1

DERIVATIVES AND ITS APPLICATION

Unit Structure

1.0 OBJECTIVE: -

- Derive the Newton-Raphson method formula,
- Develop the algorithm of the Newton-Raphson method,
- Use the Newton-Raphson method to solve a nonlinear equation.
- Discuss the drawbacks of the Newton-Raphson method.
- Understanding of Mathematical concepts like limit, continuity, derivative, integration of functions

1.1 INTRODUCTION: -

The course is designed to have a grasp of important concepts of Calculus in a scientific way. It covers topics from as basic as definition of functions to partial derivatives of functions in a gradual and logical way. The learner is expected to solve as many examples as possible to get a compete clarity and understanding of the topics covered.

1.2 REVIEW OF FUNCTIONS: -

Definitions Let f be a function with domain D. Then f has an absolute maximum value on D at a point c

If $f(x) \leq f(c)$ for all x in D and an absolute minimum value on D at c if $f(x) \geq f(c)$ for all x in D.

Maximum and minimum values are called extreme values of the function f. Absolute maxima or minima are also referred to as global maxima or minima.

Theorem 1—The Extreme Value Theorem If f is continuous on a closed interval [a, b]. Then f attains both an absolute maximum value M and an absolute minimum value m in [a, b] . That is, there are numbers x_1 and x_2 in [a, b] with $f(x_1) = m$, $f(x_2) = M$, and $m \leq f(x) \leq M$ for every other x in [a, b] The proof of the Extreme Value Theorem requires a detailed knowledge of the real number system (see Appendix 7) and we will not give it here. Figure 4.3 illustrates possible locations for the absolute extrema of a continuous function on a closed interval [a, b] . As we observed for the function y = cos x, it is possible that an absolute minimum (or absolute maximum) may occur at two or more different points of the interval

1.3 INCREASING AND DECREASING FUNCTIONS: -

Let f be some function defined on an interval.

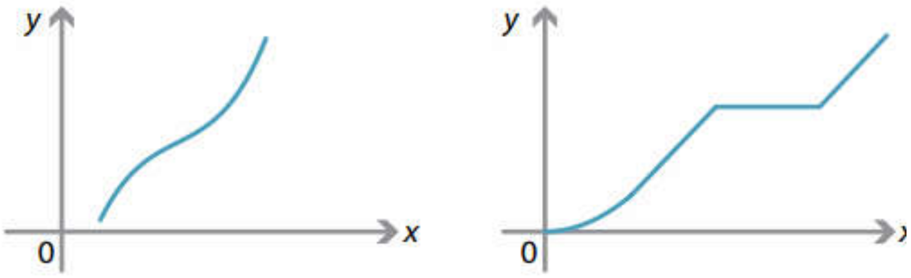
Definition

The function f is increasing over this interval if, for all points x_1 and x_2 in the interval,

$$x_1 \leq x_2 \Rightarrow f(x_1) \leq f(x_2)$$

This means that the value of the function at a larger number is greater than or equal to the value of the function at a smaller number.

The graph on the left shows a differentiable function. The graph on the right shows a piecewise-defined continuous function. Both these functions are increasing.

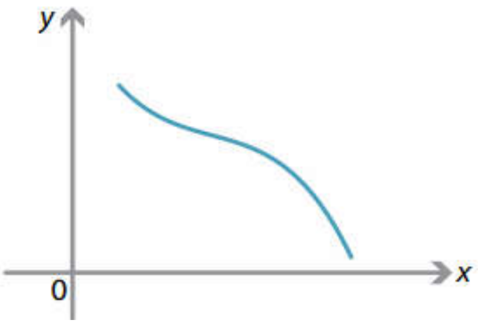


Examples of increasing functions.

The function f is decreasing over this interval if, for all points x_1 and x_2 in the interval,

$$x_1 \leq x_2 \Longrightarrow f(x_1) \geq f(x_2)$$

The following graph shows an example of a decreasing function



Note that a function that is constant on the interval is both increasing and decreasing over this interval. If we want to exclude such cases, then we omit the equality component in our definition, and we add the word strictly:

- A function is strictly increasing if $x_1 < x_2$ implies $f(x_1) < f(x_2)$.
- A function is strictly decreasing if $x_1 < x_2$ implies $f(x_1) > f(x_2)$.

We will use the following results. These results refer to intervals where the function is differentiable. Issues such as endpoints have to be treated separately.

- If $f'(x) > 0$ for all x in the interval, then the function f is strictly increasing.
- If $f'(x) < 0$ for all x in the interval, then the function f is strictly decreasing.
- If $f'(x) = 0$ for all x in the interval, then the function f is constant.

1.4 STATIONARY POINTS: -

Definitions

Let f be a differentiable function.

- A stationary point of f is a number x such that $f'(x) = 0$.

- The point c is a maximum point of the function f if and only if

$f(c) \geq f(x)$, for all x in the domain of f . The value f (c) of the function at c is called the maximum value of the function.

• The point c is a minimum point of the function f if and only if

f (c) ≤ f (x), for all x in the domain of f . The value f (c) of the function at c is called the minimum value of the function.

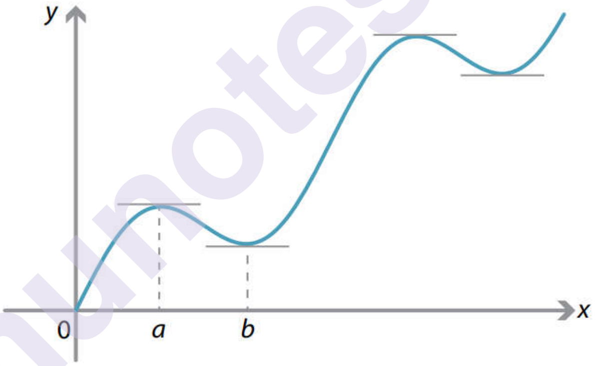
1.5 MAXIMUM AND MINIMUM PROBLEMS: -

• The point c is a local maximum point of the function f if there exists an interval (a, b) with c ∈ (a,b) such that f (c) ≥ f (x), for all x ∈ (a,b).

• The point c is a local minimum point of the function f if there exists an interval (a, b) with c ∈ (a,b) such that f (c) ≤ f (x), for all x ∈ (a,b).

These are sometimes called relative maximum and relative minimum points. Local maxima and minima are often referred to as turning points

The following diagram shows the graph of y = f (x), where f is a differentiable function. It appears from the diagram that the tangents to the graph at the points which are local maxima or minima are horizontal. That is, at a local maximum or minimum point c, we have f (c) = 0, and hence each local maximum or minimum point is a stationary point



1.6 GRAPHING POLYNOMIALS: -

In the last module, we looked at the long-term behavior of polynomials. After studying that module, you should be able to recognize that polynomials like $f(x) = x^4 - 3x^3 - 63x^2 + 27x + 486$ and $g(x) = x^4 - 12x^3 - 27x^2 + 270x + 648$ have similar long-term behavior. Since they are both 4th degree polynomials with a positive leading coefficient, we know that their graphs must have arrows pointing up at the extreme left- and right-sides (i.e., the outputs of both functions increases without bound as the inputs increase without bound and as the inputs decrease without bound). See Figure 1 below.

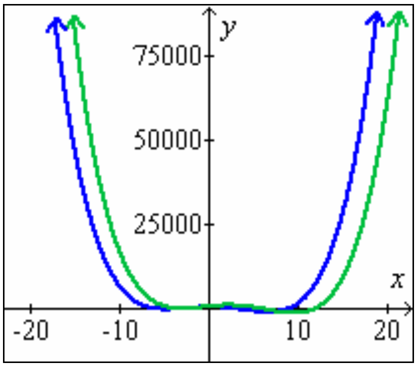


Figure 1: $y = f(x)$ and $y = g(x)$

Although the functions $f(x) = x^4 - 3x^3 - 63x^2 + 27x + 486$ and $g(x) = x^4 - 12x^3 - 27x^2 + 270x + 648$ have similar long-run behavior, they are **not** identical functions! Let's study the **short-run** behavior of their graphs to see how these functions differ. The short-run behavior of the graph of a function concerns graphical features that occur when the input values aren't very large. (It's hard to specify what "not large" means since it will be different for each function, but we'll for particular graphical features rather than look within a particular interval, so we don't need to worry about being more specific.)

Clearly, $x = 0$ isn't a large x-value, so the **y-intercept** will be part of the short-run behavior of a polynomial function's graph. (Notice that the y-coordinate of the y-intercept of a polynomial function is its the constant term.)

The y-intercept of $f(x) = x^4 - 3x^3 - 63x^2 + 27x + 486$ is $(0, 486)$.

The y-intercept of $g(x) = x^4 - 12x^3 - 27x^2 + 270x + 648$ is $(0, 648)$.

Zeros (or **roots**) are another important part of the **short-run** behavior of the graph of a polynomial function. To find the **roots** of a polynomial function, we can write it in **factored form**:

$$\begin{aligned} f(x) &= x^4 - 3x^3 - 63x^2 + 27x + 486 \\ &= (x + 3)(x - 3)(x + 6)(x - 9) \end{aligned}$$

Each factor of f gives rise to a root, since when each factor equals zero, the output for the function is zero. To find the roots, determine which numbers make the factors equal to zero.

FACTOR	$\Rightarrow$	ROOT
$x + 3$	$\Rightarrow$	$x = -3$
$x - 3$	$\Rightarrow$	$x = 3$
$x + 6$	$\Rightarrow$	$x = -6$
$x - 9$	$\Rightarrow$	$x = 9$

In order to graph f, we can plot its roots and y-intercept; see Figure 2 below.

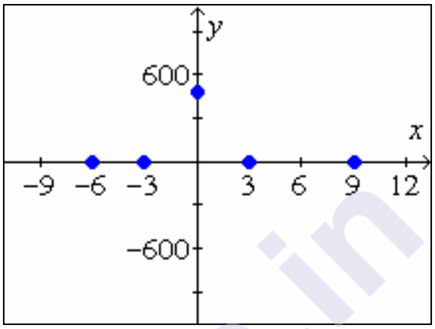


Figure 2

Now we can connect these points – making **sure that our graph has the correct long-run behavior** – to complete the graph of $f(x) = x^4 - 3x^3 - 63x^2 + 27x + 486$; see Figure 3.

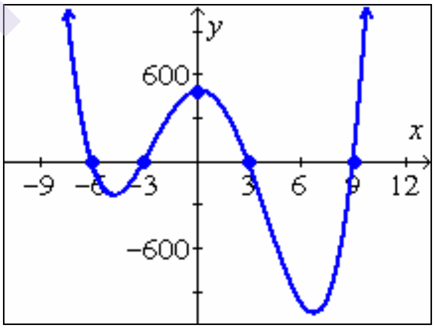


Figure 3: The graph of $y = f(x)$.

Now let's factor g to determine its roots:

$$g(x) = x^4 - 12x^3 - 27x^2 + 270x + 648$$
$$= (x + 3)^2(x - 6)(x - 12)$$

FACTOR	⇒	ROOT
$x + 3$	⇒	$x = -3$
$x - 6$	⇒	$x = 6$
$x - 12$	⇒	$x = 12$

Since the root $x = -3$ is associated with a squared (or "double") factor, it is often called a **double root** or a **root of multiplicity two**.

In order to graph g, we can plot its roots and y-intercept; see Figure 4. (The orange point at $x = -3$ is a *root of multiplicity two*.)

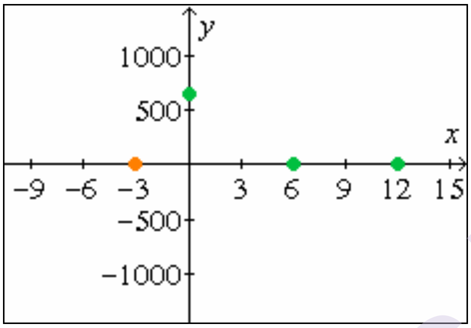


Figure 4

Now we can connect these points – **making sure that our graph has the correct long-run behavior** – to complete the graph of $g(x) = x^4 - 12x^3 - 27x^2 + 270x + 648$ (see Figure 5). Notice that the only way to connect the points without using additional roots or incorrect long-run behavior is to have the graph "bounce" of the double root at $x = -3$. (Try it yourself!)

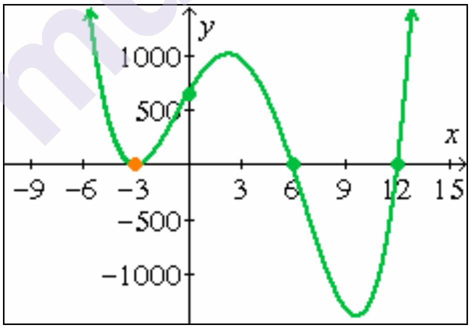


Figure 5: The graph of $y = g(x)$.

EXAMPLE: Write an algebraic rule for the polynomial function p graphed Figure 6. Note that the graph passes through the point $(-3, 18)$.

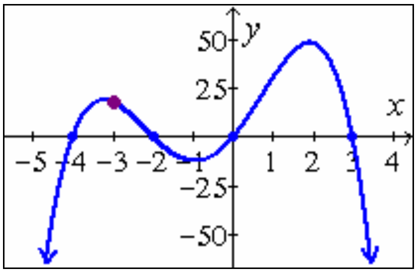


Figure 6: The graph of $y = p(x)$.

SOLUTION:

Since there are roots at $x = -4$, $x = -2$, $x = 0$, and $x = 3$, we know that p has form

$$p(x) = k \cdot x(x - 3)(x + 4)(x + 2)$$

where k is a constant. To find k we can use the point $(-3, 18)$.

$$\begin{aligned}
(-3, 18) \quad \Rightarrow \quad p(-3) = 18 &= k \cdot (-3)(-3 - 3)(-3 + 4)(-3 + 2) \\
\Rightarrow \quad 18 &= k \cdot (-3)(-6)(1)(-1) \\
\Rightarrow \quad 18 &= -18k \\
\Rightarrow \quad k &= -1
\end{aligned}$$

Therefore, $p(x) = -x(x - 3)(x + 4)(x + 2)$.

EXAMPLE: Write an algebraic rule for the polynomial function w graphed in Figure 7. Note that the y-intercept of w is $(0, 12)$.

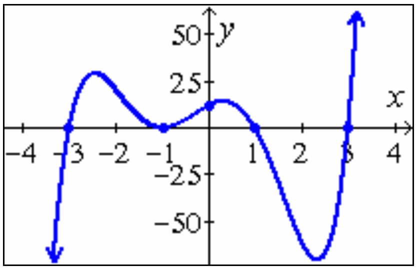


Figure 7: The graph of $y = w(x)$.

