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Sr. No. of Question Paper : 4512

Unique Paper Code : 32351601

Name of the Paper : BMATH 613 – Complex Analysis

Name of the Course : **B.Sc. (H) Mathematics**

Semester

Duration : 3 Hours   Maximum Marks : 75

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Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.

2. **All** questions are compulsory.

3. Attempt **two** parts from each question.

1. (a) Find and sketch, showing corresponding orientations, the images of the hyperbolas

$$x^2 - y^2 = c_1 \ (c_1 < 0) \text{ and } 2xy = c_2 \ (c_2 < 0)$$

under the transformation $w = z^2$. (6)

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(b) (i) Prove that the limit of the function

$$f(z)=\left(\frac{z}{\bar{z}}\right)^2$$

as z tends to 0 does not exist.

(ii) Show that

$$\lim_{z\to\infty}\frac{4z^2}{(z-1)^2}=4.$$ (3+3=6)

(c) Show that the following functions are nowhere differentiable.

(i) $f(z) = z - \bar{z}$,

(ii) $f(z) = e^y \cos x + ie^y \sin x$. (3+3=6)

(d) (i) If a function f(z) is continuous and nonzero at a point z_0, then show that $f(z) \neq 0$ throughout some neighborhood of that point.

(ii) Show that the function $f(z) = (z^2 - 2)e^{-x}e^{-iy}$ is entire. (3+3=6)

2. (a) (i) Write $|\exp(2z + i)|$ and $|\exp(iz^2)|$ in terms of x and y. Then show that

$$|\exp\exp(2z+i) + \exp(iz^2)| \leq e^{2x} + e^{-2xy}.$$

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(ii) Find the value of z such that

$$e^z = 1+\sqrt{3}\,i$$ (3.5+3=6.5)

(b) Show that

(i) $\overline{\cos(iz)} = \cos(i\bar{z})$ for all z;

(ii) $\overline{\sin(iz)} = \sin(i\bar{z})$ if and only if $z = n\pi i$ $(n = 0 \pm 1, \pm 2, \ldots)$. (3.5+3=6.5)

(c) Show that

(i) $\log\log(i^2) = 2\log i$ where

$$\log z = \ln r + i\theta\left(r > 0, \frac{\pi}{4} < \theta < \frac{9\pi}{4}\right).$$

(ii) $\log\log(i^2) \neq 2\log i$ where

$$\log z = \ln r + i\theta\left(r > 0, \frac{3\pi}{4} < \theta < \frac{11\pi}{4}\right).$$

(3.5+3=6.5)

(d) Find all zeros of sinz and cosz. (3.5+3=6.5)

3. (a) State Fundamental theorem of Calculus.

Evaluate the following integrals to test if Fundamental theorem of Calculus holds true or not :

(i) $\int_0^{\pi/2} \exp(t+it)dt$

(ii) $\int_0^1 (3t-i)^2\, dt$ (2+2+2=6)

(b) Let y(x) be a real valued function defined piecewise on the interval $0 \le x \le 1$ as

$y(x) = x^3\sin(\pi/x)$, $0 < x \le 1$ and $y(0) = 0$

Does this equation $z = x + iy$, $0 \le x \le 1$ represent

(i) an arc

(ii) A smooth arc. Justify.

Find the points of intersection of this arc with real axis. (2+2+2=6)

(c) For an arbitrary smooth curve C: $z = z(t)$, $a \le t \le b$, from a fixed point z_1 to another fixed point z_2, show that the value of the integral depends only on the end points of C.

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State if it is independent of the arc under consideration or not?

Also, find its value around any closed contour. (3+1+2=6)

(d) Without evaluation of the integral, prove that

$\left|\int_c \frac{1}{z^2+1}dz\right| \le \frac{1}{2\sqrt{5}}$ where C is the straight line segment from 2 to 2 +i. Also, state the theorem used. (4+2=6)

4. (a) Use the method of antiderivative to show that

$\int_c (z-z_0)^{n-1}\, dz = 0$, $n = \pm1, \pm2, \text{--}$ where C is any closed contour which does not pass through the point z_0. State the corresponding result used. (4+2.5=6.5)

(b) Use Cauchy Gourset theorem to evaluate :

(i) $\int_c f(z)dz$, when $f(z) = \frac{1}{z^2+2z+2}$ and C is the unit circle $|z| = 1$ in either direction.

