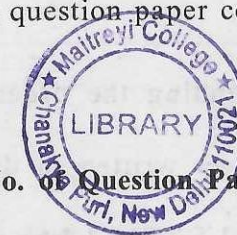


1534

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- (c) Let $G = \{1, 8, 12, 14, 18, 21, 27, 31, 34, 38, 44, 47, 51, 53, 57, 64\}$ under the operation multiplication modulo 65. Determine the isomorphism class of the group G .

[This question paper contains 8 printed pages.]



27.12.2023(M)

Your Roll No.....

Sr. No. of Question Paper : 1534

G

Unique Paper Code : 2352012301

Name of the Paper : Group Theory

Name of the Course : B.Sc. (H) Mathematics – DSC

Semester : III

Duration : 3 Hours

Maximum Marks : 90

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt all questions by selecting **two** parts from each question.
3. Part of the questions to be attempted together.
4. All questions carry equal marks.
5. Use of Calculator is not allowed.

(3000)

P.T.O.

1. (a) Derive a formula for finding the order of a permutation of a finite set written in disjoint cycle form. Let $\beta = (1,3,5,7,9) (2,4,6) (8,10)$.

What is the smallest positive integer for which

$$\beta^m = \beta^{-5}?$$

- (b) Let $\alpha, \beta \in S_n$. Prove that $\alpha\beta$ is even if and only if either both α and β are even or both α and β are odd.

- (c) Suppose H is a subgroup of S_n of odd order. Prove that H is a subgroup of A_n .

2. (a) Let H and K be normal subgroups of a group G such that $H \cap K = \{e\}$, then prove that the elements of H and K commute. Give an example of a non-Abelian group whose all subgroups are normal.

5. (a) Find all subgroups of order 3 in $\mathbb{Z}_9 \oplus \mathbb{Z}_3$.

- (b) Let G and H be finite cyclic groups. Then $G \oplus H$ is cyclic if and only if $|G|$ and $|H|$ are relatively prime.

- (c) Using the concept of external direct product, determine the last two digits of the number 23^{123} .

6. (a) Determine the number of elements of order 15 and the cyclic subgroups of order 15 in $\mathbb{Z}_{30} \oplus \mathbb{Z}_{20}$.

- (b) Define the internal direct product of n normal subgroups of a group. If a group G is the internal direct product of a finite number of normal subgroups

H_1, H_2, \dots, H_n , then show that G is isomorphic to the external direct product of H_1, H_2, \dots, H_n .

- (ii) For \mathbb{C}^* , the multiplicative group of non-zero complex numbers, prove that the mapping ϕ defined as

$$\phi(z) = z^6 \text{ for all } z \in \mathbb{C}^*$$

is a homomorphism. Also, find the kernel of ϕ .

- (c) (i) Prove that every normal subgroup of a group G is the kernel of a homomorphism of G .

- (ii) Suppose that ϕ is a homomorphism from $U(40)$ to $U(40)$ and its kernel is given by $\ker \phi = \{1, 9, 17, 33\}$. If $\phi(11) = 11$, find all elements of $U(40)$ that are mapped to 11.

- (b) Let G be group and suppose that N is a normal subgroup of G and H is any subgroup of G . Prove that $N \cap H$ is a normal subgroup of G . Justify this statement with an example too.

- (c) Suppose H and K are subgroups of a group G . If aH is a subset of bH , then prove that H is a subset of K . Is the converse true except when a and b are identity? Justify your answer with an example.

3. (a) State and prove first isomorphism theorem. Show that $\mathbb{Z}/n\mathbb{Z}$ is isomorphic to \mathbb{Z}_n , for all $n \in \mathbb{N}$.

- (b) Let ϕ be an isomorphism from a group G onto a group G' , prove that

(i) G is cyclic if and only if G' is cyclic.

(ii) for all elements $g \in G$, the order of g is equal to order of $\phi(g)$.

(c) (i) Let G be a group, prove that the mapping ϕ defined as

$$\phi(g) = g^{-1} \text{ for all } g \in G$$

is an automorphism if and only if G is an Abelian group.

(ii) Suppose ϕ is a homomorphism from a group G onto a group G' , prove that

$$\phi(Z(G)) \subseteq Z(G').$$

Here, $Z(G)$ denotes the center of G .

4. (a) (i) Suppose that G is a finite Abelian group and G has no element of order 2. Show that the mapping ϕ defined as

$$\phi(g) = g^2 \text{ for all } g \in G$$

is an automorphism. Is this mapping ϕ an automorphism when G is an infinite group and has no element of order 2? Justify your answer.

(ii) Prove that a homomorphism ϕ from a group G onto a group G' is one-one if and only if $\ker \phi = \{e\}$, where e is the identity element of G .

(b) (i) Suppose ϕ is a homomorphism from a group G to a group G' and $g \in G$ such that $\phi(g) = g'$, prove that

$$\phi^{-1}(g') = g \ker \phi.$$