SOLUTION:

Since there are roots at $x = -3$, $x = -1$, $x = 1$, and $x = 3$, and since the graph *bounces off* the x-axis at $x = -1$, this is a root of multiplicity two (i.e., a double-root). Thus, we know that w has form

$$w(x) = k \cdot (x + 3)(x + 1)^2(x - 1)(x - 3)$$

where k is a constant. To find k we can use the fact that the y-intercept of w is $(0, 12)$:

$$\begin{aligned}(0, 12) \;\Rightarrow\; f(0) = 12 &= k\cdot(0+3)(0+1)^2(0-1)(0-3)\\ \Rightarrow\; 12 &= k\cdot(3)(1)^2(-1)(-3)\\ \Rightarrow\; 12 &= 9k\\ \Rightarrow\; k &= \tfrac{12}{9} = \tfrac{4}{3}\end{aligned}$$

Thus, $w(x) = \frac{4}{3}(x+3)(x+1)^2(x-1)(x-3)$.

EXAMPLE: Write an algebraic rule for the polynomial function h graphed in Figure 8. Note that the y-intercept of h is $(0, 13)$.

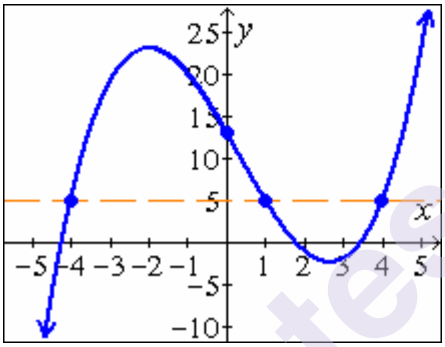


Figure 8: The graph of $y = h(x)$.

SOLUTION:

Notice that the graph of $y = h(x)$ does not have roots at integer values. But there are some "nice" points on the graph of $y = h(x)$ along the horizontal line $y = 5$. If we treat this line as the x-axis (i.e., imagine that the graph of h is shifted down 5 units) then the graph would have roots $x = 4$, $x = 1$, and $x = -4$. So to find a rule for h we can create a function that has these three roots and then shift it up 5 units:

$$h(x) = k\cdot(x-4)(x-1)(x+4) + 5$$

To find k we can use the y-intercept $(0, 13)$:

$$\begin{aligned}(0, 13) \;\Rightarrow\; h(0) = 13 &= k\cdot(0-4)(0-1)(0+4) + 5\\ \Rightarrow\; 13 &= k\cdot(-4)(-1)(4) + 5\\ \Rightarrow\; 13 &= 16k - 5\\ \Rightarrow\; 8 &= 16k\\ \Rightarrow\; k &= \tfrac{1}{2}\end{aligned}$$

Therefore, an algebraic rule for h is $h(x) = \frac{1}{2}(x-4)(x-1)(x+4) + 5$.

Properties of Polynomial Functions

- The graph of a polynomial is a smooth unbroken curve. (By "smooth" we mean that the graph does not have any sharp corners as turning points.)
- The graph of a polynomial always exhibits the characteristic that as $|x|$ gets very large, $|y|$ gets very large.
- If p is a polynomial of degree n, then the polynomial equation $p(x)=0$ has *at most* n distinct solutions; that is, p has at most n zeros. This is equivalent to saying that the graph of $y = p(x)$ crosses the x-axis at most n times. Thus a polynomial of degree five can have *at most* five x-intercepts.
- The graph of a polynomial function of degree n can have *at most* $n-1$ turning points (see Key Point below). For example, the graph of a polynomial of degree five can have *at most* four turning points. In particular, the graph of a quadratic (2^{nd} degree) polynomial function always has exactly one turning point – its vertex.

EXAMPLE: What is the minimum possible degree of the polynomial function in Figure 9?

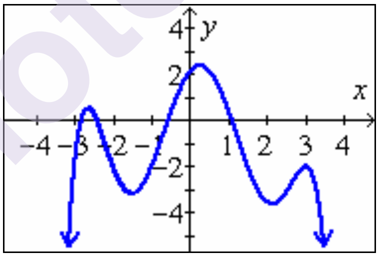


Figure 9

SOLUTION:

The polynomial function graphed in Figure 9 has four zeros and five turning points. The properties of polynomials tell us that a polynomial function with four zeros must have a degree of at least four. These properties also tell us that if a polynomial has degree n then it can have at most $n-1$ turning points. In other words the degree of a polynomial must be at least one more than the number of turning points. Since this graph has four turning points, the degree of the polynomial must be at least six.

Keep in mind that although a 6^{th} degree polynomial *may* have as many as six real zeros, it need **not** have that many. The graph in Figure 9 only has four real zeros.

Concavity

There are four graphs in Figure 10 (below). The first two graphs (**a** and **b**) have different shapes, but they are both *increasing* on the interval $(-2, 1)$. The second two graphs (**c** and **d**) also have different shapes, but they are both *decreasing* on the interval $(-2, 1)$.

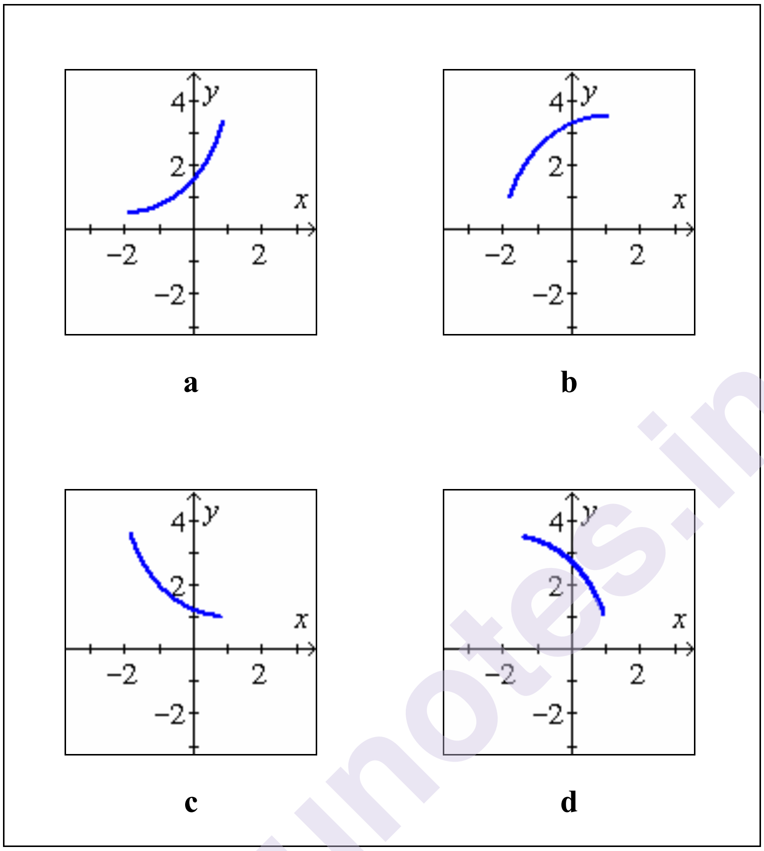


Figure 10

As the first graph (**a**) rises, it bends or curves upward; but as the second graph (**b**) rises, it bends or curves downward. Similarly, the third graph (**c**) is falling and curved upward, whereas the fourth graph (**d**) is falling and curved downward. A graph the curves upward (like the graphs **a** and **c**) is called **concave up**; a graph that curves downward (like the graphs **b** and **d**) is called **concave down**. It might help to remember thata parabola that opens upward is concave up and a parabola that opens downward is concave down.

The graph in Figure 11 is always increasing, concave down on the interval $(-\infty, 1)$, and concave up on the interval $(1, \infty)$. At the point $(1, 1)$ the concavity changes from concave up to concave down. Such a point is called a **point of inflection**.

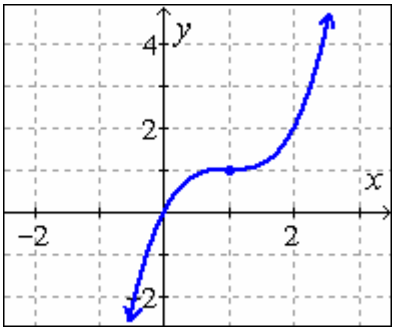


Figure 11: An inflection point occurs at $(1, 1)$.

The polynomial $y = x^3 + x^2 - 2x$ is graphed in Figure 12; the graph appears to have an inflection point at $x \approx -\frac{1}{3}$.

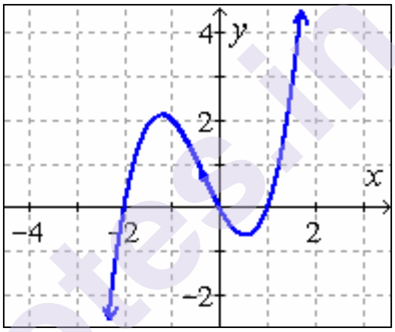


Figure 12: An inflection point occurs at $x \approx -\frac{1}{3}$.

EXAMPLE: Use the graph in Figure 13 to find …

a. the approximate intervals where the graph of f is increasing.

b. the approximate intervals where the graph of f is decreasing.

c. the approximate intervals where the graph of f is concave up.

d. the approximate intervals where the graph of f is concave down.

e. the number of inflection points on the graph of f.

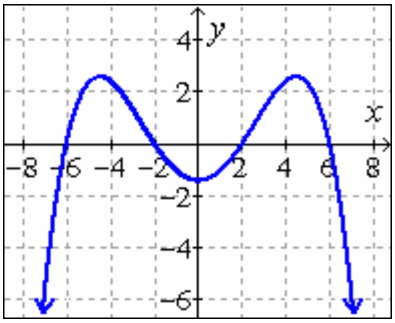


Figure 13: The graph of $y = f(x)$.

SOLUTION:

a. The approximate intervals where the graph of f is increasing are $(-\infty, -4.5)$ and $(0, 4.5)$.

b. The approximate intervals where the graph of f is decreasing are $(-4.5, 0)$ and $(4.5, \infty)$.

1.7 NEWTON'S METHOD: -

Methods such as the bisection method and the false position method of finding roots of a nonlinear equation $f(x)=0$ require bracketing of the root by two guesses. Such methods are called *bracketing methods*. These methods are always convergent since they are based on reducing the interval between the two guesses so as to zero in on the root of the equation.

In the Newton-Raphson method, the root is not bracketed. In fact, only one initial guess of the root is needed to get the iterative process started to find the root of an equation. The method hence falls in the category of *open methods*. Convergence in open methods is not guaranteed but if the method does converge, it does so much faster than the bracketing methods.

The Newton-Raphson method is based on the principle that if the initial guess of the root of $f(x)=0$ is at x_i, then if one draws the tangent to the curve at $f(x_i)$, the point x_{i+1} where the tangent crosses the x-axis is an improved estimate of the root (Figure 1).

Using the definition of the slope of a function, at $x = x_i$

$$f'(x_i) = \tan\theta$$

$$= \frac{f(x_i)-0}{x_i - x_{i+1}},$$

which gives

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \qquad (1)$$

Equation (1) is called the Newton-Raphson formula for solving nonlinear equations of the form $f(x)=0$. So starting with an initial guess, x_i, one can find the next guess, x_{i+1}, by using Equation (1). One can repeat this process until one finds the root within a desirable tolerance.

Algorithm

The steps of the Newton-Raphson method to find the root of an equation $f(x)=0$ are

1. Evaluate $f'(x)$ symbolically
2. Use an initial guess of the root, x_i, to estimate the new value of the root, x_{i+1}, as

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

3. Find the absolute relative approximate error $|\in_a|$ as

$$|\in_a| = \left|\frac{x_{i+1} - x_i}{x_{i+1}}\right| \times 100$$

4. Compare the absolute relative approximate error with the pre-specified relative error tolerance, $\in_s$. If $|\in_a| > \in_s$, then go to Step 2, else stop the algorithm. Also, check if the number of iterations has exceeded the maximum number of iterations allowed. If so, one needs to terminate the algorithm and notify the user.

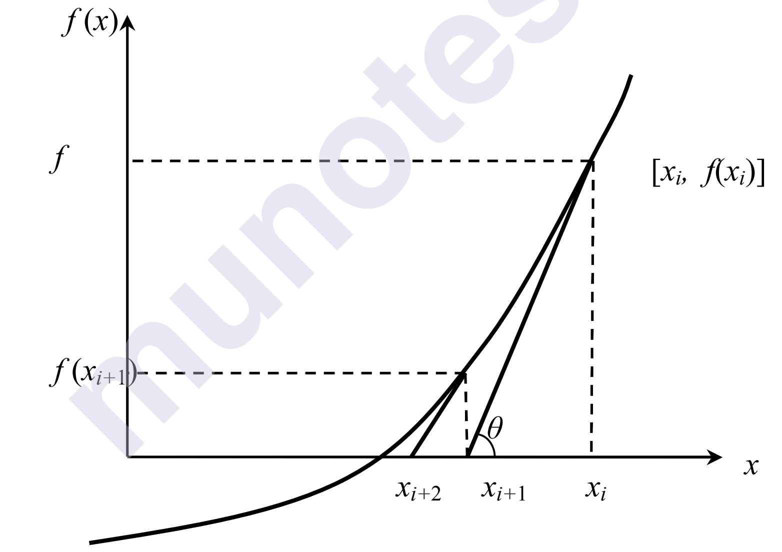


Figure 1 Geometrical illustration of the Newton-Raphson method.

Example 1

You are working for 'DOWN THE TOILET COMPANY' that makes floats for ABC commodes. The floating ball has a specific gravity of 0.6 and has a radius of 5.5 cm. You are asked to find the depth to which the ball is submerged when floating in water.

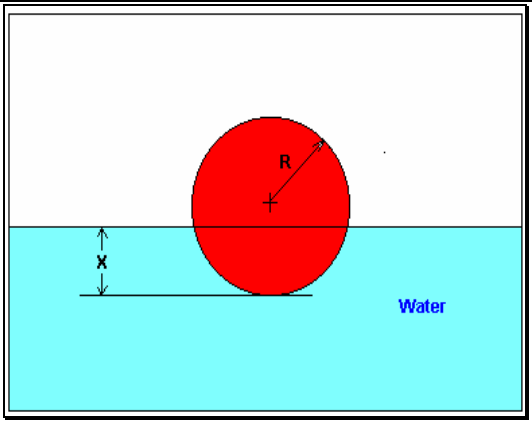


Figure 2 Floating ball problem.

