}\right)^{\frac{n}{x}}\right\}^x = e^x$

3. $a^x = 1+x\ \log\ a+\frac{x^2}{2!}(\log\ a)^2+..........$ is called Exponential series.

$$\therefore \frac{a^x-1}{x} = \log a+\frac{x}{2!}(\log a)^2+\frac{x^2(\log a)^3}{3!}+........$$

$$\boxed{\therefore \lim_{x\to 0}\frac{a^x-1}{x} = \log a} \qquad As\ x\to 0\ x, x^2........\to 0$$

4. $e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+...............$

$$\therefore \frac{e^x-1}{x} = 1+\frac{x}{2!}+\frac{x^2}{3!}+................$$

$$\boxed{\therefore \lim_{x\to 0}\frac{e^x-1}{x} = 1} \qquad \text{As } x\to 0\ \ x^2, x^3..........\to 0$$

4.4 SOLVED EXAMPLES

(i) $\lim_{x\to 2}\frac{x^2+5x-1}{2x+1}$

$=\frac{4+10-1}{4+1}=\frac{13}{5}$

(ii) $\lim_{x\to 3}\frac{x^2-7x+12}{x^2-5x+6}$

$=\lim_{x\to 3}\frac{\cancel{(x-3)}(x-4)}{\cancel{(x-3)}(x-2)}$ $\quad [\because\ x\to 3,\ x\neq 3,\ x-3\neq 0]$

$=\frac{3-4}{3-2}=-1$

(iii) $\lim_{x\to 2}\frac{x^2-4}{x^2-11x+18}$

$=\lim_{x\to 2}\frac{\cancel{(x-2)}(x+2)}{(x-9)\cancel{(x-2)}}$ $\quad [\because x\to 2,\ x\neq 2,\ x-2\neq 0]$

$=\frac{4}{-7}=-\frac{4}{7}$

(iv) $\lim_{x\to 1}\left[\frac{1}{x-1}+\frac{1}{x^2-3x+2}\right]$

$=\lim_{x\to 1}\left[\frac{1}{(x-1)}+\frac{1}{(x-1)(x-2)}\right]$

$=\lim_{x\to 1}\left[\frac{(x-2)+1}{(x-1)(x-2)}\right]$

$=\lim_{x\to 1}\frac{\cancel{(x-1)}}{\cancel{(x-1)}(x-2)}$ $\quad [\because x\to 1,\ x\neq 1,\ x-1\neq 0]$

$=\frac{1}{1-2}=-1$

(v) $\lim_{x\to 4}\left[\frac{1}{x^2-3x-4}+\frac{1}{x^2-13x+36}\right]$

$$= \lim_{x\to 4}\left[\frac{1}{(x-4)(x+1)}+\frac{1}{(x-4)(x-9)}\right]$$

$$= \lim_{x\to 4}\frac{(x-9)+(x+1)}{(x-4)(x+1)(x-9)}$$

$$= \lim_{x\to 4}\frac{2x-8}{(x-4)(x+1)(x-9)}$$

$$= \lim_{x\to 4}\frac{2\cancel{(x-4)}}{\cancel{(x-4)}(x+1)(x-9)} \qquad [\because x\to 4, x\neq 4, x-4\neq 0]$$

$$= \frac{2}{5(-5)} = -\frac{2}{25}$$

(vi) $\lim_{x\to 0}\frac{\sqrt{1+x+x^2}-1}{x}$

$$= \lim_{x\to 0}\frac{\left(\sqrt{1+x+x^2}-1\right)\left(\sqrt{1+x+x^2}+1\right)}{x\left(\sqrt{1+x+x^2}+1\right)}$$

$$= \lim_{x\to 0}\frac{1+x+x^2-1}{x\left(\sqrt{1+x+x^2}+1\right)}$$

$$= \lim_{x\to 0}\frac{x+x^2}{x\left(\sqrt{1+x+x^2}+1\right)} = \lim_{x\to 0}\frac{x(1+x)}{x\left(\sqrt{1+x+x^2}+1\right)}$$

$$= \lim_{x\to 0}\frac{(1+x)}{\left(\sqrt{1+x+x^2}+1\right)} \qquad [\because x\to 0, x\neq 0]$$

$$= \frac{1+0}{\sqrt{1+0+0}+1} = \frac{1}{2}$$

(vii) $\lim_{x\to 2}\frac{\sqrt{6+x}-\sqrt{10-x}}{x^2-4}$

$$= \lim_{x\to 2}\frac{\left(\sqrt{6+x}-\sqrt{10-x}\right)\left(\sqrt{6+x}+\sqrt{10-x}\right)}{\left(x^2-4\right)\left(\sqrt{6+x}+\sqrt{10-x}\right)}$$

$$= \lim_{x\to 2}\frac{6+x-10+x}{\left(x^2-4\right)\left(\sqrt{6+x}+\sqrt{10-x}\right)}$$

$$= \lim_{x\to 2} \frac{2x-4}{\left(x^2-4\right)\left(\sqrt{6+x}+\sqrt{10-x}\right)}$$

$$= \lim_{x\to 2} \frac{2\left(\cancel{x-2}\right)}{\left(x+2\right)\left(\cancel{x-2}\right)\left(\sqrt{6+x}+\sqrt{10-2}\right)}$$

$$\left[\because x \to 2, x \neq 2, x-2 \neq 0\right]$$

$$= \frac{2}{4\left(\sqrt{8}+\sqrt{8}\right)} = \frac{2}{8\sqrt{8}} = \frac{1}{4\sqrt{8}} = \frac{1}{8\sqrt{2}}$$

(viii) $\lim_{x\to 0} \frac{4^x+5^x-2^{x+1}}{x}$

$$= \lim_{x\to 0} \frac{4^x-1+5^x-1-2^{x+1}+2}{x}$$

$$= \lim_{x\to 0} \left[\frac{4^x-1+5^x-1-2\left(2^x-1\right)}{x}\right]$$

$$= \lim_{x\to 0} \left[\frac{4^x-1}{x}+\frac{5^x-1}{x}-\frac{2\left(2^x-1\right)}{x}\right]$$

$$= \lim_{x\to 0} \frac{4^x-1}{x}+\lim_{x\to 0} \frac{5^x-1}{x}-2\lim_{x\to 0} \frac{2^x-1}{x}$$

$$= \log 4+\log 5-2\log 2 \qquad \left[\because \lim_{x\to 0} \frac{a^x-1}{x}=\log a\right]$$

$$= \log 4+\log 5-\log 4 \qquad \left[\because 2\log 2=\log 2^2=\log 4\right]$$

$$= \log 5$$

(ix) $\lim_{x\to 0} \frac{3^{2x}-6^x}{2x}$

$$= \lim_{x\to 0} \frac{9^x-1-6^x+1}{2x}$$

$$= \frac{1}{2}\left[\lim_{x\to 0} \frac{9^x-1}{x}-\lim_{x\to 0} \frac{6^x-1}{x}\right]$$

$$= \frac{1}{2}\left(\log 9-\log 6\right) \qquad \left[\because \lim_{x\to 0} \frac{a^x-1}{x}=\log a\right]$$

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$$= \frac{1}{2} \log \frac{3}{2} \quad \left[\because \log m - \log n = \log \frac{m}{n} \right]$$

(x) $\lim_{x \to \infty} \frac{5x^2 + 8x - 3}{3x^2 + 7x + 21}$

Numerator and denominator both divided by x^2

$$= \lim_{x \to \infty} \frac{5 + \frac{8}{x} - \frac{3}{x^2}}{3 + \frac{7}{x} + \frac{21}{x^2}} \qquad \left[\text{As } x \to \infty \quad \frac{1}{x} \to 0, \frac{1}{x^2} \to 0 \right]$$

$$= \frac{5}{3}$$

(xi) $\lim_{n \to \infty} \frac{1 + 2 + 3 + \text{........} + \text{n}}{n^2}$

$$= \lim_{n \to \infty} \frac{n(n+1)}{2n^2} \qquad \left[\because \Sigma n = \frac{n(n+1)}{2} \right]$$

$$= \lim_{n \to \infty} \frac{1}{2}\left(1 + \frac{1}{n}\right) = \frac{1}{2} \qquad \text{As } n \to \infty \quad \frac{1}{n} \to 0$$

(xii) $\lim_{n \to \infty} \frac{1.3 + 3.5 + 5.7 + \text{..........}}{n^3}$

$$= \lim_{n \to \infty} \frac{\sum (2n-1)(2n+1)}{n^3}$$

$$= \lim_{n \to \infty} \frac{\sum (4n^2 - 1)}{n^3}$$

$$= \lim_{n \to \infty} \frac{4 \sum n^2}{n^3} - \lim_{n \to \infty} \frac{\sum 1}{n^3}$$

$$= \lim_{n \to \infty} \frac{4n(n+1)(2n+1)}{6n^3} - \lim_{n \to \infty} \frac{n}{n^3}$$

$$\left[\because \Sigma n^2 = \frac{n(n+1)(2n+1)}{6} \quad \Sigma 1 = n \right]$$

$$= \frac{4}{6} \lim_{n\to\infty} 1\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) - \lim_{n\to\infty} \frac{1}{n^2}$$

$$= \frac{4}{6} \times 1 \times 2 - 0 \qquad \left[\text{As } n \to \infty \ \ \frac{1}{n} \to 0, \frac{1}{n^2} \to 0\right]$$

$$= \frac{8}{6} = \frac{4}{3}$$

4.5 UNIT AND EXERCISES

i) $\lim_{x\to 2} \frac{3x^2 - x - 5}{x^2 + x - 6}$

ii) $\lim_{x\to 4} \frac{x^3 - 4x - 48}{x^2 - 7x + 12}$

iii) $\lim_{x\to 1} \frac{3x^2 + 2x + 7}{x + 1}$

iv) $\lim_{x\to 4} \left[\frac{1}{x-2} - \frac{4}{x^2 + x - 6}\right]$

v) $\lim_{x\to 0} \frac{\sqrt{3+x} - \sqrt{3-x}}{x}$

vi) $\lim_{x\to 5} \frac{x^2 - x - 20}{\sqrt{3x+1} - 4}$

vii) $\lim_{n\to\infty} \frac{1.4 + 2.5 + 3.6 + \ldots\ldots\ldots\ldots}{n^4}$

viii) $\lim_{n\to\infty} \frac{1^3 + 2^3 + 3^3 + \ldots\ldots\ldots}{n^4}$

ix) $\lim_{n\to\infty} \frac{2n^3 + 5n^2 - n}{4n^3 - 1}$

x) $\lim_{n\to\infty} \frac{n^2(n+5)}{n(n+1)(n+2)}$

xi) $\lim_{x\to 0}\frac{6^x-3^x-2^x+1}{x^2}$

xii) $\lim_{x\to 0}\frac{4^x+5^x+6^x-3^{x+1}}{x}$

xiii) $\lim_{x\to 0}(1+2x)^{2/x}$

xiv) $\lim_{n\to\infty}\left(1+\frac{5}{n}\right)^n$

xv) $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+4}$

xvi) $\lim_{n\to 0}\left(\frac{n+3}{n+1}\right)^{n+3}$

xvii) $\lim_{x\to 2}\frac{\log x-\log 2}{x-2}$

2. For what values of x will the expression $\frac{2x+3}{(x+2)(x-1)}$ tends to infinity?

3. If f(x) = x^2, show that $\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}=12$

*** <u>Continuity:</u>**

A function f(x) is said to be continuous at x = a if

$$\lim_{x\to a}f(x)=f(a);\lim_{x\to a_-}f(x)=\lim_{x\to a_+}f(x)=f(a)$$

Example

(i) Consider $f(x)=1\ if\ x\neq 0$

$=0\ if\ x=0$

Examine whether the function is continuous at x = 0

$\lim_{x\to 0}f(x)=\lim_{x\to 0}1=1\neq 0\neq f(0)$

$\therefore$ Function is not continuous at x = 0

(ii) Consider f(x) $=x+1\quad -1<x<0$

$=1,\quad 0\leq x<1$

$=x,\quad 1\leq x\leq 2$

Examine whether f(x) is continuous at x = 0 and x = 1

$$\lim_{x \to 0_-} f(x) = \lim_{x \to 0_-} x + 1 = 1$$
$$\lim_{x \to 0_+} f(x) = \lim_{x \to 0_+} 1 = 1$$

And f (0) = 1

$\therefore$ Function is continuous at $x = 0$

Try the above example at $x = 1$

Conclusion:

At the end of this chapter we have understood the following concepts.

(1) Functions, their types
(2) Functions in Business and Economics
(3) Limit of a function
(4) Left hand and right hand Limits
(5) Limit to infinity
(6) Continuity of a function

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5A

DERIVATIVE

UNIT STRUCTURE

5A.1. OBJECTIVES

After going through this chapter you will able to know:

- Find rate of change of a function with respect to one variable
- Successive differentiation
- Physical and Geometrical meaning of Derivative
- Computation of Derivative with the rules of differentiation

5A.2. INTRODUCTION

Calculus is the mathematics of change in motion. It is used to calculate change in displacement velocity with respect to time also used to change in supply, with respect to price. It helps to calculate maximize profit and minimize cost. We find slope of any curve using it.

We have already seen different type of function and there limit in previous unit let us now try to find exact rates of change at a point.

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Consider a function $y = f(x)$ of a variable x. Suppose x change to small value x_o and set x_1 i.e. $x_1 = x + x_o$ then the increment of x is given by $\Delta x = x_1 - x_o$

As x change, y change from $f(x_o)$ to $f(x_o + \Delta x)$ i.e.

$$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

The function is called difference quotient.

*** <u>Derivative :</u>**

Let $y = f(x)$ be the function of x. To measure rate at which $f(x)$ change with respect to x. i.e. rate of change of y with respect to "x", is called Derivative and it is denoted by $f'(x)$ or $\frac{dy}{dx}$.

Let x change from $x \to x + h$ then $f(x) \to f(x + h)$, where the value of h, becomes smaller and smaller, change in y is given by tending $h \to 0$

$f'(x) = \frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ exists is called the derivative of function f(x) and denoted by $f'(x)$ or $\frac{dy}{dx}$.

Note: If $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ exists, it is called first principle of derivative.

Ex.1. Find $\frac{dy}{dx}$, if $y = 3x^2 + 2$ by using first principle of derivative.

Sol^n: Given $y = f(x) = 3x^2 + 2$

$\therefore$ f (x+h) = 3(x+h)^2 + 2

$= 3x^2 + 6xh + 3h^2 + 2$

By definition of first principle of derivative,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{3x^2 + 6xh + 3h^2 + 2 - 3x^2 - 2}{h}$$

$$= \lim_{h\to 0} \frac{6xh + 3h^2}{h}$$

$$= \lim_{h\to 0} \frac{h(6x + 3h)}{h}$$

$$= 6x$$

$\therefore$ If $y = 3x^2 + 2$ then $\dfrac{dy}{dx} = 6x$

Ex.2. Find $f'(x)$ from first principles of the derivative.

If $f(x) = x\sqrt{x}$

Sol^n : $f(x) = x\sqrt{x} = x \cdot x^{1/2} = x^{3/2}$

$\therefore \ f(x+h) = (x+h)^{3/2}$

Now $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$

$$\therefore \ f'(x) = \frac{dy}{dx} = \lim_{h\to 0} \frac{(x+h)^{3/2} - (x)^{3/2}}{h}$$

Put $x + h = y \quad \therefore \ h = y - x$

$\therefore \ h \to 0 \quad \therefore x + h \to x \quad \therefore y \to x$

$$\therefore \ f'(x) = \lim_{y\to x} \frac{y^{3/2} - x^{3/2}}{y - x}$$

$$\therefore \ f'(x) = \frac{3}{2}(x)^{3/2-1} \quad \left[\therefore \lim_{x\to a} \frac{x^n - a^n}{x - a} = na^{n-1}\right]$$

$$\therefore \ f'(x) = \frac{3}{2} x^{1/2}$$

$$\therefore \ f'(x) = \frac{3}{2}\sqrt{x}$$

$$\therefore If \ \ f(x) = x^{3/2} \text{then} \ \ f'(x) = \frac{dy}{dx} = \frac{3}{2}\sqrt{x}$$

***** **<u>Physical meaning of Derivative</u>:**

We have study velocity acceleration and magnification in our school section, these all are related with derivative.

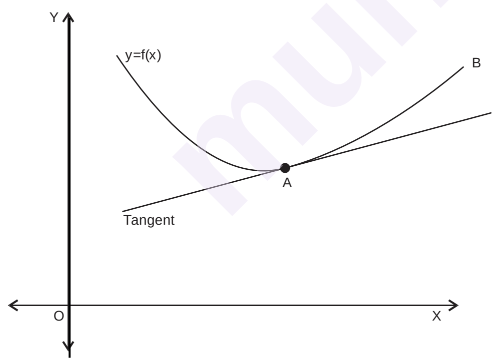
A car moves from point P to point Q, if displacement is given by f(x) in time x then $f'(x)$ is the velocity of a car. i.e.

$$f'(x) = \frac{df}{dx} = \frac{kg}{hr} \ \ km/hrs$$

Similarly, when change in velocity take place and it is f(x) in time x. Acceleration is given by

$$f'(x) = \frac{df}{dx}$$

***** **<u>Geometric meaning of Derivative</u> :**



Let y = f(x) be a curve and let A and B be two point on curve y = f(x), AB be chord on the curve y = f(x), draw tangent at point A is define by

$$\text{Slope of the Chord } AB = \frac{f(a+h) - f(a)}{h}$$

Taking $\lim h \to 0$, We get $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$

$\therefore$ Slope of tangent at point A $=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$

$$=\frac{dy}{dx}=f'(x)=\tan\theta$$

Derivative represent the slope of the curve $y = f(x)$

Note: i) If the tangent at point A is parallel to x-axis

then $\theta=0 \Rightarrow \tan\theta=0 \Rightarrow \frac{dy}{dx}=0$

ii) If the tangent at point A is parallel to y – axis

then $\theta=90^{\circ} \Rightarrow \tan\theta=\text{infinite} \Rightarrow \frac{dy}{dx}=\infty$

Basic formula for Derivatives

i) If $y=x^n$ $\therefore \frac{dy}{dx}=nx^{n-1}$

ii) If y = k where k = constant $\therefore \frac{dy}{dx}=0$

iii) If $y=e^x$ $\therefore \frac{dy}{dx}=e^x$

iv) If $y=a^x$ $\therefore \frac{dy}{dx}=a^x\log a$

v) If $y=\log x$ $\therefore \frac{dy}{dx}=\frac{1}{x}$

***Algebra of Derivatives:**

I) Sum Rule of Derivative :

If u and v are differentiable function of x and y = u + v then

$$\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}$$

Similarly, if y = u – v then

$$\frac{dy}{dx}=\frac{du}{dx}-\frac{dv}{dx}$$

II) Derivatives of Scalar multiplication :

If y = k.u where k is constant and u is function of x, then

$$\frac{dy}{dx} = k \cdot \frac{du}{dx}$$

III) Product Rule of Derivative :

If u and v are differentiable function of x and y = u · v then

$$\frac{dy}{dx} = u \cdot \frac{dv}{dx} + v\frac{du}{dx}$$

IV) Derivative and Division of two function :

If u and v are differentiable function of x and $y = \frac{u}{v}$ $v \neq 0$ then

$$\frac{dy}{dx} = \frac{v \cdot \frac{du}{dx} - u\frac{dv}{dx}}{v^2} \quad \text{when } v \neq 0$$

Ex.3. Find, $\frac{dy}{dx}$ if $y = 3x^5 + 2e^x + \frac{2}{3}\log x + 6$

Sol^n: Given function, $y = 3x^5 + 2e^x + \frac{2}{3}\log x + 6$

Differentiating with respect to x

$$\frac{dy}{dx} = \frac{d}{dx}\left(3x^5 + 2e^x + \frac{2}{3}\log x + 6\right)$$

$$= 3\frac{d}{dx}\left(x^5\right) + 2\frac{d}{dx}\left(e^x\right) + \frac{2}{3}\frac{d}{dx}(\log x) + \frac{d}{dx}(6)$$

$$= 3\left(5x^4\right) + 2e^x + \frac{2}{3}\left(\frac{1}{x}\right) + 0$$

$$\therefore \frac{dy}{dx} = 15x^4 + 2e^x + \frac{2}{3x}$$

Ex.4. Find, $\frac{dy}{dx}$ if $y = \left(x^2 + 1\right)\log x$

Sol^n: Given function, $y = \left(x^2 + 1\right)\log x$

Diff. w.r.t. x

$$\therefore \quad \frac{dy}{dx} = \left(x^2 + 1\right)\frac{d}{dx}(\log x) + \log x \cdot \frac{d}{dx}\left(x^2 + 1\right)$$

$$=\left(x^2+1\right)\times\frac{1}{x}+\log x\cdot 2x$$

$$\frac{dy}{dx}=\frac{x^2+1}{x}+2x\cdot\log x$$

Ex.5. Find, $\frac{dy}{dx}$ if $y=\frac{x^2+1}{2x+3}$

Sol^n: Given function

$$y=\frac{x^2+1}{2x+3}$$

Let $u=x^2+1$ and $v=2x+3$

Diff. w. r. t. x

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

$$\therefore\ \frac{dy}{dx}=\frac{(2x+3)\frac{d}{dx}\left(x^2+1\right)-\left(x^2+1\right)\frac{d}{dx}(2x+3)}{(2x+3)^2}$$

$$=\frac{(2x+3)(2x)-\left(x^2+1\right)(2)}{(2x+3)^2}$$

$$=\frac{4x^2+6x-2x^2-2}{(2x+3)^2}$$

$$\therefore\ \frac{dy}{dx}=\frac{2x^2+6x-2}{(2x+3)^2}$$

Ex.6. Find, $\frac{dy}{dx}$ if $y=\frac{1-\sqrt{x}}{1+2\sqrt{x}}$

Sol^n: Given function

$$y=\frac{1-\sqrt{x}}{1+2\sqrt{x}}$$

Let $u=1-\sqrt{x}$ and $v=1+2\sqrt{x}$

Diff. w. r. t. x.