(ii) $\int_c f(z)dz$, when $f(z)=\frac{5z+7}{z^2+2z-3}$ and C is the circle $|z-2| = 2$. (3+3.5=6.5)

(c) State and prove Cauchy Integral Formula. (2+4.5=6.5)

(d) Evaluate the following integrals :

(i) $\int_c \frac{\cos z}{z(z^2+8)}dz$, where C is the positive oriented boundary of the square whose sides lie along the lines x = ±2 and y = ±2.

(ii) $\int_c \frac{2s^2-s-2}{s-2}ds$, $|z| \neq 3$ at z = 2, where C is the circle $|z| = 3$. (3.5+3=6.5)

5. (a) If a series of complex numbers converges then prove that the nth term converges to zero as n tends to infinity. Is the converse true? Justify. (6.5)

(b) Find the Maclaurin series for the function $f(z) = \sinh z$. (6.5)

(c) If a series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ converges to f(z) at all points interior to some circle $|z-z_0| = R$, then prove that it is the Taylor series for the function f(z) in powers of $z-z_0$. (6.5)

(d) Find the integral of f(z) around the positively oriented circle $|z| = 3$ when $f(z)=\frac{(3z+2)^2}{z(z-1)(2z+5)}$. (6.5)

6. (a) For the given function $f(z)=\left(\frac{z}{2z+1}\right)^3$, show any singular point is a pole. Determine the order of each pole and find the corresponding residue. (6)

(b) Find the Laurent Series that represents the function $f(z)=z^2 \sin\frac{1}{z^2}$ in the domain $0 < |z| < \infty$. (6)

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(c) Suppose that $z_n = x_n + iy_n$, $(n = 1,2,3,\ldots)$ and $S = X + iY$. Then show that

$$\sum_{n=1}^{\infty} z_n = S \text{ iff } \sum_{n=1}^{\infty} x_n = X \text{ and } \sum_{n=1}^{\infty} y_n = Y.$$

(6)

(d) If a function f(z) is analytic everywhere in the finite plane except for a finite number of singular points interior to a positively oriented simple closed contour C, then show that

$$\int_c f(z)dz = 2\pi i\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right].$$ (6)

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Your Roll No..............

Sr. No. of Question Paper	:	4530 E
Unique Paper Code	:	32351401
Name of the Paper	:	BMATH408- Partial Differential Equations
Name of the Course	:	B.Sc.(H) Mathematics
Semester	:	IV

Duration : 3 Hours

Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. All sections are compulsory.
3. Marks of each part are indicated.

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Section – I

1. Attempt any two out of the following: [7.5+7.5]

(a) Find the integral surfaces of the equation $u\,u_x + u_y = 1$ for the initial data:

$$x(s,0) = s, y(s,0) = 2s, u(s,0) = s.$$

(b) Apply $\sqrt{u} = v$ and $v(x,y) = f(x) + g(y)$ to solve:

$$x^4\,u_x^2 + y^2\,u_y^2 = 4\,u.$$

(c) Find the solution of the initial–value systems

$$u_t + u\,u_x = e^{-x}v,\ \ v_t - av_x = 0,$$

with $u(x,0) = x$ and $v(x,0) = e^x$.

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Section – II

2. Attempt any one out of the following: [6]

(a) Derive the two- dimensional wave equation of the vibrating membrane

$$u_{tt} = c^2(u_{xx} + u_{yy}) + F,$$

where, $c^2 = T/\rho$, and T is the tensile force per unit length

$F = f/\rho$, and f be the external force, acting on the membrane.

(b) Drive the potential equation $\nabla^2 V = 0$, where ∇^2 is known as Laplace operator.

3. Attempt any two out of the following: [6+6]

(a) Determine the general solution of

$$4u_{xx} + 5u_{xy} + u_{yy} + u_x + u_y = 2.$$

(b) Given that the parabolic equation

$$u_{xx} = a\,u_t + b\,u_x + c\,u + f,$$

where the coefficients are constants, by the substitution $u = v\,e^{\frac{1}{2}bx}$ and for the case $c = -(b^2/4)$, show that the given equation is reduced to the heat equation

$$v_{xx} = a\,v_t + g,$$

where $g = f\,e^{-bx/2}$.

(c) Reduce the equation

$$(n-1)^2 u_{xx} - y^{2n}\,u_{yy} = n\,y^{2n-1}\,u_y,$$

to canonical form for $n = 1$ and $n = 2$ if possible and also find their solutions.

