

(3 Hours)

[Total Marks : 100]

- N.B.** 1. All questions are compulsory.
2. Figures to the right indicate marks for respective parts

Q.1 Choose correct alternative in each of the following: (20)

- i. The set $S = \{(x, y) \in \mathbb{R}^2 / 1 \leq x^2 + y^2 \leq 2\}$ is
- (a) An open set (b) Closed set
(c) Neither open nor closed (d) None of these
- ii. Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ defined as $f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$. Then,
- (a) $\lim_{(x, y) \rightarrow (0, 0)} f(x, y)$ exist. (b) f is continuous at $(0, 0)$.
(c) $|f(x, y)| \leq \frac{1}{2}, \forall (x, y) \in \mathbb{R}^2$ (d) None of these
- iii. Let $f(x, y) = |x| + |y|$, for $(x, y) \in \mathbb{R}^2$, then
- (a) $f_x(0, 0) = 0, f_y(0, 0) = 0$ (b) $f_x(0, 0)$ and $f_y(0, 0)$ do not exist.
(c) $f_x(0, 0) = 1, f_y(0, 0) = 1$ (d) None of these
- iv. Let A: Gradient of a scalar field is a scalar.
B: Every differentiable scalar field is continuous.
Then which of the following is true?
- (a) A is true, B is false. (b) A is false, B is true.
(c) Both A & B are true. (d) Both A & B are false.
- v. Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ be such that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and are bounded. Then
- (a) f may or may not be continuous at all points (b) f is continuous at all points
(c) f is differentiable at all points (d) None of these
- vi. The total derivative T_α of a scalar field is _____.
- (a) a constant (b) a linear transformation
(c) a vector (d) a real number
- vii. The linear approximation to $e^x \cos(y + z)$ near the origin is
- (a) independent of x . (b) independent of y
(c) independent of z (d) 1

viii. If $z = f(x, y)$ is differentiable and $x = g(u, v)$, $y = h(u, v)$ are also differentiable functions then $\frac{\partial z}{\partial u}$ is

- (a) $\frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v}$ (b) $\frac{\partial z}{\partial x} \frac{\partial x}{\partial u} - \frac{\partial z}{\partial y} \frac{\partial y}{\partial v}$
 (c) $\frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v}$ (d) $\frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u}$

ix. If $f(x, y) = x^2 - 2x + y^2$ then

- (a) (1,0) is a critical point of f which is a local minima. (b) (1,0) is a critical point of f which is a local maxima.
 (c) (1,0) is a saddle point of f . (d) None of these

x. Stationary point is a point where function $f(x, y)$ have

- (a) $\frac{\partial f}{\partial x} = 0$. (b) $\frac{\partial f}{\partial y} = 0$
 (c) Both (a) and (b). (d) None of these

Q.2 a) Attempt any ONE question from the following: (08)

- i. Let $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a vector valued function and let $a \in \mathbb{R}^n$. Prove that f is continuous at a if and only if each $f_i: \mathbb{R}^n \rightarrow \mathbb{R}$ is continuous at a .
 ii. State and prove mean value theorem for derivatives of scalar fields.

b) Attempt any TWO questions from the following: (12)

- i. Using definition of limit, check if $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$ exists, where

$$f(x, y) = \begin{cases} x \sin \frac{1}{y} & \text{if } y \neq 0 \\ 0 & \text{if } y = 0 \end{cases}$$

- ii. Define limit of a function $f: S \rightarrow \mathbb{R}$ where $S (\neq \emptyset) \subseteq \mathbb{R}^n$, at point a , and show that the limit of function of several variables is uniquely determined

- iii. Define directional derivative of a scalar field $f(x, y, z) = \left(\frac{x}{y}\right)^z$, $y \neq 0$ at the point $a = (1, 1, 1)$, in the direction of $u = (2, 1, -1)$ Find the directional derivative of the following functions at the indicated point in the direction (using the result $D_u f(a) = \nabla f(a) \cdot u$)

- iv. If $\sin u = \frac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}$; $(x, y, z) \neq (0, 0, 0)$. Then prove that

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} + 3 \tan u = 0$$

- Q.3 a) Attempt any ONE question from the following: (08)
- Let $f: \mathbb{R}^n \rightarrow \mathbb{R}$ be a function and $a \in \mathbb{R}^n$. Define the total derivative $Df(a)$ in terms of a linear transformation and show that $Df(a)$ when exist, is uniquely defined.
 - State and prove chain rule for the derivative of a scalar field.
- b) Attempt any TWO questions from the following: (12)
- Find total derivative as linear transformation T for the function $f(x, y, z) = x^3 + 3y^2 + 4z^2$ at point $a = (-1, -1, 1)$
 - Find directional derivative of $f(x, y, z) = 3x - 5y + 2z$ at $(2, 2, 1)$ in the direction of outward normal to the sphere $x^2 + y^2 + z^2 = 9$.
 - Find the equation of tangent plane and normal line to the surface $z + 1 = xe^y \cos z$ at $(1, 1, 0)$
 - Show that, for each of the following functions, the second order mixed partial derivatives are equal.
 - $f(x, y) = x^4 + y^4 - xy^3$
 - $f(x, y) = \sqrt{xy}$
- Q.4 a) Attempt any ONE question from the following: (08)
- Define $Df(a)$, the total derivative at $a \in \mathbb{R}^n$ for a function $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ in terms of a linear transformation. Show that if f is differentiable at a then f is continuous at a . Is the converse true ? Explain.
 - Let $f: S \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ be a scalar field. Let $(a, b) \in S$ be a stationary point of f . Suppose $f(x, y)$ possesses continuous second order partial derivatives in some neighbourhood of (a, b) . Let $A = f_{xx}(a, b)$, $B = f_{xy}(a, b)$, $C = f_{yy}(a, b)$ and $\Delta = AC - B^2$. Then prove that
 - if $\Delta > 0, A > 0$, then f has local minimum at (a, b) .
 - if $\Delta > 0, A < 0$, then f has local maximum at (a, b) .
 - if $\Delta < 0$, then f has saddle point at (a, b) .
 if $\Delta = 0$, the test is inconclusive (show this by example).
- b) Attempt any TWO questions from the following: (12)
- Let U is open set in \mathbb{R}^n and $f: U \rightarrow \mathbb{R}^m$ is differentiable at $a \in U$. Show that $\exists M > 0, \delta > 0$ such that

$$\|x - a\| < \delta \Rightarrow \|f(x) - f(a)\| \leq M\|x - a\|.$$