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$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\cdot\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx}=\frac{(1+2\sqrt{x})\frac{d}{dx}\left(1-\sqrt{x}\right)-\left(1-\sqrt{x}\right)\frac{d}{dx}\left(1+2\sqrt{x}\right)}{\left(1+2\sqrt{x}\right)^2}$$

$$=\frac{(1+2\sqrt{x})\left(\frac{-1}{2\sqrt{x}}\right)-\left(1-\sqrt{x}\right).2.\left(\frac{1}{2\sqrt{x}}\right)}{\left(1+2\sqrt{x}\right)^2}$$

$$=\frac{\frac{-1}{2\sqrt{x}}-1-\frac{1}{\sqrt{x}}+1}{\left(1+2\sqrt{x}\right)^2}$$

$$=\frac{\frac{-3}{2\sqrt{x}}}{\left(1+2\sqrt{x}\right)^2}$$

$$=\frac{-3}{2\sqrt{x}\left(1+2\sqrt{x}\right)^2}$$

Ex.7. If f(x) = $2x^3 - 21x^2 + 72x + 17$

Find the values of x, such that $f'(x)=0$

Sol^n: Given function

$$f(x)=2x^3-21x^2+72x+17$$

Diff. w.r. t. x

$$f'(x)=\frac{d}{dx}\left(2x^3-21x^2+72x+17\right)$$

$$\therefore f'(x)=2\frac{d}{dx}\left(x^3\right)-21\frac{d}{dx}\left(x^2\right)+72\frac{d}{dx}(x)+\frac{d}{dx}(17)$$

$$=2\left(3x^2\right)-21(2x)+72\times(1)+0$$

$$f'(x) = 6x^2 - 42x + 72$$

$$\therefore f'(x) = 6(x-3)(x-4)$$

but given that $f'(x) = 0$

$$6(x-3)(x-4) = 0$$

$x-3=0$ OR $x-4=0$

$x=3$ OR $x=4$

- **Check your progress :**

Q.1. Differentiate the following function w.r.t. x

a) $y = \frac{x^3}{3} - 2x^2 + 6x + 3$ $\left(Ans: \frac{dy}{dx} = x^2 - 4x + 6\right)$

b) $y = x^4 - 3x^2 + x$ $\left(Ans: \frac{dy}{dx} = 4x^3 - 6x + 6x + 1\right)$

c) $y = (x^3 - 3x^2 + 4)(x^4 - 1)$ $\left(Ans: \frac{dy}{dx} = 7x^6 - 18x^5 + 16x^3 - 3x^3 + 6x\right)$

d) $y = x(x-2)(x^2+1)$ $\left(Ans: \frac{dy}{dx} = 4x^3 - 6x^2 + 2x - 2\right)$

e) $y = \frac{x^2 - 2x}{x^2 + 7}$ $\left(Ans: \frac{dy}{dx} = \frac{2x^3 + 14x - 14}{(x^2+7)^2}\right)$

f) $y = (1+x)\sqrt{x}$ $\left(Ans: \frac{dy}{dx} = \frac{1+3x}{2\sqrt{x}}\right)$

g) $y = \frac{x}{1+x}$ $\left(Ans: \frac{dy}{dx} = \frac{1}{(1+x)^2}\right)$

Q.2. If $f(x) = \frac{\sqrt{x}}{x+1}$, show that $f'(1) = 0$

Q.3. If $y = x^{3/2} - x$, show that $\left[\frac{dy}{dx}\right]_{x=4} = 2$

*** Chain Rule:**

(Differentiation of a function of a function)

This is also called function of a function which is very useful. We can say that y is function of u and u is function of x, so that y is of the type of function of a function, it is denoted by,

$$y = f\left[g\left(x\right)\right]$$

This is also called as composite function of x. In composite function y is not directly function of x, but it is a function of x connected through sum other variable.

If y is function of u and u is function of x then derivative of y with respect to x is equal to the product of the derivative of y with respect to u and derivative of u with respect to x.

i.e. $$\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx}$$

this result is called chain rule of differentiation.

Ex.8. Find $\frac{dy}{dx}$, if $y = \left(5x-7\right)^3$

Sol^n. $y = \left(5x-7\right)^3$

Diff. w. r. t. x

$$\frac{dy}{dx} = \frac{d}{dx}\left(5x-7\right)^3$$

$$= 3\left(5x-7\right)^2 \frac{d}{dx}\left(5x-7\right)$$

$$= 3\left(5x-7\right)^2 \cdot \left[5\left(1\right)\right]$$

$$\therefore \quad \frac{dy}{dx} = 15\left(5x-7\right)^2$$

Ex.9. Find $\frac{dy}{dx}$, if $y = \sqrt{5x^2-4x+1}$

Sol^n. $y = \sqrt{5x^2-4x+1}$

Diff. w.r. t. x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{5x^2-4x+1}\right)$$

$$= \frac{1}{2\sqrt{5x^2-4x+1}} \cdot \frac{d}{dx}\left(5x^2-4x+1\right)$$

$$= \frac{1}{2\sqrt{5x^2-4x+1}} \times \left[5(2x)-4(1)\right]$$

$$= \frac{10x-4}{2\sqrt{5x^2-4x+1}}$$

$$= \frac{(5x-2)}{\sqrt{5x^2-4x+1}}$$

$$= \frac{5x-2}{\sqrt{5x^2-4x+1}}$$

$$\therefore \frac{dy}{dx} = \frac{5x-2}{\sqrt{5x^2-4x=1}}$$

Ex.10. If $y = \sqrt{x} + \frac{1}{\sqrt{x}}$

Prove that $2x\frac{dy}{dx} = \sqrt{x} - \frac{1}{\sqrt{x}}$

Sol^n. $y = \sqrt{x} + \frac{1}{\sqrt{x}}$

Diff. w.r. t. x

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)$$

$$= \frac{1}{2\sqrt{x}} - \frac{1}{2x^{3/2}}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} - \frac{1}{2x\sqrt{x}}$$

$$2x\frac{dy}{dx} = \sqrt{x} - \frac{1}{\sqrt{x}}$$

Ex.11. If = $y = \frac{e^x + e^{-x}}{e^x - e^{-x}}$ find $\frac{dy}{dx}$

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Sol^n. $y=\frac{e^{x}+e^{-x}}{e^{x}-e^{-x}}$

Diff. w.r. t. x

$$\frac{dy}{dx} = \frac{\left(e^{x}-e^{-x}\right)\frac{d}{dx}\left(e^{x}+e^{-x}\right)-\left(e^{x}+e^{-x}\right)\frac{d}{dx}\left(e^{x}-e^{-x}\right)}{\left(e^{x}-e^{-x}\right)^{2}}$$

$$= \frac{\left(e^{x}-e^{-x}\right)\left(e^{x}-e^{-x}\right)-\left(e^{x}+e^{-x}\right)\left(e^{x}+e^{-x}\right)}{\left(e^{x}-e^{-x}\right)^{2}}$$

$$= \frac{\left(e^{x}-e^{-x}\right)^{2}-\left(e^{x}+e^{-x}\right)^{2}}{\left(e^{x}-e^{-x}\right)^{2}}$$

$$\frac{dy}{dx} = \frac{-4e^{x}\cdot e^{-x}}{\left(e^{x}-e^{-x}\right)^{2}}$$

$$\therefore \frac{dy}{dx} = \frac{-4}{\left(e^{x}-e^{-x}\right)^{2}}$$

Ex.12. Find $\frac{dy}{dx}$, if $y=\log\left(\frac{2+3x}{x-3}\right)$

Solu^n. $y=\log\left(\frac{2+3x}{x-3}\right)$ $\left[\because \log\left(\frac{m}{n}\right)=\log m-\log n\right]$

$$y=\log(2+3x)-\log(x-3)$$

Diff. w. r. t. x

$$\frac{dy}{dx} = \frac{d}{dx}\left[\log(2+3x)-\log(x-3)\right]$$

$$= \frac{1}{2+3x}.\frac{d}{dx}(2+3x)-\frac{1}{x-3}.\frac{d}{dx}(x-3)$$

$$= \frac{1}{2+3x}(3)-\frac{1}{x-3}(1)$$

$$\therefore \frac{dy}{dx} = \frac{3}{2+3x}-\frac{1}{x-3}$$

Ex.13. If $y=\log\left[\frac{\left(x^{2}-2\right)\sqrt{3x+5}}{\sqrt[3]{x+1}}\right]$ find $\frac{dy}{dx}$

Sol^n. $y=\log\left[\frac{\left(x^{2}-2\right)\sqrt{3x+5}}{\sqrt[3]{x+1}}\right]$

$$y=\log\left[\left(x^{2}-2\right)\left(\sqrt{3x+5}\right)\right]-\log\left(\sqrt[3]{x+1}\right)$$

$$y=\log\left(x^{2}-2\right)+\frac{1}{2}\log(3x+5)-\frac{1}{3}\log(x+1)$$

Diff. w.r. t. x.

$$\frac{dy}{dx}=\frac{d}{dx}\log\left(x^{2}-2\right)+\frac{1}{2}\frac{d}{dx}\log\left(3x+5\right)-\frac{1}{3}\frac{d}{dx}\log\left(x+1\right)$$

$$\frac{dy}{dx}=\frac{1}{x^{2}-2}.\frac{d}{dx}\left(x^{2}-2\right)+\frac{1}{2}\cdot\frac{1}{3x+5}\cdot\frac{d}{dx}(3x+5)-\frac{1}{3}\frac{1}{x+1}\cdot\frac{d}{dx}\left(x+1\right)$$

$$\therefore\frac{dy}{dx}=\frac{2x}{x^{2}-2}+\frac{3}{2\left(3x+5\right)}-\frac{1}{3\left(x+1\right)}$$

- **Check your progress :**

  Q.1.Find $\frac{dy}{dx}$

  a) If $y=\log\left[e^{x}\left(\frac{x-2}{x+3}\right)^{3/4}\right]$ $\left[Ans.\frac{dy}{dx}=1+\frac{3}{4}\left(\frac{1}{x-2}-\frac{1}{x+3}\right)\right]$

  b) If $y=\log\left(x+\sqrt{x^{2}+a^{2}}\right)$ $\left[Ans.\frac{dy}{dx}=\frac{1}{\sqrt{x^{2}+a^{2}}}\right]$

  c) If $y=\log\left(\frac{x+\sqrt{x^{2}+a^{2}}}{-x+\sqrt{x^{2}+a^{2}}}\right)$ $\left[Ans.\frac{dy}{dx}=\frac{2}{\sqrt{x^{2}+a^{2}}}\right]$

d) If $y=\log\left(e^{mx}+e^{-mx}\right)$ $\left[Ans.\dfrac{dy}{dx}=\dfrac{m\left(e^{mx}+e^{-mx}\right)}{\left(e^{mx}+e^{-mx}\right)}\right]$

e) If $y=\log_5 x$ $\left[Ans.\dfrac{dy}{dx}=\log_5 e\cdot\dfrac{1}{x}\right]$

f) If $y=3^{x\log x}$ $\left[Ans.\dfrac{dy}{dx}=3^{x\log x}\cdot\log 3\cdot\left(1+\log x\right)\right]$

g) If $y=\left(2x^2-5x+3\right)^4$ $\left[Ans.\dfrac{dy}{dx}=4\left(4x-5\right)\left(2x^2-5x+3\right)^3\right]$

h) If y= $e^{\frac{x^2}{1+x^2}}$ $\left[Ans.\dfrac{dy}{dx}=e^{\frac{x^2}{1+x^2}}\cdot\dfrac{2x}{\left(1+x^2\right)^2}\right]$

i) If $y=3^{x^2+2x}$ $\left[Ans.\dfrac{dy}{dx}=3^{x^2+2x}\cdot\log 3\cdot\left(2x+2\right)\right]$

j) If $y=\sqrt{a+\sqrt{a+x}}$ $\left[Ans.\dfrac{dy}{dx}=\dfrac{1}{4\sqrt{a+x}\left(\sqrt{a+\sqrt{a+x}}\right)}\right]$

k) If $y=\sqrt{1-x^2}\cdot e^{5x^2}$ $\left[Ans.\dfrac{dy}{dx}=10x\cdot e^{5x^2}\sqrt{1-x^2}-\dfrac{xe^{5x^2}}{\sqrt{1-x^2}}\right]$

l) If $y=\sqrt{\dfrac{1+e^x}{1-e^x}}$ $\left[Ans.\dfrac{dy}{dx}=-\dfrac{e^x}{\left(1-e^x\right)^{3/2}\cdot\left(1+e^x\right)^{1/2}}\right]$

Q.2. If $y=\log\left(\sqrt{x-1}-\sqrt{x+1}\right)$

Show that $\dfrac{dy}{dx}+\dfrac{1}{\sqrt{x^2+1}}=0$

Q.3. If $y=\log\left[\sqrt{\dfrac{x-1}{x+1}}\right]$

Prove that: $\dfrac{dy}{dx}=\dfrac{1}{x^2-1}$

***Implicit Functions:**

An equation in the form $y = f(x)$ defines y as an explicit function of x and an equation in form of $x = g(y)$ defines x as an explicit function of y.

A function or relation in which the dependent variable is not isolated on one side of the equation i.e. an equation is in the form $f(x, y) = 0$, defines y as an implicit function of x.

Ex.14. If $\sqrt{x}+\sqrt{y}=\sqrt{a}$ find $\dfrac{dy}{dx}$

Sol^n. $\sqrt{x}+\sqrt{y}=\sqrt{a}$

Diff. w. r. t. x on both sides,

$$\frac{d}{dx}\left(\sqrt{x}+\sqrt{y}\right)=\frac{d}{dx}\left(\sqrt{a}\right)$$

$$\therefore\ \frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\ \frac{dy}{dx}=0$$

$$\therefore\ \frac{1}{2\sqrt{y}}\cdot\frac{dy}{dx}=-\frac{1}{2\sqrt{x}}$$

$$\therefore\ \frac{dy}{dx}=\frac{-\sqrt{y}}{\sqrt{x}}$$

$$\therefore\ \frac{dy}{dx}=-\sqrt{\frac{y}{x}}$$

Ex.15. Find $\frac{dy}{dx}$

If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Solu^n.

Given: $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Diff. w.r.t. x

$$\frac{d}{dx}\left[ax^2 + 2hxy + by^2 + 2gx + 2fy + c\right] = 0$$

$$\therefore\ 2ax + 2h\frac{d}{dx}(xy) + 2by\frac{d}{dx}(y) + 2g + 2f\frac{d}{dx}(y) + 0 = 0$$

$$\therefore\ 2ax + 2h\left[x\frac{dy}{dx} + y(1)\right] + 2by\frac{dy}{dx} + 2g + 2f\frac{dy}{dx} = 0$$

$$\therefore\ ax + hx\frac{dy}{dx} + hy + by\frac{dy}{dx} + g + f\frac{dy}{dx} = 0$$

$$\therefore\ \frac{dy}{dx}(hx + by + f) = -(ax + hy + g)$$

$$\therefore\ \frac{dy}{dx} = \frac{-(ax + hy + g)}{(hx + by + f)}$$

*** <u>Higher order derivative OR successive differentiation:</u>**

Let y = f(*x*) be a function of *x*, defined over an interval, then 1st order derivative is given by $f'(x)$ while differentiate with respect to *x*.

i.e. $y = f(x)$

$$\therefore\ \frac{dy}{dx} = f'(x)$$

Now $\phi = f'(x)$

again diff. w.r.t. x

We get $\frac{d^2y}{dx^2} = \frac{d\phi}{dx} = f''(x)$

It is called second order derivative of *f*(*x*)

In this manner by successive differentiation of the function is given by

$$\frac{d^n y}{dx^n} = f^n(x)$$

Ex.16. Find $\frac{d^2 y}{dx^2}$

If $y = 5x^7 - 4x^3 + 7x^2 + 20$

Sol^n.

Given: $y = 5x^7 - 4x^3 + 7x^2 + 20$

Diff. w.r.t. x

$$\frac{dy}{dx} = \frac{d}{dx}\left(5x^7 - 4x^3 + 7x^2 + 20\right)$$

$$= 5\frac{d}{dx}\left(x^7\right) - 4\frac{d}{dx}\left(x^3\right) + 7\frac{d}{dx}\left(x^2\right) + \frac{d}{dx}(20)$$

$$= 5\left(7x^6\right) - 4\left(3x^2\right) + 7(2x) + 0$$

$$\therefore \frac{dy}{dx} = 35x^6 - 12x^2 + 14x$$

$$\therefore \frac{d^2 y}{dx^2} = \frac{d}{dx}(35x^6 - 12x^2 + 14x)$$

$$= 35\frac{d}{dx}\left(x^6\right) - 12\frac{d}{dx}\left(x^2\right) + 14\frac{d}{dx}(x)$$

$$= 35\left(6x^5\right) - 12(2x) + 14(1)$$

$$\therefore \frac{d^2 y}{dx^2} = 210x^5 - 24x + 14$$

Note:

1) $\frac{d^2 y}{dx^2}$ is the derivative of $\frac{dy}{dx}$ w. r. t. x

2) $\frac{d^2 y}{dx^2} \neq \left(\frac{dy}{dx}\right)^2$

Ex.17. If $y = \left[x + \left(\sqrt{x^2 + 1}\right)\right]^p$

Prove that $\left(1 + x^2\right)\frac{d^2 y}{dx^2} + x \cdot \frac{dy}{dx} = p^2 y = 0$

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Sol^n. $y=\left[x+\left(\sqrt{x^2+1}\right)\right]^p$

Diff. w.r.t. x

$$\frac{dy}{dx}=p\left(x+\sqrt{x^2+1}\right)^{p-1}\frac{d}{dx}\left(x+\sqrt{x^2+1}\right)$$

$$=p\left(x+\sqrt{x^2+1}\right)^{p-1}\left[1+\frac{2x}{2\sqrt{x^2+1}}\right]$$

$$=p\left(x+\sqrt{x^2+1}\right)^{p-1}\left[1+\frac{x}{\sqrt{x^2+1}}\right]$$

$$=p\left(x+\sqrt{x^2+1}\right)^{p-1}\left[\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}\right]$$

$$=\frac{p\left(x+\sqrt{x^2+1}\right)^{p-1}\left(x+\sqrt{x^2+1}\right)}{\sqrt{x^2+1}}$$

$$\therefore \frac{dy}{dx}=\frac{p\left(x+\sqrt{x^2+1}\right)^{p}}{\sqrt{x^2+1}}$$

$$\therefore \quad \sqrt{x^2+1}\ \frac{dy}{dx}=p\left(x+\sqrt{x^2+1}\right)^{p}$$

Again Diff. w.r.t. x

$$\therefore \frac{x}{\sqrt{x^2+1}}\frac{dy}{dx}+\sqrt{x^2+1}\frac{d^2y}{dx^2}=p^2\left[x+\sqrt{x^2+1}\right]^{p-1}\cdot\left[1+\frac{x}{\sqrt{x^2+1}}\right]$$

$$\therefore \frac{x}{\sqrt{x^2+1}}\cdot\frac{dy}{dx}+\sqrt{x^2+1}\frac{d^2y}{dx^2}=p^2\left[x+\sqrt{x^2+1}\right]^{p-1}\left[\frac{x+\sqrt{x^2+1}}{\sqrt{x^2+1}}\right]$$

$$\therefore \frac{x}{\sqrt{x^2+1}}\frac{dy}{dx}+\sqrt{x^2+1}\frac{d^2y}{dx^2}=\frac{p^2\left[x+\sqrt{x^2+1}\right]^{p}}{\sqrt{x^2+1}}$$

$$\therefore \frac{x}{\sqrt{x^2+1}}\frac{dy}{dx}+\sqrt{x^2+1}\frac{d^2y}{dx^2}=\frac{p^2y}{\sqrt{x^2+1}}$$

Multiplying by $\sqrt{x^2+1}$ and re-arranging

$$\left(1+x^2\right)\frac{d^2y}{dx^2}+x\cdot\frac{dy}{dx}-p^2y=0$$

Ex.18. If $y=x^3\ \log\frac{1}{x}$ *then*

Prove that $\frac{d^2y}{dx^2}-\frac{2}{x}\frac{dy}{dx}+3x=0$

Sol^n^. $y=x^3\log\frac{1}{x}$

$$\therefore \quad y=-x^3\log x\left[\because \log\frac{1}{x}=-\log x\right]$$

Diff. w.r.t. x

$$\therefore \frac{dy}{dx}=-\frac{d}{dx}(x^3\log x)$$

$$\frac{dy}{dx}=-\left[3x^2\cdot\frac{d}{dx}\left(\log x\right)+\log x\frac{d}{dx}\left(x^3\right)\right]$$

$$=-\left[x^3\left(\frac{1}{x}\right)+\log x\left(3x^2\right)\right]$$

$$=-\left[x^2+3x^2\log x\right]$$

$$=-x^2-3x^2\log x$$

$$=-x^2+\frac{3}{x}\left(-x^3\log x\right)\left[\because y=-x^3\log x\right]$$

$$\frac{dy}{dx}=-x^2+\frac{3}{x}y$$

$$\therefore x\frac{dy}{dx}=3y-x^3$$

Again differentiating both sides with respect to x

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}=3\frac{dy}{dx}-3x^2$$

OR $x\frac{d^2y}{dx^2}-2\frac{dy}{dx}+3x^2=0$

Now dividing throughout by x, we get the required equation,

$$\frac{d^2y}{dx^2}-\frac{2}{x}\cdot\frac{dy}{dx}+3x=0$$

Ex.19. If $y=ax+b\sqrt{x}$ (a, b are constants)

Prove that $2x^2\frac{d^2y}{dx^2}-x\cdot\frac{dy}{dx}+y=0$

Sol^n^. Given: $y=ax+b\sqrt{x}$

$$\left[\text{Substituting value of } y,\frac{dy}{dx},\frac{d^2y}{dx^2}\right]$$

$$\therefore \frac{dy}{dx}=a+\frac{b}{2\sqrt{x}} \quad =a+\frac{bx^{-1/2}}{2} \qquad \ldots\ldots \quad \text{(i)}$$

Differentiating again

$$\therefore \frac{d^2y}{dx^2}=\frac{b}{2}\cdot\left(-\frac{1}{2}x^{-3/2}\right)=-\frac{b}{4x^{3/2}} \qquad \ldots\ldots \quad \text{(ii)}$$

$$\therefore 2x^2\frac{d^2y}{dx^2}-x\frac{dy}{dx}+y$$

$$=2x^2\left[\frac{-b}{4x^{3/2}}\right]-x\left[a+\frac{b}{2\sqrt{x}}\right]+\left(ax+b\sqrt{x}\right)$$

$$\left[\text{Substituting value of } y,\frac{dy}{dx},\frac{d^2y}{dx^2}\right]$$

$$=\frac{-bx^2}{2x^{3/2}}-ax-\frac{b\sqrt{x}}{2}+ax+b\sqrt{x}$$

$$=\frac{-b\sqrt{x}}{2}-\cancel{ax}-\frac{b\sqrt{x}}{2}+\cancel{ax}+b\sqrt{x}$$

$$=\frac{-2b\sqrt{x}}{2}+b\sqrt{x}$$

$$=-b\sqrt{x}+b\sqrt{x}$$

$= 0$, Hence proved.

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5A.3. SUMMARY

In this chapter we have learned.