Section – III

4. Attempt any three parts out of the following: [7+7+7]

(a) Determine the solution of the given below initial-value problem

$$u_{tt} - c^2 u_{xx} = x, \qquad u(x,0) = 0, \quad u_t(x,0) = 3.$$

(b) Obtain the solution of the initial boundary-value problem

$u_{tt} = 9u_{xx}$, $0 < x < \infty$, $t > 0$,

$u(x,0) = 0$, $0 \leq x < \infty$,

$u_t(x,0) = x^3$, $0 \leq x < \infty$,

$u_x(0,t) = 0$, $t \geq 0$.

(c) Solve:

$u_{tt} = c^2 u_{xx}$,

$u(x,t) = f(x)$ on $t = t(x)$,

$u(x,t) = g(x)$ on $x + ct = 0$,

where $f(0) = g(0)$.

(d) Determine the solution of the initial boundary-value problem:

$u_{tt} = c^2 u_{xx}$, $0 < x < l$, $t > 0$

$u(x,0) = f(x)$, $0 \leq x \leq l$,

$u_t(x,0) = g(x)$, $0 \leq x \leq l$,

$u(0,t) = 0$, $u(l,t) = 0$, $t \geq 0$.

Section -- IV

5. Attempt any three out of the following: [7+7+7]

(a) Determine the solution of the initial boundary value problem:

$u_t = 4\,u_{xx}$, $0 < x < 1$, $t > 0$

$u(x,0) = x^2\,(1 - x)$, $0 \leq x \leq 1$,

$u(0,t) = 0$, $u(l,t) = 0$, $t \geq 0$.

(b) Determine the solution of the initial boundary value problem by the method of separation of variables:

$u_{tt} = c^2 u_{xx}$, $0 < x < \pi$, $t > 0$

$u(x,0) = 0$, $0 \leq x \leq \pi$,

$u_t(x,0) = 8\sin^2 x$, $0 \leq x \leq \pi$,

$u(0,t) = 0$, $u(\pi,t) = 0$, $t \geq 0$.

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(c) Solve by using method of separation of variables:

$$u_{tt} - u_{xx} = h, \qquad 0 < x < 1,\ t > 0,\ h \text{ is a constant}$$
$$u(x,0) = x^2, \qquad 0 \le x \le 1,$$
$$u_t(x,0) = 0, \qquad 0 \le x \le 1,$$
$$u(0,t) = 0, \quad u(1,t) = 0, \quad t \ge 0.$$

(d) State and prove the uniqueness of solution of the heat conduction problem.

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Sr. No. of Question Paper : 4548 E

Unique Paper Code : 32351201

Name of the Paper : Real Analysis (CBCS-LOCF)

Name of the Course : B.Sc. (Hons) Mathematics

Semester : II

Duration : 3 Hours Maximum Marks : 75

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Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. **All** questions are compulsory.
3. Attempt any two parts from each question.

1. (a) If x and y are positive real numbers with $x < y$, then prove that there exists a rational number $r \in \mathbb{Q}$ such that $x < r < y$. (6.5)

(b) Define Infimum and Supremum of a nonempty set of $\mathbb{R}$. Find infimum and supremum of the set

$$S = \left\{1 - \frac{(-1)^n}{n} : n \in \mathbb{N}\right\}.$$ (6.5)

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(c) State the completeness property of $\mathbb{R}$, hence show that every nonempty set of real numbers which is bounded below, has an infimum in $\mathbb{R}$. (6.5)

2. (a) Prove that there does not exist a rational number $r \in \mathbb{Q}$ such that $r^2 = 2$. (6)

(b) Define an open set and a closed set in $\mathbb{R}$. Show that if a, b $\in \mathbb{R}$, then the open interval (a, b) is an open set. (6)

(c) Let S be a nonempty bounded set in $\mathbb{R}$. Let $a > 0$, and let $aS = \{as: s \in S\}$. Prove that inf (aS) = a(inf S) and sup (aS) = a(supS). (6)

3. (a) Define limit of a sequence. Using definition show that $\lim_{n\to\infty}\left(\frac{3n+1}{2n+5}\right)=\frac{3}{2}$. (6.5)

(b) Prove that every convergent sequence is bounded. Is the converse true? Justify. (6.5)

(c) Let $x_1 = 1$ and $x_{n+1} = \frac{1}{4}(2x_n + 3)$ for $n \in \mathbb{N}$. Show that $\langle x_n \rangle$ is bounded and monotone. Find the limit. (6.5)

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4. (a) If $\langle a_n \rangle$ and $\langle b_n \rangle$ converges to a and b respectively, prove that $\langle a_n b_n \rangle$ converges to ab. (6)

(b) Show that $\lim_{n\to\infty} n^{1/n} = 1$. (6)

(c) State Cauchy Convergence Criterion for sequences. Hence show that the sequence $\langle a_n \rangle$, defined by $a_n = 1+\frac{1}{2}+\cdots\cdots+\frac{1}{n}$, does not converge. (6)

5. (a) Prove that if an infinite series $\sum_{n=1}^{\infty} a_n$ is convergent then $\lim_{n\to\infty} a_n = 0$. Hence examine the convergence of $\sum_{n=1}^{\infty}\frac{n}{2n+3}$. (6.5)

(b) Examine the convergence or divergence of the following series.

