(3 Hours) **[Total Marks: 100]**

Note: (i) All questions are compulsory.
(ii) Figures to the right indicate marks for respective parts.

Q.1 Choose correct alternative in each of the following **(20)**

i. The order and degree of differential equation $y = x^2\frac{dy}{dx} + \sqrt{1+\left(\frac{dy}{dx}\right)^5}$ is

(a) 1 and 2 (b) 1 and 5
(c) 2 and 5 (d) 5 and 5

ii. Orthogonal trajectories of the family of curves $xy = k$ represents a family of
(a) Parabolas (b) Straight lines
(c) Hyperbolas (d) Circles

iii. The integrating factor of $\frac{dy}{dx} + \frac{y}{1+x^2} = \frac{e^{tan^{-1}(x)}}{1+x^2}$ is
(a) $\frac{1}{1+x^2}$ (b) $\frac{e^{tan^{-1}(x)}}{1+x^2}$
(c) $e^{tan^{-1}(x)}$ (d) None of these

iv. Which of the following is an exact differential equation?
(a) $(x^2+1)dx - xydy = 0$ (b) $xdy + (3x-2y)dx = 0$
(c) $2xydx + (x^2+2)dy = 0$ (d) $x^2ydy - ydx = 0$

v. Which of the following functions are linearly independent?
(a) $y_1 = 2x; y_2 = 1-x$ (b) $y_1 = logx; y_2 = logx^3$
(c) $y_1 = x; y_2 = x^2$ (d) $y_1 = x^2; y_2 = 5x^2$

vi. The general solution of $4y'' + 12y' + 9y = 0$ is
(a) $c_1e^{-\frac{3}{2}x} + c_2xe^{-\frac{3}{2}x}$ (b) $c_1 + c_2e^{-\frac{3}{2}x}$
(c) $c_1cosx + c_2sin\left(-\frac{3}{2}x\right)$ (d) None of these

vii. If $y_1(x) = cos\ \pi x$ and $y_2(x) = sin\ \pi x$, value of Wronskian $W(y_1, y_2)$ is
(a) π (b) 2π
(c) 0 (d) None of these

viii. The Wronskian of two solutions $(e^{2t}, 2e^{2t})$ and $(e^{3t}, 2e^{3t})$ of a homogeneous linear system of differential equations, is equal to

(a) 0 (b) $4e^{5t}$
(c) $2e^{2t}$ (d) None of these

ix. One of the solutions of the system $\frac{dx}{dt} = y - x, \frac{dy}{dt} = 3x + y$ is

(a) $x = 3e^{2t}, y = e^{2t}$ (b) $x = e^{2t}, y = 3e^{2t}$
(c) $x = e^{t}, y = 3e^{t}$ (d) None of these

x. If (e^{4t}, e^{4t}) and $(e^{-2t}, -e^{-2t})$ are linearly independent solutions of $\frac{dx}{dt} = x + 3y$, $\frac{dy}{dt} = 3x + y$ then particular solution for which $x(0) = 5$, $y(0) = 1$ is

(a) $(3e^{4t} + 2e^{4t}, 3e^{-2t} - 2e^{-2t})$ (b) $(4e^{4t} + e^{4t}, 4e^{-2t} - 3e^{-2t})$
(c) $(3e^{4t} + 2e^{-2t}, -2e^{4t} + e^{-2t})$ (d) $(3e^{4t} + 2e^{-2t}, 3e^{4t} - 2e^{-2t})$

Q2. Attempt any **ONE** question from the following: **(08)**

a) i. When a differential equation $M(x,y)dx + N(x,y)dy = 0$ is said to be exact? State and prove the necessary and sufficient condition for the above differential equation to be exact.

ii. Obtain a method to solve the Bernoulli's differential equation $\frac{dy}{dx} + P(x)y = Q(x)y^n (n \neq 0,1)$ where $P(x), Q(x)$ are functions of x and hence solve $\frac{dy}{dx} - ytanx = sin(x)cos^2(x)y^2$.

Q.2 Attempt any **TWO** questions from the following: **(12)**

b) i. When a differential equation $M(x,y)dx + N(x,y)dy = 0$ is said to be homogeneous? Solve the homogenous differential equation

$$x\frac{dy}{dx} - y = \sqrt{x^2 - y^2}.$$

ii. The population of a village increases at a rate proportional to the population at that time. In a period of 10 years, the population grew from 10 thousand to 15 thousand. What will be the population after 20 years? ($ln1.5 = 0.405$, $e^{0.81} = 2.248$)

iii. Find the solution to the differential equation $3x^2y^2dx + 4(x^3y - 3)dy = 0$

iv. Solve $(1 + y^2)dx = (tan^{-1}(y) - x)dy$

Q3. Attempt any **ONE** question from the following: **(08)**

$a)$ i. Show that a second order homogenous differential equation $y'' + P(x)y' + Q(x)y = 0$ is completely determined by two linearly independent solutions $y_1(x)$ and $y_2(x)$. However, the converse is not true, that is $y_1(x)$ and $y_2(x)$ are not uniquely determined by the differential equation.

ii. Let $y'' + ay' + b = 0$ be a homogeneous differential equation with constant coefficients and let $m^2 + am + b = 0$ be the corresponding auxiliary equation with roots $m_1 and\ m_2$. Discuss the general solution of the differential equation when:
$(p) m_1 and\ m_2$ are real and equal $(q) m_1 and\ m_2$ are complex.