- First principal of derivative
- Physical and geometrical meaning of derivative
- Rules of Differentiation
- Chain rules of differentiation
- Differentiation of Implicit function
- Successive differentiation

5A.4. UNIT END EXERCISE:

(a) Find $\frac{dy}{dx}$ for the following :

1) $y = 5x^6 - 7x^3 + 9x^2 + 11$ $\left[Ans. 30x^5 - 21x^2 + 18x\right]$

2) $y = \left(\sqrt{x} + \frac{1}{x}\right)\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)$ $\left[Ans. 1 + \frac{1}{x^2}\right]$

3) $y = 6a^x - 7\log_e x + 6x^{-2/3} + 21$ $\left[Ans. 6a^x \log a - \frac{7}{x} - 4x^{-5/3}\right]$

4) $y = \frac{\sqrt{a} + \sqrt{x}}{\left(\sqrt{a} - \sqrt{x}\right)^2}$ $\left[Ans. \frac{\sqrt{a}}{\sqrt{x}\left(\sqrt{a} - \sqrt{x}\right)^2}\right]$

5) $y = \log\left(x + \sqrt{x^2 + a^2}\right)$ $\left[Ans. \frac{1}{\sqrt{x^2 + a^2}}\right]$

6) $y = e^x \log\left(1 + x^2\right)$ $\left[Ans.\ \ e^x\left(\log\left(1 + x^2\right) + \frac{2x}{\left(1 + x^2\right)}\right)\right]$

7) $y = \sqrt{1 - x^2}.e^{5x^2}$ $\left[Ans. 10xe^{5x^2}\left(\sqrt{1 - x^2}\right) - \frac{xe^{5x^2}}{\sqrt{1 - x^2}}\right]$

b) $y = 6x^5 - 4x^4 - 2x^2 + 5x - 9$. Find $\frac{dy}{dx}$ at x = -1 [Ans: 55]

c) Find the slope of the curve $y = \left(x^2 - 3x + 2\right)$ Hint: (find $\frac{dy}{dx}$ at x = 1)]

d) Find the rate at which the function f(x) = $x^5+3x^3-7x^2+9$ changes with respect to x.[Ans. $5x^4+9x^2-14x$]

e) If $y=\log\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)$, prove that $\frac{dy}{dx}=\frac{x-1}{2x(x+1)}$

$\left[\text{Hint: y = }\log(x+1)-1/2\log x\right]$

f) If $y=\frac{x}{(x+2)}$ prove that $x\frac{dy}{dx}=(1-y)y$

g) If $y=e^x+e^{-x}$ prove that $\frac{dy}{dx}=\sqrt{y^2-4}$

h) If $y=\sqrt{a^2-x^2}$ prove that $y\frac{dy}{dx}+x=0$

i) Find $\frac{dy}{dx}$, when,

1) $x^5+y^5-5ax^2y^2=0$ $\left[Ans.\frac{2axy^2-x^4}{(y^4-2ax^2y)}\right]$

2) $y+\sqrt{xy}=x^2$ $\left[Ans.\frac{\sqrt{y}\left(4x\sqrt{x}-\sqrt{4}\right)}{\sqrt{x}\left(2\sqrt{y}+\sqrt{x}\right)}\right]$

3) $\log(xy)=x^2+y^2$ $\left[Ans.\frac{y(2x^2-1)}{x(1-2y^2)}\right]$

4) $e^{x-y}=\log\left(\frac{x}{y}\right)$

j) If $x\sqrt{1+y}+y\sqrt{1+x}=0$ prove that $(1+x^2)\frac{dy}{dx}+1=0$

k) If $xy^2=1$ prove that $2\frac{dy}{dx}+y^2=0$

l) If $y=\left(x+\sqrt{x^2+1}\right)^m$ show that $(x^2+1)\frac{d^2y}{dx^2}+x\frac{dy}{dx}-m^2y=0$

♦♦♦♦♦

5B

APPLICATIONS OF DERIVATIVE

UNIT STRUCTURE

5B.1. OBJECTIVES

After going through this chapter you will able to know:

- The relation between average revenue and marginal revenue
- Relationship between average cost and marginal cost
- Relation between marginal product and marginal cost.
- Relation between production, cost and revenue functions.
- Relation between marginal propensity to consume and marginal propensity to save.
- Calculate concavity, convexity and point of Inflection of function.

- Calculate maximum and minimum value of function at a particular point.

5B.2 INTRODUCTION

In the study of all economic problems it is essentially a problem of finding out the rate of change. It may be the rate of change in the dependent variable (demand) with respect to the change in the explanatory variable (price) or it may be the rate of change in the endogenous variable (national income) with respect to change in the exogenous variable (government expenditure) or a particular parameter (marginal propensity to consume). The mathematical tool which is used to find out the magnitude and direction of change in a particular variable due to change in the value of other variables of parameters is broadly known as the technique of derivative or differentiation. The concept of derivative is used to deal with a variety of economic problems.

***Demand Function:**

It is relation between demand and price of commodity. Let P be the price and D be demand of commodity. Then we can write:

$$D = f(P)$$

Also write as $P = f(D)$

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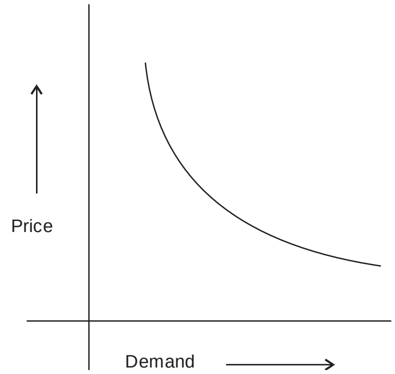
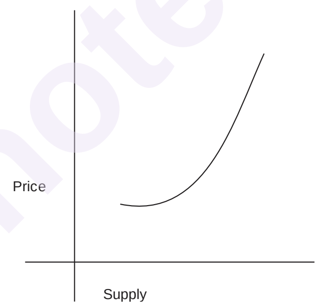


Note: Demand function is always decreasing function.

***Supply Function:**

It is relationship between supply and price of commodity. Let P be the price and S be supply commodity. Then we can write

S = *f* (p)

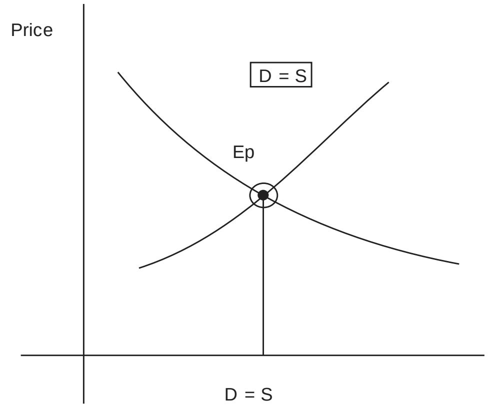


Also write as P = *f* (s)

Note: Supply function is always increasing function.

***Equilibrium Price:**

When demand and supply of goods are equal that condition at point is called Equilibrium point. For Demand D and Supply S for particular goods, curves are intersect each other, then the point of intersection is called Equilibrium point on price.



***Cost Function**:*

The amount required to produce x units of goods, is called cost. It C denotes the production cost at x units of goods, then it can be written as

C(x) = Fixed cost + Variable Cost

i.e. C(x) = a + $b\,x$ when $x > 0$

***Revenue Function**:*

The amount received from selling the product. It is denoted by R let x units are sold at price P, then Revenue is R = p x = demand × price

R(x) = P x

***Profit Function**:*

Let R be revenue and C be cot function for x unit of goods.

Profit = Revenue – Cost

$$\boxed{\pi = R - C}$$

$$\pi(x) = R(x) - C(x)$$

***Average Cost:**

The cost per unit is called Average Cost. Let C be cost function of x units then,

$$(AC)\ \text{Average Cost} = \frac{\text{Total Cost}}{\text{No. of Units}} = \frac{C}{x}$$

***Marginal Cost:**

The rate of change of cost (C) with respect to quantity : 'x' is called Marginal Cost it is denoted by MC.

$$MC = \frac{d}{dx}\left[C(x)\right] = \frac{dc}{dx}$$

***Marginal Average Cost:**

The rate of change at Average cost (AC) with respect to quantity x is called Marginal Average Cost.

$$MAC = \frac{d}{dx}(AC) = \frac{d(AC)}{dx}$$

***Marginal Revenue:**

The rate of total Revenue with respect to the quantity x i.e.

$$MR = \frac{d}{dx}\left[R(x)\right] = \frac{dR}{dx}$$

In marginal analysis at firm operating under.

i) Perfect competition and

ii) Monopoly

I) Perfect Competition:

i) In this case the price is constant then

Average Revenue = $AR = \frac{R}{x} = \frac{Px}{x} = P$

= Constant

ii) Marginal Revenue= $MR = \frac{dR}{dx} = \frac{d}{dx}(P\ x) = P$

= Constant

i.e. In Perfect Competition AR = MR = P = Constant

II) Monopoly –

In this case a monopoly is a sale suppliers of goods produced be fixed the price ‘P’ of good accordingly to the demand in the market. Thus price is not constant.

i) $AR = \frac{Px}{x} = P$, When P is function of x

ii) $MR = \frac{d}{dx}R = \frac{d}{dx}(Px) = P + x\frac{dP}{dx}$

Thus $AR \neq MR$

*** <u>Marginal Revenue of Product</u>** <u>(MRP):</u>

Let R be the revenue of x units with price P when P is function x and $\frac{dR}{dt}$ is Rate of change of Revenue with respect to employee t’ is called Marginal Revenue.

Ex-1. For a certain product, cost function is given by $C = 3x^4 - 5x^2 + 50x + 20$. Find Average Cost, Marginal Cost and Marginal Average Cost when x = 5.

Solu^n: Given Cost Function, $C = 3x^4 - 5x^2 + 70x + 20$

$$AC = \frac{\text{Total Cost}}{x} = \frac{3x^4 - 5x^2 + 70x + 20}{x} = 3x^2 - 5x + 70 + \frac{20}{x}$$

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$$(\text{AC})_{x=5} = (5)^3 - 5(5) + 70 + \frac{20}{5}$$

$$= 3(125) - 25 + 70 + 4$$

$$= 375 - 25 + 74$$

$$= 424$$

$$MC = \frac{dc}{dx} = \frac{d}{dx} \cdot \left(3x^4 - 5x^2 + 70x + 20\right)$$

$$MC = 12x^3 - 10x + 70$$

$$(MC)_{x=5} = 12(5)^3 - 10(5) + 70 = 1520$$

$$MAC = \frac{d}{dx}(AC)$$

$$= \frac{d}{dx}\left(3x^3 - 5x + 70 + \frac{20}{x}\right)$$

$$= 9x^2 - 5 + 0 - \frac{20}{x^2}$$

$$= 9x^2 - 5 - \frac{20}{x^2}$$

$$(MAC)_{x=5} = 9(5)^2 - 5 - \frac{20}{(5)^2} = 219.2$$

Ex-2. If the demand function is given by

$P = 3x^2 - 5x + 25$ When x is the demand

Find: i) Revenue function

ii) Average revenue function

iii) Marginal Revenue when x =10

Solu^n:

Curve demand function $P = 3x^2 - 5x + 25$

Revenue function $R = pD$

$$= \left(3x^2 - 5x + 25\right)(x)$$

$$= 3x^3 - 5x^2 + 25x$$

Average Revenue function

$$AR \quad = \frac{R}{D} = \frac{PD}{D} = P$$

$$= 3x^2 - 5x + 25$$

Marginal Revenue = $\dfrac{dR}{dx} = \dfrac{d}{dx}(R)$

$$= \frac{d}{dx}\left(3x^3 - 5x^2 + 25x\right)$$

$$= 9x^2 - 10x + 25$$

$$(MR)_{x=10} = 9(10)^2 - 10(10) + 15$$

$$= 900 - 100 + 25 = 825$$

Ex-3. The manufacturer x units of articles at a cost (12 x + 95) per units and the demand function if P = 47x – 45, when P is price and x is demand. Find x for which the total profit is increasing.

Solu^n :

Let no. of units = x

Cost per units = 12x + 95

∴ Total Cost = C = (12x +95) x = 12x^2 + 95x

Demand function P = 47x – 45

∴ Revenue Function = R = PD

= (47 x – 45) × x

= 47x^2 - 45x

Profit Function $f(x)$ = $\pi = R - C$

$$= (47x^2 - 45x) - \left(12x^2 + 95x\right)$$

$$= 35x^2 - 140x$$

$$\therefore \pi = 35x^2 - 140x$$

Diff. w. r. t. x

$$f'(x) = \frac{d\pi}{dx} = 70x - 140$$

For increasing Function $f'(x) > 0$

$$i.e. \frac{d\pi}{dx} > 0$$

$$\therefore\ 70x - 140 > 0$$

$$\therefore\ 70x > 140$$

$$\therefore\ x > 2$$

The total profit is increasing for $R = 5x^3 - 45x^2 + 120x + 30$

5B.3. INCREASING AND DECREASING FUNCTION

***Increasing Function:**

If Y = *f*(*x*) is a function of *x* in the interval (a, b) and if Y increases as *x* increases in (a, b) then Y is called the increasing function of *x* in the interval (a, b).

Let *f*(*x*) be increasing function in the interval (a, b) if $x_1, x_2 \in (a, b)$ such that $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

***Decreasing function:**

If Y = *f*(*x*) is a function of x in the interval (a, b) and if Y, decreases as x is increases and vice versa in (a, b) then Y is called the decreasing function of x in the interval (a, b)

Let *f* (*x*) be decreasing function in the interval (a, b) if $x_1, x_2 \in (a, b)$ such that $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$

5B.4. MAXIMA AND MINIA

***Maximum Points:**

A function f(*x*) is said to have a maximum value in an interval I around *x* = a, if a $\in$ I, and if *f* (a) $\Rightarrow$ *f*(*x*), *x* $\in$ I then *f* (a) is called the maximum value of $f(x)$ $\in$ I and a is called the point of maximum at f(x) in I.

***Minimum Point**:*

A function $f(x)$ is said to have a minimum value in interval I around x = a, if a $\in$I and if f (a) $\leq f$ (x), x $\in$ I, then f (x) is called the minimum value of f (x) in I and a is called the point of minimum of f (x) in I.

Note:

If Y = f (x) where the slope of f (x) is neither positive nor negative but it is zero i.e. at this point target is parallel to x-axis.

i.e. $\frac{dy}{dx} = 0$ or $f'(x) = 0$ at x = 0

*** Second Derivative Test for Maxima and Minima:**

If Y = f (x) is continuous differential function at neighbourhood of a point 'a'

i) $f(x)$, is Maximum at x = a

If $f'(a) = 0$ and $f''(a) < 0$

ii) f(x) is minimum at x = a, $f'(a) = 0$ and $f''(a) > 0$

Following steps are to be followed

Step – I First find under derivative or $f(x)$ i.e. $f'(x)$

Step – II taking $f'(x) = 0$ find x = a, b, c root

Step – II find second order derivative i. e. $f''(x)$

Step - IV

(a) $f(x)$ is maximum at x = a if

f'' (x = a) < 0

(b) $f(x)$ is minimum at x = b if f'' (x = b) > 0

Ex. 4. Find values of x for which function

$f(x) = 2x^3 - 3x^2 - 72x + 100$ is

(i) Increasing (ii) Decreasing.

Solu^n:

Given: $f(x) = 2x^3 - 3x^2 - 72x + 100$

Diff. w. r. t. x

$$f'(x) = 6x^2 - 6x - 72$$

i) $f(x)$ is increasing if $f'(x) > 0$

i.e. $6x^2 - 6x - 72 > 0$

$6(x^2 - x - 12) > 0$

$\therefore \quad 6(x - 4)(x + 3) > 0$

i.e. $x > 4$ an $x > -3$

$\therefore \quad f(x)$ is increasing when $x > 4$ or $x > -3$

ii) $f(x)$ is decreasing if $f'(x) < 0$

$6x^2 - 6x - 72 < 0$

$6(x^2 - x - 12) < 0$

$6(x-4)(x+3) < 0$

$x < 4$ as $x < -3$

$\therefore \quad f(x)$ is decreasing when $x < 4$ or $x < -3$

Ex.5. Find the local maxima and local Minima for the function

$$f(x) = x^3 - 6x^2 + 9x + 7$$

Solu^n:

Let given $f(x) = x^3 - 6x^2 + 9x + 7$ …. I

Diff. w.r.t. x

$f'(x) = 3x^2 - 12x + 9$

$f''(x) = 6x - 12$

Taking $f'(x)=0$ we get

$3x^2-12x+9=0$

$3(x^2-4x+3)=0$

$(x-3)(x-1)=0$

$x=3$ or $x=1$

$[f''(x)]_{x=1}=6(1)-12=-6<0$

$\therefore f(x)$ is maximum at $x=1$

$\therefore f(1)=(1)^3-6(1)^2+9(1)+7=11$

$\therefore f(x)$ is maximum at x = 1 & Maximum value = $f(1)$

$\therefore f(1)=(1)^3-6(1)^2+9(1)+7=11$

$\therefore f(x)$ is maxima at $x=1$ & Maximum value is 11

$[f''(x)]_{x=3}=6(3)-12=18-12=6>0$

$\therefore f(x)$ is minimum at $x = 3$ and minimum value = $f(3)$

$\therefore f(3) = (3)^3 - 6(3)^2 + 9(3) + 7 = 7$

$\therefore f(x)$ is minimum at $x = 3$ and minimum value is 7

Ex.6. For the certain product total cost $c=x^3-9x^2+24x+17$ Total Revenue $R=5x^3-45x^2+120x+30$. Find x for which the profit is maximum and minimum.

Solu^n^:

Here $C=x^3-9x^2+24x+17$

$R=5x^3-45x^2+120x+30$

Profit function:

$\pi(x)=R-C$

$=(5x^3-45x^2+120x+30)-(x^3-9x^2+24x+17)$

$=5x^3-45x^2+120x+30-x^3+9x^2-24x+17$

$=4x^3-36x^2+96x+13$ …. I

$\pi'(x) = 12x^2 - 72x + 96$ …. II

$\pi''(x) = 24x - 72$

Taking $\pi'(x) = 0$ we get

$\therefore 12x^2 - 72x + 96 = 0 \quad \therefore 12(x^2 - 6x + 8) = 0$

$\therefore (x-4)(x-2) = 0$

$\therefore x = 4$ OR $x = 2$

For $x = 2$

$\pi''(x=2) = 24(2) - 72 = 48 - 72 = -24 < 0$

$\therefore \pi(x)$ is maximum at $x = 2$ and maximum value = π $(x = 2)$

$\pi(x=2) = (2)^3 - 36(2)^2 + 96(2) + 13 = 93$

$\therefore \pi(x)$ is maximum at $x = 2$ and maximum profit is 93

$\pi''(x=4) = 24(4) - 72 = 24 > 0$

$\therefore \pi(x)$ is minimum at $x = 4$ and minimum value = $\pi(x=4)$

$\pi(x=4) = 4(4)^3 - 36(4)^2 + 96(4) + 13 = 77$

$\therefore \pi(x)$ is minimum at x = 4 & minimum value is 77

Ex.7. A manufacturer determines that the employees will produce a total of x units of a product per day, where x = 5t, if the demand equation for the product is P = -2 x + 100 to find the Marginal Revenue Product when t = 3. Interpret your result ….

Solu^n:

Here, $P = -2x + 100$ & $x = 5t$

$R = Px = (-2x + 100)x$

$R = -2x^2 + 100x$

$R = -2(5t)^2 + 100(5t)$

$$R = -50t^2 + 500t$$

$$R = -50t^2 + 500t$$

Diff. w. r. t. **t**

$$\frac{dR}{dt} = -100t + 500$$

$$\therefore \left(\frac{dR}{dt}\right)_{t=3} = -100(3) + 500 = 200$$

If 4th employee is hired the extra revenue generated is approximate 200.

Ex.8. Given a consumption function.

$$\mathbf{C}(x) = 2000 - \frac{6000}{10 + x}$$

1) Find out marginal propensity to consume (MPC) and marginal propensity to save when x = 90
2) Also show that MPC & MPS move in the opposite direction when income (x) changes

Solu^n:

I. Given $C = 2000 - \frac{6000}{(10 + x)}$

$$MPC = \frac{dC}{dx} = \frac{6000}{(10 + x)^2}$$

$$(MPC)_{x=90} = \frac{6000}{(10 + 90)^2} = \frac{3}{5} = 0.6$$

S is saving function defined as

$$\therefore S = x - C$$

$$= x - 2000 + \frac{6000}{(10 + x)}$$

$$MPS = \frac{d}{dx}(S) = \frac{d}{dx}\left[x - 2000 + \frac{6000}{10 + x}\right]$$

$$= 1 - 0 + (-1)\frac{6000}{(10+x)^2}$$

$$[MPS]_{x=90} = 1 - 0.6$$

$$= 0.4$$

II. $$\frac{d}{dx}(MPC) = \frac{d^2C}{dx^2} = (-2)\frac{6000}{(10+x)^3} = -\frac{12000}{(10+x)^3} < 0$$

$$\frac{d}{dx}(MPS) = \frac{d^2S}{dx^2} = -(-2)\frac{6000}{(10+x)^3} = \frac{12000}{(10+x)} > 0$$

Since $\frac{d^2C}{dx^2} < 0 \text{ and } \frac{d^2S}{dx^2} > 0. MPC \text{ and } MPS$ move in the opposite direction as x changes.

Ex.9. Given the total Cost Function

$$C = 1000 + 100x - 10x^2 + \frac{x^3}{3}$$

Find:

i) Marginal Cost Function

ii) The slope of marginal cost function

iii) Output at which marginal cost is equal to average variable cost.

Solu^n :

i) Given the total cost function

$$C = 1000 + 100x - 10x^2 + \frac{x^3}{3}$$

$$MC = \frac{d}{dx}[C] = \frac{d}{dx}\left[1000 + 100x - 10x^2 + \frac{x^3}{3}\right]$$

$$= 100 - 20x + x^2$$

ii) Slope of MC = $\frac{f(b) - f(a)}{b-a} \geq f'(x), x - 1$

$\therefore$ Slope of MC = $-20+2x$

iii) $C = 1000 + 100x - 10x^2 + \frac{x^3}{3}$

Where $V(x) = 100x - 10x^2 + \frac{x^3}{3}$ and F.C. = 1000

$$\therefore \quad AVC = \frac{V(x)}{x} = \frac{100x - 10x^2 + x^3/3}{x}$$

$$= 100 - 10x + \frac{x^2}{3}$$

As per given condition we want to find our output at which

MC = AVC

$$\therefore \ 100 - 20x + x^2 = 100 - 10x + \frac{x^2}{3}$$

$$\therefore \ \frac{2}{3}x^2 - 10x = 0$$

$$\therefore \ \left(\frac{2}{3}x - 10\right)x = 0$$

$$\therefore x = 0 \ OR \ \frac{2}{3}x - 10 = 0$$

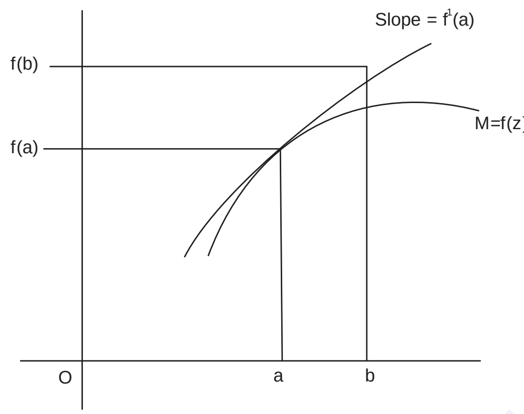
$$\therefore \ x = 0 \text{ OR } x = \frac{30}{2} = 15$$

$$\therefore x = 0 \text{ OR } x = 15$$

Concavity and convex pointed production function.

The concavity and convexity are used widely in economic theory and are also central to optimization, thereby A function of a single variable is concave if every line segment joining, two points on its graph does not lie above the graph at any point. Similarly, a function of a single variable is convex if every line segment joining two point on it graph does not live below the graph at any point.

The concavity and convexity are important in optimization theory because as we see, a simple condition is sufficient and necessary for maximize of a differentiable concave function and for a minimize of a differentiable function i.e. every point at which the derivative at a concave differentiable function is zero is a maximize of the function and every point at which the derivative of convex differentiable function is zero is minimized the function.



From the graph we say that:

The differentiable function of a single variable defined as an open interval I is convex on I if and only if

$$\frac{f'(b)-f(a)}{b-a} \leq f'(x) \text{ for } x \in I$$

The differentiable function of a single variable defined as an open interval I is convex if and only if

$$\frac{f(b)-f(a)}{b-a} \geq f'(x), x \in 1$$

***<u>Inflexion Point:</u>**

The point C is an inflexion point of successive differentiable function f of a simple variable if f ' '(c) = 0, for some value of a and b with a < c < b

Test of concavity and convexity of a curve by second order derivative.

i) If $f''(x) \leq 0$ then it is convex

ii) If $f''(x) \geq 0$ then it is convex

iii) If $f''(x) = 0$ then it is point of inflexion.

Ex.10. Suppose that the total consumption expenditure (C) in (thousand Rs.) of a family is given by the function $C = 30x + 60x^2 - 2x^3$ when x denotes the family monthly disposable income in (thousand Rs) Does the curvature of the consumption function change and at what level of income it change?