The equation that gives the depth x in meters to which the ball is submerged under water is given by

$$x^3 - 0.165x^2 + 3.993 \times 10^{-4} = 0$$

Use the Newton-Raphson method of finding roots of equations to find

a) the depth x to which the ball is submerged under water. Conduct three iterations to estimate the root of the above equation.

b) the absolute relative approximate error at the end of each iteration, and

c) the number of significant digits at least correct at the end of each iteration.

Solution

$$f(x) = x^3 - 0.165x^2 + 3.993 \times 10^{-4}$$

$$f'(x) = 3x^2 - 0.33x$$

Let us assume the initial guess of the root of $f(x) = 0$ is $x_0 = 0.05$ m. This is a reasonable guess (discuss why $x = 0$ and $x = 0.11$ m are not good choices) as the extreme values of the depth x would be 0 and the diameter (0.11 m) of the ball.

Iteration 1

The estimate of the root is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 0.05 - \frac{(0.05)^3 - 0.165(0.05)^2 + 3.993 \times 10^{-4}}{3(0.05)^2 - 0.33(0.05)}$$

$$= 0.05 - \frac{1.118 \times 10^{-4}}{-9 \times 10^{-3}}$$

$$= 0.05 - (-0.01242)$$

$$= 0.06242$$

The absolute relative approximate error $|\in_a|$ at the end of Iteration 1 is

$$|\in_a| = \left| \frac{x_1 - x_0}{x_1} \right| \times 100$$

$$= \left| \frac{0.06242 - 0.05}{0.06242} \right| \times 100$$

$$= 19.90\%$$

The number of significant digits at least correct is 0, as you need an absolute relative approximate error of 5% or less for at least one significant digit to be correct in your result.

Iteration 2

The estimate of the root is

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.06242 - \frac{(0.06242)^3 - 0.165(0.06242)^2 + 3.993 \times 10^{-4}}{3(0.06242)^2 - 0.33(0.06242)}$$

$$= 0.06242 - \frac{-3.97781 \times 10^{-7}}{-8.90973 \times 10^{-3}}$$

$$= 0.06242 - (4.4646 \times 10^{-5})$$

$$= 0.06238$$

The absolute relative approximate error $|\in_a|$ at the end of Iteration 2 is

$$|\in_a| = \left| \frac{x_2 - x_1}{x_2} \right| \times 100$$

$$= \left| \frac{0.06238 - 0.06242}{0.06238} \right| \times 100$$

$$= 0.0716\%$$

The maximum value of m for which $|\in_a| \le 0.5 \times 10^{2-m}$ is 2.844. Hence, the number of significant digits at least correct in the answer is 2.

Iteration 3

The estimate of the root is

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 0.06238 - \frac{(0.06238)^3 - 0.165(0.06238)^2 + 3.993\times 10^{-4}}{3(0.06238)^2 - 0.33(0.06238)}$$

$$= 0.06238 - \frac{4.44\times 10^{-11}}{-8.91171\times 10^{-3}}$$

$$= 0.06238 - \left(-4.9822\times 10^{-9}\right)$$

$$= 0.06238$$

The absolute relative approximate error $|\in_a|$ at the end of Iteration 3 is

$$|\in_a| = \left|\frac{0.06238 - 0.06238}{0.06238}\right| \times 100$$

$$= 0$$

The number of significant digits at least correct is 4, as only 4 significant digits are carried through in all the calculations.

Drawbacks of the Newton-Raphson Method

1. **Divergence at inflection points**

If the selection of the initial guess or an iterated value of the root turns out to be close to the inflection point (see the definition in the appendix of this chapter) of the function $f(x)$ in the equation $f(x) = 0$, Newton-Raphson method may start diverging away from the root. It may then start converging back to the root. For example, to find the root of the equation

$$f(x) = (x-1)^3 + 0.512 = 0$$

the Newton-Raphson method reduces to

$$x_{i+1} = x_i - \frac{(x_i^3 - 1)^3 + 0.512}{3(x_i - 1)^2}$$

Starting with an initial guess of $x_0 = 5.0$, Table 1 shows the iterated values of the root of the equation. As you can observe, the root starts to diverge at Iteration 6 because the previous estimate of 0.92589 is close to the inflection point of $x = 1$ (the value of $f'(x)$ is zero at the inflection point). Eventually, after 12 more iterations the root converges to the exact value of $x = 0.2$.

Table 1 Divergence near inflection point.

Iteration Number	x_i
0	5.0000
1	3.6560
2	2.7465
3	2.1084
4	1.6000
5	0.92589
6	–30.119
7	–19.746
8	–12.831
9	–8.2217
10	–5.1498
11	–3.1044
12	–1.7464
13	–0.85356
14	–0.28538
15	0.039784
16	0.17475
17	0.19924
18	0.2

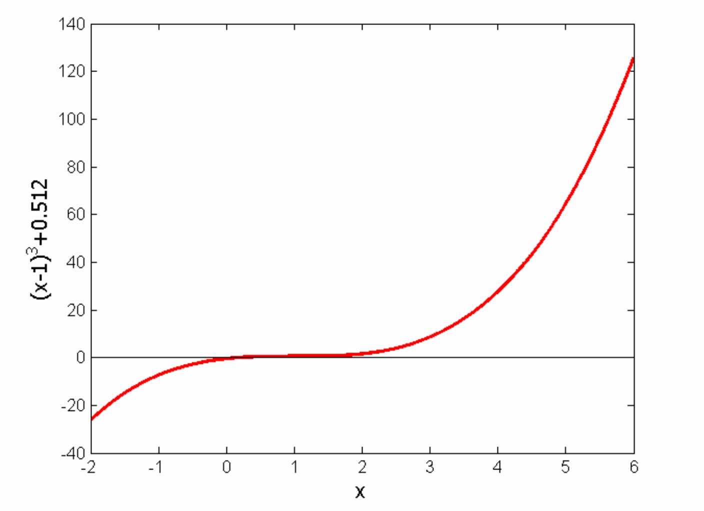


Figure 3 Divergence at inflection point for $f(x)=(x-1)^3=0$.

2. **Division by zero**

For the equation

$$f(x)=x^3-0.03x^2+2.4\times10^{-6}=0$$

the Newton-Raphson method reduces to

$$x_{i+1}=x_i-\frac{x_i^{\,3}-0.03x_i^{\,2}+2.4\times10^{-6}}{3x_i^{\,2}-0.06x_i}$$

For $x_0=0$ or $x_0=0.02$, division by zero occurs (Figure 4). For an initial guess close to 0.02 such as $x_0=0.01999$, one may avoid division by zero, but then the denominator in the formula is a small number. For this case, as given in Table 2, even after 9 iterations, the Newton-Raphson method does not converge.

Table 2 Division by near zero in Newton-Raphson method.

Iteration Number	x_i	$f(x_i)$	$\lvert\in_a\rvert\%$
0	0.019990	-1.60000×10^{-6}	——
1	–2.6480	18.778	100.75
2	–1.7620	–5.5638	50.282
3	–1.1714	–1.6485	50.422
4	–0.77765	–0.48842	50.632
5	–0.51518	–0.14470	50.946
6	–0.34025	–0.042862	51.413
7	–0.22369	–0.012692	52.107
8	–0.14608	–0.0037553	53.127
9	–0.094490	–0.0011091	54.602

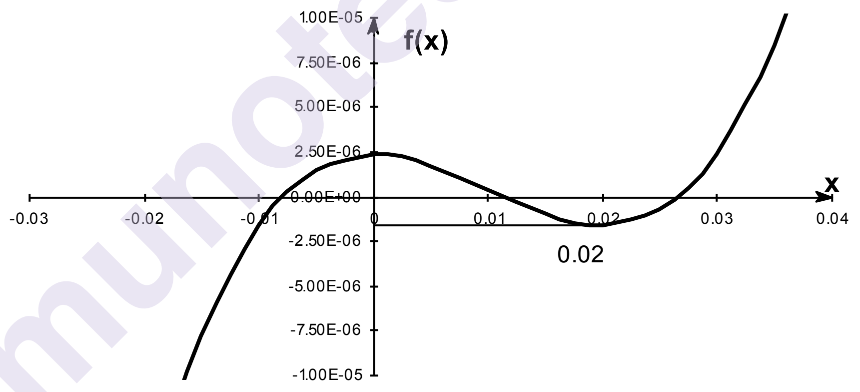


Figure 4 Pitfall of division by zero or a near zero number.

3. Oscillations near local maximum and minimum

Results obtained from the Newton-Raphson method may oscillate about the local maximum or minimum without converging on a root but converging on the local maximum or minimum. Eventually, it may lead to division by a number close to zero and may diverge.

For example, for

$$f(x) = x^2 + 2 = 0$$

the equation has no real roots (Figure 5 and Table 3).

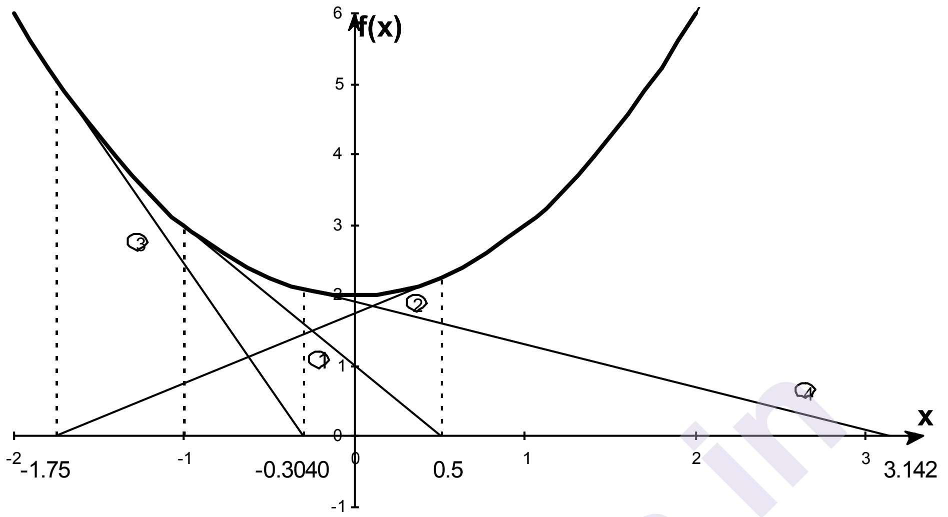


Figure 5 Oscillations around local minima for $f(x) = x^2 + 2$.

Table 3 Oscillations near local maxima and minima in Newton-Raphson method.

Iteration Number	x_i	$f(x_i)$	$\lvert\in_a\rvert\%$
0	–1.0000	3.00	—
1	0.5	2.25	300.00
2	–1.75	5.063	128.571
3	–0.30357	2.092	476.47
4	3.1423	11.874	109.66
5	1.2529	3.570	150.80
6	–0.17166	2.029	829.88
7	5.7395	34.942	102.99
8	2.6955	9.266	112.93
9	0.97678	2.954	175.96

4. Root jumping

In some case where the function $f(x)$ is oscillating and has a number of roots, one may choose an initial guess close to a root. However, the guesses may jump and converge to some other root. For example for solving the equation $\sin x = 0$ if you choose $x_0 = 2.4\pi = (7.539822)$ as an initial guess, it converges to the root of $x = 0$ as shown in Table 4 and Figure 6. However, one may have chosen this as an initial guess to converge to $x = 2\pi = 6.2831853$.

Table 4 Root jumping in Newton-Raphson method.

Iteration Number	x_i	$f(x_i)$	$\lvert\in_a\rvert\%$
0	7.539822	0.951	——
1	4.462	–0.969	68.973
2	0.5499	0.5226	711.44
3	–0.06307	–0.06303	971.91
4	8.376×10^{-4}	8.375×10^{-5}	7.54×10^{4}
5	-1.95861×10^{-13}	-1.95861×10^{-13}	4.28×10^{10}

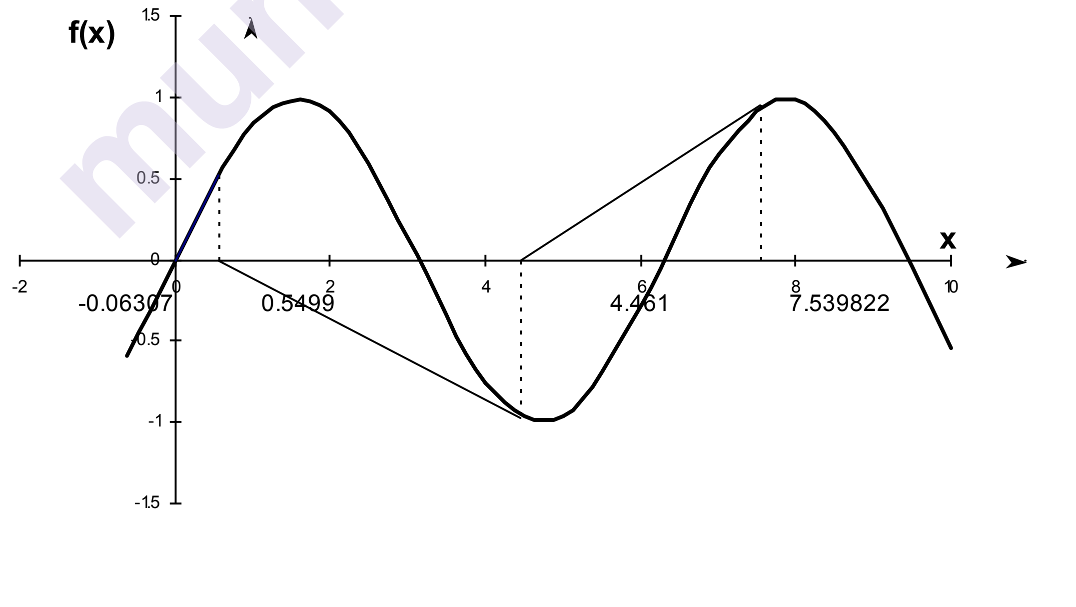


Figure 6 Root jumping from intended location of root for $f(x) = \sin x = 0$.

Appendix A. What is an inflection point?

For a function $f(x)$, the point where the concavity changes from up-to-down or down-to-up is called its inflection point. For example, for the function $f(x)=(x-1)^3$, the concavity changes at $x=1$ (see Figure 3), and hence (1,0) is an inflection point.

