Duration:[2½Hours] **[Total Marks: 75]**

N.B. 1) All questions are compulsory.
2) Figures to the right indicate full marks.

1. (a) Attempt any **ONE** question: (8)
   i. If G is $k-$ critical graph then show that
      I) G is connected
      II) Every vertex v of graph G has atleast $k-1$ degree.
      III) Graph G cannot be partitioned into subgraphs.
   ii. Prove that a graph G with $p \geq 2$ is 2-connected if and only if any two vertices are connected by at least two internally disjoint paths.

   (b) Attempt any **TWO** questions: (12)
   i. Show that vertex connectivity of a graph G is always less or equal to the edge connectivity of G.
   ii. Let $\pi_k(G)$ denote the chromatic polynomial of the graph G. If G is simple graph then prove that $\pi_k(G) = \pi_k(G-e) - \pi_k(G.e)$ where e is an edge of G.
   iii. Show that a connected graph G on p vertices is a tree if and only if the chromatic polynomial of G is $k(k-1)^{p-1}$.
   iv. For any graph G, prove that $\chi(G) \leq \Delta(G)+1$ where $\chi(G)$ represents vertex chromatic number of a graph G and $\Delta(G)$ denotes the maximum degree of G. Give an example of graphs for which $\chi(G) > \Delta(G)$.

2. (a) Attempt any **ONE** question: (8)
   i. State and prove Max Flow - Min Cut Theorem.
   ii. Show that there are exactly five regular polyhedra.

   (b) Attempt any **TWO** questions: (12)
   i. State and prove Euler theorem for planar graph.
   ii. Show that if G is a planar (p, q) graph in which every face is bounded by a cycle of length at least n then show that $q \leq \frac{n(p-2)}{n-2}$.
   iii. If G is a graph with p vertices and q edges, and let every vertex of G has degree at least 6 then prove that $q \geq 3p$. Hence prove that every planar graph contains a vertex of degree at most 5.
   iv. Show that the complete graph K_5 and complete bipartite graph $K_{3,3}$ are nonplanar.

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3. (a) Attempt any **ONE** question: (8)
   i. State and prove Hall's (Marriage) Theorem for a System of Distinct Representatives.
   ii. An elf has a staircase of n stairs to climb. Each step it takes can cover either one stair or two stairs. Find a recurrence relation for a_n, the number of different ways for the elf to ascend the $n-$stair staircase and solve it by using generating function.

   (b) Attempt any **TWO** questions: (12)
   i. Let B denotes a forbidden chess board in which a special square $*$ has been identified and let D denote the board obtained from the original board by deleting the row and column containing the special square and E denote the board obtained from the original board where only the special square $*$ is removed from the board, then prove that $R(x, B) = xR(x, D) + R(x, E)$.
   ii. Show that a matching M in G is a maximum matching if and only if G contains no $M-$augmenting path.
   iii. Show that the number of non-negative integer solutions of the equation $x_1 + x_2 + \cdots + x_k = r$ is given by $\begin{pmatrix} r+k-1 \\ r \end{pmatrix}$.
   iv. Solve recurrence relation $a_n = 5a_{n-1} - 6a_{n-2}$, $(n \geq 2)$ subject to initial conditions with $a_0 = 1, a_1 = -2$ using generating function.

4. Attempt any **THREE** questions: (15)
   (a) Determine the chromatic polynomial and chromatic number of a graph G obtained by deleting an edge from K_4.
   (b) If G is a bipartite graph, then show that $\chi'(G) = \Delta(G)$, where $\chi(G)$ represents vertex chromatic number of a graph G and $\Delta(G)$ denotes the maximum degree of G.
   (c) Prove that for any flow f and any cut $(S, \overline{S})$, $val(f) = f^+(S) - f^-(S)$.
   (d) Show that the edge e is a loop in G if and only if e^* is a bridge in G^*.
   (e) Let h_n denote the number of non-negative integral solutions of the equation $3e_1 + 4e_2 + 2e_3 + 5e_4 = n$. Find the generating function $f(x)$ for $h_0, h_1, \ldots, h_n, \ldots$
   (f) Find the rook polynomial for the following $\{(1,1), (2,5), (3,3), (4,2), (4,4), (5,1), (5,3)\}$.

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