Solu^n : Given consumption function

Let $Y = C = 30x + 60x^2 - 2x^3$

diff w.r.t. x

$$\frac{dY}{dx} = 30 + 120x - 6x^2$$

again diff w.r.t. x

$$\frac{d^2Y}{dx^2} = 120 - 12x$$

Taken at Second order derivative zero i.e. $\frac{d^2y}{dx^2} = 0$

$$\therefore 120 - 12x = 0$$

$$\therefore x = 10$$

Now for all $0 < x < 10$ the second order derivative is positive i.e. The consumption function is strictly convex until $x = 10$

For all $x > 10$ the second order derivative is negative i.e. the consumption function is strictly concave after $x = 10$.

Which implies that at x = 10 the inflexion point occurs or the curvature of the consumption function changes at $x = 10$.

Elasticity of demand:

We are well known with demand law in economics that "If the price increases the quality demanded will decrease". Thus the ratio of the proportionate change in the quantity demand to the proportionate change in the price is called elasticity of demand.

Let the demand change from D to D+ BD, where price changes from P to P + BP, there elasticity of demand is given by

$$\frac{BD/D}{BP/P} = \frac{P}{D} \quad \frac{BD}{BP}$$

Taking limit $BP \to 0$ we get

Elasticity of demand $= \lim_{BP \to 0} \left[\frac{P}{D} \frac{BD}{BP} \right]$

$$= \frac{P}{D} \; \frac{dD}{dP}$$

But according to demand law if price increases then demand falls i.e.

Elasticity of demand = $-\frac{p}{D} \cdot \frac{dD}{dp}$

Elasticity of demand is denoted by η

$$\therefore \; \eta = \frac{-p}{D} \quad \frac{dD}{dp}$$

Note:

i) If $|\eta d| = 0$ then it is perfectly inelastic

ii) If $0 < |\eta d| < 1$ then it is inelastic

iii) If $0 < |\eta d| = 1$ then it is unit elastic

iv) If $nd > 1$ then it is elastic

v) If $|\eta d| = 1$ then it is perfectly elastic

Ex.11. Find elastic of demand, if Demand function

D = 25 – 11p + p^2 when p = 4 and comment on it.

Solu^n : Given demand function

$$D = 45 - 11P + p^2$$

Diff w.r.t. to p

$$\frac{dD}{dp} = 0 - 11 + 2P = 2P - 11$$

$$\therefore \eta = \frac{-P}{D}\frac{dD}{dp} = \frac{-P}{45 - 11p + p^2} \times 2P - 11$$

$$= \frac{-2P^2 + 11P}{45 - 11P + P^2}$$

When p = 4 then elastic at demand is

$$\eta P = 4 = \frac{-2(4)^2 + 11(4)}{45 = 11(4) + (4)^2} = \frac{12}{17} = 0.7058$$

Here $O < \eta \le 1$ Hence it is inelastic demand.

- **Relation between marginal revenue, average revenue and elasticity of demand is,**

Let η = elasticity of demand

MR = Marginal Revenue

AR = Average Revenue then,

By definition of elasticity of demand,

$$\eta = \frac{-P}{D} - \frac{dD}{dp}$$

& $AR = \frac{TR}{D} \Rightarrow TR = AR \times D$

$$MR = \frac{d}{dD}(TR) = \frac{d}{dD}[D \times AR]$$

$$= D\frac{d}{dD}AR + AR\frac{d}{dD}(D)$$

$$= AR + D\frac{d}{dD}A.R$$

$$[\because TR = D \times P = DP \Rightarrow P = \frac{TR}{D} = AR \quad \therefore \boxed{AR = P}]$$

$$MR = AR + D\frac{d}{dD}(AR)$$

$$= P + D\frac{dP}{dD}$$

$$= P + \frac{D}{P} \times P \times \frac{dP}{dD}$$

$$= P[1 + \frac{D}{P} \times \frac{dP}{dD}]$$

$$[\because n = -\frac{P}{D} \times \frac{dD}{dP}]$$

$$= P(1 - \frac{1}{n})$$

$$\therefore \boxed{MR = AR\left(1 - \frac{1}{\eta}\right)}$$

- **Impact of Excise Tax:**

An excise tax is a tax changed on each unit of a good or service that is sold. This is not the same as sales tax in that it is received per unit of the good rather than as a percentage of the sales. The excise tax can be received on either the buyer or the seller of the commodity.

Increase the tax rate can either increase or decrease total tax Revenue depending upon,

i) The elasticity of demand

ii) The elasticity of supply

iii) The size of tax base

A subsidy shift either the demand or supply cause to the right depending upon whether the buyer or seller receives the subsidy. If it is the buyer receiving the subsidy the demand curve shift right, lending to an increase in the quantity of demand increasing while equilibrium price decrease.

- **<u>Impact of Subsidy:</u>**

Marginal listening on production will shift the supply curve to the right until the vertical distance between the two supply cure is equal to the per unit subsidy when other things remain equal this will decrease price paid by the consumer and increase the price received by the producers. Similarly, a marginal subsidy on consumption will shift the demand curve to the right when other things remain equal. This will increase the price received by producers by the same amounts, as if subsidy had been granted to producer. However in this case new market price will be the price received by producers.

- **<u>Sales Tax effect on supply and demand:</u>**

Most states impose sales tax on some goods and services as a means of generating revenue. However, sales taxes also influence consumer behaviour.

As sales tax causes the supply curve to shift inward it has a secondary effect on the equilibrium price for a product equilibrium price is the price at which the producers supply matches. Consumer demand at a stable price since sales tax increases the price of goods, it causes the equilibrium price to fall. This may mean that it becomes more difficult consumers change their buying behaviour to purchase less of the more expensive goods.

While sales tax affects supply directly it only has an indicate effect on consumer demand. Sales tax also impact consumers buying power.

- **<u>Inventory Control:</u>**

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It is the quantity Q, which when purchased in each under minimizes the total cost T, incurred in obtaining and storing material of a certain time period to fulfil given date of demand for the material during the time period. The cost of the stock is called Inventory cost and the management of inventory is called inventory control.

Ex.12. The material demanded 1000 units per year the cost price of material ` 2 per unit; and it cost ` 50 to make the factory ready for production run of the items regardless of the number of units x producers in a run; and the cost of staring material is 25% yearly on the rupee value of the average inventory x on hand.

i) Find the economic under quantity and total cost corresponding to that

ii) Find the total cost when each under places for 10000 units.

Solu^{n}:

No of production per year = $\frac{1000}{x}$

Cost for production = $\frac{1000}{x} \times 25 = \frac{25000}{x}$

Variable cost = 2 per units : 20,000

Inventory cost on 100 units = 25

$\therefore$ Inventory cost of x units

$$x \times 25 \times \frac{1}{100} = \frac{x}{4}$$

Total cost = $C = 1000 + \frac{25000}{x} + \frac{x}{4}$

Diff w.r. t. x

$$\frac{dC}{dx} = \frac{-25000}{x^2} + \frac{1}{4} = 0$$

Taking $\frac{dC}{dx} = 0$ we get

$\therefore \frac{-25000}{x^2} + \frac{1}{4} = 0$

$\therefore$ X = 6250

$$\frac{d^2C}{dx^2} = \frac{50{,}000}{x^3}$$

$$\left[\frac{d^2c}{dx^2}\right]_{x=6250} = \frac{50{,}000}{(6250)^3} > 0$$

Which is positive

C is minimum when x = 6250

Minimum cost $C = 1000 + \frac{25000}{6250} + \frac{6250}{4}$

= Rs. 62566.5

When x = 10000 units then,

$$C = 1000 + \frac{25000}{10000} + \frac{10000}{4}$$ = Rs. 3502.5

5B.5. SUMMARY

- In this chapter we have learned
- Marginal Cost, Marginal Revenue
- Marginal Average Cost, Marginal Average Revenue
- Marginal Analysis of a film operating
- Marginal Revenue of products
- Maxima, Minima, increasing and decreasing function
- Marginal propensity to consume, marginal propensity to save, concavity and convexity and point of inflection for profit function
- Optimal trade in time, effect of taxation and subsidies, effect of excise tax, imposition of sales tax

- Inventory control

5B.6. UNIT END EXERCISE

1) Show that the scope of average cost curve is equal to $\frac{1}{x}\left[MC - AC\right]$ for the total cost function $C = ax^3 + bx^2 + cx + d$.

2) The total revenue received from the sale of x units if a products is given by. Find (a) the Average Revenue (b) Marginal Revenue at x = 5 (c) the actual Revenue from selling 51^{st} Unit.

   Ans. : (a) $R(x) = 2x^2 + 12x + 6/x$ (b) 212 (c) 214

3) The manufacturing cost of an item consists of ` 6000 as over heads, material cost ` 5 per unit and labour cost ` $x^2/60$ units produced. Find how many units must be produced so that the average cost is minimum.

   Ans: $x =$ 600

4) A company charges `700 for a mobile hand set on an order of 60 or less sets. The charges Reduced by `10per set for each set ordered in excess of 60. Find the largest size order company should allow so as to received a maximum revenue.

   Ans: x = 65 Sets.

5) Give the price equation P = 100 – 2Q where Q is quantity demanded, find (a) the marginal Revenue (b) point elasticity of demand when a = 10 (c) nature of the commodity.

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6) A monopolist's demand function is P = 300 – 5x. Find (a) the marginal Revenue function (b) the relation between the slope of MR and AR and (c) at what price is the marginal Revenue zero.

7) A manufacturer determine t employees will produce a total x units of a product per day, where x – (200 t – t^2). If the demand equation for the produce is P = -0.1 x + x + 70, determine the marginal revenue product where t = 40.

8) If c = 7 + 0.6I – 0.25 is a consumption function, find marginal propensity to consume and marginal propensity to save. When I = 16.

9) Given the production function Q = SL1/2 and the price equation P = 200 – 2Q, obtain the marginal revenue product of labour (L) when L = 25

10) Prove that marginal cost is 1/x (MC – AC) for the total cost function c(*x*) = $3x^2 + 2x^2 + 4x + 7$

11) The total function is given by $C(x) = 5000 + 1000x - 500x^2 + \frac{2}{3}x^3$

12) A company has examined its cost structure and revenue structure and have determined that C the total cost, R total revenue and x the number of units produced are related as,

$C = 100 + 0.015x^2$ & $R = 3x$

Find the production rate x that will maximise profits of the company. Find that profit also find profit when x = 120.

13) The cost of producing x units is given by

$C(x) = 0.001x^3 - 0.3x^2 + 30x + 42$

Determine where this cost function if concave up and where it is concave down also find the inflexion point.

14) If the price per unit (p) is given by p = 5(2 – x) and the total cost by c(x) = 10 + 3x2 – 2x3, where x is the number of units produced. Determine the optimum level of price and output for profit maximization.

15) If the total cost function is $C = 3q^3 - 4q^2 + 2q$. Find at what level of output average cost will be minimum and what level will it be?

16) The total profit y(in rupees) of a company from the manufacture and sale of x bottles is given by if $= \frac{x^2}{400} + 2x - 80$

17) The cost of function c(x) of a firm is given by $C(x) = 300x - 10x^2 + \frac{1}{2}x^3$

(1) Outpur at which marginal cost (MC) is minimum
(2) Output at which average cost (AC) is minimum
(3) Output at which AC=MC

18) For a firm under perfect competition, it is given that $C = \frac{x^2}{3} - 5x^2 + 28x + 27$, where p is price per unit, x is the units of output, c is the total cost of x units

(1) Find the quantity produced at which profit will be maximum amount of maximum profit.
(2) What happens to equilibrium output and maximum profit if P = 12.

19) A manufacturer determines that his total cost function is $C = \frac{x^2}{3} - 2x + 300$, where x is the no. of units produced. At which level of output will average cost per unit be minimum?

20) A firm produces 36,000 items per year. It costs Rs.250 to make the machine min, regardless of the number (x) of items produced in a nw. The cost of storage is 50 paisa per year an average inventory x/2 on hand. The cost of material per item is Rs.5. Find economic lot size.

21) For a particular process, the average cost is given by $C = 80 - 12x + x^2$, where C is the average cost (per unit) x the number of units produced. Find the minimum value of the average cost the no. of units to be produced.

22) If the demand law is $x = \frac{25}{P = 1}$ find the elasticity of demand with respect to price at the point p = 3.

23) A demand function is given by $x = 25 - 4p - p^2$, where x is the demand of the goods at price p. Calculate the elasticity of demand at price p = 5 and p = 8.

24) The demand y for a commodity when its price x is given by

$$y = \frac{x+2}{x-1}$$

Find the elasticity of demand when the price is 3 units.

♦♦♦♦♦

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5C

PARTIAL DERIVATIVE

UNIT STRUCTURE

5C.1. OBJETIVES

After going through this chapter you will be able to.

- Partial derivative with total differentiation, second order partial derivative.
- Elasticity of demand with application of partial derivative

➢ **Production function:-**

Marginal productivity of labour and capital, average product of labour and capital

Some economics application

5C.2. INTRODUCTION

In previous chapter we have learnt differential calculus and its application with one variable of the form y = f(x) but in real life, there are so many case of multivariable. For example production may be treated as a function of labour & capital and price may be a function of demand and supply. In such a case we use partial derivative two or more variables are these but we will study here two and three variable function.

5C.3 DEFINITIONS

Partial derivative:-

Z = f(x, y) be a function of two independent variables x and y. The derivative of f(x, y) with respect to x, keeping y constant is called partial derivative of z with respect to x and is denoted by $\frac{\partial z}{\partial x}$ or $\frac{\partial f}{\partial x}$ or f_x.

Similarly when the derivative of f(x, y) with respect to y, keeping x constant is called partial derivative of z with respect to y and is denoted by $\frac{\partial z}{\partial y}$ or $\frac{\partial f}{\partial y}$ or f_y.

Thus in terms of limit we can write,

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$ limit exists

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

Successive partial derivative:

Let $z = f(x,y)$ be two variable function and $\frac{\partial z}{\partial x} \& \frac{\partial z}{\partial y}$ the first order partial derivative & both are function of x & y respectively further be differentiate partially with respect to x & y, we get second order partial derivative

i.e. $$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial z}\right) = \frac{\partial^2 z}{\partial x^2} = fxx \qquad \text{.............} \quad (\text{I})$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = fyy \qquad \text{.............} \quad (\text{II})$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial^2 z}{\partial x\partial y}= fyy \qquad .... \quad (III)$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial x\partial y}= fyy \qquad .... \quad (IV)$$

III & IV are called mixed partial differentiation.

NOTE:If f, f_x & f_y are continuous then $f_{xy} = f_{yx}$

Ex. 1) Find all second order partial derivative for

$$f(x,y)= 30-x^2-y^2-4xy+7x$$

Sol^n: Given function $f(x,y)= 30-x^2-y^2-4xy+7x$

$$f_x=\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}\left[30-x^2-y^2-4xy+7x\right]$$

$$=0-2x-0-4y+7$$

$$f_x=-2x-4y+7$$

$$f_y=\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}\left[30-x^2-y^2-4xy+7x\right]$$

$$=0-0-2y-4x+0$$

$$f_y=-2y-4x$$

$$f_{xx}=\frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right]=\frac{\partial}{\partial x}\left[-2x-4y+7\right]$$

$$=-2-0+0=-2$$

$$\therefore\ f_{xx}=-2$$

$$f_{yy}=\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right]=\frac{\partial}{\partial y}\left[-2y-4x\right]$$

$$=-2-0=-2$$

$$\therefore\ f_{yy}=-2$$

$$f_{yx}=\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right]=\frac{\partial}{\partial y}\left[-2x-4y+7\right]$$

$$=0-4+0$$

$= -4$

$\therefore f_{yx} = -4$

$$f_{xy} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial}{\partial x}\left[-2y - 4x\right]$$

=0-4

$= -4$

$\therefore f_{xy} = -4$

Ex 2) If $z = \dfrac{1}{\sqrt{1-2xy+y^2}}$

Prove that $x\dfrac{\partial z}{\partial x} - y.\ \dfrac{\partial z}{\partial y} = y^2 z^3$

Sol^n:

Given function $z = \dfrac{1}{\sqrt{1-2xy+y^2}}$

i.e $z = \left(1-2xy+y^2\right)^{-1/2}$

$$\frac{\partial z}{\partial x} = \frac{-1}{2}\left(1-2xy+y^2\right)^{-3/2}\frac{\partial}{\partial x}\left(1-2xy+y^2\right)$$

$$= \frac{-1}{2}\left(1-2xy+y^2\right)^{-3/2}\left(-2y\right)$$

$$\therefore x.\frac{\partial z}{\partial x} = xy\left(1-2xy+y^2\right)^{-3/2} \qquad \text{(I)}$$

Now,

$$\frac{\partial z}{\partial y} = \frac{-1}{2}\left(1-2xy+y^2\right)^{-\frac{3}{2}}\frac{\partial}{\partial y}\left(1-2xy+y^2\right)$$

$$= \frac{-1}{2}\left(1-2xy+y^2\right)^{-\frac{3}{2}}\left(-2x+2y\right)$$

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$$= \frac{-1}{2}\left(1-2xy+y^2\right)^{-\frac{3}{2}}(-2)(x-y)$$

$$y.\frac{\partial z}{\partial y} = \left(xy-y^2\right)\left(1-2xy+y^2\right)^{-3/2} \qquad \text{(II)}$$

Subtracting (II) from (I) we get,

$$x\frac{\partial z}{\partial x} - y.\frac{\partial z}{\partial y}$$

$$= (xy)\left(1-2xy+y^2\right)^{-3/2} - \left(xy-y^2\right)\left(1-2xy+y^2\right)^{-3/2}$$

$$= (xy)\left(1-2xy+y^2\right)^{-3/2} - xy\left(1-2xy+y^2\right)^{-3/2} + y^2\left(1-2xy+y^2\right)^{-3/2}$$

$$= y^2\left(1-2xy+y^2\right)^{-3/2}$$

$$= y^2 z^3$$

Hence, $x\frac{\partial z}{\partial x} - y.\frac{\partial z}{\partial y} = y^2 z^3$ proved.

Homogenous function:

A function $f(x,y)$ of two variables is said to be homogeneous function in which the power of each term is same. i.e. x & y of same degree.

If f (x, y) is homogeneous function of two variables x & y with degree n,

If $f(x,y) = \lambda^n f(x,y)$ for $\lambda > 0$

$eg:$ $f(x,y) = \frac{1}{x^2} + \frac{1}{xy}$

Here $f(x,y)$ is a homogeneous function of degree – 2.

Euler's theorem on homogeneous functions:

Statement:

Let $f(x,y)$ be a homogeneous function of degree n, with $x_1, x_{2,}.........., x_n$ variables then

$$x_1.\frac{\partial f}{\partial x_1} + x_2.\frac{\partial f}{\partial x_2} + ... + x_n\frac{\partial f}{\partial x_n} = n.f(x_1, x_2, ...x_n)$$

If z is homogeneous function of *x, y*, of degree n and $z = f(u)$, then

$$x.\frac{\partial z}{\partial x} + y.\frac{\partial z}{\partial y} = \frac{n.f(u)}{f(u)}$$

Ex. 1) If $f(x,y) = \frac{1}{x^2} + \frac{1}{xy} + \frac{\log x - \log y}{x^2 + y^2}$

then prove that $x.\frac{\partial f}{\partial x} + y.\frac{\partial f}{\partial y} + 2f = 0$

Sol^n^: $f(x,y) = \frac{1}{x^2} + \frac{1}{xy} + \frac{\log x - \log y}{x^2 + y^2}$

$$= \frac{1}{x^2} + \frac{1}{x^2}\frac{1}{\left(y/x\right)} - \frac{1}{x^2}.\frac{\log\left(y/x\right)}{\left[1 + \left(y/x\right)^2\right]}$$

$f(x,y)$ is a homogeneous function of degree – 2

$\therefore$ By Euler's theorem.

$$x.\frac{\partial f}{\partial x} + y.\frac{\partial f}{\partial y} = nf$$

$$x.\frac{\partial f}{\partial x} + x.\frac{\partial f}{\partial y} = -2f$$

$$x.\frac{\partial f}{\partial x} + f.\frac{\partial f}{\partial y} + 2f = 0$$

Ex. 2) Verify Euler's theorem for the function

$$z = \frac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}$$

Sol^n: Here $z = \dfrac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}$

The numerator is a homogeneous function of degree $1/4$ and denominator is homogeneous function of degree $1/5$

By Euler's theorem,

$$x.\frac{\partial z}{\partial x} + y.\frac{\partial z}{\partial y} = \frac{1}{20} z.$$

Now, $\dfrac{\partial z}{\partial x} = \dfrac{\partial}{\partial x}\left[\dfrac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}\right]$

$$= \frac{\left(x^{1/5} + y^{1/5}\right).\frac{d}{dx}\left(x^{1/4} + y^{1/4}\right) - \left(x^{1/4} + y^{1/4}\right)\frac{d}{dx}\left(x^{1/5} + y^{1/5}\right)}{\left(x^{1/5} + y^{1/5}\right)^2}$$

$$= \frac{\left(x^{1/5} + y^{1/5}\right).\frac{1}{4}x^{-3/4} - \left(x^{1/4} + y^{1/4}\right)\frac{1}{5}x^{-4/5}}{\left(x^{1/5} + y^{1/5}\right)^2}$$

$$\therefore x.\frac{\partial z}{\partial x} = \frac{\frac{1}{4}.x^{1/4}\left(x^{1/5} + y^{1/5}\right) - \frac{1}{5}.\,x^{1/5}\left(x^{1/4} + y^{1/4}\right)}{\left(x^{1/5} + y^{1/5}\right)^2}$$

Similarly,

$$y.\frac{\partial z}{\partial y} = \frac{\frac{1}{4}.\,y^{1/4}\left(x^{1/5} + y^{1/5}\right) - \frac{1}{5}.\,y^{1/5}\left(x^{1/4} + y^{1/4}\right)}{\left(x^{1/5} + y^{1/5}\right)^2}$$

$$\therefore x.\ \frac{\partial z}{\partial x} + y.\ \frac{\partial z}{\partial y}$$

$$= \frac{\frac{1}{4}\left(x^{1/5} + y^{1/5}\right)\left(x^{1/4} + y^{1/4}\right) - \frac{1}{5}.\left(x^{1/4} + y^{1/4}\right)\left(x^{1/5} + y^{1/5}\right)}{\left(x^{1/5} + y^{1/5}\right)^2}$$

$$= \frac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}\left(\frac{1}{4} - \frac{1}{5}\right)$$

$$= \frac{1}{20} z.$$

Here, Euler's theorem,

$$x.\frac{\partial z}{\partial x} + y.\frac{\partial z}{\partial y} = \frac{1}{20} z \text{ is verified.}$$

Ex. 3)

If $z = \log\left(\frac{x^3 + y^3}{x + y}\right)$. Show that $x.\frac{\partial z}{\partial x} + y.\frac{\partial z}{\partial y} = 2$

Sol^n:

$$z = \log\left(\frac{x^3 + y^3}{x + y}\right)$$

Here z is not homogeneous function but if

$$u = e^z = \frac{x^3 + y^3}{x + y} = \frac{x^3\left[1 + \left(y/x\right)^3\right]}{x\left[1 + \left(y/x\right)\right]} = x^2 f\left(y/x\right)$$

Here z is homogenous function of degree 2.

$$x.\frac{\partial z}{\partial x} + y.\frac{\partial z}{\partial y} = \frac{n.f(z)}{f^1(z)} = 2.\frac{e^z}{e^z} = 2$$

Here $x.\frac{\partial z}{\partial x}+y.\frac{\partial z}{\partial y}=2$

- **Check your progress:**

(1) If $z=\frac{x^3y^3}{x^3+y^3}$ show that $x.\frac{\partial z}{\partial x}+y.\frac{\partial z}{\partial y}=3z$

(2) If $f=\left(\frac{x^2+y^2}{\sqrt{x+y}}\right)$, prove that $x.\frac{\partial z}{\partial x}+y.\frac{\partial z}{\partial y}=\frac{3}{2}z$

(3) If $u=f\left(\frac{y}{x}\right)$, prove that $x.\frac{\partial u}{\partial x}+y.\frac{\partial u}{\partial y}=0$

(4) Verify Euler's theorem for

(1) $ax^2+24xy+by^2$

(2) $\frac{x^4+y^4}{x+y}$

(5) Verify Euler's theorem for function

$$z=\frac{x^{3/4}+y^{3/4}}{x^{3/5}+y^{3/5}}$$

- **Maximum and minimum values of two variable function $z=f(x,y)$**

Let $z=f(x,y)$ be two variables function for which continuously partial derivative exists in interval (a, b).

