

(3 Hours)

[Total Marks : 100]

- N.B.** 1. All questions are compulsory.
 2. Figures to the right indicate marks for respective parts
 3. Use of Calculator is not allowed.

Q.1 Choose correct alternative in each of the following: (20)

i. Which of the following equation has a root in (1,2)?

- (a) $4x^2 - 1 = 0$ (b) $\sin x - x = 0$
 (c) $2x^3 - 3x^2 + 2x - 3 = 0$ (d) None of these

ii. If A is countable set and B is uncountable set then the most we can say about $A \cap B$ is _____.

- (a) Finite (b) At most countable
 (c) Uncountable (d) Countable

iii. If the decimal representation of a number is non terminating and non repeating then the number is _____.

- (a) A whole number (b) An irrational number
 (c) A natural number (d) A rational number

iv. The norm of the partition $P = \{ -6, -5.8, -5.2, -4.8, -3.2, -3 \}$ is

- (a) 2.6 (b) 1.4
 (c) 1.6 (d) None of these

v. Let P and Q be any two Partitions of interval $[a, b]$. Then the statement that is always true is

- (a) $L(P, f) \leq U(Q, f)$ (b) $U(P, f) \leq U(Q, f)$
 (c) $U(P, f) \leq L(Q, f)$ (d) None of these

- vi. Let f and g be functions such that the function $f + g$ is integrable on I , then
- (a) Both f and g must be integrable on I (b) At least one of f and g must be integrable on I
- (c) f and g may or may not be integrable on I (d) None of these

vii. Let $F: [\pi, 2\pi] \rightarrow \mathbb{R}$ such that $F(x) = \int_{\pi}^x \sin^2 t dt$ then $F'(x) = \dots$

- (a) $\sin^2 x$ (b) $\cos^2 x$
- (c) $\cos x \sin x$ (d) $2\sin x$

viii. Let $f(x)$ and $g(x)$ be two positive \mathbb{R} -integrable functions on $[a, \infty]$ such that $f(x) \leq g(x), \forall x \in [a, \infty]$ then

- (a) $\int_a^{\infty} g(x)dx$ is convergent if $\int_a^{\infty} f(x)dx$ is convergent
- (b) $\int_a^{\infty} g(x)dx$ is divergent if $\int_a^{\infty} f(x)dx$ is divergent
- (c) $\int_a^{\infty} f(x)dx$ is divergent if $\int_a^{\infty} g(x)dx$ is divergent
- (d) both(a) and (b) are true.

ix. $\Gamma(n + 1)$ is

- (a) $(n + 1)!$ (b) $(n + 1)$
- (c) $n!$ (d) n

x. The equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ represents

- (a) ellipsoid (b) sphere
- (c) Hyperboloid of one sheet (d) Hyperboloid of two sheets

Q.2 a) Attempt any ONE question from the following: (08)

i. State and prove Nested Interval Theorem.

ii. Using Nested Interval Theorem prove that if $f: [a, b] \rightarrow \mathbb{R}$ is a continuous function with $f(a)f(b) < 0$ then there exists $c \in (a, b)$ s.t. $f(c) = 0$.

b) Attempt any TWO questions from the following: (12)

i. If $I_n = \left(0, \frac{1}{n}\right)$ for all $n \in \mathbb{N}$ then prove that $\bigcap_{n=1}^{\infty} I_n = \emptyset$

- ii. Show that a real number is rational iff it has repeating decimal representation.
- iii. Show that the equation $x^3 - 15x + 1 = 0$ has three solutions in the interval $[-4, 4]$
- iv. Find convergent subsequence of sequence $\sin\left(\frac{n\pi}{2}\right)$.

Q.3 a) Attempt any ONE question from the following: (08)

- i. Let $f : [a, b] \rightarrow \mathbb{R}$ be a bounded function. Prove that f is R-integrable on $[a, b]$ iff for any $\epsilon > 0$, there exists a partition P_ϵ of $[a, b]$ such that $U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon$.
- ii. Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function. Then show that f is Riemann integrable on $[a, b]$.

b) Attempt any TWO questions from the following: (12)

- i. Let $f : [a, b] \rightarrow \mathbb{R}$ be a bounded function with $m = \text{Inf}(f)$ and $M = \text{Sup}(f)$ on $[a, b]$. With usual notations, define $L(P, f)$ and $U(P, f)$ where P is a partition of $[a, b]$. Hence prove that $m(b - a) \leq L(P, f) \leq U(P, f) \leq M(b - a)$.

- ii. Prove that the function $f : [0, 4] \rightarrow \mathbb{R}$ defined by $f(x) = 2x^2 + 1$ is Riemann integrable and evaluate $\int_0^4 f(x) dx$.

- iii. Let $f : [3, 6] \rightarrow \mathbb{R}$ be defined by $f(x) = 12$ for $3 \leq x < 6$
 $= 500$ for $x = 6$

Prove that f is Riemann integrable and $\int_3^6 f(x) dx = 36$.

- iv. Using Riemann Criterion, show that the function $f : [0, 1] \rightarrow \mathbb{R}$ defined by $f(x) = x^2$ is Riemann integrable.

Q.4 a) Attempt any ONE question from the following: (08)

- i. Let $f : [a, b] \rightarrow \mathbb{R}$ be R-integrable on $[a, b]$ and $F(x) = \int_a^x f(t) dt, \forall x$. If f is continuous on $[a, b]$ then show that F is differentiable and $F'(x) = f(x)$

ii. Show that $\int_0^1 x^{m-1} (1-x)^{n-1} dx$ exists iff $m > 0, n > 0$.

b) Attempt any TWO questions from the following:

(12)

i. Solve the improper integral $\int_{-1}^1 \frac{1}{x^2} dx$.

ii. By using “integration by parts” solve the integral $\int_0^2 xe^x dx$.

iii. Express $\int_0^1 \frac{xdx}{\sqrt{1-x^5}}$ in terms of beta function.

iv. Evaluate $\iint_D ydA$ where D is the region bounded by the line $y = x$ and the parabola $y = 4x - x^2$.

Q.5 Attempt any FOUR questions from the following:

(20)

a) Show that $G = \{(\frac{1}{n+4}, \frac{1}{n}) / n \in \mathbb{N}\}$ covers $A = (0,1)$ but has no finite sub-cover for A .

b) Show that set of real numbers \mathbb{R} is uncountable.

c) Let $P = \{0, 1, 2, 3, 4\}$ be a partition of $[0, 4]$ and $f : [0, 4] \rightarrow \mathbb{R}$ is a function such that $f(x) = 3 - x^2$ then find the lower sum $L(P, f)$ and upper sum $U(P, f)$.

d) Show that the function $f : [1, 3] \rightarrow \mathbb{R}$ is Riemann integrable, where

$$f(x) = 5 \quad \text{for } 1 \leq x < 2$$

$$= -9 \quad \text{for } 2 \leq x \leq 3$$

e) Check the convergence of improper integral $\int_2^\infty \frac{1}{\sqrt{x^2-1}} dx$ by comparison test.

f) Reverse the order of integration and evaluate $\int_0^1 \int_2^{4-2x} xydydx$.
