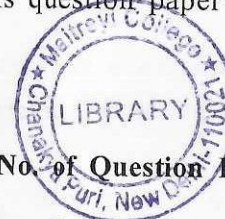


- (c) Let $T: X \rightarrow X$ be a contraction of a complete metric space (X, d) . Show that T has a unique fixed point. (6.5)
5. (a) Show that the subset $A \subseteq \mathbb{R}^2$, where (6.5)
 $A = \{(x, y) \in \mathbb{R}^2 : x^2 - y^2 \geq 9\}$ is disconnected.
- (b) Let $I = [-1, 1]$ and let $f: I \rightarrow I$ be continuous, then show that there exists a point $c \in I$ such that $f(c) = c$. Discuss the result if $I = [-1, 1)$. (4+2.5)
- (c) Let (X, d_X) be a connected metric space and f be a continuous mapping from (X, d_X) onto (Y, d_Y) . Prove that (Y, d_Y) is also connected. Does there exist an onto continuous map $g: [0, 1] \rightarrow [2, 3] \cup [4, 5]$? Justify your answer. (6.5)
6. (a) Let f be a continuous function from a compact metric space (X, d_X) to a metric space (Y, d_Y) , then prove that f is uniformly continuous on X . (6.5)
- (b) Let (X, d) be a metric space and Y be a compact subset of (X, d) . Then prove that Y is closed and bounded. Give an example of a closed and bounded subset of a metric space which fails to be compact. (4+2.5)
- (c) State finite intersection property. Show by using the finite intersection property that (\mathbb{R}, d) with usual metric is not compact. (2+4.5)

(3200)

[This question paper contains 4 printed pages.]



27.12.2023(M)
 Your Roll No.....

Sr. No. of Question Paper : 4332

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Unique Paper Code : 32351501

Name of the Paper : BMATH511 – Metric Spaces

Name of the Course : B.Sc. (Hons) Mathematics (LOCF)

Semester : V

Duration : 3 Hours

Maximum Marks : 75

Instructions for Candidates

- Write your Roll No. on the top immediately on receipt of this question paper.
 - Attempt any **two** parts from each question.
- (a) Let (X, d) be a metric space. Show that (X, d^*) is a metric space where

$$d^*(x, y) = \min\{1, d(x, y)\}, \forall x, y \in X. \quad (6)$$
 - (b) (i) Let (X, d) be a metric space. Let $\langle x_n \rangle$ and $\langle y_n \rangle$ be sequences in X such that $\langle x_n \rangle$ converges to x and $\langle y_n \rangle$ converges to y . Prove that $d(x_n, y_n)$ converges to $d(x, y)$. (2)

P.T.O.

- (ii) Prove that if a Cauchy sequence of points in a metric space (X, d) contains a convergent subsequence, then the sequence converges to the same limit as the subsequence. (4)
- (c) (i) Let $X = \mathbb{N}$, the set of natural numbers. Define $d(m, n) = \left| \frac{1}{m} - \frac{1}{n} \right|$; $m, n \in X$. Show that (X, d) is an incomplete metric space. (4)
- (ii) Is the metric space (X, d) of the set X of rational numbers with usual metric d a complete metric space? Justify. (2)
2. (a) (i) Define an open set in a metric space (X, d) . Show that every open ball in (X, d) is an open set. Is the converse true? Justify. (4)
- (ii) Let $S(x, r)$ be an open ball in a metric space (X, d) . Let A be a subset of X such that diameter of A , $d(A) < r$ and $S(x, r) \cap A \neq \emptyset$. Show that $A \subseteq S(x, 2r)$. (2)
- (b) Let (X, d) be a metric space and A_1 and A_2 be subsets of X . Prove that $\overline{(A_1 \cup A_2)} = \overline{A_1} \cup \overline{A_2}$. Is the closure of the union of an arbitrary family of the subsets of X equal to the union of the closures of the members of the family? Justify. (6)

- (c) Prove that a subspace of a complete metric space is complete if and only if it is closed. (6)
3. (a) Let (X, d_X) and (Y, d_Y) be two metric spaces. Show that a mapping $f: X \rightarrow Y$ is continuous if and only if for every subset F of Y , $(f^{-1}(F))^{\circ} \supseteq f^{-1}(F^{\circ})$. (6)
- (b) (i) Let (X, d) be a metric space and A be a non-empty subset of X . Let $f(x) = d(x, A) = \inf \{d(x, a), a \in A\}$, $x \in X$. Show that f is uniformly continuous over X . (4)
- (ii) Is a continuous function over a metric space always uniformly continuous? Justify. (2)
- (c) Let (X, d) be a metric space and $f: X \rightarrow \mathbb{R}^n$ be a function defined by $f(x) = (f_1(x), f_2(x) \dots f_n(x))$, where $f_k: X \rightarrow \mathbb{R}$, $1 \leq k \leq n$ is a function. Show that f is continuous on X if and only if for each k , f_k is continuous on X . (6)
4. (a) Define homeomorphism between two metric spaces. Show that the image of a complete metric space under homeomorphism need not be complete. (6.5)
- (b) Let d_1 and d_2 be two metrics on a non-empty set X . Show that d_1 and d_2 are equivalent if and only if the identity mapping $I: (X, d_1) \rightarrow (X, d_2)$ is a homeomorphism. (6.5)