Consider $\quad A=\frac{\partial^2 f}{\partial x^2}=f_{xx}(a,b)$

$$B = \frac{\partial^2 f}{\partial x . \partial y} = f_{xy} \quad (a,b)$$

$$C = \frac{\partial^2 f}{\partial y^2} = f_{yy} \quad (a,b)$$

II[nd] derivative test conditions for maxima and minima.

(1) If $AC - B^2 > 0$ & $A < 0$ then z is relative minimum.

(2) If $AC - B^2 < 0$ relative maximum.

(3) If $AC - B^2 < 0$ it is neither minimum nor maximum. It is called saddle point.

(4) If $AC - B^2 < 0$ then it is maximum, minimum or saddle point.

Ex. 1) Find the points of maximum & minimums for the function

$$z = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$$

$$\frac{\partial z}{\partial x} = 3x^2 + 3y^2 - 6x$$

$$\frac{\partial z}{\partial y} = 6xy - 6y$$

Now, $\frac{\partial z}{\partial x} = 0$ & $\frac{\partial z}{\partial y} = 0$

$3x^2 + 3y^2 - 6x = 0$ .... (I) $\qquad 6xy - 6y = 0$ .....(II)

Solving (I) & (II) we get,

$x = 0 \quad x = 1 \quad x = 2$

$y = 0 \quad y = \pm 1$

$\therefore (0,0)(2,0),(1,1),(1,-1)$ are the stationary points.

$A = f_{xx} = 6x - 6$

$f_{xx}(0,0) = -6 \qquad f_{xx}(2,0) = 6$

$f_{xx}(1,1) = 0 \qquad f_{xx}(1,-1) = 0$

$B = f_{xy} = 6y$

$f_{xy}(0,0) = 0 \qquad f_{xy}(2,0) = 0$

$f_{xy}(1,1)=6 \quad f_{xy}(1,-1)=-6$

$C = f_{yy} = -6+6x$

$f_{yy}(0,0)=-6 \quad f_{yy}(2,0)=6$

$f_{yy}(1,1)=0 \quad f_{yy}(1,-1)=0$

For (0, 0)

$AC\text{-}B^2 = (-6)\times(-6)-(0)^2 = 36>0 \;\&\; A<0$

$\therefore z$ is relatively maximum at (0,0)

$z(0,0)=4$

For (2, 0)

$AC\text{-}B^2 = (6\times 6)-(0)^2 = 36>0 \;\&\; A>0$

$\therefore z$ is relatively minimum at (2,0)

$z(2,0)=0$

For (1,1)

$AC-B^2 = (0\times 0)-(6)^2 = 0-36 = -36<0$

Which is saddle points.

For (1,-1)

$AC - B^2 = (0\times 0)-(-6)^2 = 0-36 = -36<0$

Which is saddle points.

➢ **Examine maxima & minima for the following functions:**

(1) $z = 3x^2+2y^2-xy-4x-7y+12$ z is minimum at (1,2) z = 3

(2) $z = 2+2x+2y-x^2-y^2$ z is maximum at (1,1) z = 4

(3) $z = x^3+y^3-3xy$ z is maximum at (1,1) & z = -1

(4) $z = x^3+y^3 = 6x-63y+12xy$

z is maximum (3,3) z = 540

z is maximum (-7,-7) z = 784

***Production function**:

The Cobb - Dougles production function of the economy as a whole is given by $P = AL^{\alpha}K^{\beta}, \ \alpha + \beta = 1$

Where P is total production, L is the quantity of labour, k is the quantity of capital and A, α, β are constants.

Marginal productivity of labour & capital

Assume Q = f (L, K) is the production function where the amount produced is given as a function of the labour and capital used.
Where Q (Total output) = f (L, K).

The partial derivative $\frac{\partial Q}{\partial L} \& \frac{\partial Q}{\partial K}$ gives marginal product of labour & capital respectively for the Cobb-Dougles production function.

$$MPK = \frac{\partial Q}{\partial K} = b.AL^{a}K^{b-1} = \frac{bQ}{k} \quad \text{and}$$

$$MPL = \frac{\partial Q}{\partial L} = aAL^{a-1}K^{b} = \frac{aQ}{k}$$

Thus for the Cobb-Dougles production function, the marginal product of capital respect to labour is a constant times the average product capital with respect to labor.

$$\in = \frac{K}{Q}\frac{\partial Q}{\partial K} = \frac{\frac{\partial Q}{\partial K}}{Q/K}$$

Which is called the Capital elasticity of product, it is equal to the ratio of the marginal product of capital to the average product of capital.

Ex 1) The following is a linear homogenous production function $x=\sqrt{aL^2+2hLK+bk^2}$ where X, L, K represent output, labour and capital respectively. Show that

$$L.\frac{\partial X}{\partial L}+K.\frac{\partial X}{\partial L}=X$$

Sol^n: Given homogeneous & production function

$$X=\left(aL^2+2hKL+bK^2\right)^{1/2}$$

Marginal production labour

$$\frac{\partial X}{\partial L}=\frac{\partial}{\partial L}\left[\left(aL^2+2hKL+bK^2\right)^{1/2}\right]$$

$$=\frac{1}{2}\left(aL^2+2hKL+bK^2\right)^{-\frac{1}{2}}.\frac{\partial}{\partial L}\left(aL^2+2hKL+bK^2\right)$$

$$=\frac{1}{2}\frac{2aL+2hK}{\sqrt{\left(aL^2+2hKL+bK^2\right)}}=\frac{aL+hK}{\sqrt{\left(aL^2+2hKL+bK^2\right)}}$$

Marginal product capital

$$\frac{\partial X}{\partial K}=\frac{\partial}{\partial K}\left[\left(aL^2+2hKL+bK^2\right)^{1/2}\right]$$

$$=\frac{1}{2}\frac{1}{\sqrt{\left(aL^2+2hKL+bK^2\right)}}\frac{\partial}{\partial K}\left(aL^2+2hKL+bK^2\right)$$

$$=\frac{2bK+2hL}{2\sqrt{aL^2+2hKL+bK^2}}=\frac{bK+hL}{\sqrt{aL^2+2hKL+bK^2}}$$

$$\therefore L.\frac{\partial X}{\partial L}+K\frac{\partial X}{\partial L}=\frac{L\left(aL+hK\right)}{\sqrt{aL^2+2hKL+bK^2}}+\frac{K\left(bK+hL\right)}{\sqrt{aL^2+2hKL+bK^2}}$$

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$$= \frac{aL^2 + 2hKL + bK^2}{\sqrt{aL^2 + 2hKL + bK^2}}$$

$$= \left(aL^2 + 2hKL + bK^2\right)^{1/2}$$

$$= \quad X$$

$$\therefore L.\frac{\partial X}{\partial L} + K.\frac{\partial X}{\partial K} = X \,. \qquad \text{Hence proved}$$

Principle of marginal rate of technical substitution (MRTS)

The principle of marginal rate of technical substitution is based on the production function where two functions can be substituted in variable proportions in such a way as to produce a constant level of output.

Definition: "The marginal rate of technical substitution is the amount of an output that a firm can give up by increasing the amount of the other input by one unit and still remain on the came isoquant".

The marginal rate of technical substitution between two functions capital can be substituted for capital in the production of goal x without changing the quantity of output. As we move along an isoquant downwards to the right each point on it represents the substitution of labour for capital.

The marginal rate of technical substitution of labour for capital is the slope or gradient of the isoquant at a point accordingly,

$$\text{Slope of } MRTS_k \quad = \frac{-\Delta K}{\Delta L} = \frac{-\partial K}{\partial L}$$

The marginal rate of technical substitution can also be expressed as the ratio of the marginal physical product labour to the marginal physical product of capital.

i.e. $MRTS_{LK} = \frac{MPL}{MPK}$

Though the output remains constant the process of substitution being change.

Isoquants:-Isoquant is the curve respectively all possible efficient combination of inputs needed to produce a certain quantity of specific output or output combination.

The main assumptions of iso-quant curves are as follows:

(1) Only two functions are used to product into small points.
(2) Factors of production can be divided into small parts.
(3) Technique of production is constant.
(4) The substitution between the two factors is technically possible.
(5) Under the given technique, factors of production can be used with maximum efficiency.

- **<u>Properties of Isoquant curve:</u>**

(1) Isoquant curve slope downwards from left to right because MRTS of labour for capital diminishes.
(2) Isouants are coming to the origin because of the MRTS diminishes along the isoquant.
(3) Two isoquant curves represent higher level of output.
(4) Isoquants need not be parallel to each other.
(5) No Isoquants can touch either axis.
(6) Each Isoquant is oval-shaped.

- **<u>Utility function:</u>**

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Let $u = f(x, y, z)$ be a function of three variables *x*, *y* & *z* then partial derivatives of *u*, i.e $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \& \frac{\partial u}{\partial z}$ are the marginal functions of *x*, *y*, *z* respectively.

Let $u = f(x, y)$ be the total utility function of a consumer, where *x* & *y* are the quantities of two commodities (or goods) q_1, q_2 which be consumes. Then $\frac{\partial u}{\partial x}$ is the (partial) marginal utility of *x* and $\frac{\partial u}{\partial y}$ is the (partial) marginal utility of y.

- **Marginal Utility:**

In the theory of economics behaviours utility functions relates to total utility (u) obtained from the consumptions of a given quantity (u). Thus given the utility function u = u(Q), the additional derivative from an additional infinitesimal consumption of Q is given by the derivative

$$\frac{\partial u}{\partial Q} = u'(Q)$$

Which is called marginal utility. Furthers, the change in marginal utility due to infinitesimal change in Q is given by the second order derivative

$$\frac{\partial^2 u}{\partial Q^2} = \frac{\partial}{\partial Q}\left(\frac{\partial u}{\partial Q}\right) = u''(Q)$$

If marginal utility (mu) declines as Q increases, thus $u''(Q) < 0$ indicating the operation of the law of diminishing marginal utility.

- **Derivative of shape of indifference curve:**

The concept of partial derivative and total derivative can be used to find out the shape of an indifference curve in connection with consumer's behaviours as isoquant in connection with a production function. Both the indifference curve and the isoquant are always complex to the origin in order that utility on total output should remain constant.

For setting convexity of a curve, the first derivative should be negative and the second order derivative should be positive.

Consider a utility function when the consumer consumes two commodities x & y such that

$$u = u(x, y) \qquad \text{..... (I)}$$

The convexity of indifference curve given by

$$\frac{\partial y}{\partial x} < 0 \,\&\, \frac{\partial^2 y}{\partial x^2} > 0$$

By taking total differential of utility function we get,

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = 0$$

$$\therefore \frac{dy}{dx} = \frac{-\partial u / \partial x}{\partial y / \partial y} = \frac{-u_x}{u_y} \qquad \text{.... (II)}$$

When u_x & u_y is marginal utility of *x* & *y* respectively

$$\therefore u_x = u_x(x, y) \,\&\, u_y = u_y(x, y)$$

again differentiate eq^n II with respect to *x*

$$\frac{d^2 y}{dx^2} = \frac{\partial}{\partial x}\left(\frac{dy}{dx}\right) = \frac{\partial}{\partial x}\left(\frac{-u_x}{u_y}\right)$$

$$= \frac{2u_x u_y u_{xy} - u_y^2 u_{xx} - u_x^2 u_{yy}}{(u_y{}^3)}$$

The sign of above equation indicate whether the difference curve is convex or concave. We can show that the value of the expression in the numerator is positive $u_y > 0$ and $u_y^3 > 0$.Hence $\frac{d^2 y}{dx^2} > 0$

Ex Chick whether the indifference curve of convex or concave.

$u = \frac{1}{2} x^2 \sqrt{y}$ is convex or concave.

Sol^n: Given utility function:

$$u = \frac{1}{2} x^2 \sqrt{y}$$

$$u_x = \frac{1}{2} 2x\sqrt{y} = x\sqrt{y}$$

$$u_y = \frac{1}{2} x^2 \frac{1}{2\sqrt{y}} = \frac{1}{4} \frac{x^2}{\sqrt{y}}$$

$$u_{xx} = \sqrt{y} \qquad u_{xy} = \frac{1}{2} \frac{x}{\sqrt{y}}$$

$$u_{yy} = -\frac{1}{2} \cdot \frac{1}{4} x^2 y^{-3/2} = \frac{-x^2}{8\left(\sqrt{y}\right)^3}$$

$$\therefore \frac{d^2 y}{dx^2} = \frac{2u_x u_y u_{xy} - u_y^2 u_{xx} - u_x^2 u_{yy}}{u_y^{\,3}}$$

$$= \frac{2\left(x\sqrt{y}\right)\left(\frac{1}{4}\frac{x^2}{\sqrt{y}}\right)\left(\frac{x}{2\sqrt{y}}\right) - \left(x\sqrt{y}\right)^2 \left(\frac{-1}{8}\frac{x^2}{\left(\sqrt{y^3}\right)}\right) - \left(\frac{1}{4}\frac{x^2}{\sqrt{y}}\right)^2 \sqrt{y}}{\left(\frac{1}{4}\left(\frac{x^2}{\sqrt{y}}\right)\right)^3}$$

$$= \frac{\frac{x^4}{4\sqrt{y}} + \frac{x^4}{8\sqrt{y}} - \frac{x^4}{16\sqrt{y}}}{\frac{x^6}{64y\sqrt{y}}}$$

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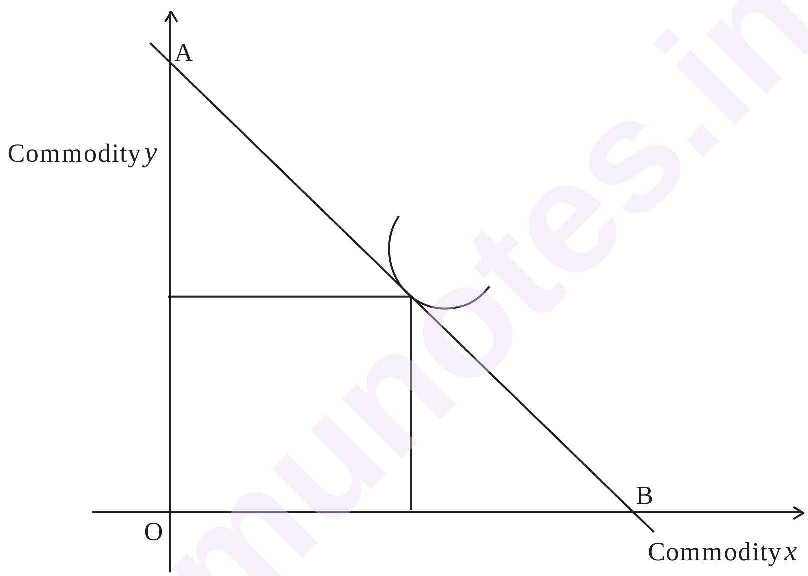
$$= \frac{20y}{x^2}$$

Since both x & y are positive

Therefore, the indifference curve of the utility function is convex to the origin.

***Marginal Rate Substitution**:*

If the utility function is given level of utility or satisfaction from various combinations a_1 & a_2 respectively. Havens, the locus of all such combination is an indifference curve.



The total differential of the utility function

$$U = u(x, y) \text{ is}$$

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy.$$

But U along the indifference curve is zero.

$$du = o$$

$$\frac{du}{\partial x} dx + \frac{\partial u}{\partial y} dy = 0$$

$$\therefore \frac{dy}{dx} = -\frac{u_x}{u_y}$$

The negative value of $\frac{dy}{dx}$ is called the marginal rate of Substitution of commodity a_1 for a_2.

The Hessian matrix and determining optimization.

The Hessian matrix of $z = f(x, y)$ is defined to be

$$H_f(x, y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

at any point at which all the second partial derivative of f exists.

Determinant of $H_f(x, y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$

$$= (f_{xx} \cdot f_{yy}) - (f_{xy} \cdot f_{yx})$$

Test for optimization:-

1st find boundary point by using $\nabla f(x, y) = 0$. To identity, if a point $(x, y,)$ with zero gradient is local maximum and minimum, check the Hassian determinant.

(a) If H (x_1, y_1) is positive then (x_1, y_1) is a local minimum.

(b) If H (x_1, y_1) is negative definite then (x_1, y_1) is a local maximum.

Ex Find the local extreme of $f(x, y) = x^3 + y^3 - 3xy$

$$\nabla f(x, y) = \begin{bmatrix} 3x^2 - 3y \\ 3y^2 - 3x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$3x^2 - 3y = 0$ ....(I) $\qquad 3y^2 - 3x = 0$ .....(II)

Solving (I) & (II) we get,

$$(x, y) = (0,0) \text{ or } (1,1)$$

$$H(x, y) = \begin{bmatrix} 6x & -3 \\ -3 & 6y \end{bmatrix}$$

$$Det\ \ H(0,0) = \begin{vmatrix} 0 & -3 \\ -3 & 0 \end{vmatrix} = -9$$

1st minor = det (H_1) = 36 & 2nd minor = det |H_2| = -9

$$H(1,1) = \begin{vmatrix} 6 & -3 \\ -3 & 6 \end{vmatrix}$$

1st minor = det (H_1) = 36 & 2nd minor = det |H_2| = 36 - 9 = 25 > 0

$\therefore H(1,1)$ is positive definite which implies that (1,1) is local maximum f(x, y)

- **Budget line:-**

A graphical depiction of the various combinations of two selected products that a consumer can afford at specified prices for the products given their particular income level. When a typical business is analysing a two product budget line, the amount of the first product are plotted on the horizontal *x* axis and the amounts of the second product are plotted on the vertical *y* axis.

The problem is about how much goods a person can buy with limited income. Assume: on saving, with income I, only spend money on goods *x* & *y* with the price P_x & P_y.

Thus, budget constraint is $P_x - x + P_y.\ y \le I$

Suppose $P_x = 2\ \ P_y + 1,\ \ I = 8$ then

$$2x + y \le 8$$

The slope of budget line is

$$\frac{-dy}{dx} = \frac{Px}{Py}$$

Bundles below the line are affordable.

- **Constrained Optimization with lagrange's multipliers:**

Suppose $f(x,y)$ is a given two variable function and $g(x, y) = 0$ is constraints on the variable x & y.

Using the Lagrange's multipliers method to find the constrained maxima or minima of f (x,y) by using Lagrange's function

$$L(x, y, \lambda) = f(x, y) - \lambda g(x, y)$$

The following steps are to be followed:

Step (1) Taking lagrange's multiplier

$$L(x, y, \lambda) = f(x, y) - \lambda g(x, y)$$

Step (2) Find L_x, L_y *and* L_λ

Step (3) Taking $L_x = 0$ & $L_y = 0$ & $L_\lambda = 0$ we get stationary points and value of λ

Step (4) Find L_{xx}, L_{xy}, L_{yy} & g_x, g_y

Step (5) Taking determinant

$$D = \begin{vmatrix} 0 & 9x & 9y \\ 9x & L_{xx} & L_{xy} \\ 9y & L_{xy} & L_{yy} \end{vmatrix}$$

Step (6)

(i) D > 0, (a, b) then given function f has a maximum at the stationary point.

(ii) If D < 0 then the given function f has minimum at that stationary point.

Ex 1) The cost function of a product is made from two row material x and y. The profit function is given by $115x + 117y - x^2 - 2y^2$. If we want to manufacture 98 units of both products, taken together per day, find the number of units of each type to be manufactured by the company to get maximum profit.

Sol^n:

Given cost function $f(x, y) = 115x + 117y - x^2 - 2y^2$ and constraints is $g(x, y) = x + y - 98$

$\therefore$ Lagrange's function

$$L(x,y,\lambda)=115x+117y-x^2-2y^2-\lambda(x+y-98)$$

taking 1st order partial derivative

$L_x=115-2x-\lambda$ $\qquad$ $L_y=117-4y-\lambda$

$L_\lambda=-x-y+98$

taking $L_x=0,\ L_y=0\ \&\ L_\lambda=0$ we get,

$115-2x-\lambda=0 \quad 117-4y-\lambda=0 \quad -x-y+98=0$

$x=\dfrac{115-\lambda}{2}$ ..... (I) $\qquad$ $y=\dfrac{117-\lambda}{4}$ .....(II)

$x+y=98$ ..... (III)

Substituting I & II in III we get

$$\frac{115-\lambda}{4}+\frac{117-\lambda}{4}=98$$

$$230-2\lambda+117-\lambda=392$$

$$227=3\lambda$$

$$-45=3\lambda$$

$$\lambda=-15$$

Substituting $\lambda=-15$ in I & II we get

x = 65 & y = 33

$\therefore(65,33)$ is the stationary point.

