(c) Suppose that ϕ is a homorphism from U/(30) to U(30) and that Kerϕ = {1,11}. If $\phi(7) = 7$, find all the elements of U(30) that are mapped to 7. State and prove the result used.

(2×6.5=13)

[This question paper contains 8 printed pages.]

02.01.2024(M)

Your Roll No..............

Sr. No. of Question Paper : 4404 G

Unique Paper Code : 32351302

Name of the Paper : Group Theory – I

Name of the Course : **B.Sc. (Hons) Mathematics**

Semester : III

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. **All** questions are compulsory.
3. Attempt any **two** parts from each question from **Q2** to **Q6**.
4. In the question paper, given notations have their usual meaning unless until stated otherwise.

P.T.O.

1. Give **short** answers to the following questions. Attempt any **six**.

(i) Find an element X in D_4 such that $R_{90}VXH = D'$.

Where R_{90} = Rotation of 90°, V = Flip about a vertical axis, H = Flip about a horizontal axis, D' = Flip about the other diagonal.

(ii) Is G = {1,2,3,4,5} a group under multiplication modulo 6? In general when is G = {1,2,...,n-1}; $n \geq 2$, a group under multiplication modulo n? Answer both in a few lines.

(iii) Can a non-Abelian group have a non-trivial Abelian subgroup? Give short answer in few lines.

(b) Determine the possible homomorphisms from Z_{20} to Z_{10}. Also, find which of the homomorphisms are onto.

(c) Prove or disprove the following by justifying them :

(i) $U(8) \approx Q_8$, the group of Quaternions.

(ii) $U(20) \approx D_4$

(iii) $(Q, +) \approx (Z, +)$ (2×6=12)

6. (a) If ϕ is an isomorphism from a group G onto a group $\overline{G}$, then prove that

$|\phi(g)| = |g|$ for all $g \in G$.

(b) Let $\mathbb{C}$ be the set of complex numbers and

$M = \left\{\begin{bmatrix} a & -b \\ b & a \end{bmatrix} : \ a, b \in \mathbb{R}\right\}$. Prove that $\mathbb{C}$ and M are isomorphic under addition and that $\mathbb{C}^*$ and M^*, the non-zero elements of M, are isomorphic under multiplication.

(b) (i) Let $H = \{I, (12)(34)\}$, $G = A_4$. Show that H is not a normal subgroup of G.

(ii) Is the order of a factor group of an infinite group is infinite? Give example or counter example to support your answer.

(3+3.5=6.5)

(c) (i) Prove that Z (G), the centre of a group G, is always a normal subgroup of G.

(ii) Let G = Z, the group of integers under addition. Write all the elements of factor group Z/20Z of Z. Is this factor group cyclic? Give explanation in support of your answer.

(3+3.5=6.5)

5. (a) If H is a subgroup of a group G and K is a normal subgroup of G, then prove that $H/(H \cap K)$ is isomorphic to HK/K.

(iv) Let G be a group such that $x = x^{-1}$, for all $x \in G$. Prove that G is Abelian.

(v) Give an example of a non-cyclic group, whose every proper subgroup is cyclic.

(vi) Prove that a group of order 4 is Abelian.

(vii) List all the generators of (Z, +), Z_7 and Z_8.

(viii) For any integer $n > 2$, show that there are at least two elements in U(n) that satisfy $x^2 = 1$.

(6×2=12)

2. (a) Prove that

$$G = \left\{ \begin{bmatrix} \cos(a) & -\sin(a) \\ \sin(a) & \cos(a) \end{bmatrix} : a \in \mathbb{R} \right\}$$

is an infinite Abelian group under matrix multiplication.

(b) Define a cyclic subgroup of a group. Is it Abelian or Non-Abelian? Justify your answer. Prove that

$$H = \left\{\begin{bmatrix}1 & n\\ 0 & 1\end{bmatrix} : n \in Z\right\}$$

is a cyclic subgroup of $GL(2, \mathbb{R})$.

(c) Prove that the subgroup of a cyclic group is cyclic. Find the smallest subgroup of (Z, +) containing 8 and 14. (2×6.5=13)

3. (a) Prove that the order of a permutation of a finite set written in disjoint cycle form is the least common multiple of the lengths of the cycles.

(6)

(b) (i) Show that S_7 has an element of order 12. Find one such element.

(ii) Give two reasons why the set of odd permutations in S_n is not a subgroup.

(3+3=6)

(c) (i) Let G = U (24), H = {1,7}. Write all the distinct left cosets of H in G.

(ii) Prove that a group of order 98 can have at the most one subgroup of order 49.

(3+3=6)

4. (a) (i) Let H be a subgroup of G and a and b belongs to G. Then prove that

$aH = bH$ iff $a^{-1}b \in H$

(ii) State Lagrange's Theorem for finite groups and prove that every group of prime order is cyclic. (3+3.5=6.5)