

- N.B.:** 1. All questions are compulsory.  
2. Figures to the right indicate full marks.

Q.1 Choose correct alternative in each of the following: (20)

- i. The series  $\sum_{n=1}^{\infty} \frac{5n+2}{4^n}$  of real numbers
- (a) is not convergent (b) Converges to  $\frac{5}{4}$   
(c) Converges to 0 (d) None of the above
- ii. If  $\sum x_n$  and  $\sum y_n$  are two series of real numbers such that  $\sum(x_n - y_n)$  and  $\sum y_n$  are both convergent then  $\sum x_n$
- (a) Is convergent (b) Is divergent  
(c) Conditionally convergent (d) None of the above
- iii. The series  $\sum_{n=1}^{\infty} (6r)^n, r \in \mathbb{R}$  is
- (a) Convergent for any  $r \in \mathbb{R}$  (b) Divergent for any  $r \in \mathbb{R}$   
(c) Convergent if  $|r| < \frac{1}{6}$  (d) None of the above
- iv. The function  $y = \frac{1}{x+1}$  has  $\frac{d^4 y}{dx^4}$  equal to
- (a)  $\frac{1}{(x+1)^5}$  (b)  $\frac{64}{x+1}$   
(c)  $\log(x+1)$  (d) None of the above
- v. The function  $f(x) = |x+5|, x \in \mathbb{R}$
- (a) Is differentiable at  $x = -5$  (b) Is not differentiable at  $x = 5$   
(c) Is differentiable at every  $x \in \mathbb{R}$  (d) None of the above

- vi. The value of  $\lim_{x \rightarrow 0} \frac{(8^x - 3^x)}{x}$  is
- (a)  $\log_e \left( \frac{8}{3} \right)$  (b)  $\log_{10} 5$
- (c)  $\frac{8}{3}$  (d) None of the above
- vii. The function  $f(x) = \frac{1}{x-5}$ ,  $\forall x \in (5,6)$  is
- (a) Continuous and bounded (b) Continuous but not bounded
- (c) Discontinuous (d) None of the above
- viii. The function  $f(x) = \log x$ ,  $x > 0$  is
- (a) Concave upwards (b) Concave downwards
- (c) Decreasing function (d) None of the above
- ix. The function  $f(x) = 3x^2 - 7x + 2$  is
- (a) Increasing for all  $x \in \mathbb{R}$  (b) Decreasing for all  $x \in \mathbb{R}$
- (c) Increasing for all  $x > \frac{7}{6}$  (d) None of the above
- x. If  $f, g: \mathbb{R} \rightarrow \mathbb{R}$  are such that  $f \cdot g$  is differentiable then
- (a) Both  $f, g$  are differentiable (b) At least one of  $f, g$  is differentiable
- (c)  $f + g$  is differentiable (d) None of the above

Q.2 a) Attempt any ONE question from the following: (08)

i. Let  $(s_n)$  is the sequence of partial sums for series  $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ .

Prove that  $(s_{2n})$  is decreasing sequence and  $(s_{2n+1})$  is increasing sequence. Further prove that

$$\lim_{n \rightarrow \infty} s_{2n} = \lim_{n \rightarrow \infty} s_{2n+1} = \lim_{n \rightarrow \infty} s_n.$$

- ii. If  $\lim_{n \rightarrow \infty} \left( \frac{x_n}{y_n} \right) = 0$  and  $\sum_{n=1}^{\infty} y_n$  is absolutely convergent series then prove that series  $\sum_{n=1}^{\infty} x_n$  is also convergent.

b) Attempt any TWO questions from the following: (12)

- i. Prove that  $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k+3}$  is conditionally convergent series.  
 ii. State the ratio test and use it to test the convergence of  $\sum_{k=1}^{\infty} \frac{2^n}{n!}$ .  
 iii. Prove that for each non-negative integer  $n$ ,

$$\frac{1}{(2^n + 1)^2} + \frac{1}{(2^n + 2)^2} + \dots + \frac{1}{(2^n + 2^n)^2} \leq \frac{1}{2^n}.$$

And deduce that the sequence of partial sums for series  $\sum_{k=1}^{\infty} \frac{1}{k^2}$  is bounded.

- iv. Let  $A, B \in \mathbb{R}$ ,  $\sum_{n=1}^{\infty} a_n = A$  and  $\sum_{n=1}^{\infty} b_n = B$ . Then prove that  $\sum_{n=1}^{\infty} (Ba_n - Ab_n) = 0$ .

Q.3 a) Attempt any ONE question from the following: (08)

- i. Let  $f$  be a real valued continuous function on  $[a, b]$  such that  $f(a) \neq f(b)$ . Then for each  $k$ ,  $f(a) < k < f(b)$ , prove that there exists  $c \in (a, b)$  such that  $f(c) = k$ .  
 ii. State and prove Chain Rule for the derivatives of a composite function.

b) Attempt any TWO questions from the following: (12)

- i. Find  $\frac{dy}{dx}$  if  $x^3 + y^3 = 3xy$ .  
 ii. For  $y = e^x \sin x$ , prove that  $y_2 - 2y_1 + 2y = 0$ . Hence prove that  $y_{n+2} - 2y_{n+1} + 2y_n = 0$ .  
 iii. Prove that  $x^3 - 15x + 1 = 0$  has at least one root in  $[-4, 4]$ .  
 iv. Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be given by  $f(x) = \cos x$ . Show that  $f$  is differentiable on  $\mathbb{R}$ .

Q.4 a) Attempt any ONE question from the following: (08)

- i. State and prove Rolle's theorem.
- ii. If  $f$  is a differentiable function defined on an open interval  $(a, b)$  and  $f'(x) < 0 \forall x \in (a, b)$ , then prove that  $f$  is decreasing on  $(a, b)$ .

b) Attempt any TWO questions from the following: (12)

- i. Find the local maximum and minimum of the function  $f(x) = x + \frac{1}{x}$  if they exist.
- ii. Use Rolle's theorem to show that the equation  $x^3 + x - 1 = 0$  has exactly one real root.
- iii. For what values of  $x$  is the curve  $y = x^4 - 6x^3 + 12x^2 + 5x + 7$  concave upwards and when is it concave downwards? Also find a point of inflection.
- iv. Verify Cauchy's Mean Value Theorem for  $f(x) = x^2$  and  $g(x) = x^3, x \in [1, 2]$ .

Q.5 Attempt any FOUR questions from the following: (20)

- a) Test for convergence of the series  $\sum \frac{n^2}{2^n}$  stating the result used.
- b) Prove that  $\sum x^n$  is convergent if and only if  $|x| < 1$ .
- c) Find  $n^{\text{th}}$  derivative of  $y = x^2 \cos x$ .
- d) If  $f : \mathbb{R} \rightarrow \mathbb{R}$  is an even function and differentiable on  $\mathbb{R}$  then prove that  $f'$  is an odd function.
- e) Expand  $3x^3 - 2x^2 + 4x + 1$  in powers of  $(x - 1)$  using Taylor's theorem.
- f) Evaluate  $\lim_{x \rightarrow 0^+} \frac{\log \tan x}{\log x}$ .

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