$L_{xx}=-2\ \ L_{yx}=0\ \ L_{yy}=-4\ \ g_x=1,\ \ g_y=1$

$$\therefore D=\begin{vmatrix}0 & 9x & 9y\\ 9x & L_{xx} & L_{xy}\\ 9y & L_{xy} & L_{yy}\end{vmatrix}=\begin{vmatrix}0 & 1 & 1\\ 1 & -2 & 0\\ 1 & 0 & -4\end{vmatrix}$$

$$=0\begin{vmatrix}-2 & 0\\ 0 & -4\end{vmatrix}-1\begin{vmatrix}1 & 0\\ 1 & -4\end{vmatrix}+1\begin{vmatrix}1 & -2\\ 1 & 0\end{vmatrix}$$

$$=0-1(-4)+1(0+2)$$

$= 4 + 2 = 6 > 0$

$\therefore$ The function f(x, y) is maximum at (65, 33) & maximum value of f is

$$f(65, 33) = 115(65) + 177(33) - (65)^2 - 2(33)^2$$

$$= 4933$$

Ex 2) Find the maximum and minimum distance of the point (3, 4, 12) from the sphere $= x^2 + y^2 + z^2 = 1$

Sol^n: Let the co-ordinates of the given point be (x, y, z) then its distance D for (3, 4, 12)

$$D = \sqrt{(x-3)^2 + (y-4)^2 + (z-12)^2}$$

$$f(x, y, z) = (x-3)^2 + (y-4)^2 + (z-12)^2$$

$$x^2 + y^2 + z^2 = 1$$

$$g(x) = x^2 + y^2 + z^2 - 1$$

$$L(x, y, z, \lambda) = f(x, y, z) - \lambda g(x, y, z)$$

$$= (x-3)^2 + (y-4)^2 + (z-12)^2 - \lambda(x^2 + y^2 + z^2 - 1)$$

$$L_x = 2(x-3) - 2x\lambda \quad \text{..... (I)}$$

$$L_y = 2(y-4) - 2y\lambda \quad \text{..... (II)}$$

$$L_z = 2(z-12) - 2z\lambda \quad \text{..... (III)}$$

taking $L_x = 0,\ L_y = 0,\ L_z = 0$

$$x = \frac{3}{1-\lambda} \quad \text{..... (IV)} \qquad y = \frac{4}{1-\lambda} \quad \text{..... (V)}$$

$$z = \frac{12}{1-\lambda} \quad \text{..... (VI)}$$

$$\left(\frac{3}{1-\lambda}\right)^2 + \left(\frac{4}{1-\lambda}\right)^2 + \left(\frac{12}{1-\lambda}\right)^2 = 1$$

$$\frac{9}{(1-\lambda)^2} + \frac{16}{(1-\lambda)^2} + \frac{144}{(1-\lambda)^2} = 1$$

$$(1-\lambda)^2 = 169$$

$1-\lambda=\pm 13$

Substituting $1-\lambda=\pm 13$ in IV,V&IVI

$$\left(\frac{\pm 3}{13,}\pm\frac{4}{13,}\pm\frac{12}{13}\right)\Rightarrow\left(\frac{3}{13},\frac{4}{13},\frac{12}{13}\right)$$

and $\left(\frac{-3}{13,}\frac{-4}{13},\frac{-12}{13}\right)$

The optimum distance $D=\sqrt{\left(3-\frac{3}{13}\right)^2+\left(4-\frac{4}{13}\right)^2+\left(12-\frac{12}{13}\right)^2}$

$=12$

The maximum distance = $\sqrt{\left(3+\frac{-3}{13}\right)^2+\left(4+\frac{4}{13}\right)^2+\left(12+\frac{12}{13}\right)^2}$

$=14$

5C.4. SUMMARY

In this chapter we have learned:

- Partial derivative of Ist & IInd order.
- Elasticity of demand with the partial derivative.
- Production function, MPL and MPK.
- Utility function, Marginal utility
- Marginal rate of technical substitution
- Isoquant properties of Isoquant
- Indifference curve
- Optimization test y Hassian matrix
- Budget line
- To optimization by Lagrange's multipliers

5C.5. UNIT END EXERCISE:

(1) Calculate $\frac{\partial z}{\partial x}\ \&\ \frac{\partial z}{\partial y}$, function $z=\frac{x^2}{x-y+1}$

Ans: $\frac{\partial z}{\partial x}=\frac{x(x-2y+z)}{(x-y+1)^2}$; $\frac{\partial z}{\partial y}=\frac{x^2}{(x-y+1)^2}$

(2) If $u = f(x)$ where $r = \sqrt{x^2 + y^2}$ prove that

$$\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial y^2} = f''(\partial) + \frac{1}{\partial} f'(\partial)$$

(3) Verify Euler's theorem for the function

(I) $z = \frac{1}{x^2 + xy + y^2}$ (II) $z = \frac{x^4 + y^4}{x + y}$

(4) If $z = \log\left(\frac{x^2 + y^2}{x + y}\right)$ prove that $z = \frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 1$

(5) The production function of a firm is given by $Q = 4L^{3/4}K^{1/4}, L > 0, K > 0.$ Find the marginal productivity of labor and capital. Also show that $L\frac{\partial Q}{\partial L} + K.\frac{\partial Q}{\partial K} = Q$

(6) If $z = 6\log(xy) - 18x^5y^2$. Find the value $\frac{\partial^2 z}{\partial x.\partial y}$ & $\frac{\partial^2 z}{\partial y.\partial x}$

State the conclusion to be drawn from the result.

(7) If utility function is $u = \log\left(ax_1 + bx_2 + \left(\sqrt{x_1x_2}\right)\right)$. Find the marginal utility.

(8) If $u = f\left(y/x\right)$. Show that $x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} = 0$

(9) If $f(x, y) = 2x^3 - 11x^2 + 3y^3$ prove that $x.\frac{\partial f}{\partial x} + y.\frac{\partial f}{\partial y} = 3f(x, y)$

(10) If $y = x^2 \log\left(y/x\right)$ prove that $x.\frac{\partial y}{\partial x} + y.\frac{\partial y}{\partial y} = 24$

(11) Given the production function $Q = 1.08L^2 - 0.03L^3 + l.68K^2 - 0.08K^3$. Find the quantities of labor and capital that maximize output Q. Ans $= (L = 24, K = 24)$

(12) Given a total cost function $c = 7x^2 - 2xy + 64$. Find combination of input x & y must be produced to meet the requirement of 77 units that maximizes the cost of fulfilling the requirement.

(13) Find the all 2nd order partial derivative at given point.

(1) $z = x^2 + \frac{x}{y} + \frac{y}{x} + y^2$

(2) $z = x^3 - 2x^2y + 3xy^2 + y^3$

(14) If the production function is given as $Q = 100L^{0.4}K^{0.7}$. Find the marginal productivity of labor & capital.

(15) Find the marginal productivity of labor and capital at (5, 1) of production.

$$Q = L^3 - 6L - 9K + LK + K^3$$

(16) Examine for maxima & minima for

(1) $f(x, y) = x^2 - 6x + y^2 + 4y$

(2) $f(x, y) = x^3 - 12x + 17y + y^3$

(17) Optimize the cost function subject $2x^2 - 26x + y^2 - ky + 10$ to the constraint $x + y = 20$.

(18) A monopolist charges different prices in the two markets where his functions are: $x_1 = 21 - 0.1P_1$, & $x_2 = 50 - 0.4P_2$; P_1, P_2 being prices and x_1, x_2 be quantities demanded. His total cost function is TC = 10x + 2000, where x is total output Find the prices that the monopolist should charge to maximize his profit.

Ans : max profit = 67.5.

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6A

INDEFINITE and DEFINITE INTEGRATION

UNIT STRUCTURE:

6A.1. OBJECTIVES

After studying this chapter, you should be able to understand:

- To find the indefinite integral of a given function
- To state the standard indefinite integrals
- To evaluate definite integrals

6A.2. INTRODUCTION

After having learnt what is meant by differentiation, we come to the reverse process of it, namely integration.

Consider the following examples:

(1) If $f(x)=x$, then $f'(x)=1$,Question: What is the function whose derivative is 1? Ans: x

(2) If $f(x)=x^3$, then $f'(x)=3x^2$,Question: What is the function whose derivative is $3x^2$? Ans: x^3

(3) If $f(x)=x^{3/2}$,then $f'(x)=\frac{3}{2}x^{1/2}$,Question: What is the function whose derivative is $\frac{3}{2}x^{1/2}$? Ans: $x^{3/2}$

The answers which we find (the functions x, x^3, $x^{3/2}$) are called primitives or anti-derivatives or integrals of the given function.

* **Definition of integral of a function:**

If f(x) and g(x) are two functions such that $\frac{d}{dx}(g(x))=f(x)$ then we define integral of f(x), w.r.t x to be the function g(x).This is put in notation form as

$$\int f(x)dx=g(x),$$

read as integral of f(x) w.r.t. x is g(x).The function f(x) is called the integrand. Presence of dx indicates that the integration is to be taken with respect to the variable 'x'. The process of finding the primitive or integral of a function is called integration. Thus integration is the inverse process of a differentiation.

$\frac{d}{dx}\left(x^3\right)=3x^2 \therefore \int 3x^2dx=x^3,$

But $\frac{d}{dx}\left(x^3+7\right)=3x^2, \frac{d}{dx}\left(x^3-5\right)=3x^2,$

In general, $\frac{d}{dx}\left(x^3+c\right)=3x^2$, *where c is any real number*

Hence, in general, we write,

$\int 3x^2dx=x^3+c$.

The number c is called the constant of integration.

Hence, if $\frac{d}{dx}\left(g(x)\right)=f(x)$ *then*

$$\int f(x)dx=g(x)+c,$$

For different values of c, we get different integrals of g(x).

$\therefore \int f(x)\ dx=g(x)+c$ *is called indefinite integral.*

***Integrals of standard functions:**

If a and b are any non-zero real numbers,

$$1.\int x^n dx = \frac{x^{n+1}}{n+1} + c, n \neq -1 \therefore \int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c, n \neq -1$$

$$2.\int \frac{1}{x} dx = \log|x| + c \therefore \int \frac{1}{(ax+b)} dx = \frac{1}{a}\log|ax+b| + c$$

$$3.\int e^x dx = e^x + c \therefore \int e^{ax+b} dx = \frac{1}{a} e^{ax+b} + c$$

$$4.\int a^x dx = \frac{a^x}{\log a} + c \therefore \int a^{bx+k} dx = \frac{a^{bx+k}}{b\log a} + c$$

$$5.\int \frac{1}{x^2+a^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$$

$$6.\int \frac{1}{x^2-a^2} dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$$

$$7.\int \frac{1}{a^2-x^2} dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c$$

$$8.\int \frac{1}{\sqrt{x^2-a^2}} dx = \log\left|x+\sqrt{x^2-a^2}\right| + c$$

$$9.\int \frac{1}{\sqrt{x^2+a^2}} dx = \log\left|x+\sqrt{x^2+a^2}\right| + c$$

$$10.\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\frac{x}{a} + c$$

$$11.\int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c$$

$$12.\int \sqrt{a^2+x^2}\, dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\log\left|x+\sqrt{a^2+x^2}\right| + c$$

$$13.\int \sqrt{x^2-a^2}\, dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + c$$

$$14.\int \frac{f'(x)}{f(x)} dx = \log|f(x)| + c$$

$$15.\int \frac{f'(x)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)} + c$$

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***Rules of Integration:**

If f(x) and g(x) are two real valued functions such that

$\int f(x)dx$ *and* $\int g(x)dx$ *exist, then*

1. $\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$

2. $\int k\ f(x)dx = k\int f(x)dx$ *, where k is a real constant.,*

Examples:

Integrate the following w.r.t. where x is given by:

(i) $\int (3x+4)^3\,dx = \dfrac{(3x+4)^{3+1}}{3(3+1)} + C = \dfrac{(3x+4)^4}{12} + C$

(ii) $\int (3x^2-5)^2\,dx = \int (9x^4 - 30x^2 + 25)dx$

$$= 9\int x^4dx - 30\int x^2dx + 25\int 1dx$$

$$= 9\left[\frac{x^5}{5}\right] - 30\left[\frac{x^3}{3}\right] + 25x + c$$

$$= \frac{9}{5}x^5 - 10x^3 + 25x + c$$

(iii) If $f'(x) = 4x^3 - 3x^2 + 2x + k$ and $f(0) = 1;\ f(1) = 4$,

Find $f(x)$.

$f'(x) = 4x^3 - 3x^2 + 2x + k$

$\therefore f(x) = \int f'(x)dx = \int (4x^3 - 3x^2 + 2x + k)dx$

$$= 4\int 3x^2dx - 3\int x^2dx + 2\int x\ dx + k\int 1dx$$

$$= 4\left[\frac{x^4}{4}\right] - 3\left[\frac{x^3}{3}\right] + 2\left[\frac{x^2}{2}\right] + kx + c$$

$$\therefore f(x) = x^4 - x^3 + x^2 + kx + c$$

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Now

$$1 = f(0) = (0)^4 - (0)^3 + (0)^2 + k(0) + c$$

$\therefore \boxed{c = 1}$

$$4 = f(1) = (1)^4 - (1)^3 + (1)^2 + k(1) + c$$

$$= \cancel{1} - \cancel{1} + 1 + k + c$$

$$= k + 1 + 1 \qquad (\because c = 1)$$

$$4 = k + 2 \qquad \therefore 4 - 2 = k$$

$\therefore \boxed{k = 2}$

$$\therefore \boxed{f(x) = x^4 - x^3 + x^2 + 2x + 1}$$

***Methods of Integration:**

In this method, we reduce the given function to standard form by changing variable *x* to *t*, using some suitable substitution $x = \phi(t)$.

Result:

If $x = \phi(t)$ is differentiable function of *t*

Then,

$$\int f(x)\,dx = \int f[\phi(t)]\phi'(t)\,dt$$

Corollary: 1 $\int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} + c, n+1 \neq 0$

Corollary: 2 $\int \frac{f'(x)}{f(x)}dx = \log|f(x)| + c$

Corollary: 3 $\int \frac{f'(x)}{\sqrt{f(x)}}dx = 2\sqrt{f(x)} + c$

***Evaluate the following:**

(1) $\int 7^{x\log x}(1 + \log x)\,dx$

Sol^n: $Let\ I = \int 7^{x\log x}(1 + \log x)\,dx$

Put $x \log x = t$. Different w.r.t. x

$$\therefore \left(x \cdot \frac{1}{x} + \log x \cdot 1 \right) = \frac{dt}{dx}$$

$$\therefore (1 + \log x)\, dx = dt$$

$$\therefore I = \int 7^t\, dt = \frac{7t}{\log_e 7} + c = \frac{7^{x \log x}}{\log_e 7} + c$$

2) $\int \frac{1}{x + x^{-n}}\, dx$

Sol^{n}:

$$Let\ I = \int \frac{1}{x + \frac{1}{x^n}} dx$$

$$= \int \frac{x^n}{x^{n+1} + 1} dx$$

$$= \frac{1}{n+1} \int \frac{(n+1) x^n}{x^{n+1} + 1}\, dx$$

$$= \frac{1}{n+1} \int \frac{\frac{d}{dx}\left(x^{n+1} + 1\right)}{\left(x^{n+1} + 1\right)}\, dx$$

$$= \frac{1}{n+1} \log | x^{n+1} + 1 | + c \qquad [\because \int \frac{f'(x)}{f(x)} = \log | f(x) | + c]$$

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3) $\int \frac{e^{5x}}{\sqrt{e^{5x}+1}}dx$

Sol^n:

Let $I = \int \frac{e^{5x}}{\sqrt{e^{5x}+1}}dx$

Put $e^{5x}+1=t$. Differentiate w.r.t. *x*,

$$5e^{5x}dx = dt \qquad \therefore e^{5x}dx = \frac{dt}{5}$$

$$\therefore I = \int \frac{1}{5}\frac{1}{\sqrt{t}}dt = \frac{1}{5}\int \frac{1}{\sqrt{t}}dt = \frac{1}{5}\int t^{-1/2}dt = \frac{1}{5}\frac{t^{-1/2+1}}{-\frac{1}{2}+1}+c$$

$$= \frac{1}{5}\frac{t^{1/2}}{1/2}+c = \frac{2}{5}t^{1/2}c = \frac{2}{5}\sqrt{t}+c = \frac{2}{5}\sqrt{e^{5x}+1}+c$$

(4) $\int \frac{e^{x-1} + x^{e-1}}{e^x + x^e}\, dx$.

Sol^n:

Let $I = \frac{e^{x-1} + x^{e-1}}{e^x + x^e}\, dx$

$$I = \int \frac{e^x \cdot e^{-1} + x^{e-1}}{e^x + x^e}\, dx$$

$$= \int \frac{\frac{e^x}{e} + x^{e-1}}{e^x + x^e}\, dx$$

$$= \int \frac{e^x + ex^{e-1}}{e\left(e^x + x^e\right)}\, dx$$

$$= \frac{1}{e}\int \frac{\frac{d}{dx}(e^x + x^e)}{\left(e^x + x^e\right)}\, dx$$

$$= \frac{1}{e}\log\left|e^x + x^e\right| + c \quad [\because \int \frac{f'(x)}{f(x)} = \log|f(x)| + c]$$

* **Some Special Integrals :**

1) $\int \frac{1}{x^2 - a^2}\, dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c$

2) $\int \frac{1}{x^2 - a^2}\, dx = \frac{1}{2a} \log\left|\frac{x - a}{x + a}\right| + c$

3) $\int \frac{1}{a^2 - x^2}\, dx = \frac{1}{2a} \log\left|\frac{a + x}{a - x}\right| + c$

4) $\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \sin^{-1} \frac{x}{a} + c$

5) $\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \log\left|x + \sqrt{x^2 + a^2}\right| + c$

6) $\int \frac{1}{\sqrt{x^2 - a^2}}\, dx = \log\left|x + \sqrt{x^2 - a^2}\right| + c$

7) $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) + c$

8) $\int \sqrt{a^2 + x^2}\, dx = \frac{x}{2} \sqrt{a^2 + x^2} + \frac{a^2}{2} \log\left|x + \sqrt{a^2 + x^2}\right| + c$

9) $\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2} \sqrt{x^2 - a^2} + \frac{a^2}{2} \log\left|x + \sqrt{x^4 - a^2}\right| + c$

Ex.: Find the following integrals:

1) $I = \int \frac{1}{9x^2 + 49}\, dx = \frac{1}{9} \int \frac{1}{x^2 + \frac{49}{9}}\, dx = \frac{1}{9} \int \frac{1}{(x)^2 + \left(\frac{7}{3}\right)^2}\, dx$

$= \frac{1}{9} \times \frac{1}{7/3} \tan^{-1}\left(\frac{x}{7/3}\right) + c \left[\because \int \frac{1}{x^2 + a^2}\, dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c\right]$

$= \frac{1}{9} \times \frac{3}{7} \tan^{-1}\left(\frac{3x}{7}\right) + c = \frac{1}{21} \tan^{-1}\left(\frac{3x}{7}\right) + c$

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2) $$I = \int \frac{1}{4x^2 - 1}\, dx = \frac{1}{4}\int \frac{1}{x^2 - \frac{1}{4}}\, dx = \frac{1}{4}\int \frac{1}{(x)^2 - \left(\frac{1}{2}\right)^2}\, dx$$

$$= \frac{1}{4} \cdot \frac{1}{2\left(\frac{1}{2}\right)} \log\left|\frac{x - \frac{1}{2}}{x + \frac{1}{2}}\right| + c$$

$$\left[\because \int \frac{1}{x^2 - a^2}\, dx = \frac{1}{2a} \log\left|\frac{x-a}{x+a}\right| + c\right]$$

$$= \frac{1}{4} \log\left|\frac{2x-1}{2x+1}\right| + c$$

3) $$I = \int \frac{1}{\sqrt{25 + x^2}}\, dx = \int \frac{1}{\sqrt{(5)^2 + (x)^2}}\, dx$$

$$= \log\left|x + \sqrt{x^2 + 25}\right| + c \quad \left[\therefore \int \frac{1}{\sqrt{a^2 + x^2}}\, dx = \log\left|x + \sqrt{x^2 + 25}\right| + c\right]$$

4) $$I = \int \frac{x^2}{\sqrt{1 + x^6}}\, dx = \int \frac{x^2}{\sqrt{1 - (x^3)^2}}\, dx$$

Put $x^3 = t \qquad \therefore 3x^2\, dx = dt \qquad \therefore x^2\, dx = \frac{dt}{3}$

$$\therefore I = \frac{1}{3}\int \frac{dt}{\sqrt{1 - t^2}} = \frac{1}{3}\sin^{-1} t + c$$

$$= \frac{1}{3}\sin^{-1}(x^3) + c \qquad \left[\therefore \int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \sin^{-1}\left(\frac{x}{a}\right) + c\right]$$

5) $$I = \int \frac{1}{x^2 + 4x + 8}\, dx$$

$$= \int \frac{1}{x^2 + 4x + 4 - 4 + 8}\, dx = \int \frac{1}{(x+2)^2 + 2^2}\, dx$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{x+2}{2}\right) + c \qquad \left[\therefore \int \frac{1}{x^2 + a^2}\, dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c\right]$$

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6) $\int \sqrt{4 - 9x^2}$

Let $I = \int \sqrt{4-9x^2}\ dx$

$= \int \sqrt{(2)^2 - (3x)^2}\ dx$

$= \frac{3x}{2} \sqrt{4-9x^2} + \frac{9x^2}{2} \sin^{-1}\left(\frac{3x}{2}\right) + c$

$\left[\because \int \sqrt{a^2 - x^2}\ dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) + c\right]$

7) $\int \sqrt{9x^2 - 4}\ dx$

Let $I = \int \sqrt{3(x)^2 - (2)^2}$

$= \frac{3x}{2} \sqrt{9x^2 - 4} - \frac{9x^2}{2} \log\left|3x + \sqrt{9x^2 - 4}\right| + c$

$\left[\because \int \sqrt{x^2 - a^2}\ dx = \frac{x}{2} \sqrt{x^2 - a^2} + \frac{a^2}{2} \log\left|x + \sqrt{x^2 - a^2}\right| + c\right]$

8) $\int \sqrt{x^2 - 4x + 5}\ dx$

Let $I = \int \sqrt{x^2 - 4x + 5}\ dx$

$= \int \sqrt{x^2 - 4x + 4 + 1}\ dx$

$= \int \sqrt{(x-2)^2 + (1)^2}\ dx$

$= \frac{x-2}{2} \sqrt{x^2 - 4x + 5} + \frac{1}{2} \log\left|(x-2) + \sqrt{x^2 - 4x + 5}\right| + c$

9) $\int \sqrt{2ax - x^2}\ dx$

Let $I = \int \sqrt{2ax - x^2}\ dx$

$= \int \sqrt{a^2 - a^2 + 2ax - x^2}\ dx$

$= \int \sqrt{a^2 \left(a^2 - 2ax + x^2\right)}\ dx$

$= \int \sqrt{a^2 - (a-x)^2}\ dx$

$$= \frac{a-x}{2}\sqrt{2ax - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{a-x}{a}\right) + c$$

10) $\int \sqrt{(x-3)(5-x)}\, dx$

$$= \int \sqrt{-x^2 + 8x - 15}\, dx$$

$$= \int \sqrt{(-1)\left(x^2 - 8x + 15\right)}\, dx$$

$$= \int \sqrt{(-1)\{x^2 - 8x + 16 - 16 + 1\}}\, dx$$

$$= \int \sqrt{(-1)\left\{(x-4)^2 - (1)^2\right\}}\, dx$$

$$= \int \sqrt{(-1)^2 - (x-4)^2}\, dx$$

$$= \frac{1}{2}(x-4)\sqrt{1^2 - (x-4)^2} + \frac{(1)^2}{2}\sin^{-1}\left|\frac{x-4}{1}\right| + c$$

$$= \frac{x-4}{2}\sqrt{(x-3)(5-x)} + \frac{1}{2}\sin^{-1}(x-4) + c\,.$$

* **<u>Integration by parts:</u>**

The method of integration by parts is used when the integrand is expressed as a product of two functions, one of which can be differentiated and the other can be integrated conveniently.

If u and v are both functions of x, then

$$\boxed{\int u.v dx = u\int v dx - \int\left\{\frac{d}{dx}(u)\int v dx\right\}dx}$$

<u>Note:</u>

(1) When integrand is a product of two functions, out of which the second has to be integrated (who's integral is known), hence we should make the proper choice of first and second function.

(2) We can also choose the first function as the function which comes first in the word '**LIATE**' where

L-logarithmic function

I-the inverse function

A-the algebraic function

T-the trigonometry function

E-the exponential function

Examples.: Find the following integrals.

