(Time: 3 Hours) **[Total Marks : 100]**

N.B. **1**. All questions are compulsory.

2. Figures to the right indicate marks for respective parts.

3. Use of Calculator is not allowed.

Q.1 Choose correct alternative in each of the following: (20)

i. A linear system of equations may have

(a) No solution (b) Unique solution

(c) infinitely many solutions (d) Any of (a), (b), (c)

ii. A diagonal matrix $D = diag(d_1, d_2, \dots, d_n)$ is invertible if and only if

(a) $d_k \neq 0$ for some k (b) $d_k = 1$ for some k

(c) $d_k \neq 0$ for all k (d) $d_k = 1$ for all k

iii. Let A = $\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$. Then A^n (where n is positive integer) is

(a) I_2 for all n (b) A for all n

(c) I_2 if n is even and A if n is odd (d) None of these

iv. Which of the following is a subspace of $\mathbb{R}^3$ over $\mathbb{R}$?

(a) $\{(x, y, z) \in \mathbb{R}^3 / x = -y, z = 2x\}$ (b) $\{(x, y, z) \in \mathbb{R}^3 / x = -y + 1, z = 2x\}$

(c) $\{(x, y, z) \in \mathbb{R}^3 / x = -y, z = x^2\}$ (d) None of the above

v. S = {(0, 0, 1), (0, 1, 0)} then the linear span of S is

(a) YZ-plane (b) $\mathbb{R}^3$

(c) XY-plane (d) $\mathbb{R}^2$

vi. The singleton set {0} in any real vector space V is

(a) linear independent (b) linear dependent

(c) a basis of the empty set φ (d) none of the above

vii. Which of the following set is a generating set of $\mathbb{R}^3$?

(a) {(1, 2, 0), (0, 1, 1),(-1, 0, 1)} (b) {(1, 2, 1), (2, 4, 2), (-1, 0, 1)}

(c) {(1, 1, 0), (0, 1, 1)} (d) {(-1, 0, 1), (0, 1, 1), (0, 0, 0)}

viii. The dimension of the vector space of all real matrices of order 2×3 is

(a) 5 (b) 6

(c) 13 (d) None of these

ix. Which of the following is a linear transformation?

(a) $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $T(x,y) = (x+2, y+3)$

(b) $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $T(x,y) = (x+y, 3y)$

(c) $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $T(x,y) = (x, y^2)$

(d) None of the above

x. If for a linear transformation $T: \mathbb{R}^7 \rightarrow \mathbb{R}^9$, Rank T = 4 then the nullity of T is

(a) 3

(b) 5

(c) 0

(d) None of these

Q.2 a) Attempt any ONE question from the following: (08)

i. Prove that the homogeneous system $a_1x + b_1y = 0, a_2x + b_2y = 0$ has a non-trivial solution if and only if $a_1b_2 - a_2b_1 = 0$.

ii. Define trace of a square matrix A (Tr(A)) of order n. Show that if A and B are square matrices of order *n* then Tr(A) = Tr(A^T) and Tr(A + B) = Tr(A) + Tr(B).

b) Attempt any TWO questions from the following: (12)

i. For any square matrix A of order n, prove that

(p) $A + A^T$ is symmetric (q) $A - A^T$ is skew symmetric.

ii. Show that the following system of equations has non-trivial solution for all non-zero real numbers *a*, *b* and *c*:

$x + ay + (b + c)z = 0, x + by + (c + a)z = 0, x + cy + (a + b)z = 0$

iii. If A is a square matrix then prove that

(p) If $A^2 = O$ then $I - A$ is invertible. (q) If $A^3 = O$ then $A + I$ is invertible.

iv. Show that the following system of linear equations is consistent using Gauss elimination method: $x + y + z = 3$, $x - 2y - 3z = 1$, $2x + 3y + z = 9$

Q.3 a) Attempt any ONE question from the following: (08)

i. Let V be vector space over $\mathbb{R}$ and $V = \{v_1, v_2, \dots \dots v_n\}$ be linear independent set V. For $u \in V$, prove that $S \cup \{u\}$ is linear dependent if and only if $u \in L(S)$ where $L(S)$ is linear span of S.

ii. Verify all properties of scalar multiplication of the vector space $M_{2\times 3}(\mathbb{R}) = \left\{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \mid a_{ij} \in \mathbb{R}\right\}$, where the operation of matrix addition (+) and scalar multiplication (.) are defined as usual.

b) Attempt any TWO questions from the following: (12)

i. Show that in any vector space the subset of a linear independent set is linear independent.

ii. Prove that the set $W = \{A \in M_2(\mathbb{R}) | AB = BA\}$is a subspace of $M_2(\mathbb{R})$, where $B$is a fixed matrix in $M_2(\mathbb{R})$.

iii. Express $\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 4 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

iv. Let u, v and w be linear independent vectors in real vector space V, show that $u + v - 2w, u + v - w$ and $u + v$ are linear dependent.

Q.4 a) Attempt any ONE question from the following: (08)

i. Prove that in a vector space of dimension *n* any set containing *n* + 1 vectors is linear dependent.

ii. Define kernel and image of a linear transformation. If $T: V \to V'$ is a linear transformation, prove that Im(T) is a subspace of V'.

b) Attempt any TWO questions from the following: (12)

i. Prove that the vectors (1,1,0), (1,2,3) and (2,-1,5) form a basis for $\mathbb{R}^3$.

ii. Extend $S = \{(1,0,-1)\}$ to a basis of $\mathbb{R}^3$.

iii. Let $T_1, T_2 : V \to W$be two linear transformations, Prove that $T_1 + T_2 : V \to W$ is also a linear transformation.

iv. State Rank-Nullity theorem and verify it for the linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$defined as $T(x, y, z) = (x, y, 0)$.

Q.5 Attempt any FOUR questions from the following: (20)

a) Find parametric representation for the plane passing through point $P = (1,1,1)$, $Q = (2,1,2), R = (2,2,-3)$

b) Reduce the matrix $\begin{bmatrix} 2 & 1 & 1 \\ 3 & 1 & -1 \\ 5 & 2 & 0 \end{bmatrix}$ to it echelon form.

c) Let S = {(1,0), (0,1)} and T = {(1,1), (1,-1)}. Prove that L(S) = L(T).

d) Show that any two bases of a finite dimensional vector space have the same number of vectors.

e) Find a linear map $T: \mathbb{R}^3 \to \mathbb{R}^2$ such that $T(1,0,1) = (2,3)$, $T(0,1,0) = (0,1)$, $T(0,1,1) = (1,0)$.

f) Find the matrix of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x, y) = (x - 2y, 3x + 4y, x - 6y)$ w.r.t bases {(1,0), (0,1)} of $\mathbb{R}^2$ and {(1,0,0), (0,1,0), (0,0,1)} of $\mathbb{R}^3$.

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