An inflection points MAY exist at a point where $f''(x)=0$ and where $f''(x)$ does not exist. The reason we say that it MAY exist is because if $f''(x)=0$, it only makes it a possible inflection point. For example, for $f(x)=x^4-16$, $f''(0)=0$, but the concavity does not change at $x=0$. Hence the point (0, –16) is not an inflection point of $f(x)=x^4-16$.

For $f(x)=(x-1)^3$, $f''(x)$ changes sign at $x=1$ ($f''(x)<0$ for $x<1$, and $f''(x)>0$ for $x>1$), and thus brings up the *Inflection Point Theorem* for a function $f(x)$ that states the following.

"If $f'(c)$ exists and $f''(c)$ changes sign at $x=c$, then the point $(c, f(c))$ is an inflection point of the graph of f."

Newton-Raphson method can also be derived from Taylor series. For a general function $f(x)$, the Taylor series is

$$f(x_{i+1})=f(x_i)+f'(x_i)(x_{i+1}-x_i)+\frac{f''(x_i)}{2!}(x_{i+1}-x_i)^2+\Lambda$$

As an approximation, taking only the first two terms of the right hand side,

$$f(x_{i+1})\approx f(x_i)+f'(x_i)(x_{i+1}-x_i)$$

and we are seeking a point where $f(x)=0,$ that is, if we assume

$$f(x_{i+1})=0,$$

$$0\approx f(x_i)+f'(x_i)(x_{i+1}-x_i)$$

which gives

$$x_{i+1}=x_i-\frac{f(x_i)}{f'(x_i)}$$

This is the same Newton-Raphson method formula series as derived previously using the geometric method.

a. It appears that the graph of f is concave up on the interval $(-2, 2)$.

b. It appears that the graph of f is concave down on the intervals $(-\infty, -2)$ and $(2, \infty)$.

c. As we move from left to right, the graph changes from concave down to concave up and then back to concave down. Each change occurs at an inflection point, so the graph of f has two inflection points.

EXAMPLE: Suppose p is a polynomial function that satisfies the following conditions: The graph of p has exactly three turning points : $(-2.5, -5)$, $(0, 2)$, $(2.5, -5)$ and the graph of p has exactly two inflection points: $(-1, 0)$ and $(1, 0)$. Sketch a graph of p based upon this information. How many real zeros does p have?

SOLUTION:

Let's start by plotting the given points; see Figure 14.

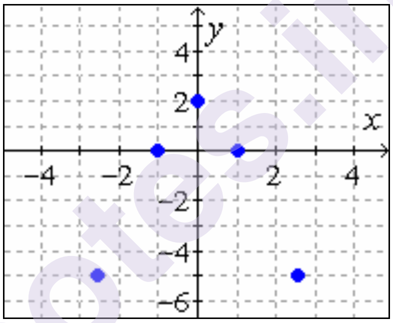


Figure 14

There is only one way to connect the points to create a polynomial function without adding turning points or inflection points; see Figure 15. You should verify this for yourself.

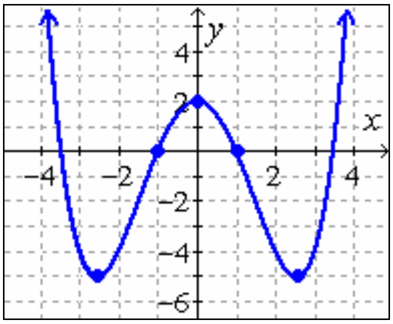


Figure 15: The graph of $y = p(x)$.

Since it has four x-intercepts, p has four real zeros.

1.8 SUMMARY: -

This chapter is mainly focusing on basic concepts in derivatives.

1.9 EXERCISE: -

- The accompanying figure shows some level curves of an unspecified function f(x, y). Which of the three vectors shown in the figure is most likely to be ∇f ? Explain

1.10 REFERENCES: -

- Calculus: Early transcendental (10th Edition): Howard Anton, Irl Bivens, Stephen Davis, John Wiley & sons, 2012.

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2

INTEGRATION AND ITS APPLICATIONS

Unit Structure

2.0 OBJECTIVE

- Derive the Simpson's method formula,
- Develop the algorithm of the Simpson's method,

2.1 INTRODUCTION

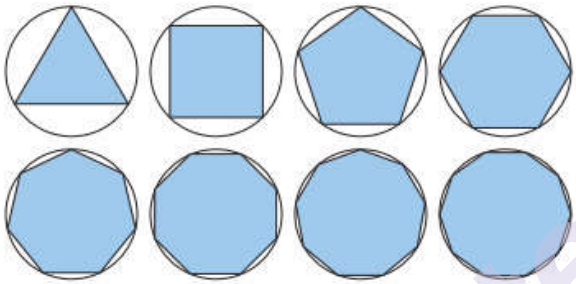
In this chapter we will begin with an overview of the problem of finding areas—we will discuss what the term "area" means, and we will outline two approaches to defining and calculating areas. Following this overview, we will discuss the Fundamental Theorem of Calculus, which is the theorem that relates the problems of finding tangent lines and areas, and we will discuss techniques for calculating areas. We will then use the ideas in this chapter to define the average value of a function, to continue our study of rectilinear motion, and to examine some consequences of the chain rule in integral calculus. We conclude the chapter by studying functions defined by integrals, with a focus on the natural logarithm function.

2.2 AN OVERVIEW OF THE AREA PROBLEM

Formulas for the areas of polygons, such as squares, rectangles, triangles, and trapezoids, were well known in many early civilizations. However, the problem of finding formulas for regions with curved boundaries (a circle being the simplest example) caused difficulties for early mathematicians.

The first real progress in dealing with the general area problem was made by the Greek Mathematician Archimedes, who obtained areas of regions bounded by circular arcs, parabolas, spirals, and various other curves using an ingenious procedure that was later called the method of exhaustion. The method, when applied to a circle, consists of inscribing a succession of regular polygons in the circle and allowing the number of sides to increase indefinitely (Figure 5.1.1). As the number of Sides increases, the polygons tend to “exhaust” the region inside the circle, and the areas of the polygons become better and better approximations of the exact area of the circle.

To see how this works numerically, let A_n denote the area of a regular n-sided polygon inscribed in a circle of radius 1. Table 5.1.1 shows the values of A(n) for various choices of n. Note that for large values of n the area A(n) appears to be close to π (square units),



▲ Figure 5.1.1

n	$A(n)$
100	3.13952597647
200	3.14107590781
300	3.14136298250
400	3.14146346236
500	3.14150997084
1000	3.14157198278
2000	3.14158748588
3000	3.14159035683
4000	3.14159136166
5000	3.14159182676
10,000	3.14159244688

as one would expect. This suggests that for a circle of radius 1, the method of exhaustion is equivalent to an equation of the form

$$\lim_{n\to\infty} A(n) = \pi$$

Since Greek mathematicians were suspicious of the concept of “infinity,” they avoided its use in mathematical arguments. As a result, computation of area using the method of exhaustion was a very cumbersome procedure. It remained for Newton and Leibniz to obtain a general method for finding areas that explicitly used the notion of a limit. We will discuss their method in the context of the following problem

THE RECTANGLE METHOD FOR FINDING AREAS

One approach to the area problem is to use Archimedes’ method of exhaustion in the following way:

• Divide the interval [a, b]inton equal subintervals, and over each subinterval construct a rectangle that extends from the x-axis to any point on the curve y = f(x) that is above the subinterval; the particular point does not matter—it can be above the center, above an endpoint, or above any other point in the subinterval. In Figure 5.1.3y = f(x)it is above the center.

• For each n, the total area of the rectangles can be viewed as an approximation to the exact area under the curve over the interval [a, b]. Moreover, it is evident intuitively that as n increases these approximations will get better and better and will approach the exact area as a limit

(Figure 5.1.4). That is, if A denotes the exact area under the curve and An denotes the approximation to A using n rectangles, then

$$A = \lim_{n \to +\infty} An$$

We will call this the rectangle method for computing A

to illustrate this idea, we will use the rectangle method to approximate the area under the curve $y = x2$ over the interval [0, 1] (Figure 5.1.5). We will begin by dividing the interval [0, 1] into *n*equal subintervals, from which it follows that each subinterval has length 1/*n*; the endpoints of the subintervals occur at,

$$0\ ,\frac{1}{n},\ ,\frac{2}{n},,\frac{3}{n},\dots,\frac{n-1}{n},1,$$

(Figure 5.1.6). We want to construct a rectangle over each of these subintervals whose height is the value of the function $f(x) = x^2$ at some point in the subinterval. To be specific, let us use the right endpoints, in which case the heights of our rectangles will be

$$\left(\frac{1}{n}\right)^2,\ \left(\frac{2}{n}\right)^2,\ \left(\frac{3}{n}\right)^2 \dots,\ 1^2$$

and since each rectangle has a base of width 1/*n*, the total area *An*of the *n* rectangles will be

$$A_n = \left[\left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \left(\frac{3}{n}\right)^2 + \cdots + 1^2\right]\left(\frac{1}{n}\right)$$

For example, if $n = 4$, then the total area of the four approximating rectangles would be

$$A_4 = \left[\left(\tfrac{1}{4}\right)^2 + \left(\tfrac{2}{4}\right)^2 + \left(\tfrac{3}{4}\right)^2 + 1^2\right]\left(\tfrac{1}{4}\right) = \tfrac{15}{32} = 0.46875$$

Table 5.1.2 shows the result of evaluating (1) on a computer for some increasingly large values of *n*. These computations suggest that the exact area is close to 1 3. Later in this chapter we will prove that this area is exactly 1 3 by showing that

$$\lim_{n \to \infty} A_n = \frac{1}{3}$$

2.3 THE INDEFINITE INTEGRAL ANTIDERIVATIVES:

THE INDEFINITE INTEGRALANTIDERIVATIVES

Definition: A function F is called an ant derivative of a function f on a given open interval if
$F'(x) = f(x)$ for all x in the interval.

For example, the function $F(x) = \frac{1}{3}x^3$ is an ant derivative of $f(x) = x^2$ on the interval $(-\infty, +\infty)$ because for each *x* in this interval.

$$F'(x) = \frac{d}{dx}\left[\tfrac{1}{3}x^3\right] = x^2 = f(x)$$

However, F(x) = $\frac{1}{3}x^3$ is not the only ant derivative of f on this interval. If we add any constant C to $\frac{1}{3}x^3$, then the function G(x) = $\frac{1}{3}x^3$ + C is also an ant derivative of f on (−∞, +∞), since

$$G'(x) = \frac{d}{dx}\left[\tfrac{1}{3}x^3 + C\right] = x^2 + 0 = f(x)$$

In general, once any single antiderivative is known, other antiderivatives can be obtained by adding constants to the known antiderivative. Thus,

$$\tfrac{1}{3}x^3, \quad \tfrac{1}{3}x^3 + 2, \quad \tfrac{1}{3}x^3 - 5, \quad \tfrac{1}{3}x^3 + \sqrt{2}$$

are all ant derivatives of $f(x) = x^2$.

2.4 THE INDEFINITE INTEGRAL: -

The process of finding antiderivatives is called ant differentiation or integration. Thus, if

$$\frac{d}{dx}[F(x)] = f(x)$$

then integrating (or ant differentiating) the function f(x) produces an ant derivative of the form F (x) + C. To emphasize this process, Equation (1) is recast using integral notation,

$$\int f(x)\,dx = F(x) + C$$

then integrating (or ant differentiating) the function f(x) produces an ant derivative of the form F (x) + C. To emphasize this process, Equation (1) is recast using integral notation,

$$\int x^2\,dx = \tfrac{1}{3}x^3 + C \quad \text{is equivalent to} \quad \frac{d}{dx}\left[\tfrac{1}{3}x^3\right] = x^2$$

Note that if we differentiate an ant derivative of f(x), we obtain f(x) back again. Thus,

$$\frac{d}{dx}\left[\int f(x)\,dx\right] = f(x)$$

The expression ∫ f(x) dx is called an indefinite integral. The adjective "indefinite" emphasizes that the result of ant differentiation is a "generic" function, described only up to a constant term. The "elongated s" that appears on the left side of (2) is called an integral sign,∗ the function f(x) is called the integrand, and the constant C is called the constant of integration. Equation (2) should be read as: *The integral of f(x) with respect to x is equal to F (x) plus a constant*.

THE DEFINITION OF AREA AS A LIMIT; SIGMA NOTATION

SIGMA NOTATION

To simplify our computations, we will begin by discussing a useful notation for expressing lengthy sums in a compact form. This notation is called sigma notation or summation notation because it uses the uppercase Greek letter Σ (sigma) to denote various kinds of sums. To illustrate how this notation works, consider the sum

$1^2+2^2+3^2+4^2+5^2$

in which each term is of the form k^2 , where k is one of the integers from 1 to 5. In sigma notation this sum can be written as,

$$\sum_{k=1}^{5} k^2$$

Which is read "the summation of k^2, where k runs from 1 to 5." The notation tells us to form the sum of the terms that result when we substitute successive integers for k in the expression k^2, starting with k = 1 and ending with k = 5.