1) $\int x^2 e^{2x} dx$

Let $I = \int x^2 e^{2x} dx$

$u = x^2 \ \ v = e^{2x}$ & by using $\boxed{\int u.v\,dx = u\int v\ dx - \int\left(\frac{du}{dx}\int v\,dx\right)dx}$

$$\text{I} = x^2 \int e^{2x}\ dx - \int\left(\frac{d}{dx}\left(x^2\right)\int e^{2x}\ dx\right)dx$$

$$= x^2 \cdot \frac{e^{2x}}{2} - \int\left[\left(2x\right)\left(\frac{e^{2x}}{2}\right)\right]dx$$

$$= \frac{x^2 e^{2x}}{2} - \int x e^{2x}\ dx$$

Again by using integration by parts method for $\int x e^{2x}\ dx$

$$= \frac{1}{2}x^2 e^{2x} - \left[x\int e^{2x}dx - \int\left(\frac{d}{dx}(x)\int e^{2x}dx\right)dx\right]$$

$$= \frac{1}{2}x^2 e^{2x} - \left[x\int e^{2x}dx - \int\left[(1)\left(\frac{e^{2x}}{2}\right)\right]dx\right]$$

$$= \frac{1}{2}x^2 e^{2x} - \left[\frac{1}{2}\ x e^{2x} - \frac{1}{2}\int e^{2x}\ dx\right]$$

$$= \frac{1}{2}x^2 e^{2x} - \left[\frac{1}{2}\ x e^{2x} - \frac{1}{2}\left[\frac{e^{2x}}{2}\right]\right] + c$$

$$= \frac{1}{2}x^2 e^{2x} - \left[\frac{1}{2}\ x e^{2x} - \frac{1}{4}\ e^{2x}\right] + c$$

$$\text{I} = \frac{1}{2}x^2 e^{2x} - \frac{1}{2}\ x e^{2x} + \frac{1}{4}\ e^{2x} + c$$

2) $\int \log x\ dx$

Let $I = \int \log x \, dx$

$$= \int 1.\log x \, dx$$

Let $u = \log x$, $v = 1$ & $\int uv \, dx = u \int v \, dx - \int \left(\frac{du}{dx} \int v \, dx \right) dx$

$$\therefore I = \log x \int 1 \, dx - \int \left[\frac{d}{dx} (\log x) \int 1 \, dx \right] dx$$

$$= \log x \, (x) - \int \left[\left(\frac{1}{x} \right) (x) \right] dx$$

$$= x \log x - \int 1 \, dx$$

$$= x \log x - x + c$$

3) $\int x \cdot 2^{-3x} \, dx$

Let $I = \int x \cdot 2^{-3x} \, dx$

Taking $u = x$ & $v = 2^{-3x}$ by using integration by parts

$$= x \int 2^{-3x} \, dx - \int \left[\frac{d}{dx}(x) \int 2^{-3x} \, dx \right] dx$$

$$= x \cdot \frac{2^{-3x}}{-3(\log 2)} - \int 1 \cdot \left[\frac{2^{-3x}}{-3 \log 2} \right] dx$$

$$I = \frac{x \, 2^{-3x}}{3 \log 2} + \frac{1}{3 \log 2} \int 2^{-3x} \, dx$$

$$= -\frac{1}{3 \log 2} x 2^{-3x} + \frac{1}{3 \log 2} \left[\frac{2^{-3x}}{(-3) \log 2} \right] + c$$

$$= -\frac{1}{3 \log 2} x \cdot 2^{-3x} - \frac{2^{-3x}}{9(\log 2)^2} + c$$

4) $\int x^n \log x \, dx$

Let $I = \int x^n \log x \, dx$

Taking $u = \log x$ & $v = x^n$ integrating by parts

$$I = \log x \int x^n \ dx - \int \left[\frac{d}{dx} (\log x) \int x^n \ dx \right] dx$$

$$= \log \ x \left[\frac{x^{n+1}}{n+1} \right] - \int \left[\left(\frac{1}{x} \right) \left(\frac{x^{n+1}}{n+1} \right) \right] dx$$

$$= \frac{x^{n+1}}{n+1} \log x - \frac{1}{n+1} \int x^n \, dx$$

$$= \frac{x^{n+1}}{n+1} \log x - \frac{1}{n+1} \left[\frac{x^{n+1}}{n+1} \right] + c$$

$$\therefore I = \frac{x^{n+1}}{n+1} \left[\log x - \frac{1}{n+1} \right] + c$$

5) $\int \log(1 + x^2) \, dx$

Let $I = \int 1 \cdot \log(1 + x^2) \, dx$

Taking $u = \log(1+x^2)$ $v = 1$ integrating by parts

$$I = \log(1+ x^2) \int 1 \, dx - \int \left[\frac{d}{dx} \left(\log(1 + x^2) \right) \int 1 \, dx \right] dx$$

$$= \log \left(1 + x^2\right) (x) - \int \left[\frac{2x}{1 + x^2} \times x \right] dx$$

$$= x \log \left(1 + x^2\right) - 2 \int \frac{x^2}{1 + x^2} \, dx$$

$$= x \ \log \left(1 + x^2\right) - 2 \int \frac{1 + x^2 - 1}{1 + x^2} \, dx$$

$$= x \ \log \left(1 + x^2\right) - 2 \int 1 \, dx + 2 \int \frac{1}{1 + x^2} \, dx$$

$$= x \ \log \left(1 + x^2\right) - 2x + 2 \ \tan^{-1} x + c$$

6) $\int \frac{\log(\log x)}{x} \, dx$

$$I = \int \frac{\log(\log x)}{x}\, dx = \int \log(\log x) \cdot \frac{1}{x}\ dx$$

Taking $u = \log(\log x)$ & $v = \frac{1}{x}$, Integrating by parts

$$I = \log(\log x) \int \frac{1}{x}\, dx - \int \left[\frac{d}{dx} \left(\log(\log x)\right) \int \frac{1}{x}\, dx \right] dx$$

$$= \log\,(\log\ x)\ \log|x| - \int \left[\frac{1}{x \log\ x}\ \log|x| \right] dx$$

$$= \log\,(\log\ x)\ (\log|x| - \int \frac{1}{x}\, dx$$

$$= \log\,(\log\ x)\ \log|x| - \log|x| + c$$

$$= \log|x| \left[\log\,(\log\ x) - 1 \right] + c$$

***Integration by Partial Function:**

If f(x) and g(x) are two polynomials, then $\frac{f(x)}{g(x)}$ is a rational function, where g(x) $\neq$ 0.

If deg of f(x) < deg of g(x), then $\frac{f(x)}{g(x)}$ is a proper rational function.

It can be expressed by partial fractions using following table.

Where A, B, C and D used in the table are real numbers.

	Rational Form	Partial Form
1	$\frac{Px^2 + qx + c}{(x-a)(x-b)(x-c)}$	$\frac{A}{(x-a)} + \frac{B}{(x-b)} + \frac{c}{(x-c)}$
2	$\frac{Px^2 + qx + r}{(x-a)^2 (x-b)}$	$\frac{A}{(x-a)} + \frac{B}{(x-a)^2} + \frac{c}{x-b}$
3	$\frac{Px^2 + qx + r}{(x-a)^3 (x-b)}$	$\frac{A}{(x-a)} + \frac{B}{(x-a)^2} + \frac{c}{(x-a)^3} + \frac{D}{x-b}$
4	$\frac{Px^2 + qx + r}{(x-a)(x^2 + bx + c)}$	$\frac{A}{(x-a)} + \frac{Bx + c}{x^2 + bx + c}$

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		Where $x^2 + bx + c$ cannot be factorized further

Examples:

i) Evaluate $\int \frac{x}{(x-1)(2x+1)} dx$

Let $\frac{x}{(x-1)(2x+1)} = \frac{A}{x-1} + \frac{B}{(2x+1)}$

Multiplying both sides by $(x-1)(2x+1)$, we have

$$x = A(2x+1) + B(x-1) \qquad \text{.... (i)}$$

Putting $x = 1$ we get $1 = A(2(1)+1) + B(1-1)$

$$\therefore 1 = A(3) \qquad \therefore \boxed{A = \frac{1}{3}}$$

Putting $x = -\frac{1}{2}$ in eq^n (i) we get

$$\therefore -\frac{1}{2} = A\left(2\left(-\frac{1}{2}\right)+1\right) + B\left(-\frac{1}{2}-1\right) \quad \therefore -\frac{1}{2} = B\left(-\frac{3}{2}\right)$$

$$\therefore B = \frac{-1/2}{-3/2} = 1/3 \qquad \therefore \boxed{B = 1/3}$$

$$\therefore \int \frac{x}{(x-1)(2x+1)} dx = \int \left[\frac{1}{3(x-1)} + \frac{1}{3(2x+1)}\right] dx$$

$$= \frac{1}{3}\int \frac{1}{x-1} dx + \frac{1}{3}\int \frac{1}{2x+1} dx$$

$$= \frac{1}{3}\log(x-1) + \frac{1}{3}\cdot\frac{1}{2}\log(2x+1)$$

$$= \frac{1}{3}\log|x-1| + \frac{1}{6}\log|2x+1| + c$$

ii) Evaluate $\int \frac{dx}{x-x^3}$

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$$\int \frac{dx}{x-x^3} = \int \frac{dx}{x\left(1-x^2\right)} = \int \frac{dx}{x(1-x)\,(1+x)}$$

Let $\dfrac{1}{x(1-x)\,(1+x)} = \dfrac{A}{x} + \dfrac{B}{1-x} + \dfrac{c}{1+x}$

Multiplying both sides by $x(1-x)\,(1+x)$, we have

$$1 = A(1-x)\,(1+x) + Bx(1+x) + c\;x(1-x)$$

Putting $x = 0, 1$ and -1 we have

$$1 = A \Rightarrow A = 1$$

$$1 = 2B \Rightarrow B = \frac{1}{2}$$

$$1 = -2c \Rightarrow c = -\frac{1}{2}$$

$$\therefore \int \frac{dx}{x - x^3} = \int \left\{ \frac{1}{x} + \frac{1}{2(1-x)} - \frac{1}{2(1+x)} \right\} dx$$

$$= \log|x| + \frac{1}{2}\,\frac{\log|1-x|}{-1} - \frac{1}{2}\log|1+x|$$

$$= \log|x| - \frac{1}{2}\log|1-x| - \frac{1}{2}\log|1-x| + c$$

$$= \frac{1}{2}\left[2\log|x| - \log|1-x| - \log|1+x|\right] + c$$

$$= \frac{1}{2}\left[\log x^2 - \left(\log|1-x| + \log|1+x|\right)\right] + c$$

$$= \frac{1}{2}\left[\log x^2 - \log\left|1 - x^2\right|\right] + c$$

$$= \frac{1}{2}\log\left|\frac{x^2}{1-x^2}\right| + c$$

iii) Evaluate $\displaystyle\int \frac{2x}{\left(x^2+1\right)\left(x^2+2\right)}\,dx$

Putting $x^2 = t$

$\therefore 2x\;dx = dt$

$$\therefore I = \int \frac{dt}{(t+1)(t+2)}$$

Now $\frac{1}{(t+1)(t+2)} = \frac{A}{t+1} + \frac{B}{t+2}$

$$\therefore 1 = A(t+2)\ B(t+1) \qquad \text{……….} (1)$$

Putting $t = -1$ in eq^n (i) we get $A = 1$

Putting $t = -2$ in eq^n (i) we get $B = -1$

Substituting the values A & B in eq^n (*1*) we get

$$I = \int \frac{1}{(t+1)(t+2)}\, dx = \int \frac{dt}{t+1} - \int \frac{dt}{t+2}$$

$$= \log|t+1| - \log|t+2| + c$$

$$= \log|x^2+1| - \log|x^2+2| + c$$

iv) Evaluate $\int \frac{x+5}{(x+1)(x+2)^2}\, dx$

Let $\frac{x+5}{(x+1)(x+2)^2} = \frac{A}{x+1} + \frac{B}{(x+2)} + \frac{c}{(x+2)^2}$ …. (1)

$$\therefore x+5 = A(x+2)^2 + B(x+1)(x+2) + C(x+1)$$

Putting $x = -2$ we get $-2+5 = A(0) + B(0) + c(-2+1)$

$$\therefore 3 = -c \qquad \therefore \boxed{c = -3}$$

Putting $x = -1$ we get $-1+5 = A(-1+2)^2 + B(0) + c(0)$

$$\therefore 4 = A(1)^2 \qquad \therefore \boxed{A = 4}$$

Comparing coeff of x^2 on both sides of equation

$$\therefore\ x+5 = A(x^2+4x+4) + B(x^2+3x+2) + c(x+1)$$

$$= x^2(A+B) + x(4A+3B+C) + (4A+2B+C)$$

$$\therefore A+B = 0 \quad \therefore A = -B$$

$$\therefore B = -A$$

$\therefore \boxed{B = -4}$

Substituting value of A, B & C in eq^n (1)

$$\int \frac{x+5}{(x+1)(x+2)^2}\,dx = \int \frac{4}{x+1}\,dx + \int \frac{-4}{x+2}\,dx + \int \frac{-3}{(x+2)^2}\,dx$$

$$= 4\int \frac{1}{x+1}\,dx - 4\int \frac{1}{x+2}\,dx - 3\int \frac{1}{(x+2)^2}\,dx$$

$$= 4\log|x+1| - 4\log|x+2| - 3\left(-\frac{1}{x+2}\right) + c$$

$$= 4\log|x+1| - 4\log|x+2| - \frac{3}{x+2} + c$$

v) Evaluate $\int \frac{x^2+1}{(x^2+4)(x^2+25)}\,dx$

Consider the integrated & replace x^2 by t only.

(Here we are not substituting t for x)

Then the integrated becomes

$$\frac{t+1}{(t+4)(t+25)} = \frac{A}{t+4} + \frac{B}{t+25} \qquad \text{.......... }(x)$$

$$\therefore t+1 = A(t+25) + B(t+4) \qquad \text{.......... (i)}$$

Putting $t = -4$ in eq^n (i), we get $-3 = A(21) \Rightarrow \boxed{A = -\frac{1}{7}}$

Putting $t = 25$ in eq^n (i), we get $-25+1 = A(0) + B(-21)$

$$\therefore -24 = -21B \therefore \boxed{B = \frac{24}{21}}.$$

Substituting the values A & B and replacing t by x in eq^n (x);

We get,

$$\int \frac{x^2+1}{\left(x^2+4\right)\left(x^2+25\right)}\, dx$$

$$= \int \frac{-\frac{1}{7}}{x^2+4}\, dx + \int \frac{\frac{24}{21}}{x^2+25}\, dx$$

$$= -\frac{1}{7}\int \frac{1}{x^2+4}\, dx + \frac{24}{21}\int \frac{1}{x^2+25}\, dx$$

$$= -\frac{1}{14}\tan^{-1}\left(\frac{x}{2}\right) + \frac{24}{105}\tan^{-1}\left(\frac{x}{5}\right) + c$$

$$= -\frac{1}{7}\cdot\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + \frac{24}{21}\cdot\frac{1}{5}\tan^{-1}\left(\frac{x}{5}\right) + c$$

vi) Evaluate $\int \frac{8}{(x+2)\left(x^2+4\right)}\, dx$

Let $\int \frac{8}{(x+2)\left(x^2+4\right)}\, dx = \frac{A}{x+2} + \frac{Bx+c}{x^2+4}$ .... (x)

Then $8 = A\left(x^2+4\right) + (Bx+c)(x+2)$ .... (i)

Putting $x = -2$ in eq^n (i) we get $8 = 8A \Rightarrow \boxed{A=1}$

Comparing coeff of x^2, x and constant number on both sides of (i)

$$8 = Ax^2 + 4A + Bx^2 + cx + 2Bx + 2c$$

$$= x^2(A+B) + x(C+2B) + 4A + 2c$$

Comparing coeff of $x^2 \Rightarrow A + B = 0 \Rightarrow A = -B \Rightarrow \boxed{B=-1}$

Comparing coeff of $x \Rightarrow 2B + c = 0 \Rightarrow 2(-1) + c = 0 \Rightarrow \boxed{C=2}$

Compare constant form $4A + 2C = 8 \Rightarrow 4(1) + 2(2) = 8$

Substituting value of A, B & C in equation (x), we get

$$\int \frac{8}{(x+2)(x^2+4)}\, dx = \int \frac{1}{x+2}\, dx + \int \frac{-x+2}{x^2+4}\, dx$$

$$= \int \frac{1}{x+2}\, dx - \int \frac{x}{x^2+4}\, dx + 2\int \frac{1}{x^2+4}\, dx$$

$$= \int \frac{1}{x+2}\, dx - \frac{1}{2}\int \frac{2x}{x^2+4}\, dx + 2\int \frac{1}{x^2+4}\, dx$$

$$= \log|x+2| - \frac{1}{2}\log|x^2+4| + 2.\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$$

$$= \log|x+2| - \frac{1}{2}\log|x^2+4| + \tan^{-1}\left(\frac{x}{2}\right) + c$$

$$\left[\text{For } \int \frac{2x}{x^2+4} dx \text{ using } \int \frac{f'(x)}{f(x)}\, dx = \log|f(x)| + c\right]$$

and

$$[For \ \frac{1}{x^2+4} dx \ \text{ using } \int \frac{1}{x^2+a^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c]$$

Definite Integration:

In geometrical and other applications of integral calculus, it becomes necessary to find the difference in the values of the integral of a function $f(x)$ between two assigned values of an independent variable x, say a, b. The difference is called the definite integral $f(x)$,
where a & be are finite number [a, b], over the interval [a, b] and is denoted by $\int_a^b f(x)\, dx$. Thus $\int_a^b f(x)\, dx = \phi(b) - \phi(a)$ where $\phi(x)$ is an integral of $f(x)$.

The number a is called lower limit and b is called upper limit of definite integral.

$\int_a^b f(x)\,dx$ is called definite integral because the indefinite constant of integration does not appear in it.

Since

$$\int_a^b f(x)\,dx = \left[\phi(x) + c\right]_a^b = \{\phi(b) + c\} - \{\phi(a) + c\} = \phi(b) - \phi(a)$$

Here arbitrary constant c disappears in the process.

*<u>Fundamental theorem of Integral Calculus:</u>

Let f be the continuous function defined on [a ,b].

If $\int f(x)dx = \phi(x) + c, then \int_a^b f(x)dx = \phi(b) - \phi(a)$

There is no need of taking the constant of integration c, because c gets eliminated.

*<u>EXAMPLES:</u>

Evaluate the following integrals:

$$(1)\int_2^3 e^{2x}\,dx$$

$$= \left[\frac{e^{2x}}{2}\right]_2^3$$

$$= \frac{1}{2}\left[e^{2x}\right]_2^3$$

$$= \frac{1}{2}\left[e^6 - e^4\right]$$

$$= \frac{1}{2}e^4\left(e^2 - 1\right)$$

$$(2)\int_{-1}^{1}(3x-2)^2\ dx$$

$$=\int_{-1}^{1}\left(9x^2-12x+4\right)dx$$

$$=\left[9\left(\frac{x^3}{3}\right)-12\left(\frac{x^2}{2}\right)+4x\right]_{-1}^{1}$$

$$=\left[3x^3-6x^2+4x\right]_{-1}^{1}$$

$$=\left[3(1)^3-6(1)^2+4(1)\right]-\left[3(-1)^3-6(-1)^2+(-1)\ \right]$$

$$=\left[3(1)-6+4\right]-\left[3(-1)-6(1)+4(1)\right]$$

$$=(3-6+4)-(-3-6-4)$$

$$=1-(-13)$$

$$=1+13=14$$

$$3)\quad\int_{-1}^{4}x^{-1/2}\ dx$$

$$=\left[\frac{x^{-1/2}+1}{-1/2+1}\right]_{1}^{4}$$

$$=\left[\frac{x^{1/2}}{1/2}\right]_{1}^{4}$$

$$=2\left[4^{1/2}-1^{1/2}\right]$$

$$=2[2-1]$$

$$=2(1)=2$$

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(4) $\int_{-2}^{3} e^{-x/2}\, dx$

$$= \left[\frac{e^{-x/2}}{-1/2}\right]_{-2}^{3}$$

$$= -2\left[e^{-3/2} - e^{+1}\right]$$

$$= -2\left(e^{3/2} - e\right)$$

(5) $\int_{0}^{1} \frac{1}{\sqrt{1+x} + \sqrt{x}}\, dx$

$$= \int_{0}^{1} \frac{\left(\sqrt{1+x} - \sqrt{x}\right)}{\left(\sqrt{1+x} + \sqrt{x}\right)\left(\sqrt{1+x} - \sqrt{x}\right)}\, dx$$

$$= \int_{0}^{1} \frac{\sqrt{1+x} - \sqrt{x}}{1}\, dx$$

$$= \int_{0}^{1} \left(\sqrt{1+x} - \sqrt{x}\right) dx$$

$$= \int_{0}^{1} (1+x)^{1/2}\, dx - \int_{0}^{1} x^{1/2}\, dx$$

$$= \left[\frac{(1+x)^{3/2}}{3/2}\right]_{0}^{1} - \left[\frac{x^{3/2}}{3/2}\right]_{0}^{1}$$

$$= \frac{2}{3}\left[(1+x)^{3/2}\right]_{0}^{1} - \frac{2}{3}\left[x^{3/2}\right]_{0}^{1}$$

$$= \frac{2}{3}\left[(1+1)^{3/2} - (1+0)^{3/2}\right] - \frac{2}{3}\left[1^{3/2} - 0^{3/2}\right]$$

$$= \frac{2}{3}\left[(2)^{3/2} - 1\right] - \frac{2}{3}\left[1^{3/2}\right]$$

$$= \frac{2}{3}\left[2^{3/2} - 1 - 1\right]$$

$$= \frac{2}{3}\left[2^{3/2} - 2\right]$$

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$$= \frac{2}{3} \times 2 \left[2^{1/2} - 1 \right]$$

$$= \frac{4}{3} \left[\sqrt{2} - 1 \right]$$

(6) If $\int_0^a 3x^2 \; dx = 8$, find the value of a

$$3\left[\frac{x^3}{3}\right]_0^a = 8$$

$$\therefore \left[x^3\right]_0^a = 8$$

(7) $\int_1^2 \log x \; dx$

$$\int_1^2 \log x = \log x \int 1 \, dx - \int \left[\frac{d}{dx} (\log x) \int 1 \, dx \right] dx$$

(using integration by parts)

$$= x \log x - \int \left(\frac{1}{x} \times x \right) dx$$

$$\therefore \int_1^2 \log x \cdot 1 \, dx = \left[\log x \cdot x\right]_1^2 - \int_1^2 \frac{1}{x} x \, dx$$

$$= \left[\log x \cdot x\right]_1^2 - \left[x\right]_1^2$$

$$= \left[2 \log 2 - 1 \log 1\right] - \left[2 - 1\right]$$

$$= \left[2 \log 2 - 0\right] - (1)$$

$$= \log 2^2 - 1$$

$$= \log 4 - 1.$$

(8) $\int_1^3 \frac{1}{x\left(1 + x^2\right)} \, dx$

Let $\dfrac{1}{x\left(1+x^2\right)} = \dfrac{A}{x} + \dfrac{Bx+c}{1+x^2}$

$\therefore 1 = A\left(1+x^2\right) + Bx + C(x)$ .... (i)

Putting $x = 0$ $\boxed{A=1}$

Comparing coeff of x^2 of x on both sides

$A + B = 0$ & $\boxed{C=0}$, $\boxed{B=1}$

$$\therefore \int_1^3 \frac{1}{x\left(1+x^2\right)}\,dx$$

$$= \int_1^3 \frac{1}{x}\,dx + \int_1^3 \frac{-x}{1+x^2}\,dx$$

$$= \int_1^3 \frac{1}{x}\,dx - \frac{1}{2}\left[\log\left|1+x^2\right|\right]_1^3$$

$$= \left[\log|x|\right]_1^3 - \frac{1}{2}\left[\log\left|1+x^2\right|\right]_1^3$$

$$= \left(\log 3 - \log 1\right) - \frac{1}{2}\left(\log 10 - \log 2\right)$$

$$= \log 3 - \frac{1}{2}\log\left(\frac{10}{2}\right) = \log 3 - \frac{1}{2}\log 5.$$

UNIT END EXERCISE:

[A] Evaluate the following integral:

[1] $\int x.\sqrt{x}dx$ [2] $\int\left(x+\dfrac{1}{x}\right)^2 dx$ [3] $\int\left(e^{2x}+e^{-2x}\right)dx$ [4] $\int\dfrac{x^2}{x+1}dx$

[5] $\int\dfrac{e^x}{e^{2x}-4}dx$ [6] $\int\dfrac{1}{x+\sqrt{x^2-1}}dx$ [7] $\int\dfrac{(3x^2-2x+5)}{(x-1)^2(x^2+5)}dx$ [8] $\int\dfrac{(3x+2)}{(x-2)^2(x-3)}dx$

[9] $\int \frac{(3x+2)}{(x-2)(x-3)}dx$ [10] $\int x^2.e^{ax}dx$ [11] $\int (\log x)^2 dx$ [12] $\int \frac{1}{\sqrt{x^2-2x-3}}dx$

[13] $\int \frac{1}{x^2+8x+20}dx$ [14] $\int \frac{x}{x^4+x^2+1}dx$ [15] $\int \frac{x^2+2}{(x^2+4)(x^2+9)}dx$ [16] $\int \sqrt{9x^2+5}dx$

[17] $\int \sqrt{x^2-4x+5}dx$ [18] $\int \sqrt{4+3x-2x^2}dx$ [19] $\int \frac{e^x-1}{e^x+1}dx$ [20] $\int \frac{1}{\sqrt{x}+\sqrt{x+1}}dx$

[21] $\int (x+1)^2 e^x dx$ [22] $\int \frac{\log(logx)}{x}dx$ [23] $\int \frac{x+2}{(x+1)^3}dx$ [24] $\int \frac{e^x}{e^{2x}+6e^x+5}dx$ [25] $\int \frac{1}{x\log x}dx$

[B] Evaluate the following:

[1] $\int_0^1 \frac{1}{2x+5}dx$ [2] $\int_0^1 \frac{1}{2x^2+2x+1}dx$ [3] $\int_0^4 \frac{1}{\sqrt{x^2+2x+3}}dx$ [4] $\int_1^2 \frac{\log x}{x^2}dx$

[5] $\int_2^3 \frac{x}{x^2-1}dx$ [6] $\int_4^9 \frac{1}{\sqrt{x}}dx$ [7] $\int_1^2 x\log x dx$ [8] $\int_0^4 \frac{(x+1)(x+4)}{\sqrt{x}}dx$

[9] $\int_0^1 (2x^2-x^3)dx$ [10] $\int_0^1 \frac{1}{(x+2)(x+1)}dx$

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6A

Application of Integration

UNIT STRUCTURE

6B.1. INTRODUCTION

Applications of Integration:

***To find the cost function when Marginal cost is given:**

If C represent the total cost of producing an output x, then marginal cost is given by

$$MC = \frac{dc}{dx} \qquad \therefore C = \int (MC)\, dx + k$$

The constant of integration k can be evaluated if the fixed cost (i.e. the cost when x = 0) is given further, average cost AC can be obtained from the relation: $AC = \frac{C}{x}$

Ex: The marginal cost function of a product is given by $\frac{dc}{dq} = 100 - 10q + 0.1q^2$, where q is the output obtain the total and the average cost function of the firm under the assumption that is fixed cost is

Rs. 500.

