Duration 2 $\frac{1}{2}$Hrs | REVISED COURSE | Marks: 75

N.B. : (1) All questions are compulsory.
(2) Figures to the right indicate marks.

1. (a) Attempt any One from the following: (8)
   (i) Define an open ball $B(x,r)$ in a metric space (X,d) and show that every open ball is an open set. Also, give an example to show that the converse need not be true.
   (ii) Show that for a subset F of a metric space (X,d), the following statements are equivalent:
   (I) F is closed
   (II) F contains all its limit points.

   (b) Attempt any Two from the following: (12)
   (i) Show that in a discrete metric space (X,d), every subset is both open and closed.
   (ii) Define a metric space (X,d) and give an example of a metric space. Let (X,d) be a metric space, prove that $|d(x,y)-d(x,z)| \leq d(y,z)\ \forall x,y,z \in X$.
   (iii) Let (X,d) be a metric space and $A \subseteq X$. Show that A is open if and only if $A = A^{\circ}$ (Interior of A). Hence decide whether the set of rationals is an open subset of $\mathbb{R}$ under usual metric of $\mathbb{R}$.

2. (a) Attempt any One from the following: (8)
   (i) State and prove Cantor's Intersection Theorem for a metric space (X,d).
   (ii) Let (X,d) be a metric space and Y be a non-empty subset of X. Prove that a subset G of Y is open in the subspace (Y,d) if and only if $G = V \cap Y$ where V is an open set in (X,d) .

   (b) Attempt any Two from the following: (12)
   (i) Let (X,d) be a metric space and (x_n) be a Cauchy sequence in X. If (x_n) has a convergent subsequence then prove that sequence (x_n) itself is convergent.
   (ii) Show that $A = \{x \in \mathbb{Q} : -\sqrt{2} < x < \sqrt{2}\}$ is both open and closed in the subspace $\mathbb{Q}$ of $\mathbb{R}$ with usual distance.
   (iii) Let $x, y \in \mathbb{R}$ be such that $x < y$. Show that there exists a number $r \in \mathbb{Q}$ such that $x < r < y$.

3. (a) Attempt any One from the following: (8)
   (i) Show that a compact subset of a metric space is closed and bounded. Give an example to show that a closed and bounded subset of a metric space need not be compact.
   (ii) Consider a metric space $(\mathbb{R},d)$ where d is usual metric, $\emptyset \neq A \subset \mathbb{R}$. Prove that if A is closed and bounded then A is sequentially compact.

   (b) Attempt any Two from the following: (12)
   (i) Prove that a subset K in a discrete metric space (X,d) is compact if and only if K is a finite set.

(ii) Suppose (X, d) is a metric space and $\mathcal{C}$ is a non-empty, finite collection of compact subsets of X then show that $\bigcup_{K \in \mathcal{C}} K$ is a compact subset of X.

(iii) Prove or disprove :

(I) A closed ball $B[x, r]$ in a metric space is compact.

(II) If A, B be compact subsets of $(\mathbb{R}, d)$ (d being usual), then the set $A \setminus B$ is also compact.

4. Attempt any Three from the following: (15)

(a) Let d_1, d_2 be metrics on X. Define $d : X \times X \longrightarrow \mathbb{R}$ as $d(x, y) = \max \{d_1(x, y), d_2(x, y)\}$. Show that d is a metric on X.

(b) Let (X, d) be a metric space and A be any non empty subset of X. Define diameter of set A. Further, find diameter of the following sets in $\mathbb{R}$ with usual metric:

(I) $A = [2, 4)$ (II) $B = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$

(c) Show that the equation $\cos x = x$ has atleast one solution in $\mathbb{R}$.

(d) Prove that in a discrete metric space every Cauchy sequence is eventually constant. Hence deduce that a discrete metric space is complete.

(e) If A, B are compact subsets of $\mathbb{R}$ with respect to usual distance, show that $A \times B$ is a compact subset of $\mathbb{R}^2$ with Euclidean metric.

(f) Determine which of the following subsets of $(\mathbb{R}^2, d)$, where d is Euclidean distance is compact. Justify your answer.

(i) $C = \{(x, y) \in \mathbb{R}^2 : |x| + |y| \leq 1\}$

(ii) $D = \{(x, y) \in \mathbb{R}^2 : |x| \leq 1\}$

****