(i) $\frac{2}{5}+\frac{4}{8}+\frac{6}{11}+\cdots$ (6.5)

(ii) $\sum_{n=1}^{\infty}\left(\frac{3n+5}{2n+1}\right)^{n/2}$

(c) Prove that $\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^p}, p>0$ is convergent for $p>1$ and divergent for $p \le 1$. (6.5)

6. (a) State and prove ratio test (limit form). (6)

(b) Examine the convergence or divergence of the following series. (6)

(i) $\sum_{n=1}^{\infty}\frac{n^3+1}{n^4+3n^2+2n}$

(ii) $3+\frac{3^2}{2!}+\frac{3^3}{3!}+\frac{3^4}{4!}+\cdots$

(c) Prove that the series $\frac{1}{\ln 2}-\frac{1}{\ln 3}+\frac{1}{\ln 4}-\frac{1}{\ln 5}+\cdots$ is conditionally convergent. (6)

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Your Roll No..............

24 MAY 2023

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Sr. No. of Question Paper : 4749

Unique Paper Code : 32357614

Name of the Paper : DSE-3 MATHEMATICAL FINANCE

Name of the Course : **B.Sc. (H) Mathematics CBCS (LOCF)**

Semester : VI

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt any **two** parts from each question.
3. **All** questions are compulsory and carry equal marks.
4. Use of Scientific calculator, Basic calculator and Normal distribution tables all are allowed.

1. (a) Explain Duration of a zero-coupon bond. A 5-year bond with a yield of 12% (continuously compounded) pays a 10% coupon at the end of each year.

(i) What is the bond's price?

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(ii) Use duration to calculate the effect on the bond's price of a 0.1% decrease in its yield? (You can use the exponential values: e^x = 0.8869, 0.7866, 0.6977, 0.6188, and 0.5488 for x = –0.12, –0.24, –0.36, –0.48, and –0.60, respectively)

(b) Portfolio A consists of 1-year zero coupon with a face value of ₹2000 and a 10-year zero coupon bond with face value of ₹6000. Portfolio B consists of a 5.95-year zero coupon bond with face value of ₹5000. The current yield on all bonds is 10% per annum.

(i) Show that both portfolios have the same duration.

(ii) What are the percentage changes in the values of the two portfolios for a 5% per annum increase in yields?

(You can use the exponential values: e^x = 0.905, 0.368, 0.552, 0.861, 0.223 and 0.409 for x = –0.1, –1.0, –0.595, –0.15, –1.5 and –0.893 respectively)

(c) Explain difference between Continuous Compounding and Monthly Compounding. What rate of interest with continuous compounding is equivalent to 15% per annum with monthly compounding?

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(d) (i) "When the zero curve is upward sloping, the zero rate for a particular maturity is greater than the par yield for that maturity. When the zero curve is downward sloping the reverse is true." Explain.

(ii) Why does loan in the repo market involve very little credit risk?

2. (a) Explain Hedging. How is the risk managed when Hedging is done using?

(i) Forward Contracts; (ii) Options

(b) (i) Suppose that a March call option to buy a share for ₹50 costs ₹2.50 and is held until March. Under what circumstances will the holder of the option make a profit? Under what circumstances will the option be exercised?

(ii) It is May, and a trader writes a September put option with a strike price of ₹20. The stock price is ₹18, and the option price is ₹2. Describe the trader's cash flows if the option is held until September and the stock price is ₹25 at that time.

(c) Write a short note on European put options. Explain the payoffs in different types of put option positions with the help of diagrams.

(d) (i) A trader writes an October put option with a strike price of ₹35. The price of the option is ₹6. Under what circumstances does the trader make a gain.

(ii) A company knows that it is due to receive a certain amount of a foreign currency in 6 months. What type of option contract is appropriate for hedging?

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3. (a) Draw the diagrams illustrating the effect of changes in volatility and risk-free interest rate on both European call and put option prices when $S_0 = 50$, $K = 50$, $r = 5\%$, $\sigma = 30\%$, and $T = 1$.