Sol^n.

$$MC = \frac{dc}{dq} = 100 - 10q + 0.1q^2$$

Integrating both sides w. r. t. q we have

$$C = \int \left(100 - 10q + 0.1q^2\right) dq$$

$$= 100q - 10\frac{q^2}{2} + 0.1\frac{q^3}{3} + k$$

Now the fixed cost is 500 i.e. when $q = 0$, $C = 500 \therefore k = 500$

Hence, total cost function is

$$C = 100q - 5q^2 - \frac{q^3}{30} + 500$$

Average cost is

$$AC = \frac{C}{q} = 100 - 59 + \frac{q^2}{30} + \frac{500}{q}$$

Ex.: The marginal cost function of manufacturing x shares is $6 + 10x - 6x^2$. The total cost of producing a part of shares is 12. Find the total and average cost function.

Sol^n.

$$MC = \frac{d}{dx}(c) = 6 + 10x - 6x^2$$

$$\therefore C = \int \left(6 + 10x - 6x^2\right) dx$$

$$= 6x + 10\frac{x^2}{2} - 6\frac{x^3}{3} + k$$

Where k is the constant of integration

Now $C = 12$ when $x = 2$

$$\therefore 12 = 6(2) + 10\frac{(2)^2}{2} - 6\frac{(2)^3}{3} + k$$

$$\therefore 12 = 12 + 20 - 16 + k$$

$$\therefore k = 12 - 12 - 20 + 16 = -4$$

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$\therefore$ The total cost function is $c = 6x + 5x^2 - 2x^3 - 4$.

To find the total revenue function and demand function when marginal revenue function is given.

If R is the total revenue when the output is *x*, then the marginal revenue MR is given by $MR = \dfrac{dR}{dx}$.

Hence, if the MR is given, then the total revenue R is the indefinite integral of MR w.r.t. *x*

i.e. $R = \int (MR)\, dx + k$, where k is the constant of integration, which can be evaluated for the fact that the total revenue R is zero when output x is zero.

Since $R = px$, the demand function can be easily obtained as $p = \dfrac{R}{x}$.

Ex: If the marginal revenue function for output is given by $MR = \dfrac{6}{(q+2)^2} - 5$, find the total revenue functions by integration. Also deduce the demand function.

Sol^n :

$$MR = \frac{6}{(q+2)^2} - 5 \qquad \therefore \boldsymbol{R = \int (MR)\, dq}$$

$$\therefore \ \boldsymbol{R = \int \left(\frac{6}{(q+2)^2} - 5\right) dq = -\frac{6}{q+2} - 5q + k}$$

Since total revenue is zero at $q = 0$, we get

$$0 = -\frac{6}{2} - 5(0) + k \qquad \therefore 0 = -3 + k \qquad \therefore \boxed{k = 3}$$

$$\therefore R = -\frac{6}{q+2} - 5q + 3 \qquad \boxed{R = 3 - \frac{6}{q+2} - 5q}$$

Also we know, $R = P \times q$

$$\therefore P = \frac{R}{q} = \frac{3 - \frac{6}{q+2} - 5q}{q} = \frac{3}{q} - \frac{6}{q(q+2)} - 5$$

$$= \frac{3(q+2) - 6}{q(q+2)} - 5 = \frac{3q + 6 - 6}{q(q+2)} - 5$$

$P = \dfrac{3q}{q(q+2)} - 5$ is the required demand function.

Ex: If the marginal revenue function is $MR = \dfrac{dR}{dq} = \dfrac{1}{2} q^{-1/2}$ where R stands for total revenue .what is the demand function?

Sol^n :

$$MR = \frac{dR}{dq} = \frac{1}{2} q^{-1/2} \qquad \therefore R = \int \frac{1}{2} q^{-1/2}\, dq = \frac{1}{2}\left[\frac{q^{-1/2+1}}{-1/2+1}\right] + k$$

$$\therefore R = \frac{1}{2}\left[\frac{q^{1/2}}{1/2}\right]$$

$$\therefore R = q^{1/2} + k$$

Total revenue R = 0 at q = 0

$$\therefore R = 0 = 0^{1/2} + k \qquad \therefore k = 0$$

$\therefore \boxed{R = q^{1/2}}$ is the Total Revenue function.

$$P = \frac{R}{q} = \frac{q^{1/2}}{q} = q^{-1/2} \qquad \therefore \boxed{p = q^{-1/2}}$$

is the required demand function.

***Maximum Profit:**

Suppose we want to find out the maximum profit of a firm when only the marginal cost and the marginal revenue functions are given equating marginal cost to marginal revenue we can find the output that maximizes total profits.

Profit $P = R - C$, where P = total profit, R = total revenue, C = total cost.

$$\therefore \frac{dP}{dx} = \frac{dR}{dx} - \frac{dC}{dx} \text{; } x = \text{output} \qquad \ldots\text{ (i)}$$

Integrating eq^n (i), $P = \int \frac{dR}{dx}\, dx - \int \frac{dc}{dx}\, dx + k = R - C + k$

when k = constant of integration, can be found from the additional information given.

Remarks:

(i) If may be noted that profit is maximized when marginal revenue equals marginal cost given the assumption of pure competition total profit is the integral of marginal revenue minus marginal cost from zero quantity for which profit is maximized.

(ii) To determine profit maximizing output, first find second derivative of $(MR - MC)$ i.e. second derivative of total profit i.e. $p''(x)$

If $p''(x) < 0$ then maximum profit at *x*.

(iii) Total profit zero indicates no profit and total profit negative signify a loss.

Ex:

The ABC Co. Ltd. has approximated the marginal revenue function for one of its product by $MR = 20x - 2x^2$. The marginal cost function is approximated by $MC = 81 - 16x + x^2$. Determine the profit maximize output and the total profit at the optimal output.

Sol^n:

Solving for profit maximizing output, set $MR = MC$

$$MR = MC \Rightarrow MR - MC \Rightarrow \left(20x - 2x^2\right) - \left(81 - 16x + x^2\right) = 20x - 2x^2 - 81 + 16x - x^2 = 9$$

$$\Rightarrow -81 + 36x - 3x^2 = 0 \Rightarrow (x-3)(x-9) = 0 \Rightarrow x = 3 \ or \ x = 9$$

$$\frac{d}{dx}(MR - MC) = \frac{d}{dx}(-81 + 36x - 3x^2) = 36 - 6x = p''(x)$$

$$p''(3) = 36 - 6(3) = +18 > 0$$

$\therefore At\ x = 3$ Profit is minimum.

$$p''(9) = 36 - 6(9) = -18 < 0$$

$\therefore At\ x = 9$ Profit is maximum.

$$\text{Total Profit } = \int_0^9 \left(-81 + 36x - 3x^2\right) dx$$

$$= \left[-81.x + 36.\frac{x^2}{2} - 3.\frac{x^3}{3}\right]_0^9$$

$$= \left[-81x + 18x^2 - x^3\right]_0^9$$

$$= \left[-81(9) + 18(9)^2 - (9)^3\right]$$

$$= 0; \textit{Which indicates no profit.}$$

To find the consumption function when the marginal propensity to consume (MPC) is given:

If p is the consumption when the disposable income of a person is x, the marginal propensity to consume (MPC) is given by $MPC = \frac{dp}{dx}$

Hence if MPC is given, the consumption p is given the indefinite integral of M.P.C. w.r.t. *x*. i.e.

$$p = \int (MPC)\,dx + k$$

The constant of integration k, can be evaluated if the value of p is known for some *x*.

Ex. If the marginal propensity to save (MPS) is 1.5 + 0.2x^{-2}, when x is the income. Find the consumption function, given that the consumption is 4.8 when income is ten.

Sol^{n}. Now "derivative of consumption function w.r.t. output represents marginal propensity to consume".

$$MPS = 1.5 + 0.2x^{-2} = \frac{dp}{dx}$$

$$p = \int \left(1.5 + 0.2x^{-2}\right) dx = 1.5x + 0.2\left(\frac{x^{-2+1}}{-2+1}\right) + k = 1.5 + 0.2\frac{x^{-1}}{-1} + k$$

$$\therefore\ p = 1.5x\frac{0.2}{x} + k$$

Now p = 4.8 when *x* = 10

$$4.8 = 1.5 \times 10 - \frac{0.2}{x} + k$$

$$\therefore k = -10.18$$

Hence the consumption function is

$$p = 1.5x - \frac{0.2}{x} - 10.18$$

In order to find CS under monopoly, i.e. to maximise profit we must have MR = MC

$$\Rightarrow\ 144 - 96x + 12x^2 = 56 + 4x$$

$$\Rightarrow\ 12x^2 - 100x + 88 = 0$$

$$\Rightarrow\ 3x^2 - 25x + 22 = 0$$

$$\Rightarrow\ x = 1 = x_0\ \ OR\ \ x = \frac{22}{3} = x_0$$

When $x_0 = 1, D(x_0) = p_0 = (12-2)^2 = 100$

$$CS = \int_0^1 \left(144 - 48x + 4x^2\right) dx - 1 \times 100$$

$$_0\left[144x - 48.\frac{x^2}{2} + 4.\frac{x^3}{3}\right]_0^1 - 100 = 144 - 24 + \frac{4}{3} - 100 = \frac{64}{3} \text{ units}$$

Again when $x_0 = \frac{22}{3}; \; p_0 = D(x_0) = \left(12 - \frac{44}{3}\right)^2 = \frac{64}{9}$

$$CS = \int_0^{22/3} \left(144 - 48x + 4x^2\right) dx - \frac{22}{3} \times \frac{64}{9}$$

$$= \left[144x - \frac{48}{2}x^2 + 4\frac{x^3}{3}\right]_0^{22/3} - \frac{1408}{27}$$

$$= 144\left(\frac{22}{3}\right) - 24\left(\frac{22}{3}\right)^2 + \frac{4}{3}\left(\frac{22}{3}\right)^3 - \frac{1408}{27}$$

$$= \frac{3168}{3} - \frac{11616}{9} + \frac{4}{3} \times \frac{10648}{27} - \frac{1408}{27}$$

$$= \frac{85536 - 104544 + 42592 - 4224}{81}$$

$$= \frac{19360}{81} \textit{units}$$

- **<u>Consumer's Surplus :</u>**

Consumer's Surplus $CS = \int_0^{x_0} D(x)\,dx = x_0 \times p_0$

Ex.: The demand law for a commodity is $p = 20 - D - D^2$. Find the consumer's surplus when the demand is 3.

$p = f(D) = 20 - D - D^2$

When demand $D_0 = 3$ the price $p_0 = 20 - 3(3)^2 = 8$

$\therefore$ Consumer's Surplus $= \int_0^{D_1} f(D)\,dD - p_0 D_0$

$$=\int_0^3 \left(20-D-D^2\right)dD-\left(8\times 3\right)$$

$$=\left[20D\left(3\right)-\frac{9}{2}-\frac{27}{3}\right]-24$$

$$=60-\frac{9}{2}-9-24$$

$$=\frac{120-9-18-48}{2}$$

$$=\frac{45}{2}$$

Ex: Demand and supply functions are $D\left(x\right)=\left(2-2x\right)^2$ and $S\left(x\right)=56+4x$ respectively. Determine CS under monopoly (so as to maximise the profit) and the supply function is identified with the marginal cost function.

Sol^n.

$$TR = x\times D(x)$$

$$= x\times\left(12-2x\right)^2$$

$$= x\times\left(144-48x+4x^2\right)$$

$$= TR = 144x-48x^2+4x^3$$

$$\therefore MR=\frac{d}{dx}\left(TR\right)=144-96x+12x^2$$

Since the supply price is identified with MC, we have

$$M.C.=56+4x$$

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- **Producer's Surplus :**

Producer's Surplus $PS = x_0 \times p_0 - \int_0^{x_0} S(x)\,dx$

Ex. Find the producer surplus under the pure competition for demand function $P = \frac{8}{x+1} - 2$ and supply function $P = \frac{1}{2}(x+3)$ where p is price and x is quantity.

Sol^n.: Under pure competition, market equilibrium conditions can be obtained by equating the demand and supply.

$$\frac{8}{x+1} - 2 = \frac{1}{2}(x+3)$$

$$\therefore 16 - 4(x+1) = (x+3)(x+1)$$

$$\therefore 16 - 4x - 4 = x^2 + 4x + 3$$

$$\therefore x^2 + 8x - 9 = 0$$

$$\therefore (x+9)(x-1) = 0$$

$$\therefore \boxed{x=1} \text{ or } \boxed{x=-9}$$

$x=-9$ is inadmissible as quantity cannot be negative

$\therefore x = 1$

When $x = 1$

$$p = \frac{8}{x+1} - 2 = \frac{8}{1+1} - 2 = 4 - 2 = 2$$

Producer surplus: $p_0 x_0 - \int_0^{x_0} S(x)\,dx$

$$= 1 \times 2 - \int_0^1 \left(\frac{x+3}{2}\right) dx$$

$$= 2-\frac{1}{2}\left[\frac{x^2}{2}+3x\right]_0^1$$

$$=2-\frac{1}{2}\left[\frac{1}{2}+3\right]$$

$$=2-\frac{1}{2}\left(\frac{7}{2}\right)$$

$$=2\left(-\frac{7}{4}\right)$$

$$=2-\frac{7}{4}=\frac{1}{4}$$

Ex. The demand and supply function under perfect competition are $y=16-x^2$ and $y=2x^2+4$ respectively. Find the market price consumer's surplus and producer's surplus.

Sol^n:

Demand function: $y=16-x^2$ …. (1) Substracting (1) from 2

Supply function: $y=2x^2+4$ …. (2) $0=12-3x^2$

$$\boxed{x=2=x_0}$$

when $y=16-(2)^2-12=y_0$

Thus when the quantity demanded or supplied is 2 units the price is 12 units.

$$CS=\int_0^2\left(16-x^2\right)dx-2\times 12$$

$$=\left[16x-\frac{x^3}{3}\right]_0^2-24$$

$$= 32-\frac{8}{3}-24$$

$$= \frac{16}{3}=533$$

Producer's surplus:

$$= 2\times 12-\int_0^2 \left(2x^2+4\right)dx$$

$$= 24-\left[2\frac{x^3}{3}+4x\right]_0^2$$

$$= 24-\left[\frac{2}{3}(8)+4(2)\right]$$

$$= 24-\frac{16}{3}+8$$

$$= \frac{32}{3}=10.67$$

*__The Learning Curve:__

In any environment if a person is assigned to do the same task, then after a period of time, there is an improvement in his performance. If data points are collected over a period of time, the curve constructed on the graph will show a decrease in effort per unit for repetitive operations. This curve is very important in cost analysis, cost estimation and efficiency studies. This curve is called the learning curve. The learning curve shows that if a task is performed over and over than less time will be required at each iteration.

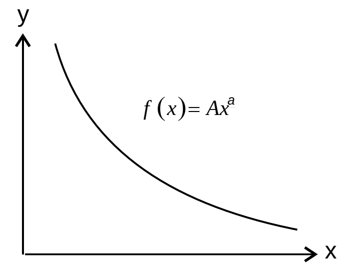
The rate of reduction in direct labour requirements is described by a curve called Learning curve. The general form of the function is usually taken as:

$f(x)=A.x^{\alpha}$

Where f(x) is the number of hours direct labour required to produce the x^{th} unit,$-1\leq\alpha<0$ and $A>0$.

The total number of labour hours required to produce units numbered 'a' through 'b' is

$$N=\int_a^b f(x)dx=\int_a^b A.x^{\alpha}dx$$



Ex.:

After producing 35 units the production manager of x company determines that its production facility is following a learning curve of the form $f(x) = 100x^{-0.5}$ where $f(x)$ is the rate of labor hours required to assemble the x[th] unit. How many total labor hours should they estimated are required to produce an additional 25 units.

Sol[n].:

$$N = \int_{35}^{60} 1000\ x^{-0.5}\ dx$$

$$= 1000\int_{35}^{60} x^{-0.5}\ dx\ 1000\left[\frac{x^{0.5}}{0.5}\right]_{35}^{60} = \frac{1000}{0.5}\left[x^{1/2}\right]_{35}^{60}$$

$$= 2000\left[60^{1/2} - 35^{1/2}\right] = 2000\ (7.746 - 5.916) = 3660 \text{ hours}$$

Rate of Sales:

When the rate of sales of a product is a known function of x, say $f(x)$ where x is a time measure, the total sales of this product over a time period T is $\int_{0}^{T} f(x)\, dx$.

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Ex.:

Suppose the rate of sales of a new product is given by $f(x) = 200 - 90e^{-x}$ where x is the number of days the product is on the market. Find the total sales during the first 4 days. Given: $\boxed{e^{-4} = (0.018)}$.

Sol^n.:

The total sales

$$= \int_0^4 f(x)\, dx = \int_0^4 \left(200 - 90e^{-x}\right) dx = \left[200x - \frac{90e^{-x}}{-1}\right]$$

$$= \left[200x - 90e^{-x}\right]_0^4 = \left[200(4) + 90e^{-4}\right] - \left[0 + 90e^{-0}\right]$$

$$= 800 + 90e^{-4} - 90 = 710 + 90(0.018) = 711.62 \text{ units}.$$

6B.2. UNIT END EXERCISE:

[1] The marginal cost function of a firm is given by MC=3000.$e^{0.3x}$+50, when x is quantity produced. If fixed cost is Rs. 80,000, find the total cost function of the firm. [TC=10000$e^{0.3x}$+50x+70,000]

[2] Find the total cost function and demand function if marginal revenue is MR=7-4x-x^2.

[3] A company determines that the marginal cost of producing x units of a particular commodity during a one-day operation is MC=16x-1591, where the commodity is fixed at Rs. 9 per unit and the fixed cost is Rs. 1800 per day.(a)Find the cost function.(b)Find the revenue function.(c)find the profit function. (d)What is the maximum profit that can be obtained in a one-day operation?

[Hint:(a)C(X)=$\int (MC)dx = \int (16x - 1591)dx = 8x^2 - 1591x + k$

(a)Now the fixed cost is 1800. i.e. when x=0, c=1800. Therefore, k=1800.

Hence the total cost function is C(x)= $8x^2 - 1591x + 1800$.

(b)R(x)=9 x

(c) P(x)=R(x)-C(X)= $-8x^2 + 1600x - 1800$

(d)P'(x)=-16 x +1600=0 $\Rightarrow$ x =100.

Also p''(x) = -16<0 $\therefore$ The maximum profit can be obtained in one day is P(100)=Rs.78,200]

[4] The marginal cost of a production of a firm is given as C'(x) =5 +0.13x. Further, the marginal revenue is R'(x)=18. Also it is given that C(0)=Rs.120. Compute the total profit.

[Since profit is maximum, where MC=MR.

i.e. 5+0.13x=18 $\therefore$ x=100

$R(x) = \int R'(x)dx = \int 18dx = 18x + k_1$, where put k_1=0,asunder pure competition,

TR = Output x Price $\therefore$ R(x)=18 x.

$C(x) = \int C'(x)dx = \int (5 + 0.13x)dx = 5x + 0.13\frac{x^2}{2} + k_2$ .But given that C(0)=120

$\therefore$ k_2=120.

$\therefore$ C(x)=5x+0.065 x^2+120.and P(x)=R(x)-C(x)

=13x-0.065 x^2-120.

$\therefore$ Total profit when x =100,P(100)=Rs. 530]

[5] Find the consumer surplus and producer surplus under pure competition for demand function

$p = \frac{8}{x+1} - 2$ and supply function $p = \frac{x+3}{2}$, where p is price and x is quantity.

[Under pure competition, market equilibrium conditions can be obtained by equating the demand and supply. CS=8log2-4 and PS=1/4.]

[6] Find the consumer surplus and producer surplus defined by demand curve D(x)=20-5x

And supply curve S(x)=4x+8.

$[CS = \int_0^{4/3} (20-5x)dx - \frac{40}{3} X \frac{4}{3}$, and $PS = \frac{160}{9} - \int_0^{4/3} (4x+8)dx$]

[7] Under a monopoly, the quantity sold and market price are determined by the demand function. If the demand function for a profit maximizing monopolist is P=274-x^2 and MC=4+3 x, find consumer's surplus.[TR=P.x=274x-x^3,MR=274-3x^2,the monopolist Maximizes profit at MR=MC. X_0=9and p_0=193

And

$$CS = \int_0^9 (274 - x^2)dx - 193X9 = 486]$$

[8] The production manager of an electronic company obtained the following function $f(x) = 1356.4x^{-0.5}$, where f(x) is the rate of labour hours required to assemble the x^{th} unit of a product. The function is based on the experience of assembling the first 50 units of the product. The company was asked to bid on a new order of 100 additional units.

Find the total labour hours required to assemble 100 units

$$[N = \int_a^b f(x)dx = \int_a^b A.x^{\alpha} dx = \int_{50}^{150} 1356.4x^{-0.5}]$$

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