Q3. Attempt any **TWO** questions from the following: **(12)**

$b)$ i. Show that $y = ax^2 + bx + 3$ is the general solution of $x^2y'' - 2xy' + 2y = 6$. Further find a particular solution of this differential equation satisfying $y(1) = 0, y'(1) = 1$.

ii. Check whether the following functions are linearly dependent (on any interval not containing 0) by finding their Wronskian.
(p) $y_1 = x^2; y_2 = \sqrt{x}$ (q) $y_1 = x^4; y_2 = x^4 logx$

iii. Using method of variation of parameters find a particular integral of the differential equation $y'' + y = secx$.

iv. Solve the differential equation $y'' + 3y' - 10y = 6e^{4x}$, by the method of undetermined coefficients.

Q4. Attempt any **ONE** question from the following: **(08)**

$a)$

i. Find the general solution of system $\begin{cases} \frac{dx}{dt} = a_1x + b_1y \\ \frac{dy}{dt} = a_2x + b_2y \end{cases}$ where $a_1, a_2, b_1 and\ b_2$ are constants when the roots of auxiliary equation are complex.

ii. Define Wronskian $W(t)$ of the two solutions $\begin{cases} x = x_1(t) \\ y = y_1(t) \end{cases}$ and $\begin{cases} x = x_2(t) \\ y = y_2(t) \end{cases}$ of the homogeneous system $\begin{cases} \frac{dx}{dt} = a_1(t)x + b_1(t)y \\ \frac{dy}{dt} = a_2(t)x + b_2(t)y \end{cases}$ where $a_1(t), a_2(t), b_1(t), b_2(t)$ are continuous functions on $[a, b]$. Show that their Wronskian is either identically zero or nowhere zero on $[a, b]$.

Q4. Attempt any **TWO** questions from the following: **(12)**

b) i. Show that $x = e^{3t}, y = e^{3t}$ and $x = e^{2t}, y = 2e^{2t}$ are linearly independent solutions to the homogeneous system $\frac{dx}{dt} = 4x - y$, $\frac{dy}{dt} = 2x + y$. Hence find the general solution.

ii. Find the general solution to the following system $\frac{dx}{dt} = 7x + 6y$, $\frac{dy}{dt} = 2x + 6y$.

iii. Find the particular solution to the following system $\frac{dx}{dt} = 2x - 8y$, $\frac{dy}{dt} = x + 6y$ with $x(0) = 4, y(0) = 1$.

iv. Let $\begin{cases} x = x_1(t) \\ y = y_1(t) \end{cases}$ and $\begin{cases} x = x_2(t) \\ y = y_2(t) \end{cases}$ of the homogeneous system $\begin{cases} \frac{dx}{dt} = a_1(t)x + b_1(t)y \\ \frac{dy}{dt} = a_2(t)x + b_2(t)y \end{cases}$ where $a_1(\mathrm{t}), a_2(\mathrm{t}), b_1(\mathrm{t}), b_2(\mathrm{t})$ are continuous functions on $[a, b]$.Then $\begin{cases} x(t) = c_1x_1(t) + c_2x_2(t) \\ y(t) = c_1y_1(t) + c_2y_2(t) \end{cases}$, is also a solution of the above system of equation for $t \in [a, b]$

Q5. Attempt any **FOUR** questions from the following: **(20)**

a) Solve the differential equation $(4x + 3y + 1)dx + (3x + 2y + 1)dy = 0$

b) Find the orthogonal trajectories of $y = ke^x$ where k is a parameter.

c) Solve $y'' - 4y' + 4y = 0$.

d) Find another linearly independent solution for the differential equation $(1 + x^2)y'' - 2xy' + 2y = 0$, given that $y_1 = x$ is a solution.

e) Show that $x = e^{3t}, y = e^{3t}$ and $x = e^{-t}, y = -e^{-t}$ are solutions of the system $\begin{cases} \frac{dx}{dt} = x + 2y \\ \frac{dy}{dt} = 2x + y \end{cases}$.

f) Define a system of homogeneous linear differential equations of order 1. State the condition for two solutions (x_1, y_1) and (x_2, y_2) to be linearly independent. Also write the general solution.

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