(c) Let $T: X \to X$ be a contraction of a complete metric space (X, d). Show that T has a unique fixed point. (6.5)

5. (a) Show that the subset $A \subseteq \mathbb{R}^2$, where (6.5)

$A = \{(x, y) \in \mathbb{R}^2 : x^2 - y^2 \geq 9\}$ is disconnected.

(b) Let $I = [-1, 1]$ and let $f: I \to I$ be continuous, then show that there exists a point $c \in I$ such that $f(c) = c$. Discuss the result if $I = [-1,1)$. (4+2.5)

(c) Let (X, d_X) be a connected metric space and f be a continuous mapping from (X, d_X) onto (Y, d_Y). Prove that (Y, d_Y) is also connected. Does there exist an onto continuous map $g: [0,1] \to [2, 3] \cup [4, 5]$? Justify your answer. (6.5)

6. (a) Let f be a continuous function from a compact metric space (X, d_X) to a metric space (Y, d_Y), then prove that f is uniformly continuous on X. (6.5)

(b) Let (X, d) be a metric space and Y be a compact subset of (X, d). Then prove that Y is closed and bounded. Give an example of a closed and bounded subset of a metric space which fails to be compact. (4+2.5)

(c) State finite intersection property. Show by using the finite intersection property that $(\mathbb{R}, d)$ with usual metric is not compact. (2+4.5)

[This question paper contains 4 printed pages.]

27.12.2023(M)

Your Roll No..............

Sr. No. of Question Paper : 4332 **G**

Unique Paper Code : 32351501

Name of the Paper : BMATH511 – Metric Spaces

Name of the Course : **B.Sc. (Hons) Mathematics (LOCF)**

Semester : V

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt any **two** parts from each question.

1. (a) Let (X, d) be a metric space. Show that (X, d^*) is a metric space where

$d^*(x, y) = \min\{1, d(x,y)\}, \forall x, y \in X.$ (6)

(b) (i) Let (X, d) be a metric space. Let $\langle x_n \rangle$ and $\langle y_n \rangle$ be sequences in X such that $\langle x_n \rangle$ converges to x and $\langle y_n \rangle$ converges to y. Prove that $d(x_n, y_n)$ converges to $d(x, y)$. (2)

(ii) Prove that if a Cauchy sequence of points in a metric space (X, d) contains a convergent subsequence, then the sequence converges to the same limit as the subsequence. (4)

(c) (i) Let X = $\mathbb{N}$, the set of natural numbers. Define $d(m, n) = \left|\frac{1}{m} - \frac{1}{n}\right|$; m, n $\in$ X. Show that (X, d) is an incomplete metric space. (4)

(ii) Is the metric space (X, d) of the set X of rational numbers with usual metric d a complete metric space? Justify. (2)

2. (a) (i) Define an open set in a metric space (X, d). Show that every open ball in (X, d) is an open set. Is the converse true? Justify. (4)

(ii) Let S(x, r) be an open ball in a metric space (X, d). Let A be a subset of X such that diameter of A, d(A) < r and $S(x, r) \cap A \neq \emptyset$. Show that $A \subseteq S(x, 2r)$. (2)

(b) Let (X, d) be a metric space and A_1 and A_2 be subsets of X. Prove that $\overline{(A_1 \cup A_2)} = \overline{A_1} \cup \overline{A_2}$. Is the closure of the union of an arbitrary family of the subsets of X equal to the union of the closures of the members of the family? Justify. (6)

(c) Prove that a subspace of a complete metric space is complete if and only if it is closed. (6)

3. (a) Let (X, d_X) and (Y, d_Y) be two metric spaces. Show that a mapping f: X $\to$ Y is continuous if and only if for every subset F of Y, $(f^{-1}(F))^{\circ} \supseteq f^{-1}(F^{\circ})$. (6)

(b) (i) Let (X, d) be a metric space and A be a non-empty subset of X. Let f(x) = d(x, A) = inf {d(x, a), a $\in$ A}, x $\in$ X. Show that f is uniformly continuous over X. (4)

(ii) Is a continuous function over a metric space always uniformly continuous? Justify. (2)

(c) Let (X, d) be a metric space and f: X $\to \mathbb{R}^n$ be a function defined by $f(x) = (f_1(x), f_2(x) \ldots f_n(x))$, where f_k: X $\to \mathbb{R}$, $1 \leq k \leq n$ is a function. Show that f is continuous on X if and only if for each k, f_k is continuous on X. (6)

4. (a) Define homeomorphism between two metric spaces. Show that the image of a complete metric space under homeomorphism need not be complete. (6.5)

(b) Let d_1 and d_2 be two metrics on a non-empty set X. Show that d_1 and d_2 are equivalent if and only if the identity mapping I: $(X, d_1) \to (X, d_2)$ is a homeomorphism. (6.5)