(3 Hours) **[Total Marks: 100]**

Note: (i) All questions are compulsory.

(ii)Figures to the right indicate marks for respective parts.

Q.1 Choose correct alternative in each of the following (20)

$i.$ Let $a,b \in D_3$, where a and b denotes rotation and reflection then $|ab| =$

(a) 2 (b) 3

(c) 6 (d) None of the above

$ii.$ Let H and K be the subgroups of a group G. Then $H \cup K$

(a) Is always a subgroup of G

(b) Is never a subgroup of G

(c) Is a subgroup of G if and only if $H \subseteq K$ or $K \subseteq H$

(d) None of the above

$iii.$ The set $\mathbb{Z}_n$ forms a group under the binary operation

(a) '+' (b) $'-'$

(c) '•' (d) None of the above

$iv.$ In the group $(\mathbb{Z}_{18}, +)$, order of $\overline{10}$ is

(a) 10 (b) 9

(c) 6 (d) 18

$v.$ Let H is a proper subgroup of $\mathbb{Z}$ under addition and 12, 14, 18 ∈ H then

(a) $H = 756\mathbb{Z}$ (b) $H = 2\mathbb{Z}$

(c) $H = 4\mathbb{Z}$ (d) $H = \mathbb{Z}$

vi. Let a be an element of a group G and let order of a in G be infinite then how many generators does the group $< a >$ have?

(a) Only one (b) Exactly 2

(c) Infinitely many (d) None

vii. If $G = (\mathbb{Z}, +)$ and $H = \{0, \pm 3, \pm 6, \pm 9, \cdots\cdots\}$ then

(a) $11 + H = 17 + H$ (b) $11 + H = 7 + H$

(c) $7 + H = 23 + H$ (d) None of these

viii. Let G be a group of order 8 then G must have an element of order

(a) 2 (b) 4

(c) 8 (d) None of these

ix. Let $\phi: \mathbb{C}^* \longrightarrow \mathbb{C}^*$ given by $\phi(x) = x^4$ be a homomorphism then $ker\phi =$

(a) $\{1, -1\}$ (b) $\{1, -1, i, -i\}$

(c) $\{i, -i\}$ (d) None of these

x. Let G be an abelian group which has no element of order 2 and $\phi: G \longrightarrow G$ given by $\phi(x) = x^2$, then

(a) ϕ is an automorphism.

(b) ϕ is a group homomorphism which may not be one $-$one.

(c) ϕ is an automorphism if G is finite.

(d) ϕ is not a group homomorphism.

Q2. Attempt any **ONE** question from the following: (08)

a) i. Show that $U(\text{n}) = \{\bar{a} \in \mathbb{Z}_n \,|\, 1 \le a \le n-1, (a,n) = 1\}$, form a group under the binary operation '•'.

ii. Define Centre of Group G. Hence or otherwise prove that the Centre of any group is a subgroup of the group.

Q.2 Attempt any **TWO** questions from the following: (12)

$b)$ i. Let G be a group. Prove that $(aba^{-1})^n = ab^n a^{-1}, \quad \forall\, a, b \in G$ and $\forall\, n \in \mathbb{Z}$

ii. Let G be a group and $a \in G$. Show that $H = \{a^{2n} | n \in \mathbb{Z}\}$ is a subgroup of G.

iii. Let G be a group and $a \in G$ with $O(a) = n$ then show that if and only if $a^m = e$ then $n|m$.

iv. Let $\alpha = (1\ 2\ 5)(6\ 13\ 5)$ and $\beta = (1\ 3\ 4)(2\ 6\ 5)(2\ 3\ 4)$ .Write α and β as a product of disjoint cycles. Further, verify the following.

p) $o(\alpha) = o(\alpha^{-1})$

q) $o(\alpha\beta) = o(\beta\alpha)$

r) $o(\alpha\beta\alpha^{-1}) = o(\beta)$

Q3. Attempt any **ONE** question from the following: (08)

$a)$ i. Prove that $\mathbb{Z}_n$ the set of residue classes modulo n is a group under addition. Also determine all the generators for $\mathbb{Z}_n$

ii. Let G be a finite cyclic group of order n then prove that G has a unique subgroup of order d for every divisor d of n.

Q3. Attempt any **TWO** questions from the following: (12)

$b)$ i. Let $G = \langle a \rangle$ be a finite cyclic group of order 12 then what are all the generators of G. Also determine all the generators of the subgroup $H = \langle a^3 \rangle$.

ii. Determine all the subgroups of the cyclic group $\mathbb{Z}_{11}^*$.

iii. Show that H = $\left\{\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} / n \in \mathbb{Z}\right\}$ is a cyclic subgroup of $GL_2(\mathbb{R})$.

iv. Consider the set $\{\overline{4}, \overline{8}, \overline{12}, \overline{16}\}$. Show that this set is a group under multiplication modulo 20 by constructing a Cayley table. What is the identity element?

Q4. Attempt any **ONE** question from the following: (08)

$a)$ i. Let H is a subgroup of a group G then $aH = H$ if and only if $a \in H$. Further aH is subgroup of G if and only if $a \in H$.

ii. Let $f: G \longrightarrow G'$ is onto group homomorphism. Prove that

(p) If H is subgroup of G then $f(H) = \{f(h)/h \in H\}$ is subgroup of G'.

(q) If H' is subgroup of G' then $f^{-1}(H') = \{a \in G/f(a) \in H'\}$ is subgroup of G and $kerf \subseteq f^{-1}(H')$.

Q4. Attempt any **TWO** questions from the following: (12)

$b)$ i. State Lagrange's theorem for finite group. If H and K are subgroups of G such that $o(H) = 12$ and $o(K) = 35$ then show that $H \cap K = \{e\}$.

ii. Let G be a finite group then show that

(p) $o(a)|o(G), \forall a \in G$ (q) $a^{o(G)} = e, \forall a \in G$

iii. Show that $f: G \longrightarrow G$ given by $f(x) = x^{-1}$ is a automorphism if and only if G is abelian.

iv. Show that $G = \{a + b\sqrt{2} \,/\, a, b \in \mathbb{Q}\}$ and $H = \left\{\begin{pmatrix} a & 2b \\ b & a \end{pmatrix} / a, b \in \mathbb{Q}\right\}$ are isomorphic groups under addition.

Q5. Attempt any **FOUR** questions from the following: (20)

$a)$ Construct composition table of $\mathbb{Z}_5^*$ under multiplication modulo 5. Also find the order of each of its elements.

$b)$ Define Abelian group. If $(ab)^2 = a^2b^2$ for every a, b in a group G, show that G is Abelian.

c) Prove that a group of order 3 must be cyclic.

d) Let G be a group and let 'a' be an element of G.

(i) If $a^{12} = e$, what can you say about order of a.

(ii) Suppose that G is cyclic and o(G) = 24 . Further if $a^8 \neq e$ and $a^{12} \neq e$ then show that $< a >=$G.

e) Give an example of a group G and a subgroup H of G such that $aH = bH$ but $Ha \neq Hb$ for some $a, b \in G$.

f) Find the number of group homomorphism from $\mathbb{Z}_{12}$ to $\mathbb{Z}_{30}$.

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