(3 Hours) **[Total Marks: 100]**

Note: (i) **All** questions are **compulsory**.
(ii) **Figures** to the right **indicate marks** for **respective parts**.

Q.1 Choose correct alternative in each of the following : **(20)**

i. If $f: [a, b] \to IR$ be bounded function and P ,Q be partitions of $[a, b]$ then

(a) $L(P, f)) \leq U(Q, f)$ (b) $L(P, f) \geq U(Q, f)$

(c) $L(P, f)) = U(Q, f)$ (d) None of the above

ii. The norm of a partition $P = \{0 < \frac{1}{2} < 1 < \frac{4}{3} < \frac{7}{3} < 3\}$ is

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$

(c) 1 (d) None of the above

iii. If $f: [a, b] \to IR$ is R- integrable then which of the following is true

(a) f must be continuous (b) f must be differentiable

(c) f must be monotonic (d) None of the above

iv. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Then $\int_{-a}^{a} f(t)dt = 0, \forall a > 0$ if and only if

(a) $f \equiv 0$ (b) f is an odd function

(c) $f \neq 0$ for only finitely many real numbers. (d) None of the above.

v. If $f, g: [a, b] \to \mathbb{R}$ are continuous functions such that $\int_{a}^{b} f(x)dx = \int_{a}^{b} g(x)dx$ then

(a) $f \equiv g$ on $[a, b]$ (b) $f(x) = g(x)$ is a constant.

(c) $\exists\, c \in [a, b]$ such that $f(c) = g(c)$ (d) None of the above.

vi. The type 2 integral $\int_{0}^{2} \frac{1}{x-1}\, dx$

(a) Diverges (b) Converge to 0

(c) Converge to $\frac{1}{2}\ln 3$ (d) Converges to $\frac{8}{9}$

vii. Integral $\int_1^\infty \frac{1}{x^p}\, dx$ converges if

(a) p > 1 (b) p < 1

(c) p = 1 (d) None of the above

viii. Find $\int_0^{\frac{\pi}{2}} cos^{11}x\, sin^9 x\, dx$

(a) $\frac{1}{10!}$ (b) $\frac{5!\,4!}{2(10!)}$

(c) $\frac{10!}{5!\,4!}$ (d) 0

ix. $\int_0^\infty x^{3/2} e^{-x}\, dx=$

(a) $\frac{3\sqrt{\pi}}{4}$ (b) $\frac{\pi}{2}$

(c) $\frac{\sqrt{\pi}}{5}$ (d) None of these

x. Identify the definite integral that computes the volume of the solid generated by revolving the region bounded by the graph of $y = x^3$ and the line $y = x$, between $x = 0$ and $x = 1$ about the line $x = 1$.

(a) $\pi\int_0^1 \left(y^{\frac{2}{3}} - y^2\right) dy$ (b) $\pi\int_0^1 \left(y^{\frac{1}{3}} - y\right)^2 dy$

(c) $2\pi\int_0^1 \left((4-x^2)(4-x^6)\right) dx$ (d) $\pi\int_0^1 \left((4-y)^2(4-y^{\frac{1}{3}})^2\right) dx$

Q2. Attempt any **ONE** question from the following : **(08)**

a) i. Let $f: [a, b] \to IR$ be a bounded function. Prove that f is Riemann integrable on [a, b] if and only if for any $\in > 0$ there exist a partition P of [a, b] such that $U(f, P) - L(f, P) < \in$.

ii. If $f; g: [a, b] \to IR$ *are* R- integrable then prove that $f + g$ is R- integrable and $\int_a^b f + g = \int_a^b f + \int_a^b g$

Q.2 Attempt any **TWO** questions from the following : **(12)**

b) i. Let f be a bounded function on [a, b]. Let P and P' are two partitions of $[a,b]$ with $P \subseteq P'$. Show that $L\,(P',f) \geq L\,(P,f)$

ii. If f is an R-integrable function on $[a,b]$ then prove that $|\,f\,|$ is R-integrable on $[a,b]$.

iii. Using Riemann Criterion, prove that the function f : [0, 1] ⟶ ℝ defined by $f(x) = x$ is Riemann integrable.

iv. If f , g : [a, b]→ ℝ are integrable functions such that $f(x) \leq g(x)$, ∀ x ∈ [a, b] then prove that $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$.

Q3. Attempt any **ONE** question from the following : **(08)**

a) i. State and prove the Fundamental Theorem of Calculus.

ii. State and prove Comparison Test for improper integrals of type-I.

Q3. Attempt any **TWO** questions from the following : **(12)**

b) i. Let $F:\ [0,1] \rightarrow\ \mathbb{R}$be defined by $F(x) = \begin{cases} x^2 \sin\left(\frac{\pi}{x}\right) & if\ 0 < x \leq 1 \\ 0 & if\ x = 0 \end{cases}$

Show that is differentiable over [0, 1].Let f : $[0,1] \rightarrow\ \mathbb{R}$be given by $f(x) = F'(x)$. Find $\int_0^1 f(t)dt$.

ii. Evaluate $\lim_{x\to\infty} \frac{1}{x^3} \int_0^x \frac{t^2}{1+t^4}\,dt$

iii. Prove that $\int_a^b \frac{1}{(b-x)^p}\,dx$ converges if and only if $p < 1$.

iv. State Abel's and Dirichlet's Tests for the conditional convergence of type 1 improper integral and discuss convergence of $I = \int_0^\infty \sin x^2\,dx$

Q4. Attempt any **ONE** question from the following : **(08)**

a) i. Prove that $\int_0^1 x^{m-1}(1-x)^{n-1}dx$ converges if and only if m and n are both positive.

ii. With usual notations for beta and gamma functions prove that

(p) $\beta(m,n) = \beta(n,m)$

(q) $\frac{\beta(m,n+1)}{n} = \frac{\beta(m+1,n)}{m} = \frac{\beta(m,n)}{m+n}$.

Q4. Attempt any **TWO** questions from the following : **(12)**

b) i. Prove that $\beta(m,n)=\int_0^{\infty}\frac{t^{m-1}}{(1+t)^{m+n}}dy.$

ii. Show that $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}.$

iii. Find the volume of the solid whose base is the disk $x^2+y^2\leq 1$ and the cross sections by the planes perpendicular to the $y-axis$ between $y=-1$ and $y=1$ are isosceles right triangles with one leg in disk by the method of slicing.

iv. Find the volume of the solid generated by revolving the regions bounded by the lines $y=2x$, $y=x$, $x=1$ and about $x-axis$ by the Washer method.

Q5. Attempt any **FOUR** questions from the following : **(20)**

a) If $f(x) = 1+2x, x \in$ IR and P be a partition such that $0 < 0.25 < 0.5 < 0.75 < 1$, then find $U(P,f)$.

b) If f is Riemann integrable on $[a,b]$ then for any $k\in$ IR prove that kf is also Riemann integrable on $[a,b]$.

c) Show that if $F'(x)=0, \forall x\in[a,b]$ then f is a constant function.

d) Identify the type and discuss the convergence of each of the following integrals

$(I)\int_0^1\frac{dx}{x^2(1+x)^3}$ (II)$\int_1^{\infty}\frac{sin^2x}{x^2}dx$

e) Prove that $\int_0^{\frac{\pi}{2}}\sqrt{\sin x}\ dx\int_0^{\frac{\pi}{2}}\frac{1}{\sqrt{\cos x}}\ dx=\pi.$

f) Find the area of the surface generated by revolving the curves about $x=2\sqrt{4-y},$ $0\leq y\leq\frac{15}{4}$ about y −axis.

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