(b) Derive the put-call parity for European options on a non-dividend-paying stock. Use put-call parity to derive the relationship between the vega of a European call and the vega of a European put on a non-dividend-paying stock.

(c) A European call option and put option on a stock both have a strike price of ₹20 and an expiration date in 3 months. Both sell for ₹3. The risk-free interest rate is 10% per annum, the current stock price is ₹19, and a ₹1 dividend is expected in 1 month. Identify the arbitrage opportunity open to a trader? ($e^{-0.0083} = 0.9917$)

(d) Find lower bound and upper bound for the price of a 1-month European put option on a non-dividend-paying stock when the stock price is ₹30, the strike price is ₹34, and the risk-free interest rate is 6% per annum? Justify your answer with no arbitrage arguments, ($e^{-0.005} = 0.9950$)

4. (a) Consider the standard one-period model where the stock price goes from S_0 to S_0u or S_0d with $d < 1 < u$, and consider an option which pays f_u or f_d in each case, and assume that the interest rate is r and time to maturity is T. Derive the formula for the no-arbitrage price of the option.

(b) A stock price is currently ₹40. It is known that at the end of one month it will be either ₹42 or ?38. The risk-free interest rate is 6% per annum with continuous compounding. Consider a portfolio consisting of one short call and Δ shares of the stock. What is the value of Δ which makes the portfolio riskless? Using no-arbitrage arguments, find the price of a one-month European call option with a strike price of ₹39? (You can use exponential value: $e^{0005} = 1.005$)

(c) Construct a two-period binomial tree for stock and European call option with

$S_0 = ₹100, u = 1.3, d = 0.8, r = 0.05, T = 1$ year, $K = ₹95$

and each period being of length $\Delta t = 0.5$ year. Find the price of the European call. If the call was American, will it be optimal to exercise the option early? Justify your answer. ($e^{-0.025} = 0.9753$)

(d) What do you mean by the volatility of a stock? How can we estimate volatility from historical prices of the stock?

5. (a) Let S_0 denote the current stock price, σ the volatility of the stock, r be the risk-free interest rate and T denote a future time. In the Black-Scholes model, the stock price S_T at time T in the risk-neutral world satisfies

$$\ln S_T \sim \phi\left[\ln S_0 + \left(r - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right]$$

where $\phi(m, v)$ denotes a norma distribution with mean m and variance v.

Using risk-neutral valuation, derive the Black-Scholes formula for the price of a European call option on the underlying stock S, strike price K and maturity T.

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(b) A stock price follows log normal distribution with an expected return of 16% and a volatility of 35%. The current price is ₹38. What is the probability that a European call option on the stock with an exercise price of ₹40 and a maturity date in six months will be exercised? (You can use values: ln(38) = 3.638, ln(40) = 3.689)

(c) What is the price of a European call option on a non-dividend-paying stock when the stock price is ₹69, the strike price is ₹70, the risk-free interest rate is 5% per annum, the volatility is 35% per annum, and the time to maturity is six months?

(You can use exponential values: $e^{-0.0144} = 0.9857$, $e^{-0.025} = 0.9753$)

(d) A stock price is currently ₹40. Assume that the expected return from the stock is 15% and that its volatility is 25%. What is the probability distribution for the rate of return (with continuous compounding) earned over a 2-year period?

6. (a) Discuss theta of a portfolio of options and calculate the theta of a European call option on a non-dividend-paying stock where the stock price is ₹49, the strike price is ₹50, the risk-free interest rate is 5% per annum and the time to maturity is 20 weeks, and the stock price volatility is 30% per annum. (ln(49/50) = –0.0202)

(b) (i) Explain stop-loss hedging scheme.

(ii) What does it mean to assert that the delta of a call option is 0.7? How can a short position in 1,000 options be made delta neutral when the delta of each option is 0.7?

(c) Find the payoff from a butterfly spread created using call options. Also draw the profit diagram corresponding to this trading strategy.

(d) Companies X and Y have been offered the following rates per annum on a ₹5 million 10-year investment :

	Fixed rate	Floating rate
Company X	8.0%	LIBOR
Company Y	8.8%	LIBOR

Company X requires a fixed-rate investment; Company Y requires a floating-rate investment. Design a swap that will net a bank, acting as intermediary, 0.2% per annum and that will appear equally attractive to X and Y.

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19 MAY 2023

Sr. No. of Question Paper : 4668 E

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Unique Paper Code : 32351202

Name of the Paper : Differential Equations

Name of the Course : B.Sc. (Hons.) Mathematics

Semester : II

Duration : 3 Hours

Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Use of non-programmable scientific calculator is allowed.

