(c) Let G = {1,8,12,14,18,21,27,31,34,38,44,47,51,53, 57,64} under the operation multiplication modulo 65. Determine the isomorphism class of the group G.

[This question paper contains 8 printed pages.]

27.12.2023(M)

Your Roll No..............

Sr. No. of Question Paper :	**1534**	**G**
Unique Paper Code	: 2352012301	
Name of the Paper	: Group Theory	
Name of the Course	: **B.Sc. (H) Mathematics – DSC**	
Semester	: III	

Duration : 3 Hours Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt all questions by selecting **two** parts from each question.
3. Part of the questions to be attempted together.
4. **All** questions carry equal marks.
5. Use of Calculator is not allowed.

P.T.O.

1. (a) Derive a formula for finding the order of a permutation of a finite set written in disjoint cycle form. Let $\beta = (1,3,5,7,9)\ (2,4,6)\ (8,10)$. What is the smallest positive integer for which $\beta^m = \beta^{-5}$?

(b) Let $\alpha, \beta \in S_n$. Prove that $\alpha\beta$ is even if and only if either both α and β are even or both α and β are odd.

(c) Suppose H is a subgroup of S_n of odd order. Prove that H is a subgroup of A_n.

2. (a) Let H and K be normal subgroups of a group G such that $H \cap K = \{e\}$, then prove that the elements of H and K commute. Give an example of a non-Abelian group whose all subgroups are normal.

5. (a) Find all subgroups of order 3 in $\mathbb{Z}_9 \oplus \mathbb{Z}_3$.

(b) Let G and H be finite cyclic groups. Then $G \oplus H$ is cyclic if and only if $|G|$ and $|H|$ are relatively prime.

(c) Using the concept of external direct product, determine the last two digits of the number 23^{123}.

6. (a) Determine the number of elements of order 15 and the cyclic subgroups of order 15 in $\mathbb{Z}_{30} \oplus \mathbb{Z}_{20}$.

(b) Define the internal direct product of n normal subgroups of a group. If a group G is the internal direct product of a finite number of normal subgroups

$H_1, H_2, \ldots, H_n$, then show that G is isomorphic to the external direct product of $H_1, H_2, \ldots, H_n$.

(ii) For $\mathbb{C}^*$, the multiplicative group of non-zero complex numbers, prove that the mapping ϕ defined as

$$\phi(z) = z^6 \text{ for all } z \in \mathbb{C}^*$$

is a homomorphism. Also, find the kernel of ϕ.

(c) (i) Prove that every normal subgroup of a group G is the kernel of a homomorphism of G.

(ii) Suppose that ϕ is a homomorphism from U(40) to U(40) and its kernel is given by $\ker \phi = \{1, 9, 17, 33\}$. If $\phi(11) = 11$, find all elements of U(40) that are mapped to 11.

(b) Let G be group and suppose that N is a normal subgroup of G and H is any subgroup of G. Prove that $N \cap H$ is a normal subgroup of G. Justify this statement with an example too.

(c) Suppose H and K are subgroups of a group G. If aH is a subset of bH, then prove that H is a subset of K. Is the converse true except when a and b are identity? Justify your answer with an example.

3. (a) State and prove first isomorphism theorem. Show that $\mathbb{Z}/n\mathbb{Z}$ is isomorphic to $\mathbb{Z}_n$, for all $n \in \mathbb{N}$.

(b) Let ϕ be an isomorphism from a group G onto a group G', prove that

(i) G is cyclic if and only if G′ is cyclic.

(ii) for all elements $g \in G$, the order of g is equal to order of $\phi(g)$.

(c) (i) Let G be a group, prove that the mapping ϕ defined as

$$\phi(g) = g^{-1} \text{ for all } g \in G$$

is an automorphism if and only if G is an Abelian group.

(ii) Suppose ϕ is a homomorphism from a group G onto a group G′, prove that

$$\phi(Z(G)) \subseteq Z(G').$$

Here, Z(G) denotes the center of G.

4. (a) (i) Suppose that G is a finite Abelian group and G has no element of order 2. Show that the mapping ϕ defined as

$$\phi(g) = g^2 \text{ for all } g \in G$$

is an automorphism. Is this mapping ϕ an automorphism when G is an infinite group and has no element of order 2? Justify your answer.

(ii) Prove that a homomorphism ϕ from a group G onto a group G′ is one-one if and only if $\ker \phi = \{e\}$, where e is the identity element of G.

(b) (i) Suppose ϕ is a homomorphism from a group G to a group G′ and $g \in G$ such that $\phi(g) = g'$, prove that

$$\phi^{-1}(g') = g \ker \phi.$$