

Duration 2  $\frac{1}{2}$ Hrs

Marks: 75

- N.B. : (1) All questions are compulsory  
(2) Figures to the right indicate marks.

1. (a) Attempt any One from the following: (8)
- State and prove Cantor's Intersection Theorem for a metric space  $(X, d)$ .
  - Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous. If  $f(a)$  and  $f(b)$  have opposite signs then using Nested Intervals Theorem, prove that there exists  $c \in (a, b)$  such that  $f(c) = 0$ .
- (b) Attempt any Two from the following: (12)
- Let  $(X, d)$  be a complete metric space and  $(Y, d_Y)$  is a subspace of  $(X, d)$ . If  $(Y, d_Y)$  is complete then show that  $Y$  is a closed subset of  $X$ .
  - Prove that a finite metric space is complete.
  - Show that  $[0, 1]$  is uncountable.
  - Prove that the set of real numbers  $\mathbb{R}$  is complete with respect to the usual distance.
2. (a) Attempt any One from the following: (8)
- Let  $f : (X, d) \rightarrow (Y, d')$  be a function. Show that  $f$  is continuous at  $p \in X$  if and only if for each sequence  $(x_n)$  in  $X$  converging to  $p$ , the sequence  $(f(x_n))$  converges to  $f(p)$  in  $Y$ .
  - Let  $(X, d)$  and  $(Y, d')$  be metric spaces. If  $(X, d)$  is compact and  $f : X \rightarrow Y$  is a continuous function, then show that  $f(X)$  is a compact subset of  $Y$ .
- (b) Attempt any Two from the following: (12)
- Let  $(X, d)$  and  $(Y, d)$  be metric spaces then show that  $f : X \rightarrow Y$  is continuous if and only if  $f^{-1}(B^\circ) \subseteq (f^{-1}(B))^\circ$ , for each subset  $B$  of  $Y$ .
  - Let  $(X, d)$  and  $(Y, d')$  be metric spaces and  $f, g : X \rightarrow Y$  be continuous on  $X$ . Show that  $\{x \in X : f(x) = g(x)\}$  is a closed subset of  $X$ .
  - Show that the identity function  $i : (\mathbb{R}, d) \rightarrow (\mathbb{R}, d_1)$ ,  $i(x) = x \ \forall x \in \mathbb{R}$  is discontinuous everywhere in  $\mathbb{R}$  where  $d$  is the usual distance and  $d_1$  is the discrete metric.
  - Let  $(X, d)$  and  $(Y, d')$  be metric spaces and  $D \subseteq X$  be a dense subset of  $X$ . If  $f : X \rightarrow Y$  is a continuous onto map, show that  $f(D)$  is dense in  $Y$ .
3. (a) Attempt any One from the following: (8)
- Prove that a subset  $E$  of  $\mathbb{R}$  is connected if and only if it is an interval. (Distance in  $\mathbb{R}$  being usual)
  - Prove that a metric space is connected if and only if every continuous function from  $X$  to  $\{1, -1\}$  is a constant function.
- (b) Attempt any Two from the following: (12)
- If  $(X, d)$  is a metric space and  $A, B$  are connected subsets of  $X$  such that  $A \cap B \neq \emptyset$  then prove that  $A \cup B$  is a connected set.
  - Prove that a convex subset of a normed linear space is path connected.

- (iii) Prove or disprove: The subset  $\{(x, y) \in \mathbb{R}^2 : y \neq 0\}$  of  $(\mathbb{R}^2, d)$  ( $d$  being Euclidean distance) is connected.
- (iv) If  $(X, d)$  be a connected metric space and  $f : X \rightarrow \mathbb{Z}$  (distance in  $\mathbb{Z}$  being usual distance) is a continuous function then prove that  $f$  is a constant function.

4. Attempt any Three from the following: (15)

- (a) Use the intermediate value property to show that there is a square whose diagonal has length between  $r$  and  $2r$  and has area equal to half the area of the circle of radius  $r$ .
- (b) Check if Cantor's Theorem is applicable in the following examples. Also, find  $\bigcap_{n \in \mathbb{N}} F_n$  in each case, where  $(F_n)$  is a sequence of subsets of  $\mathbb{R}$  and the distance in  $\mathbb{R}$  is usual.
  - (I)  $F_n = [n, \infty)$
  - (II)  $F_n = (0, \frac{1}{n})$
- (c) If  $T : \left[0, \frac{1}{3}\right] \rightarrow \left[0, \frac{1}{3}\right]$  is defined as  $T(x) = x^2$ , then show that  $T$  is a contraction map on  $\left[0, \frac{1}{3}\right]$ . Does  $T$  have any fixed points? If yes, how many? Justify your answer.
- (d) Discuss the uniform continuity of  $f : [1, \infty) \rightarrow \mathbb{R}$  (distance being usual), defined by  $f(x) = \frac{1}{x}$ .
- (e) Prove or disprove: If  $A^\circ$  and  $\partial A$  are connected then  $A$  is connected.
- (f) Show that  $E = \{(x, y) \in \mathbb{R}^2 : x > 0, x^2 - y^2 = 1\}$  is path connected.