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Section – 1

1. Attempt any **three** parts. Each part is of **5** marks.

(a) Solve the initial value problem

$$(2y\sin x\cos x + y^2\sin x)dx + (\sin^2 x - 2y\cos x)dy = 0, \qquad y(0) = 3$$

(b) Solve the differential equation

$$(x^2 - 3y^2)dx + 2xy\,dy = 0$$

(c) Solve the differential equation

$$xy'' + 2y' = 6x$$

(d) Solve the differential equation

$$(x + 2y + 3)dx + (2x + 4y - 1)dy = 0$$

2. Attempt any **two** parts. Each part is of **5** marks.

(a) In a certain culture of bacteria, the number of bacteria increased sixfold in 10 hours. How long did it take for the population to double?

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(b) An arrow is shot straight upward from the ground with an initial velocity of 160 ft/s. It experiences both the deceleration of gravity and deceleration $v^2/800$ due to air resistance. How high in the air does it go?

(c) A cake is removed from an oven at 210°F and left to cool at room temperature, which is 70°F. After 30 min the temperature of the cake is 140°F. When will it be 100°F?

Section – 2

3. Attempt any two parts. Each part is of **8** marks.

(a) The following differential equation describes the level of pollution in the lake

$$\frac{dC}{dt} = \frac{F}{V}(C_{in} - C),$$

where V is the volume, F is the flow (in and out), C is the concentration of pollution at time t and C_{in} is the concentration of pollution entering the lake. Let $V = 28 \times 10^6 m^3$, $F = 4 \times 10^6 m^3/month$. If only fresh water enters the lake,

i. How long would it take for the lake with pollution concentration $10^7 parts/m^3$ to drop below the safety threshold $4 \times 10^6 parts/m^3$?

ii. How long will it take to reduce the pollution level to 5% of its current level?

(b) In view of the potentially disastrous effects of overfishing causing a population to become extinct, some governments impose quotes which vary depending on estimates of the population at the current time. One harvesting model that takes this into account is

$$\frac{dX}{dt} = rX\left(1 - \frac{X}{K}\right) - h_0 X.$$

i. Show that the only non-zero equilibrium population is

$$X_c = K\left(1 - \frac{h}{r}\right).$$

ii. At what critical harvesting rate can extinction occur?

(c) In a simple battle model, suppose that soldiers from the red army are visible to the blue army, but soldiers from the blue army are hidden. Thus, all the red army can do is fire randomly into an area and hope they hit something. The blue army uses aimed fire.

i. Write down appropriate word equations describing the rate of change of the number of soldiers in each of the armies.

ii. By making appropriate assumptions, obtain two coupled differential equations describing this system.

iii. Write down a formula for the probability of a single bullet fired from a single red soldier wounding a blue soldier in terms of the total area A and the area exposed by a single blue soldier A_b.

iv. Hence write the rate of wounding of both armies terms of the probability and the firing rate.

Section – 3

4. Attempt any three parts. Each part is of **6** marks.

(a) Find the general solution of the differential

$$x^3y''' + 6x^2y'' + 4xy' = 0.$$

(b) Using the method of undetermined coefficients, solve the differential equation

$$y''' - 2y'' + y' = 1 + xe^x,\ y(0) = y'(0) = y''(0) = 1.$$

(c) Using the method of Variation of parameters, solve the differential equation

$$y'' + 3y' + 2y = 4e^x.$$

(d) Show that $y_1 = 1$ and $y_2 = \sqrt{x}$ are solutions of

$$yy'' + (y')^2 = 0,$$

but the sum $y = y_1 + y_2$ is not a solution. Explain why?

Section – 4

5. Attempt any **two** parts. Each part is of **8** marks.

(a) The pair of differential equations

$$\frac{dP}{dt} = rP - \gamma PT, \frac{dT}{dt} = qP,$$

where r, γ and q are positive constants, is a model for a population of microorganisms P, which produces toxins T which kill the microorganisms.

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i. Given that initially there are no toxins and P_0 microorganisms, obtain an expression relating the population density and the amount of toxins.

ii. Hence, give a sketch of a typical phase-plane trajectory.

iii. Using phase-plane trajectory, describe what happens to the microorganisms over time.