More generally, if f(k) is a function of k, and if m and n are integers such that m ≤ n, then

$$\sum_{k=m}^{n} f(k)$$

Denotes the sum of the terms that result when we substitute successive integers for k, starting with $k = m$ and ending with $k = n$

THE DEFINITE INTEGRAL

A function f is said to be integrable on a finite closed interval [a, b] if the limit

$$\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

exists and does not depend on the choice of partitions or on the choice of the points x_k^* in the subintervals. When this is the case we denote the limit by the symbol

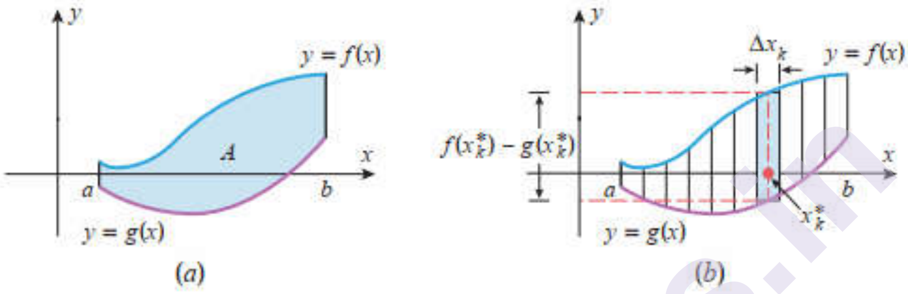
$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

Which is called the ***definite integral*** of *f* from *a* to *b*. The numbers *a* and *b* are called the ***lower limit of integration*** and the ***upper limit of integration***, respectively, and *f(x)* is called the ***integrand***

2.5 AREA BETWEEN TWO CURVES: -

Suppose that f and g are continuous functions on an interval [a, b] and

$f(x) \geq g(x)$ for $a \leq x \leq b$ [This means that the curve $y = f(x)$ lies above the curve $y = g(x)$ and that the two can touch but not cross.] Find the area A of the region bounded above by $y = f(x)$, below by $y = g(x)$, and on the sides by the lines $x = a$ and $x = b$



To solve this problem, we divide the interval [a, b] into n subintervals, which has the effect of subdividing the region into n strips (Figure 6.1.3b). If we assume that the width of the kth strip is Δx_k, then the area of the strip can be approximated by the area of a rectangle of width Δx_k and height $f(x_k^*) - g(x_k^*)$, where x_k^* is a point in the k^{th} subinterval. Adding these approximations yields the following Riemann sum that approximates the area A:

$$A \approx \sum_{k=1}^{n} [f(x_k^*) - g(x_k^*)]\Delta x_k$$

Taking the limit as n increases and the widths of all the subintervals approach zero yields the following definite integral for the area A between the curves:

$$A = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} [f(x_k^*) - g(x_k^*)]\Delta x_k = \int_a^b [f(x) - g(x)]\,dx$$

Area Formula : If f and g are continuous functions on the interval [a, b], and if $f(x) \geq g(x)$ for all x in [a, b], then the area of the region bounded above by $y = f(x)$, below by $y = g(x)$, on the left by the line $x = a$, and on the right by the line $x = b$ is

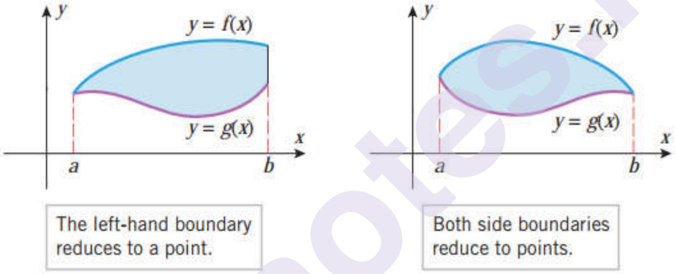
$$A = \int_a^b [f(x) - g(x)]\,dx$$

Example 1 Find the area of the region bounded above by y = x + 6, bounded below by $y = x^2$, and bounded on the sides by the lines x = 0 and x = 2.

Solution. The region and a cross section are shown in Figure 6.1.4. The cross section extends from $g(x) = x^2$ on the bottom to f(x) = x + 6 on the top. If the cross section is moved through the region, then its leftmost position will be x = 0 and its rightmost position will be x = 2. Thus, from (1)

$$A = \int_0^2 [(x+6) - x^2]\,dx = \left[\frac{x^2}{2} + 6x - \frac{x^3}{3}\right]_0^2 = \frac{34}{3} - 0 = \frac{34}{3}$$

It is possible that the upper and lower boundaries of a region may intersect at one or both endpoints, in which case the sides of the region will be points, rather than vertical line segments (Figure 6.1.5). When that occurs you will have to determine the points of intersection to obtain the limits of integration.

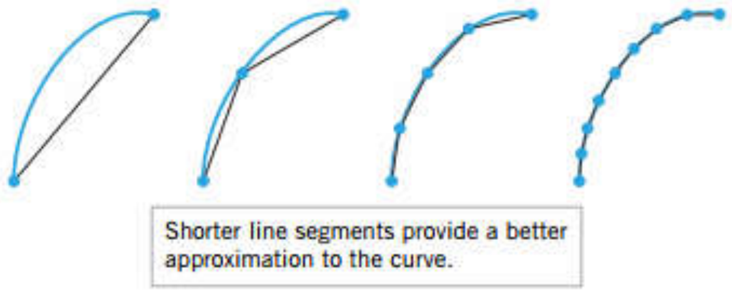


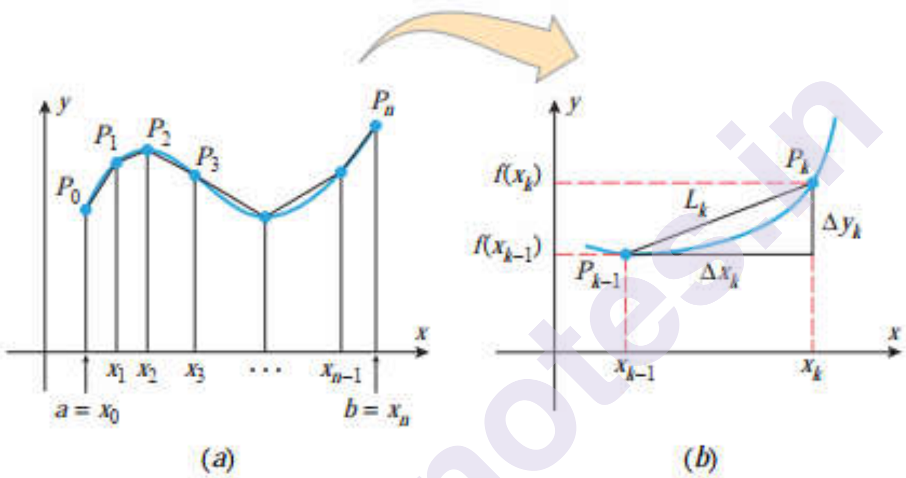
2.6 LENGTH OF A PLANE CURVE: -

Our first objective is to define what we mean by the length (also called the arc length) of a plane curve y = f(x) over an interval [a, b] (Figure 6.4.1). Once that is done we will be able to focus on the problem of computing arc lengths. To avoid some complications that would otherwise occur, we will impose the requirement that f' be continuous on [a, b], in which case we will say that y = f(x) is a smooth curve on [a, b] or that f is a smooth function on [a, b]. Thus, we will be concerned with the following problem.

arc length problem Suppose that y = f(x) is a smooth curve on the interval [a, b]. Define and find a formula for the arc length L of the curve y = f(x) over the interval [a, b].

To define the arc length of a curve we start by breaking the curve into small segments. Then we approximate the curve segments by line segments and add the lengths of the line segments to form a Riemann sum. Figure 6.4.2 illustrates how such line segments tend to become better and better approximations to a curve as the number of segments increases. As the number of segments increases, the corresponding Riemann sums approach a definite integral whose value we will take to be the arc length L of the curve. To implement our idea for solving Problem 6.4.1, divide the interval [a, b] into n subintervals by inserting points $x_1, x_2, \ldots, x_{n-1}$ between $a = x_0$ and $b = x_n$. As shown in Figure 6.4.3a, let $P_0, P_1, \ldots, P_n$ be the points on the curve with x-coordinates $a = x_0$,





$x_1, x_2, \ldots, x_{n-1}$, $b = x_n$ and join these points with straight line segments. These line segments form a polygonal path that we can regard as an approximation to the curve $y = f(x)$.

As indicated in Figure 6.4.3b, the length L_k of the kth line segment in the polygonal path is

$$L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} = \sqrt{(\Delta x_k)^2 + [f(x_k) - f(x_{k-1})]^2}$$

If we now add the lengths of these line segments, we obtain the following approximation to the length L of the curve

$$L \approx \sum_{k=1}^{n} L_k = \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + [f(x_k) - f(x_{k-1})]^2}$$

To put this in the form of a Riemann sum we will apply the Mean-Value Theorem (4.8.2). This theorem implies that there is a point x_k^* between x_{k-1} and x_k such that

$$\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} = f'(x_k^*) \quad \text{or} \quad f(x_k) - f(x_{k-1}) = f'(x_k^*)\Delta x_k$$

and hence we can rewrite (2) as

$$L \approx \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + [f'(x_k^*)]^2(\Delta x_k)^2} = \sum_{k=1}^{n} \sqrt{1 + [f'(x_k^*)]^2}\, \Delta x_k$$

Thus, taking the limit as *n* increases and the widths of all the subintervals approach zero yields the following integral that defines the arc length *L*:

$$L = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} \sqrt{1 + [f'(x_k^*)]^2}\, \Delta x_k = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

In summary, we have the following definition.

definition If y = f(x) is a smooth curve on the interval [a, b], then the arc length L of this curve over [a, b] is defined as

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

This result provides both a definition and a formula for computing arc lengths. Where convenient, (3) can also be expressed as

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Moreover, for a curve expressed in the form x = g(y), where gis continuous on [c, d], the arc length L from y = c to y = d can be expressed as

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

2.7 SIMPSON's RULE: -

SIMPSON's RULE

MODELING WITH DIFFERENTIAL EQUATIONS

A function y = y(x) is a solution of a differential equation on an open interval if the equation is satisfied identically on the interval when y and its derivatives are substituted into the equation. For example, y = e2x is a solution of the differential equation

$$\frac{dy}{dx} - y = e^{2x}$$

on the interval (−∞, +∞), since substituting y and its derivative into the left side of this equation yields

$$\frac{dy}{dx} - y = \frac{d}{dx}[e^{2x}] - e^{2x} = 2e^{2x} - e^{2x} = e^{2x}$$

for all real values of x. However, this is not the only solution on (−∞, +∞), for example, the function

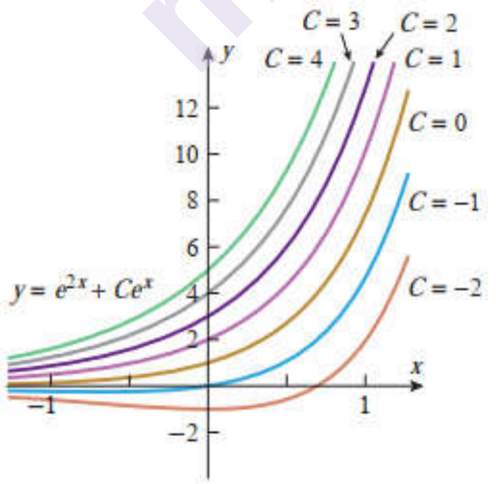
$$y = e^{2x} + Ce^{x}$$

is also a solution for every real value of the constant C, since

$$\frac{dy}{dx} - y = \frac{d}{dx}[e^{2x} + Ce^{x}] - (e^{2x} + Ce^{x}) = (2e^{2x} + Ce^{x}) - (e^{2x} + Ce^{x}) = e^{2x}$$

After developing some techniques for solving equations such as (1), we will be able to show that all solutions of (1) on (−∞, +∞), can be obtained by substituting values for the constant C in (2). On a given interval, a solution of a differential equation from which all solutions on that interval can be derived by substituting values for arbitrary constants is called a general solution of the equation on the interval. Thus (2) is a general solution of (1) on the interval (−∞, +∞),

The graph of a solution of a differential equation is called an integral curve for the equation, so the general solution of a differential equation produces a family of integral curves corresponding to the different possible choices for the arbitrary constants. For example, Figure 8.1.1 shows some integral curves for (1), which were obtained by assigning values to the arbitrary constant in (2)



Integral curves for $\frac{dy}{dx} - y = e^{2x}$

Separation of Variables

Step 1. Separate the variables in (1) by rewriting the equation in the differential form

$h(y)\,dy = g(x)\,dx$

Step 2. Integrate both sides of the equation in Step 1 (the left side with respect to y and the right side with respect to x):

$\int h(y)\,dy = \int g(x)\,dx$

Step 3. If H(y) is any antiderivative of h(y) and G(x) is any antiderivative of g(x),then the equation

$H(y) = G(x) + C$

will generally define a family of solutions implicitly. In some cases it may bepossible to solve this equation explicitly for y.

SLOPE FIELDS

In Section 5.2 we introduced the concept of a slope field in the context of differential equations of the form y = f(x); the same principles apply to differential equations of the form

$y' = f(x, y)$

To see why this is so, let us review the basic idea. If we interpret y as the slope of a tangent line, then the differential equation states that at each point (x, y) on an integral curve, the slope of the tangent line is equal to the value of f at that point (Figure 8.3.1). For example, $y' = y - x$

A geometric description of the set of integral curves can be obtained by choosing a rectangular grid of points in the xy-plane, calculating the slopes of the tangent lines to the integral curves at the gridpoints, and drawing small segments of the tangent lines through those points. The resulting picture is called a slope field or a direction field for the differential equation because it shows the “slope” or “direction” of the integral curves at the gridpoints.

The more grid points that are used, the better the description of the integral curves. For example, Figure 8.3.2 shows two slope fields for (1)—the first was obtained by hand calculation using the 49 gridpoints shown in the companying table, and the second, which gives a clearer picture of the integral curves, was obtained using 625 gridpoints and a CAS

Euler’s Method

To approximate the solution of the initial-value problem

$y' = f(x,y)$, $y(x_0) = y_0$

Proceed as follows:

Step 1. Choose a nonzero number ——————————————x to serve as an increment or step size along thex-axis, and let

$x_1 = x_0 + \Delta x, x_2 = x_1 + \Delta x, x_3 = x_2 + \Delta x, \ldots$

Step 2. Compute successively

$y_1 = y_0 + f(x_0, y_0)\Delta x$

$y_2 = y_1 + f(x_1, y_1)\Delta x$

$y_3 = y_2 + f(x_2, y_2)\Delta x$

$y_{n+1} = y_n + f(x_n, y_n)\Delta x$

The numbers $y_1, y_2, y_3, \ldots$ in these equations are the approximations of $y(x_1), y(x_2), y(x_3)$,

The Method of Integrating Factors

Step 1. Calculate the integrating factor

$\mu = e^{\int p(x)dx}$

Since any μ will suffice, we can take the constant of integration to be zero in this step.

Step 2. Multiply both sides of (3) by μ and express the result as

$\frac{d}{dx}(\mu y) = \mu q(x)$

Step 3. Integrate both sides of the equation obtained in Step 2 and then solve for y. Be sure to include a constant of integration in this step.

2.8 SUMMARY: -

This chapter basically focuses on intergration and application

2.9 EXERCISES: -

- The area A(x) under the graph of f and over the interval[a, x] is given. Find the function f and the value of a

2.10 REFERENCES -

- Calculus: Early transcendental (10th Edition): Howard Anton, Irl Bivens, Stephen Davis, John Wiley & sons, 2012.

