

- N.B. : (1) All questions are compulsory
(2) Figures to the right indicate marks.

1. (a) Attempt any One question: (8)

(i) Let f be a continuous real valued periodic function, defined on $[-\pi, \pi]$ and having period 2π . If $f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ is the Fourier series of f on $[-\pi, \pi]$

then prove that : $S_n(x) - f(x) = \frac{2}{\pi} \int_0^\pi \left[\frac{f(x+t) + f(x-t)}{2} - f(x) \right] D_n(t) dt$

where $D_n(x)$ is the Dirichlet's kernel.

(ii) State and prove Bessels Inequality.

(b) Attempt any Two questions: (12)

(i) For $f(x) = x \cos x, x \in [-\pi, \pi]$ and $f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$. find Fourier coefficients a_0, a_1, b_1 .

(ii) Let $f \in C[-\pi, \pi]$ and f has Fourier series $\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$, show that

$$\sigma_n(t) = \frac{a_0}{2} + \sum_{k=1}^{n-1} \left(1 - \frac{k}{n}\right) (a_k \cos kt + b_k \sin kt)$$

(iii) Define Fejer's Kernel $K_n(t)$. Prove that $K_n(t) = \frac{\sin^2(\frac{nt}{2})}{2n \sin^2 \frac{t}{2}} \quad -\infty < t < \infty,$
 $t \neq 2k\pi, k \in \mathbb{Z}, t \in \mathbb{R}$

(iv) Is the series $\sum_{n=1}^{\infty} \left[\frac{\cos nx + \sin nx}{n^{\frac{3}{2}}} \right]$ the Fourier series of a function $f \in C[-\pi, \pi]$? Justify your answer.

2. (a) Attempt any One question: (8)

(i) $K \subseteq \mathbb{R}^n$ (distance Euclidean), K is closed and bounded. Show that K is sequentially compact.

(ii) Let $I = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] \subseteq \mathbb{R}^n$ (distance Euclidean). Prove that I is compact.

(b) Attempt any Two questions: (12)

(i) Let (X, d) and (Y, d') be metric spaces. If (X, d) is compact and $f : X \rightarrow Y$ is a continuous function, then show that $f(X)$ is a compact subset of Y .

(ii) Show that $\{0\} \cup \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$ is a compact subset of (\mathbb{R}, d) where d is usual distance in \mathbb{R} using the definition of a compact subset.

(iii) Let (X, d) be a compact metric space and $f : X \rightarrow (0, \infty)$ is continuous. Show that there exists $\epsilon > 0$ such that $f(x) \geq \epsilon \forall x \in X$.

(P.T.O)

(iv) Prove that a subset of a discrete metric space is compact if and only if it is finite.

3. (a) Attempt any One question:

(i) Show that a subset $E \subseteq \mathbb{R}$ is connected if and only if E is an interval (distance being usual). (8)

(ii) Show that a metric space (X, d) is connected if and only if every continuous function $f : X \rightarrow \{1, -1\}$ is constant.

(b) Attempt any Two questions:

(i) Let (X, d) be a metric space and A be a connected subset of X . If $A \subseteq B \subseteq \bar{A}$, then show that B is connected. In particular, prove that \bar{A} is connected. (12)

(ii) Show that $S = \{(x, y) \in \mathbb{R}^2 : y \neq 0\}$ is not connected. Hence show that S not path connected in a Euclidean metric space \mathbb{R}^2 .

(iii) If (X, d) is a connected metric space and $f : X \rightarrow \mathbb{Z}$ a continuous function, prove that f is constant. (distance in \mathbb{Z} being usual).

(iv) Show that a convex set in \mathbb{R}^n is path connected (distance being Euclidean).

4. Attempt any Three from the following:

(a) If the series $\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ converges uniformly to f on $[-\pi, \pi]$ then prove (15)

that Fourier series of f is $\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

(b) $f(x) = \frac{x^2}{4}$, $-\pi \leq x \leq \pi$. Find the Fourier series of f . Assuming that the Fourier series of f converges to $f(x)$ at $x = 0$, find the sum $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$.

(c) Show that a closed subset of a compact set is compact in any metric space.

(d) Show that $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is a compact subset of \mathbb{R}^2 , distance being Euclidean.

(e) Let A and B be path connected subsets of a metric space (X, d) such that $A \cap B \neq \emptyset$. Show that $A \cup B$ is path connected.

(f) Let (X, d) be a connected metric space which is not bounded. Prove that for each $x_0 \in X$ and each $r > 0$, the set $\{x \in X : d(x, x_0) = r\}$ is non-empty.