(b) A model of a three species interaction is :

$$\frac{dX}{dt} = a_1X - b_1XY - c_1XZ,$$

$$\frac{dY}{dt} = a_2XY - b_2Y,$$

$$\frac{dZ}{dt} = a_3XZ - b_3Z$$

Where a_i , b_i , c_i for $i = 1, 2, 3$ are all positive constants. Here $X(t)$ is the prey density and $Y(t)$ and $Z(t)$ are the two predator species densities.

i. Find all possible equilibrium populations. Is it possible for the three populations to coexist in equilibrium ?

ii. What does this suggest about introducing an additional predator into an ecosystem ?

(c) In a long range battle, neither army can see the other, but fires into a given area. A simple mathematical model describing this battle is given by the coupled differential equations

$$\frac{dR}{dt} = -c_1RB\,, \qquad \frac{dB}{dt} = -c_2RB$$

where c_1 and c_2 are positive constants.

i. Use the chain rule to find a relationship between R and B, given the initial numbers of soldiers for the two armies are r_0 and b_0, respectively.

ii. Draw a sketch of typical phase-plane trajectories.

iii. Explain how to estimate the parameter c_1 given that the blue army fires into a region of area A.

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[This question paper contains 2 printed pages.]

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E

Sr. No. of Question Paper : **4686**

Unique Paper Code : 32351402

Name of the Paper : Riemann Integration and Series of Functions

Name of the Course : **B.Sc. (H) Mathematics**

Semester : IV

Duration : 3 Hours

Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt **two** parts from each question.

1(a) Let f be integrable on [a, b], and suppose g is a function on [a, b] such that $g(x) = f(x)$ except for finitely many x in [a, b]. Show g is integrable and $\int_a^b g = \int_a^b f$ (6)

(b) Show that if f is integrable on [a, b] then f^2 also is integrable on [a, b]. (6)

(c) (i) Let f be a continuous function on [a,b] such that $f(x) \geq 0$ for all $x \in [a, b]$. Show that if $\int_a^b f(x)\,dx = 0$ then $f(x) = 0$ for all $x \in [a, b]$ (3)

(ii) Give an example of function such that $|f|$ is integrable on [0,1] but f is not integrable on [0,1]. Justify it. (3)

2(a) State and prove Fundamental Theorem of Calculus I. (6.5)

(b) State Intermediate Value Theorem for Integrals. Evaluate $\lim_{x\to 0} \frac{1}{x}\int_0^x e^{t^2}\,dt$. (6.5)

(c) Let function $f: [0,1] \to R$ be defined as

$$f(x) = \begin{cases} x^2 & if\ x\ is\ rational \\ 0 & if\ x\ is\ irrational \end{cases}$$

Calculate the upper and lower Darboux integrals for f on the interval [0,1]. Is f integrable on [0.1] ? (6.5)

3(a) Examine the convergence of the improper integral $\int_0^\infty e^{-x}x^{n-1}dx$. (6)

(b) Show that the improper integral $\int_\pi^\infty \frac{Sinx}{x}dx$ is convergent but not absolutely convergent. (6)

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(c) Determine the convergence or divergence of the improper integral (6)

(i) $\int_0^1 \frac{dx}{x(\ln x)^2}$

(ii) $\int_1^\infty \frac{x\,dx}{\sqrt{x^3+x}}$

4(a) Show that the sequence

$$f_n(x) = \frac{nx}{1+nx}, \quad x \in [0,1], \quad n \in N$$

converges non-uniformly to an integrable function f on [0,1] such that

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)dx$$

(6.5)

(b) Show that the sequence $\{x^2 e^{-nx}\}$ converges uniformly on $[0, \infty)$. (6.5)

(c) Let $\langle f_n \rangle$ be a sequence of continuous function on $A \subset R$ and suppose that $\langle f_n \rangle$ converges uniformly on A to a function $f: A \to R$. Show that f is continuous on A. (6.5)

5(a) Let $f_n(x) = \frac{nx}{1+n^2x^2}$ for $x \geq 0$. Show that sequence $\langle f_n \rangle$ converges non-uniformly on $[0, \infty)$ and converges uniformly on $[a, \infty), a > 0$. (6.5)

(b) State and prove Weierstrass M-test for the uniform Convergence of a series of functions. (6.5)

(c) Show that the series of functions $\sum \frac{1}{n^2+x^2}$, converges uniformly *on R* to a continuous function. (6.5)

6(a) (i) Find the exact interval of convergence of the power series (3)

$$\sum_{n=0}^{\infty} 2^{-n} x^{3n}$$

(ii) Define $\sin x$ as a power series and find its radius of convergence (3)

(b) Prove that $\sum_{n=1}^{\infty} n^2 x^n = \frac{x(x+1)}{(1-x)^3}$ for $|x| < 1$ and hence evaluate $\frac{\sum_{n=1}^{\infty} n^2(-1)^n}{3^n}$. (6)

(c) Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ have radius of convergence $R > 0$. Then f is differentiable on $(-R, R)$ and

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad for \ |x| < R.$$

(6)

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Sr. No. of Question Paper : 4750 **E**

Unique Paper Code : 32357615

Name of the Paper : Introduction to Information Theory and Coding

Name of the Course : **B.Sc. (H) Mathematics**

Semester : VI

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. **All** questions are compulsory.
3. Attempt any **two** parts from each question.
4. **All** questions carry equal marks.

