

2 1/2 Hours]

Revised course

[Total Marks: 75

- N.B.: (1) All questions are compulsory.
(2) Figures to the right indicate marks for respective subquestions.

(11)

1. (a) Answer any ONE

- i. Let H be a subgroup of group G . Prove that the following statements are equivalent. (8)
- (p) $aHa^{-1} \subseteq H$ for each $a \in G$.
- (q) $aHa^{-1} = H$ for each $a \in G$.
- (r) Every left coset of H in G is also a right coset of H in G i.e. $aH = Ha$ for each $a \in G$.
- (s) $HaHb = Hab$ for each $a, b \in G$.
- ii. State and prove the Cayley's theorem for finite group. (8)

(b) Answer any TWO

- i. Let H, K be normal subgroups of G and H be a subgroup of K . Prove that $\frac{G/H}{K/H} \cong \frac{G}{K}$. (6)
- ii. If a cyclic group H of a group G is normal in G , then show that every subgroup of H is normal in G . (6)
- iii. Find the order of each element of $\mathbb{Z}_2 \times \mathbb{Z}_4$. Is $\mathbb{Z}_2 \times \mathbb{Z}_4$ isomorphic to \mathbb{Z}_8 ? Justify. (6)
- iv. Suppose G is a non-abelian group of order p^3 where p is a prime and $Z(G) \neq \{e\}$, then prove that $|Z(G)| = p$. (6)

2. (a) Answer any ONE

- i. Let R, R' be commutative rings and $f : R \rightarrow R'$ be a ring homomorphism. Show that- (8)
- (p) If f is surjective, I is an ideal of R , then $f(I)$ is an ideal of R' .
- (q) If I' is an ideal of R' , then $f^{-1}(I')$ is an ideal of R .
- ii. Show that, characteristic of a ring R is n if and only if the order of the multiplicative identity of R is n in the group $(R, +)$. Further if $\text{char } R = n$, where R is an integral domain, then show that n is a prime. (8)

(b) Answer any TWO

- i. Let A be a subring and B be an ideal of a ring R . Then prove that $A \cap B$ is an ideal of A and $A/(A \cap B) \cong (A+B)/B$. (6)
- ii. Let R be a commutative ring. Show that $I = \{a \in R : a^n = 0 \text{ for some } n \in \mathbb{N}\}$ is an ideal of R . Also show that R/I has no nilpotent element. (6)
- iii. Show that there is exactly one non-zero ring homomorphism from \mathbb{Q} to \mathbb{Q} . (6)
- iv. Show that, if R is a ring having 6 elements then R is commutative. Is R an integral domain? Justify. (6)

(P.T.O)

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3. (a) Answer any ONE

- i. Let F be a field and $p(x) \in F[x]$. Show that $\langle p(x) \rangle$ is maximal ideal of $F[x]$ if and only if $p(x)$ is irreducible.
- ii. Let R be an Integral Domain, Let $p \in R$. Then,
 - (p) p is prime iff $\langle p \rangle$ is a non zero prime ideal of R .
 - (q) If p is prime then p is irreducible.

(b) Answer any TWO

- i. State and prove the Eisenstein's criteria for irreducibility of a polynomial $f(x)$ over \mathbb{Q} .
- ii. Let F be a field. Show that every ideal of $F[x]$ is a principal ideal.
- iii. Let R be a commutative ring and I, J be any two ideals of R . Let P be a prime ideal of R such that $IJ \subseteq P$. Prove that $I \subseteq P$ or $J \subseteq P$.
- iv. Show that ideal $I = \{a + bi : a, b \in \mathbb{Z}, a \text{ mod } 2 = b \text{ mod } 2\}$ is a maximal ideal of $\mathbb{Z}[i]$.

4. Answer any THREE

- (a) Let H be a normal subgroup of a finite group G . If G/H has an element of order n then show that G has an element of order n .
- (b) Let $G = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : a, b \in \mathbb{Z}_3 \right\}$. Show that G is a subgroup of $GL_3(\mathbb{Z}_3)$.
Also Show that G is abelian of order 9 and is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$.
- (c) Let R be a commutative ring. If u is a unit and a is nilpotent in R , show that $u + a$ is a unit in R .
- (d) Show that every element of a finite commutative ring is either a unit or a zero divisor.
- (e) Let R, S be commutative rings and $f : R \rightarrow S$ be an onto ring homomorphism. Prove that if M is a maximal ideal in S , $f^{-1}(M)$ is a maximal ideal in R .
- (f) Prove that the ring $\mathbb{Z}_2[x]/(x^3 + x + 1)$ is a field, but $\mathbb{Z}_3[x]/(x^3 + x + 1)$ is not a field.
