Duration: 3 Hrs Marks: 100

N.B. : (1) All questions are compulsory.
(2) Figures to the right indicate marks.

1. Choose the correct alternative for each of the following: (20)

(i) If $f : (\mathbb{R}, d) \longrightarrow (\mathbb{R}, d)$, with d as the usual distance, is a continuous function then $f^{-1}((0,\infty))$ is

(a) a closed subset of $\mathbb{R}$.
(b) a bounded subset of $\mathbb{R}$.
(c) an open subset of $\mathbb{R}$
(d) none of the above is true.

(ii) Let (X, d) be a discrete metric space and (Y, d') be any metric space. If $f : X \longrightarrow Y$, then f is

(a) an uniformly continuous function on X.
(b) a bounded function on X.
(c) continuous but not uniformly continuous function on X.
(d) none of the above is true.

(iii) Consider the metrics d and d_1 on $\mathbb{N}$, where d is the induced distance from $\mathbb{R}$ with the usual metric and $d_1(m,n) = \left|\frac{1}{m} - \frac{1}{n}\right|$ for $m, n \in \mathbb{N}$. Let $i : \mathbb{N} \longrightarrow \mathbb{N}$ denote the identity map on $\mathbb{N}$. Then

(a) $i : (\mathbb{N}, d) \longrightarrow (\mathbb{N}, d_1)$ is continuous but $i : (\mathbb{N}, d_1) \longrightarrow (\mathbb{N}, d)$ is not continuous.
(b) $i : (\mathbb{N}, d) \longrightarrow (\mathbb{N}, d_1)$ is not continuous.
(c) $i : (\mathbb{N}, d_1) \longrightarrow (\mathbb{N}, d)$ is not continuous.
(d) None of the above.

(iv) In $(\mathbb{R}^2, d)$ where d is the Euclidean distance, the following set is not connected.

(a) $\mathbb{R}^2 \setminus \mathbb{Q} \times \mathbb{Q}$.
(b) $\mathbb{R}^2 \setminus \{(0,0)\}$
(c) $\mathbb{R}^2 \setminus \{(x,y) : y = 0\}$
(d) None of the above.

(v) Let A and B be connected subsets in a metric space (X, d) and $A \subseteq C \subseteq B$ Then,

(a) C is connected.
(b) C° is connected.
(c) $\overline{C}$ is connected.
(d) $C \cap \overline{A}$ is connected.

(vi) In $\mathbb{R}^2$ with the Euclidean metric, which of the following sets is convex?

(a) $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 4\}$
(b) $\{(x,y) \in \mathbb{R}^2 : xy = 0\}$
(c) $\{(x,y) \in \mathbb{R}^2 : y > 0\}$
(d) None of these.

(vii) Let (X, d) be a connected metric space and $f : X \longrightarrow \mathbb{Z}$ be a continuous map. Then, f is

(a) an onto function.
(b) a one-one function.
(c) a bijective function.
(d) a constant function.

(viii) Let $f_n(x) = \sin nx$ for $x \in \mathbb{R}$. and $g_n(x) = \dfrac{f_n(x)}{n}$ $\forall x \in \mathbb{R}$. Then,
   (a) $\{f_n\}$ and $\{g_n\}$ are uniformly convergent on $\mathbb{R}$.
   (b) $\{f_n\}$ and $\{g_n\}$ are not pointwise convergent on $\mathbb{R}$.
   (c) $\{g_n\}$ is uniformly convergent on $\mathbb{R}$ but $\{f_n\}$ is not.
   (d) $\{f_n\}$ is uniformly convergent on $\mathbb{R}$ but $\{g_n\}$ is not.

(ix) The series $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n+1}$ is
   (a) uniformly convergent on $\mathbb{R}$.
   (b) not uniformly convergent on $[-a, a]$ where $0 < a < 1$
   (c) uniformly convergent on $[-a, a]$ where $0 < a < 1$.
   (d) none of the above.

(x) If R is the radius of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} c_n x^n$ then the radius of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2} c_n x^n$ is
   (a) R^2 (b) R (c) 0   (d) ∞

2. (a) Attempt any One of the following: (8)
   (i) Let (X, d) and (Y, d') be metric spaces. If (X, d) is compact and $f : X \longrightarrow Y$ is a continuous function, then show that $f(X)$ is a compact subset of Y.
   (ii) Let $f : (X, d) \longrightarrow (Y, d')$ be a function. Prove that f is continuous on X if and only if for each open subset G of Y, $f^{-1}(G)$ is an open subset of X.