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3

PARTIAL DERIVATIVES AND ITS APPLICATIONS

Unit Structure

3.0 OBJECTIVE

- Understanding of Mathematical concepts like limit, continuity, derivative, integration of functions.
- Ability to appreciate real world applications which uses these concepts.
- Skill to formulate a problem through Mathematical modeling and simulation

3.1 INTRODUCTION

In this chapter we will extend many of the basic concepts of calculus to functions of two or more variables, commonly called functions of several variables. We will begin by discussing limits and continuity for functions of two and three variables, then we will define derivatives of such functions, and then we will use these derivatives to study tangent planes, rates of change, slopes of surfaces, and maximization and minimization problems. Although many of the basic ideas that we developed for functions of one variable will carry over in a natural way, functions of several variables are intrinsically more complicated than functions of one variable, so we will need to develop new tools and new ideas to deal with such functions.

3.2 FUNCTIONS

Definition A function f of two variables, x and y, is a rule that assigns a unique real number f(x, y) to each point (x, y) in some set D in the xy-plane.

13.1.2 definition A function f of three variables, x, y, and z, is a rule that assigns a unique real number f(x, y, z) to each point (x, y, z) in some set D in three dimensional space.

Solution. By substitution,

$$f(e, 0) = \sqrt{0+1} + \ln(e^2 - 0) = \sqrt{1} + \ln(e^2) = 1 + 2 = 3$$

To find the natural domain of f, we note that $\sqrt{y+1}$ is defined only when $y \geq -1$, while $\ln(x^2 - y)$ is defined only when $0 < x^2 - y$ or $y < x^2$. Thus, the natural domain of f consists of all points in the xy-plane for which $-1 \leq y < x^2$. To sketch the natural domain, we first sketch the parabola $y = x^2$ as a "dashed" curve and the line $y = -1$ as a solid curve. The natural domain of f is then the region lying above or on the line $y = -1$ and below the parabola $y = x^2$ (Figure 13.1.1). ◀

Example 1 Let f(x, y) = $\sqrt{y-1}$ + ln(x^2 − y). Find f(e, 0) and sketch the natural domain of f

3.3 LIMITS AND CONTINUITY: -

Continuity can fail in the following ways:

- The limit fails to exist. In some texts, this is called an *essential* discontinuity. Any of the examples in the section on limits apply here.
- The limit exists, but the function isn't defined at the point. $y = \frac{\sin x}{x}$ at $x = 0$ is an example.
- The limit exists and the function is defined at the point, but the function output is different from the limit. The function $f(x) = \begin{cases} \frac{\sin x}{x} & \text{for } x \neq 0 \\ 2 & \text{for } x = 0 \end{cases}$ is an example.

The latter two cases (where the limit exists as x approaches the point in question) are called *removable* discontinuities.

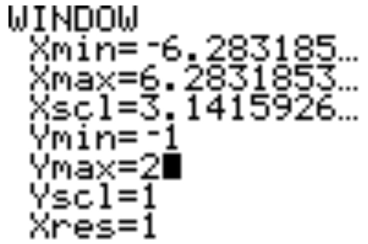
To understand these last three points we need to start taking a look at the concept of *limit* more precisely. What does it really mean when we say that a function f is continuous at $x = c$ if the values of $f(x)$ approach $f(c)$ as x approaches c? What does it mean to approach c? How close to c does x to get?

The concept of *limit* is the underpinning of calculus.

The informal definition or notation is $\lim_{x \to c} = L$ if the values of *f(x)* approach *L* as *x* approaches *c*.

We will look for trends in the values of *f(x)* as *x* gets closer to *c* but $\underline{x \neq c}$.

Example 1: $\lim_{\theta \to 0}\left(\frac{\sin\theta}{\theta}\right)$ (Use radians.)



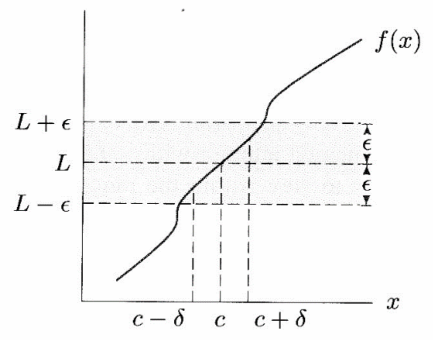
It appears from the graph that as θ approaches 0 from either side that the value of $\frac{\sin\theta}{\theta}$ appears to approach __________. The actual value of $\frac{\sin\theta}{\theta}$ when $\theta = 0$ is __________.

Therefore the limits exists but the function is not continuous at $\theta =$ 0.While it appears that θ approaches 0 from either side that the value of $\lim_{\theta \to 0}\frac{\sin\theta}{\theta} = 1$ we are still very vague about what we mean by words like "approach" and "close".

Here is the formal definition of *limit*:

> We define $\lim_{x \to c} f(x)$ to be the number L (if one exists) such that for every positive number ε (epsilon) > 0 (as small as we want), there is a positive number δ (delta) > 0 (sufficiently small) such that if $|x - c| < \delta$ and $x \neq c$ then $|f(x) - L| < \varepsilon$.

The following figure will help us with what this definition means:



What the definition of the limit means graphically

When we say "*f(x)*" is close to *L*" we measure closeness by the distance between *f(x)* and *L*. $|f(x)-L|$ = Distance between *f(x)* and *L*.

When we say "as close to *L* as we want," we use the ε (the Greek letter epsilon) to specify how close.

We write $|f(x)-L|<\varepsilon$ to indicate that we want the distance between *f(x)* and *L* to be less than ε.

Similarly, we interpret "*x* is sufficiently close to *c*" as specifying a distance between *x* and *c*: $|x-c|<\delta$, where δ (the Greek letter delta) tells us how close *x* should be to *c*.

If $\lim_{x\to c} f(x)=L$, then we know that no matter how narrow the horizontal band determined by ε, there is always a δ which makes the graph stay within that band for $c-\delta<x<c+\delta$.

Basically what we are trying to do is can we guarantee that the inputs (sufficiently close to the value we are approaching but not equal to the value) will make the outputs as close to L as we want.

We will use a graphic illustration to help make sense of this so lets go back to $f(x)=\dfrac{\sin\theta}{\theta}$.

How close should θ be to 0 $(\delta=?>0)$ in order to make $\dfrac{\sin\theta}{\theta}$ within 0.01 of 1? $(\varepsilon=0.01>0)$

First, set the y-range to go from ymin = 0.99 to ymax = 1.01. (0.99 < y < 1.01)

Making sure that the graph does not leave the window through the top or bottom (meaning it goes below 0.99 or above 1.01), change the θ range symmetrically.

Example 2 Use the definition of limit to show that the $\lim_{x\to 3} 2x=6$

We must show how, given any $\varepsilon>0$, that we can find a $\delta>0$ such that

$$\text{If } |x-3|<\delta \text{ and } x\neq 3, \text{ then } |2x-6|<\varepsilon.$$

Since $|2x-6|=2|x-3|$ the to get $|2x-6|<\varepsilon$ would require that $2|x-3|<\varepsilon$ or $|x-3|<\dfrac{\varepsilon}{2}$.

Since $|x-c|<d$ then $\delta=\dfrac{\varepsilon}{2}$.

One- and Two-Sided Limits

When we write $\lim_{x\to 2} f(x)$ we mean that the number *f(x)* approaches as *x* approaches 2 from *both sides*. This is a *Two-Sided Limit*.

If we want *x* to approach 2 only through values greater than 2 (like 2.1, 2.01, 2.003), we write $\lim_{x\to 2^+} f(x)$. This is called a *right-hand limit*. (Similar to the concept of right difference quotient)

If we want *x* to approach 2 only through values less than 2 (like 1.9, 1.99, 1.994), we write $\lim_{x\to 2^-} f(x)$. This is called a *left-hand limit*. (Similar to the concept of left difference quotient)

Right-hand limits and *left-hand limits* are examples of *One-Sided Limits*.

If both the left-hand and right-hand limits are equal, then it can be proved that $\lim_{x\to 2} f(x)$ exists.

Whenever there is no number L that $\lim_{x\to c} f(x) = L$, we say $\lim_{x\to c} f(x)$ does not exist.

***Limits have to be a number and it has to be unique for that function.**

Examples of Limits That Do Not Exist

1) Right – Hand Limit and Left-Hand Limit are different.

The one-sided limits exist but are different. At any integer, for example, the greatest integer function doesn't have a limit. Functions with split definitions can fall in this category at the point where the split occurs.

For example, with $\lim_{x\to 2}\left(\frac{|x-2|}{x-2}\right)$,

$f(x)=\begin{cases} x-1 \text{ for } x<1 \\ x^2 \text{ for } x\geq 1 \end{cases}$, $\lim_{x\to 1} f(x)$ doesn't exist.

Each of the following functions fails to have a limit at $x = 0$:

$$g(x)=\frac{|x|}{x}$$

and $h(x)=\arctan\left(\frac{1}{x}\right)$.

2) The function does not approach any finite number L as $x \to c$.

The outputs grow without bound as the inputs approaches the point from either one side or the other, or both. For example,

$$\lim_{x\to 0}\left(\frac{1}{x^2}\right),$$

$$\lim_{x\to\frac{\pi}{2}}\tan x,$$

and $\lim_{x\to 0^+}\ln x$

don't exist because in each case, the outputs from the function grow without bound.

3) The function does not settle down on a single value but oscillates madly.

The function outputs fail to settle down on a single value, instead oscillating madly.

A typical example is $\lim_{x\to 0}\sin\left(\frac{1}{x}\right)$.

3.4 PARTIAL DERIVATIVES

Given a certain multidimensional function, $A(x,y,z,t)$, a partial derivative at a specific point defines the local rate of change of that function in a particular direction. For the4-dimensional variable, $A(x,y,z,t)$, the partial derivatives are expressed as

$$\left(\frac{\partial A}{\partial x}\right)_{y,z,t} = \lim_{\substack{\Delta x\to 0\\ y,z,t \text{ constant}}} \frac{A(x+\Delta x,y,z,t)-A(x,y,z,t)}{\Delta x}$$ = slope of A in the x direction

$$\left(\frac{\partial A}{\partial y}\right)_{x,z,t} = \lim_{\substack{\Delta y\to 0\\ x,z,t \text{ constant}}} \frac{A(x,y+\Delta y,z,t)-A(x,y,z,t)}{\Delta y}$$ = slope of A in the y direction

$$\left(\frac{\partial A}{\partial z}\right)_{x,y,t} = \lim_{\substack{\Delta z\to 0\\ x,y,t \text{ constant}}} \frac{A(x,y,z+\Delta z,t)-A(x,y,z,t)}{\Delta z}$$ = slope of A in the z direction

$$\left(\frac{\partial A}{\partial t}\right)_{x,y,z} = \lim_{\substack{\Delta t\to 0\\ x,y,z \text{ constant}}} \frac{A(x,y,z,t+\Delta t)-A(x,y,z,t)}{\Delta t}$$ = the local time rate of change of A

The subscripts on the brackets indicate that those dimensions are held constant.

Notice that the definition of a partial derivative of a multi-variable function is the same as derivatives of functions of a single variable, but with the other variables of the function being held constant. Whenever you see the "backward-six" notation for the derivative, you should think about what variable you are operating on, as indicated in the denominator of the expression, while holding the other variables constant.

It is common convention that the directions being held constant are implied and not explicitely written with subscripts.

2. Higher order partial derivatives

We can apply the partial derivative multiple times on a scalar function or vector. For example, given a multivariable function, $f(x,y)$, there are four possible second order partial derivatives:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial x^2}; \quad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial y^2}; \quad \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial x\partial y}; \quad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial y\partial x}$$

The last two partial derivatives, $\frac{\partial^2 f}{\partial x\partial y}$ and $\frac{\partial^2 f}{\partial y\partial x}$ are called "mixed derivatives." An important theorem of multi-variable calculus is **the mixed derivative theorem**. The proof is beyond the scope of this course and only the results are stated.

Mixed derivative Theorem: If a function $f(x,y)$ is continous and smooth to second order, then the order of operation of the partial derivatives does not matter. In other words: $\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 f}{\partial y\partial x}$ for a continous and smooth (to second order) function $f(x,y)$

Example: For the function $f(x,y)=xy^2+\exp(x^2y)$, show $\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 f}{\partial y\partial x}$

Answer Provided:

$$\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(xy^2+\exp(x^2y)\right)\right)=\frac{\partial}{\partial x}\left(2xy+x^2\exp(x^2y)\right)=2\left(y+x\left(1+yx^2\right)\exp(x^2y)\right)$$

$$\frac{\partial^2 f}{\partial y\partial x}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\left(xy^2+\exp(x^2y)\right)\right)=\frac{\partial}{\partial y}\left(y^2+2xy\left(\exp(x^2y)\right)\right)=2\left(y+x\left(1+yx^2\right)\exp(x^2y)\right)$$

We can see that the order of operation of the partial derivative on a continous and smooth scalar function does not matter.

3. Del operator:

The del operator is a linear combination of spatial partial derivatives. In rectangular coordinates, it is expressed as

$$\vec{\nabla} = \nabla \equiv \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \qquad (1)$$

Notice the second equality above is missing the vector arrow. ∇ is always a vector operator and thus it is common convention to just leave off the vector arrow.

The analysis of the del operator on various objects such as scalar functions or vectors can be rather complex. In rectangular coordinates, however, the rules we learned about in chapter 2 on "multiplying" vectors apply to the del operator as well. It is important to notice however that the order is extremely important in the use of equation (1). The del operator acts on all objects to the right of it. **It is cruicial to note that the del operator is not commutative when applied to scalars or vectors! You only apply del operators on what is to the right in the term and never on the objects to the left.**

4. Gradient Operator

Applying the gradient operator, $\vec{\nabla}$, on a scalar function $\phi = \phi(x, y, z)$, simply requires scalar multiplication. The gradient of ϕ yields the following:

$$\vec{\nabla}\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k} \qquad (2)$$

Notice that equation (2) is a linear combination of vector components and basis vectors. In other words the gradient of a scalar yields a vector. **You will be tested on the application leading to equation (2) as well as the fact that the result of $\vec{\nabla}\phi$ is a vector.** Since the gradient of a scalar function is a vector, it obeys all the rules that we learned about in chapter 2.

