

Duration:[3 Hours]

[Total Marks: 100]

- N.B. 1) All questions are compulsory.
2) Figures to the right indicate full marks.

1. Choose correct alternative in each of the following:

(20)

- i. Let G be a connected graph. If G is neither complete nor an odd cycle then vertex chromatic number of G is
 - (a) \leq Max degree of G .
 - (b) = Maximum degree of G .
 - (c) = Max degree of $G + 1$
 - (d) None of these
- ii. Edge Chromatic number of $K_{1,n}, n \geq 2$ is
 - (a) 2
 - (b) n
 - (c) $n + 1$
 - (d) None of the above
- iii. The relation between vertex connectivity κ , edge connectivity κ' and minimum degree δ is
 - (a) $\kappa = \kappa' = \delta$
 - (b) $\kappa < \kappa' < \delta$
 - (c) $\kappa \leq \kappa' \leq \delta$
 - (d) None of these
- iv. If $G(p, q)$ be planar graph, then sum of degrees of regions is equal to
 - (a) $2p$
 - (b) $2q$
 - (c) $p + q$
 - (d) None of these
- v. Let G be a connected planar graph with q edges, p vertices and f regions then
 - (a) $2q > 3f$
 - (b) $2q \geq 3f$
 - (c) $2q < 3f$
 - (d) $2q \leq 3f$
- vi. If f is a flow in a network N and S be any subset of nodes, then (S, \bar{S}) is a cut if
 - (a) $x \in S, y$ is not in \bar{S}
 - (b) x is not in $S, y \in \bar{S}$
 - (c) x is not in S, y is not in \bar{S}
 - (d) $x \in S, y \in \bar{S}$
- vii. If the polynomial $c + 3x + x^2$ is a rook polynomial of some chess board then
 - (a) $c = 0$
 - (b) $c = 1$
 - (c) $c > 1$
 - (d) c is even
- viii. The function e^{-x} is the generating function of the sequence
 - (a) $a_n = \frac{1}{n!}$
 - (b) $a_n = \frac{-1}{n!}$
 - (c) $a_n = \frac{(-1)^n}{n!}$
 - (d) $a_n = (-1)^n$
- ix. The number of different system of distinct representatives of the family $A_1 = \{1, 2\}, A_2 = \{2, 3\}, A_3 = \{3, 4\}, A_4 = \{4, 5\}, A_5 = \{5, 1\}$
 - (a) 1
 - (b) 2
 - (c) 3
 - (d) 5
- x. A matching M in G is a maximum matching if and only if G contains
 - (a) no M -augmenting path.
 - (b) M -augmenting path.
 - (c) no M -alternating path.
 - (d) None of these

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2. (a) Attempt any **ONE** question from the following: (8)
- i. If G is k -critical graph then show that
 - I) G is connected
 - II) Every vertex v of graph G has at least $k - 1$ degree.
 - III) Graph G cannot be partitioned into subgraphs.
 - ii. Prove that a graph G with $p \geq 2$ is 2-connected if and only if any two vertices are connected by at least two internally disjoint paths.
- (b) Attempt any **TWO** questions from the following: (12)
- i. Define vertex chromatic number $\chi(G)$. Let G be the graph with n vertices. Show that $\chi(G) \geq \frac{n}{n-\delta(G)}$ where $\chi(G)$ denotes vertex chromatic number of G and $\delta(G)$ denotes minimum degree of G .
 - ii. If G is a cycle on n vertices then show that $\pi_k(G) = (k - 1)^n + (-1)^n(k - 1)$.
 - iii. If G is cubic graph, then show that $\kappa(G) = \kappa'(G)$ where $\kappa(G)$ denote the vertex connectivity and $\kappa'(G)$ denotes the edge connectivity of a graph G .
 - iv. If G is a bipartite graph, then show that $\chi'(G) = \Delta(G)$, where $\chi(G)$ represents vertex chromatic number of a graph G and $\Delta(G)$ denotes the maximum degree of G .
3. (a) Attempt any **ONE** question from the following: (8)
- i. State and prove Euler's formula for planar graphs. Hence or otherwise prove that the minimum degree of a simple planar graph is ≤ 5 .
 - ii. State and prove Max Flow - Min Cut Theorem.
- (b) Attempt any **TWO** questions from the following: (12)
- i. Define dual graph G^* of G . Show that edges in a plane graph G form a cycle in G if and only if the corresponding dual edges form a bond in G^* .
 - ii. Let f be a flow in a network N and P be any f -incrementing path then show that there exist a revised flow f' such that $val(f') = val(f) + \epsilon(p)$
 - iii. For a plane graph G , prove that G is bipartite if and only if every face of G has even length.
 - iv. Show that there is at least one face of every polyhedron is bounded by an n -cycle for some $n = 3, 4$ or 5 .
4. (a) Attempt any **ONE** question from the following: (8)
- i. State and prove Hall's (Marriage) Theorem for a System of Distinct Representatives.
 - ii. Derive the recurrence relation for number of ways of dividing a $n + 1$ -sided convex polygon into triangular regions by inserting diagonals that do not intersect in the interior and prove using generating function that the solution to this recurrence relation is a Catalan Number.

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(b) Attempt any **TWO** questions from the following: (12)

- i. Let B denotes a forbidden chess board in which a special square * has been identified and let D denote the board obtained from the original board by deleting the row and column containing the special square and E denote the board obtained from the original board where only the special square * is removed from the board, then prove that $R(x, B) = xR(x, D) + R(x, E)$.
- ii. Show that a matching M in G is a maximum matching if and only if G contains no M -augmenting path.
- iii. Determine the generating function for the number of n-combinations of apples, bananas, oranges, and pears, where, in each n-combination, the number of apples is even, the number of bananas is odd, the number of oranges is between 0 and 4, and there is at least one pear.
- iv. Solve the recurrence relation $a_n = 3a_{n-1} + 2$, with $a_0 = 1$ using generating function.

5. Attempt any **FOUR** questions from the following: (20)

- (a) Show that vertex connectivity of a graph G is always less or equal to the edge connectivity of G .
- (b) Show that if G_1, G_2, \dots, G_n are n components of graph G then $\pi_k(G) = \prod_{i=1}^n \pi_k(G_i)$.
- (c) Show that every planar graph is 6-vertex colorable.
- (d) If f is any flow and K be any cut in a network N with $val(f) = cap(K)$ then show that f is maximum flow and K is minimum cut.
- (e) Show that the number of nonnegative integer solutions of the equation $x_1 + x_2 + \dots + x_k = r$ is given by $\binom{r+k-1}{r}$.
- (f) Let $A = (A_1, A_2, A_3, A_4, A_5, A_6)$, where $A_1 = \{1, 2, 3\}, A_2 = \{1, 2, 3, 4, 5\}, A_3 = \{1, 2\}, A_4 = \{2, 3\}, A_5 = \{1\}, A_6 = \{1, 3, 5\}$. Does family A have an System of Distinct Representative? If not, what is the largest number of sets in the family with an System of Distinct Representative?