   (b) Attempt any Two of the following: (12)
   (i) Let (X, d) and (Y, d') be metric spaces then show that $f : X \longrightarrow Y$ is continuous if and only if $f^{-1}(B^\circ) \subseteq (f^{-1}(B))^\circ$, for each subset B of Y.
   (ii) (X, d) is a metric space and $f : (X, d) \longrightarrow (X, d)$ is a function such that $d(f(x), f(y)) < d(x, y)$ whenever $x \neq y$. Let $S = \{x \in X : f(x) = x\}$. Prove that
      (I) f is continuous on X.
      (II) S has at most one element.
   (iii) Let (X, d) and (Y, d') be metric spaces. When is $f : X \longrightarrow Y$ said to be uniformly continuous? Show that $f(x) = \dfrac{1}{(1 + x^2)}$ is uniformly continuous on $\mathbb{R}$ (under the usual metric).
   (iv) Let d_1 and d_2 be equivalent metrics on X and (Y, d) be any metric space. If $f : (X, d_1) \longrightarrow (Y, d)$ and $g : (Y, d) \longrightarrow (X, d_1)$ are continuous maps on X and Y respectively, then prove that $f : (X, d_2) \longrightarrow (Y, d)$ and $g : (Y, d) \longrightarrow (X, d_2)$ are also continuous.

3. (a) Attempt any One of the following: (8)

(i) Define a connected metric space and prove that a metric space (X, d) is disconnected if and only if there exists a nonempty proper subset of X which is both open and closed in X.

(ii) Prove that a metric space is connected if and only if every continuous function from X to $\{1, -1\}$ is a constant function.

(b) Attempt any Two of the following: (12)

(i) Let (X, d) be a metric space such that for any $x, y \in X$ there exists a connected subset A of X such that $x, y \in A$. Prove that X is connected.

(ii) Prove that in a normed linear space, an open ball $B(x, r)$ is a convex set.

(iii) Prove that if a subset E of $\mathbb{R}$ is connected then it is an interval. (Distance in $\mathbb{R}$ being usual)

(iv) Let $A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ and $B = \{(x, y) \in \mathbb{R}^2 : x = 1\}$ in $(\mathbb{R}^2, d)$ where d is the Euclidean metric. Show that $A \cup B$ is a path connected set.

4. (a) Attempt any One of the following: (8)

(i) Let $\{f_n\}$ be a sequence of real valued functions defined on a set $S \subseteq \mathbb{R}$ such that $f_n \longrightarrow f$ uniformly on S. If each f_n is bounded on S, then prove the following:

(I) f is bounded on S.

(II) There exists $\alpha \in \mathbb{R}^+$ such that $|f_n(x)| \leq \alpha$ for all $n \in \mathbb{N}$ and for all $x \in S$.

(ii) State and prove the Cauchy criterion for uniform convergence of a series of functions.

(b) Attempt any Two of the following: (12)

(i) Find the pointwise limit of the sequence of functions $f_n : [0, \infty) \longrightarrow \mathbb{R}$,

$f_n(x) = \begin{cases} x & \text{if } x \leq n \\ n & \text{if } x > n \end{cases}$. Is the pointwise limit bounded?

(ii) Find the radius of convergence and interval of convergence of the following power series:

(I) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{(n+1)^{\sqrt{n}}}$ (II) $\displaystyle\sum_{n=0}^{\infty} \frac{(x+3)^{n-1}}{n}$

(iii) Show that the series of functions $\displaystyle\sum_{n=1}^{\infty} \frac{e^{-nx}}{n}$ converges uniformly on $[a, \infty), a > 0$.

(iv) Consider the power series $\displaystyle\sum_{n=0}^{\infty} c_n x^n$ with integer coefficients. If $c_n \neq 0$ for infinitely many n, then show that its radius of convergence is at most 1.

5. Attempt any Four of the following: (20)

(a) Prove or disprove: Continuous image of an open ball is an open ball.

(b) Let $f : [a, b] \to [a, b]$ be continuous on $[a, b]$ and differentiable on (a, b). If $\exists\ c \in \mathbb{R}$ with $0 < c < 1$ such that $|f'(x)| \leq c$, $\forall x \in (a, b)$ then prove that f is a contraction of $[a, b]$.

(c) Let (X, d) be a connected metric space. If $f : X \longrightarrow (Y, d')$, where d' is a discrete metric, is a continuous function then prove that f is a constant function.

(d) Prove that there does not exist a continuous onto function from the set $A = \{(x, y) \in \mathbb{R}^2 : x + y = 1\}$ to the set $B = \{(x, y) \in \mathbb{R}^2 : x^2 - y^2 = 1\}$.

(e) If $\sum_{n=1}^{\infty} |a_n| < \infty$, then prove that the series $\sum_{n=1}^{\infty} a_n \cos nx$ and $\sum_{n=1}^{\infty} a_n \sin nx$ converge on $\mathbb{R}$.

(f) Let $f_n : [-1, 1] \longrightarrow \mathbb{R}$, $f_n(x) = \sqrt{x^2 + \frac{1}{n^2}}$. Given that $f_n \longrightarrow f$ uniformly on $[-1, 1]$ where $f(x) = |x|$ for $x \in [-1, 1]$. Find $\lim_{n \longrightarrow \infty} \int_{-1}^{1} f_n(x)\, dx$.

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