Example: For scalar function $\phi = xyz$ show that

$$(\nabla\phi)x \neq (\phi\nabla)x$$

Example – Given a velocity vector $\vec{u} = u\hat{i} + v\hat{j} + w\hat{k}$ and the gradient of a scalar function, $\vec{\nabla}\phi$ as defined in equation (2), expand out $\vec{u} \cdot \nabla\phi$ in rectangular coordinates:

Answer provided: Using equation (10) from chapter 2,

$$\vec{u}\cdot\nabla\phi=\left(u\hat{i}+v\hat{j}+w\hat{k}\right)\cdot\left(\frac{\partial\phi}{\partial x}\hat{i}+\frac{\partial\phi}{\partial y}\hat{j}+\frac{\partial\phi}{\partial z}\hat{k}\right)=u\frac{\partial\phi}{\partial x}+v\frac{\partial\phi}{\partial y}+w\frac{\partial\phi}{\partial z}$$

We took the dot product of the vector $\vec{u}$ with the vector $\vec{\nabla}\phi$. We could have just as well taken the dot product of the vector $\vec{u}$ with the operator $\vec{\nabla}$ and then applied that on the scalar function ϕ:

$$\left(\vec{u}\cdot\nabla\right)\phi=\left(u\hat{i}+v\hat{j}+w\hat{k}\right)\cdot\left(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)\phi=\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}\right)\phi=u\frac{\partial\phi}{\partial x}+v\frac{\partial\phi}{\partial y}+w\frac{\partial\phi}{\partial z}$$

.In other words, $\vec{u}\cdot(\nabla\phi)=\left(\vec{u}\cdot\nabla\right)\phi$. This equality is only relavent when we are operating on a ***scalar***.

In this course, we will only take **gradients** of **scalar functions**. It is possible to take gradients of vectors but you obtain a 9 element matrix called the *Dyadic product* of the vector field, $\vec{u}\left(\vec{x}\right)$. For example, given the vector $\vec{u}=u\hat{i}+v\hat{j}+w\hat{k}$, the gradient of $\vec{u}$ is

$$\vec{\nabla}\vec{u}=\frac{\partial\vec{u}}{\partial x}\hat{i}+\frac{\partial\vec{u}}{\partial y}\hat{j}+\frac{\partial\vec{u}}{\partial z}\hat{k}=\begin{bmatrix}\frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x}\\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y}\\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z}\end{bmatrix}$$

You can see why we want to avoid operations like this.

Gradient properties: magnitude

Equation (2) is a vector since it has a magnitude and direction. For a function $f=f(x,y,z)$, the magnitude of $\vec{\nabla}f$ is simply found using the rules of chapter 2.

$$\left|\vec{\nabla}f\right|=\sqrt{\vec{\nabla}f\cdot\vec{\nabla}f}=\sqrt{\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+\left(\frac{\partial f}{\partial z}\right)^2} \qquad (3)$$

Gradient properties: direction

The direction of $\vec{\nabla}f$ is a bit more complicated. From the previous chapter we can see that the direction of $\vec{\nabla}f$ can be expressed by the unit vector, $\dfrac{\vec{\nabla}f}{\left|\vec{\nabla}f\right|}$, but we also can interpret the direction of $\vec{\nabla}f$ in a more geometric or physical way. First we need to use the ***differential*** of *f*, which is labeled *df*. A differential is an infintesimal (meaning really small) change in the value of the multivariable function f and has components:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

If we define the vector line element, $d\vec{\lambda} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$, then we can see by inspection that the differential takes the simple form

$$df = \vec{\nabla} f \cdot d\vec{\lambda}$$

Now let us apply the geometric definition of the dot product $\vec{\nabla} f \cdot d\vec{\lambda}$:

$df = \vec{\nabla} f \cdot d\vec{\lambda} = \left|\vec{\nabla} f\right|\left|d\vec{\lambda}\right|\cos(\theta)$ where θ is the coplanar angle between the vector $\vec{\nabla} f$ and $d\vec{\lambda}$.

If $d\vec{\lambda}$ is perpendicular $\vec{\nabla} f$ then $\theta = 90^o$ and $df = 0$. In other words, $d\vec{\lambda}$ is along lines of *constant f* when it is perpendicular to $\vec{\nabla} f$. Alternatively, we find that df is a maximum when $d\vec{\lambda}$ is parallel to $\vec{\nabla} f$. This means that df is maximum when $d\vec{\lambda}$ is in the same direction as $\vec{\nabla} f$ (and also perpendicual to contours of constant *f*). This also means that $\vec{\nabla} f$ must always be in the direction that leads to the greatest df. The direction of $\vec{\nabla} f$ is also called the asecendantof *f*. Figure 1, on page 6, shows you a picture relating the direction of $\vec{\nabla} f$ to lines of constant *f*.

5: The change of a quantity in the direction of the velocity field (Advection)

We can find the change of a scalar, $f(x,y,z)$, in an arbitrary direction, $\hat{u} = u_1\hat{i} + u_2\hat{j} + u_3\hat{k}$ where $\sqrt{{u_1}^2 + {u_2}^2 + {u_3}^2} = 1$, by taking the dot product of $\hat{u}$ with $\vec{\nabla} f$. The results is:

$$\frac{df}{dt} = \hat{u} \cdot \nabla f \qquad (4)$$

To derive equation (4), parametrize the spatial curve,
$\vec{\lambda} = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ with respect to the variable *t*:

$$x(t) = x_o + tu_1 \Rightarrow \frac{\partial x}{\partial t} = u_1$$

$$y(t) = y_o + tu_2 \Rightarrow \frac{\partial y}{\partial t} = u_2$$

$$z(t) = z_o + tu_3 \Rightarrow \frac{\partial z}{\partial t} = u_3$$

Then, using the **chain rule**, we obtain equation (4):

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t} = \nabla f \cdot \hat{u}$$

Finding variations in a specific direction often occurs when we try to find that variation of a physical quantity *in the direction of the flow field*, $\vec{u}$. We usually discuss the rate of change of the scalar quantity, $f(x, y, z)$ due to variations in *f* along the flow field, $\vec{u}$. This is represented mathematically as:

$$\frac{df}{dt} = \vec{u} \cdot \nabla f$$

The term on the right side of the equality is called the advective term and is one of two contributions to the *total or material derivative* that we will learn more about later in the semester. Often we are interested in determining if there is any variation in the direction of flow. If one obtain's the result:

$$\vec{u} \cdot \nabla f = 0$$

We say that the function, *f*, is spatially constant along the flow field, $\vec{u}$. For example, if our scalar quantity is a time-independent pressure field, $p(x, y, z)$, then the equation $\vec{u} \cdot \nabla p = 0$, tells us that isobars are constant along the flow field which also means that isobar contours are everywhere parallel to the velocity vector field.

I. The gradient product rule of two scalar functions:

$$\nabla(fg) = g\nabla f + f\nabla g$$

II. The divergence product rule with a vector and a scalar:

$$\nabla \cdot (\rho\vec{u}) = \vec{u} \cdot (\nabla \rho) + \rho(\nabla \cdot \vec{u})$$

III. The divergence of the gradient of a scalar – The Laplacian:

$$\nabla \cdot \nabla \phi = \nabla^2 \phi$$

IV. The curl of the gradient of a scalar:

$$\nabla \times (\nabla \phi) = \vec{0} = (0,0,0)$$

Notice the solution is the vector zero, $\vec{0}$, of which each component is zero. It is common notation to imply the vector symbol of the vector zero since the curl is always a vector result.

V. The divergence of the curl of a vector:

$$\nabla \cdot \left(\nabla \times \vec{u}\right) = 0$$

Notice that this is just the scalar number 0 since the divergence always results in a scalar function or number.

VI. The cross-product product rule with a vector and a scalar:

$$\nabla \times \left(\rho \vec{u}\right) = \rho\left(\nabla \times \vec{u}\right) - \vec{u} \times (\nabla \rho)$$

VII. The divergence of the cross product:

$$\nabla \cdot \left(\vec{a} \times \vec{b}\right) = \vec{b}\left(\nabla \times \vec{a}\right) - \vec{a}\left(\nabla \times \vec{b}\right)$$

VIII. The curl of the cross product of a vector:

$$\nabla \times \left(\vec{a} \times \vec{b}\right) = \vec{b} \cdot \left(\nabla \vec{a}\right) - \vec{a} \cdot \left(\nabla \vec{b}\right) + \vec{a}\left(\nabla \cdot \vec{b}\right) - \vec{b}\left(\nabla \cdot \vec{a}\right)$$

IX. the gradient of the dot product of two vectors:

$$\nabla\left(\vec{a} \cdot \vec{b}\right) = \vec{a} \times \left(\nabla \times \vec{b}\right) + \vec{b} \times \left(\nabla \times \vec{a}\right) + \vec{a} \cdot \left(\nabla \vec{b}\right) + \vec{b} \cdot \left(\nabla \vec{a}\right)$$

X: The curl of the curl of a vector:

$$\nabla \times \left(\nabla \times \vec{a}\right) = \nabla\left(\nabla \cdot \vec{a}\right) - \nabla^2 \vec{a}$$

3.5 DIFFERENTIABILITY, DIFFERENTIALS AND LOCAL LINEARITY: -

Definition. We say that the function y = f(x) is locally linear, or differentiable, at the point x = a if the limit

$$\lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x}$$

exists.We simply say "f is locally linear" (or "differentiable") if it's locally linear at all points in a specified domain

Continuous functions We say that a function f is continuous at a point x = a if • it is defined at the point, and • we can achieve changes in the output that are arbitrarily small by restricting changes in the input to be sufficiently small. This second condition can also be expressed in the following form (due, in essence, to Augustin Cauchy in the early 1800's): Given any positive number ε (the proposed limit on the change in the output is traditionally designated by the Greek letter ε, pronounced 'epsilon'), there is always a positive number (the Greek letter 'delta'), such that whenever the change in the input is less than , then the corresponding change in the output will be less than ε. A function is said to be continuous on a set of real numbers if it is continuous at each point of the set.

3.6 CHAIN RULES

In this lesson, we will need to use the Power Rule for rational exponents. We will prove the Power Rule for rational exponents in Lesson 11. Recall that we have proved the Power Rule for positive integers in Lesson 8 and for negative integers in Lesson 9.

Recall: $\sqrt[n]{a^m} = a^{m/n}$

Examples Differentiate the following functions.

1. $y = \sqrt[3]{x^2}$

$$y = x^{2/3} \Rightarrow y' = \frac{2}{3}x^{-1/3} = \frac{2}{3x^{1/3}}$$

Answer: $y' = \dfrac{2}{3x^{1/3}}$

2. $f(x) = \dfrac{1}{\sqrt{x}}$

$$f(x) = x^{-1/2} \Rightarrow f'(x) = -\frac{1}{2}x^{-3/2} = -\frac{1}{2x^{3/2}}$$

Answer: $f'(x) = -\dfrac{1}{2x^{3/2}}$

3. $h(u) = 5\sqrt[5]{u^7} - \dfrac{3u}{\sqrt[4]{u}}$

$$h(u) = 5u^{7/5} - \frac{3u}{u^{1/4}} =$$

$$5u^{7/5} - 3u^{3/4} \Rightarrow h'(u) = 7u^{2/5} - \frac{9}{4}u^{-1/4}$$

$= 7u^{8/20} - \frac{9}{4}u^{-5/20} = \frac{1}{4}u^{-5/20}(28u^{13/20} - 9) =$

$\frac{28u^{13/20} - 9}{4u^{1/4}}$

Answer: $h'(u) = \frac{28u^{13/20} - 9}{4u^{1/4}}$

4. $s(t) = \sqrt[4]{t^3}\left(t^3 - 3t + 10\right)$

$s(t) = t^{3/4}(t^3 - 3t + 10) = t^{15/4} - 3t^{7/4} + 10t^{3/4}$

$s'(t) = \frac{15}{4}t^{11/4} - \frac{21}{4}t^{3/4} + \frac{30}{4}t^{-1/4} =$

$\frac{3}{4}t^{-1/4}(5t^3 - 7t + 10) =$

$\frac{3(5t^3 - 7t + 10)}{4t^{1/4}}$

Answer: $s'(t) = \frac{3(5t^3 - 7t + 10)}{4t^{1/4}}$

Theorem (The Chain Rule) If f and g are two differentiable functions and $k(x) = (f \circ g)(x) = f(g(x))$, then
$k'(x) = f'(g(x))\,g'(x)$.

Proof By definition, $k'(x) = \lim_{h \to 0} \frac{k(x+h) - k(x)}{h}$.

COMMENT: The Chain Rule tells us how to differentiate the composition of two functions f and g. In this form of the Chain Rule, you would have to identify both functions.

Example Differentiate $y = (2x^3 - 4x^2 + 3)^{10}$ using this form of the Chain Rule.

Let $f(x) = x^{10}$ and let $g(x) = 2x^3 - 4x^2 + 3$. Then
$f(g(x)) =$

$f(2x^3 - 4x^2 + 3) = (2x^3 - 4x^2 + 3)^{10}$. Thus, $y = f(g(x))$.

$f(x) = x^{10} \Rightarrow f'(x) = 10x^9 \Rightarrow f'(g(x)) =$ $f'(2x^3 - 4x^2 + 3) =$

$10(2x^3 - 4x^2 + 3)^9$

$g(x) = 2x^3 - 4x^2 + 3 \Rightarrow g'(x) = 6x^2 - 8x$

Thus, $y' = f'(g(x))\, g'(x) = 10(2x^3 - 4x^2 + 3)^9(6x^2 - 8x)$

Answer: $y' = 10(2x^3 - 4x^2 + 3)^9(6x^2 - 8x)$

Clearly, we need a better way than this!

Another way to state the Chain Rule: If $y = f(u)$, where $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

COMMENT: Since in the statement above, $\frac{dy}{du} = f'(g(x))$ and $\frac{du}{dx} = g'(x)$, then this is the same statement of the Chain Rule given earlier. However, in this form you only have to identify the function g, which is being called *u*. In the first statement of the Chain Rule given above, you had to identify both the functions of *f* and *g*.

<u>Example</u> Differentiate $y = (2x^3 - 4x^2 + 3)^{10}$ using this form of the Chain Rule.

Let $u = 2x^3 - 4x^2 + 3$. Then $y = u^{10}$. Thus, $\frac{dy}{du} = 10u^9 =$ $10(2x^3 - 4x^2 + 3)^9$.

NOTE: When you write the answer for the derivative $\frac{dy}{du}$, you say $10u^9$ to yourself (silently), but you write $10(2x^3 - 4x^2 + 3)^9$. Since $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$, then

$y' = 10(2x^3 - 4x^2 + 3)^9(6x^2 - 8x)$ since $\frac{du}{dx} = 6x^2 - 8x$. With this form of the Chain Rule, we are back to writing down the answer for the derivative of a function.

Answer: $y' = 10(2x^3 - 4x^2 + 3)^9(6x^2 - 8x)$

COMMENT: The fastest way to confuse a calculus student about the Chain Rule is to give them a function of *u*. So, let's address this problem. First, we need to understand that the symbols*t* and *T* are not the same. Because of this, *t* can be used to represent one expression and *T* can be used to represent another expression. In physics, it is very common for *t* to represent time and for *T* to represent temperature. Thus, for function $y = (2x^3 - 4x^2 + 3)^{10}$, that was differentiated above, we could have used *X* for the substitution variable instead of *u*. That is, let $X = 2x^3 - 4x^2 + 3$. We will call *X* "big *X*" instead of capital *X*. So, if the function above was $y = (2t^3 - 4t^2 + 3)^{10}$, then we would have used a "big *T*" for the substitution variable instead of *u*. That is, let $T = 2t^3 - 4t^2 + 3$. If the function above was $y = (2u^3 - 4u^2 + 3)^{10}$, then we would have used a "big *U*" for the substitution variable. That is, let $U = 2u^3 - 4u^2 + 3$.

<u>Examples</u> Differentiate the following functions.

1. $f(x) = (4x^3 + 2x^2 - x - 3)^3$

Let big $X = 4x^3 + 2x^2 - x - 3$. Thus, $f(x) = (big\ X)^3$. By the Power Rule and the Chain Rule, $D_x(big\ X)^3 = 3(big\ X)^2 \cdot D_x(big\ X)$. In general, we have that $D_x(big\ X)^n = n(big\ X)^{n-1} \cdot D_x(big\ X)$. Thus,

$$f(x) = (4x^3 + 2x^2 - x - 3)^3 \Rightarrow$$

$$f'(x) = 3(4x^3 + 2x^2 - x - 3)^2(12x^2 + 4x - 1)$$

Answer: $f'(x) = 3(4x^3 + 2x^2 - x - 3)^2(12x^2 + 4x - 1)$

2. $y = 3(8x - 17)^{-5}$

Let big *X* $= 8x - 17$. **Thus,** $y = 3(big\ X)^{-5}$. **Then**

$y' = -15(big\ X)^{-6} \cdot D_x(big\ X)$. **Thus,**

$y = 3(8x - 7)^{-5} \Rightarrow y' = -15(8x - 7)^{-6}8 =$
$-120(8x - 7)^{-6}$

Answer: $y' = -120(8x - 7)^{-6}$ **or** $y' = -\dfrac{120}{(8x - 7)^6}$

3. $g(w) = (3w^2 - 2w)^{3/5}$

Let big *W* $= 3w^2 - 2w$. **Thus,** $g(w) = (big\ W)^{3/5}$. **Then**

$g'(w) = \dfrac{3}{5}(big\ W)^{-2/5} \cdot D_w(big\ W)$. **Thus,**

$$g(w) = (3w^2 - 2w)^{3/5} \Rightarrow$$

$$g'(w) = \frac{3}{5}(3w^2 - 2w)^{-2/5}(6w - 2)$$

Answer: $g'(w) = \dfrac{3}{5}(3w^2 - 2w)^{-2/5}(6w - 2)$

or $g'(w) = \dfrac{6}{5}(3w^2 - 2w)^{-2/5}(3w - 1)$

or $g'(w) = \dfrac{6(3w - 1)}{5(3w^2 - 2w)^{2/5}}$

4. $s(t) = \dfrac{5}{7(4t^5 - 3t^3 + t)^4}$

First, we may write $s(t) = \dfrac{5}{7}(4t^5 - 3t^3 + t)^{-4}$.

Let big *T* $= 4t^5 - 3t^3 + t$. **Thus,** $s(t) = \dfrac{5}{7}(big\ T)^{-4}$.

Then

$$s'(t) = -\frac{20}{7}(big\ T)^{-5} \cdot D_t(big\ T)$$ **. Thus,**

$$s(t) = \frac{5}{7}(4t^5 - 3t^3 + t)^{-4} \Rightarrow$$

$$s'(t) = -\frac{20}{7}(4t^5 - 3t^3 + t)^{-5}(20t^4 - 9t^2 + 1)$$

Answer: $s'(t) = -\frac{20}{7}(4t^5 - 3t^3 + t)^{-5}(20t^4 - 9t^2 + 1)$

or $s'(t) = -\frac{20(20t^4 - 9t^2 + 1)}{7(4t^5 - 3t^3 + t)^5}$

5. $f(x) = \frac{-3}{x+15}$

We differentiate this function in Lesson 9 using the Quotient Rule.
Now, we will differentiate it using the Chain Rule.

First, we may write $f(x) = -3(x+15)^{-1}$.

Let big $X = x + 15$. Thus, $f(x) = -3(big\ X)^{-1}$. Then

$$f'(x) = 3(big\ X)^{-2} \cdot D_x(big\ X)$$. Thus,

$$f(x) = -3(x+15)^{-1} \Rightarrow$$

$$f'(x) = 3(x+15)^{-2}(1) = 3(x+15)^{-2}$$

Answer: $f'(x) = 3(x+15)^{-2}$ **or** $f'(x) = \frac{3}{(x+15)^2}$

6. $g(x) = \frac{21}{9x-17}$

First, we may write $g(x) = 21(9x-17)^{-1}$**.**

Let big $X = 9x - 17$**. Thus,** $g(x) = 21(big\ X)^{-1}$**. Then**

$$g'(x) = -21(big\ X)^{-2} \cdot D_x(big\ X)$$ **. Thus,**

$$g(x) = 21(9x - 17)^{-1} \Rightarrow$$

$$g'(x) = -21(9x - 17)^{-2} 9 =$$

$$-189(9x - 17)^{-2}$$

Answer:

$$g'(x) = -189(9x - 17)^{-2} \text{ or } g'(x) = -\frac{189}{(9x - 17)^2}$$

7. $y = \dfrac{6}{4t^2 + 3t - 22}$

First, we may write $y = 6(4t^2 + 3t - 22)^{-1}$.

Let big *T* $= 4t^2 + 3t - 22$. **Thus,** $y = 6(big\ T)^{-1}$. **Then**

$$\frac{dy}{dt} = -6(big\ T)^{-2} \cdot D_t(big\ T).$$ **Thus,**

$$y = 6(4t^2 + 3t - 22)^{-1} \Rightarrow$$

$$\frac{dy}{dt} = -6(4t^2 + 3t - 22)^{-2}(8t + 3)$$

Answer: $\dfrac{dy}{dt} = -6(4t^2 + 3t - 22)^{-2}(8t + 3)$ **or**

$$\frac{dy}{dt} = -\frac{6(8t + 3)}{(4t^2 + 3t - 22)^2}$$

8. $h(z) = 10\sqrt[3]{(z^2 + 3z - 18)^2}$

$$h(z) = 10(z^2 + 3z - 18)^{2/3}$$

Let big *Z* $= z^2 + 3z - 18$. **Thus,** $h(z) = 10(big\ Z)^{2/3}$. **Then**

$$h'(z) = \frac{20}{3}(big\ Z)^{-1/3} \cdot D_z(big\ Z).$$ **Thus,**

$$h(z) = 10(z^2 + 3z - 18)^{2/3} \Rightarrow$$

$$h'(z) = \frac{20}{3}(z^2 + 3z - 18)^{-1/3}(2z + 3)$$

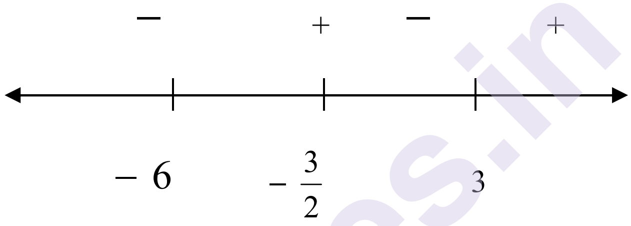
Answer: $h'(z) = \frac{20}{3}(z^2 + 3z - 18)^{-1/3}(2z + 3)$

or $h'(z) = \dfrac{20(2z + 3)}{3(z^2 + 3z - 18)^{1/3}}$

Since $z^2 + 3z - 18 = (z + 6)(z - 3)$ **,then**

$$h'(z) = \frac{20(2z + 3)}{3[(z + 6)(z - 3)]^{1/3}}$$

Sign of $h'(z)$:



NOTE: In Lesson 14, we conclude that the function h is increasing on the interval $\left(-6, -\frac{3}{2}\right)$ Y $(3, \infty)$ and is decreasing on the interval $(-\infty, -6)$ Y $\left(-\frac{3}{2}, 3\right)$. There is a local maximum occurring when $x = -\frac{3}{2}$ and since $h(z) = 10\sqrt[3]{(z + 6)^2(z - 3)^2}$, the local maximum is

$$h\left(-\frac{3}{2}\right) = 10\sqrt[3]{\left(-\frac{3}{2} + \frac{12}{2}\right)^2\left(-\frac{3}{2} - \frac{6}{2}\right)^2} =$$

$$10\sqrt[3]{\left(\frac{9}{2}\right)^2\left(-\frac{9}{2}\right)^2} =$$

$$10\sqrt[3]{\left(\frac{9}{2}\right)^2\left(\frac{9}{2}\right)^2} = 10\sqrt[3]{\left(\frac{9}{2}\right)^4} = 45\sqrt[3]{\frac{9}{2}} = 45\sqrt[3]{\frac{36}{8}} =$$

$\dfrac{45\sqrt[3]{36}}{2}$. There is a local minimum occurring when $x = -6$ and since $h(z) = 10\sqrt[3]{(z + 6)^2(z - 3)^2}$, the local minimum is $h(-6)=0$. There is a local minimum occurring when $x=3$ and the local maximum is $h(3)=0$.

9. $g(w)=\sqrt{9w+85}$

This function is from Lessons 6 and 7. In Lesson 6, we found the slope of the tangent line to the graph of $y=g(w)$ at the point $(-4, g(-4))=(-4, 7)$. In Lesson 7, we use the definition of derivative to find the derivative of this function. Now, we will use the Chain Rule to find the derivative of this function.

$$g(w)=(9w+85)^{1/2}$$

Let big *W* = $9w+85$**. Thus,** $g(w)=(big\ W)^{1/2}$**. Then**

$$g'(w)=\frac{1}{2}(big\ W)^{-1/2}\cdot D_w(big\ W)$$ **. Thus,**

$$g(w)=(9w+85)^{1/2} \Rightarrow$$

$$g'(w)=\frac{1}{2}(9w+85)^{-1/2}9=\frac{9}{2}(9w+85)^{-1/2}$$

Answer: $g'(w)=\dfrac{9}{2}(9w+85)^{-1/2}$ **or**

$$g'(w)=\frac{9}{2(9w+85)^{1/2}}$$

10. $y=(7-6x)^3(8x^2+9)^5$

NOTE: $D_x(7-6x)^3=3(7-6x)^2(-6)$

$$D_x(8x^2+9)^5=5(8x^2+9)^4 16x$$

Using the Product Rule, we obtain

$$y'=3(7-6x)^2(-6)(8x^2+9)^5+(7-6x)^3 5(8x^2+9)^4 16x=$$

$$-18(7-6x)^2(8x^2+9)^5+80x(7-6x)^3(8x^2+9)^4=$$

$2(7-6x)^2(8x^2+9)^4[-9(8x^2+9) + 40x(7-6x)] =$

$2(7-6x)^2(8x^2+9)^4(-72x^2-81+280x-240x^2) =$

$2(7-6x)^2(8x^2+9)^4(-312x^2+280x-81) =$

$-2(7-6x)^2(8x^2+9)^4(312x^2-280x+81)$

Answer: $y' = -2(7-6x)^2(8x^2+9)^4(312x^2-280x+81)$

or $y' = 2(7-6x)^2(8x^2+9)^4(-312x^2+280x-81)$

11. $y = \dfrac{(2-z)^{2/3}}{4z+5}$

12. $h(x) = \left(\dfrac{3x^2+5}{8-x^3}\right)^4$

13. $g(t) = \left(t^5 + \dfrac{6}{t^3}\right)^9$

14. $y = (3x^2-4x)(x^2+5x-3)^2$

15. $f(w) = (w^2-9)^8(16-w^4)$

3.7 DIRECTIONAL DERIVATIVES AND THE GRADIENT: -

f(x,y) be a real-valued function with domain D in R^2 , and let (a,b) be a point in D . Let v be a unit vector in R^2 . Then the directional derivative of f at (a,b) in the direction of v , denoted by $D_v f(a,b)$, is defined as

Dv f(a,b) = $\lim_{h\to 0} \dfrac{f((a,b)+hv) - f(a,b)}{h}$

For a real-valued function f(x,y) , the gradient of f , denoted by ∇f , is the vector

$\nabla f = (\partial f/\partial x, \partial f/\partial y)$

In R2 . For a real-valued function f(x,y,z) , the gradient is the vector

$\nabla f = (\partial f/\partial x, \partial f/\partial y, \partial f/\partial z)$

3.8 MAXIMA AND MINIMA

Let f(x,y)f(x,y) be a real-valued function, and let (a,b)(a,b) be a point in the domain of ff. We say that ff has a **local maximum** at (a,b) (a,b) if f(x,y)≤f(a,b)f(x,y)≤f(a,b) for all (x,y)(x,y) inside some disk of positive radius centered at (a,b) (a,b), i.e. there is some sufficiently small r>0r>0 such that f(x,y)≤f(a,b) for all (x,y) f(x,y)≤f (a,b) for all (x,y) for which (x−a)2+(y−b)2<r2(x−a)2+(y−b)2<r2.

Likewise, we say that ff has a **local minimum** at (a,b) if f(x,y)>f(a,b) for all (x,y)(a,b) if f(x,y)>f(a,b) for all (x,y) inside some disk of positive radius centered at (a,b)(a,b).

If f(x,y)≤f(a,b) for all (x,y)f(x,y)≤f(a,b) for all (x,y) in the domain of ff, then ff has a **global maximum** at (a,b)(a,b). If f(x,y)≥f(a,b) for all (x,y) in the domain of f, then ff(x,y)≥f(a,b) for all (x,y) in the domain of f, then f has a **global minimum** at (a,b)(a,b).

3.9 SUMMARY

This chapter mainly covers the basic concepts of partial derivatives.

3.10 EXERCISES

- Find (a) the equation of the tangent line to the graph of the function $y = (4x^2 - 8x + 3)^4$ at the point $(2, 81)$ and (b) the point(s) on the graph at which the tangent line is horizontal.

3.11 REFERENCES

- Calculus: Early transcendental (10th Edition): Howard Anton, Irl Bivens, Stephen Davis, John Wiley & sons, 2012.

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