1. (a) Whatis entropy? Give the expressions of entropies H(X) and H(X, Y) in terms of probability distributions.

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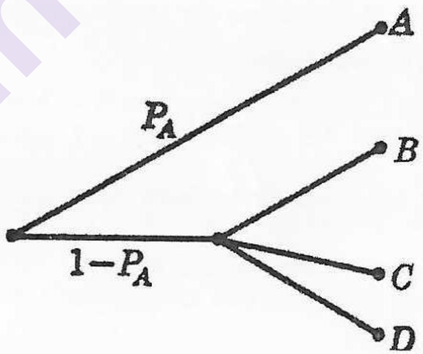
(b) Explain the term information and discuss its properties.

(c) Find the uncertainty associated with the transmission of symbols x, y, z with probabilities $\frac{1}{6}, \frac{1}{2}, \frac{1}{3}$ respectively. What is the entropy of [x, y U z]? Also, find the information of each symbol.

(d) Verify the additivity rule of entropies for the following, with.

$$P_A = \frac{1}{2},\ P_B = \frac{1}{8},\ P_C = \frac{1}{4},\ P_D = \frac{1}{8}$$



2. (a) Prove that $H(X) \leq \log(m)$, where $H(X)$ is the entropy of X and m is the number of points in X.

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(b) Consider a binary symmetric communication channel, whose input source is the alphabet $X = \{0, 1\}$ with probabilities {0.5, 0.5}; output alphabet $Y = \{0, 1\}$; and with channel matrix:

$$\begin{bmatrix} 1-\varepsilon & \varepsilon \\ \varepsilon & 1-\varepsilon \end{bmatrix}$$ where ε is the probability of transmission error.

(i) What is the entropy of the source, H(X)?

(ii) What is the probability distribution of the outputs, p(Y), and what is the entropy of this output distribution, H(Y)?

(iii) What is the joint probability distribution for the source and the output, p(X, Y) and what is the joint entropy, H(X, Y)?

(c) Let the random variable X have five possible symbols {α, β, γ, δ, σ}. Consider two probability distributions p(x) and q(x) over these symbols:

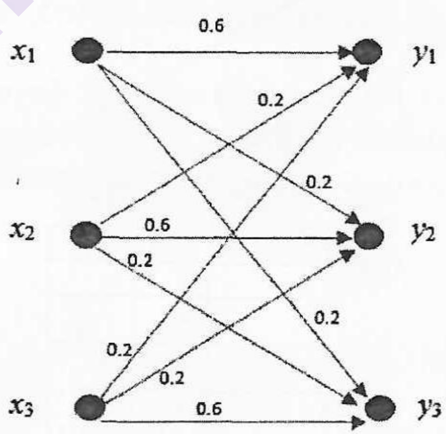
Symbol	p(x)	q(x)
α	½	½
β	$^1/_4$	$^1/_8$
γ	$^1/_8$	$^1/_8$
δ	$^1/_8$	$^1/_{16}$
σ	$^1/_8$	$^1/_{16}$

Calculate H(p) and H(q). Find the relative entropy of X for distributions p(x) and q(x). (d) For the function, $H(p_1, p_2, \ldots p_n)$, show that

$$H(p_1, p_2, \ldots p_n) = -\lambda \sum_{i=1}^{n} p_i \log p_i,\ \lambda > 0 \text{ when all}$$

$$p_i's = \frac{1}{n} \text{ for } i = 1,2,\ldots,n.$$

3. (a) A discrete source transmits messages x_1, x_2, x_3 with probabilities 1/2, 1/4, 1/4 respectively. The source is connected to receiver as shown in the diagram.



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Determine the following :

(i) Receiver entropy H(Y)

(ii) Conditional entropy H(Y|X)

(iii) Channel capacity

(b) If (X, Y) be the joint random variable of the random variables X and Y with (X, Y) ~ p(x, y). Then prove the following :

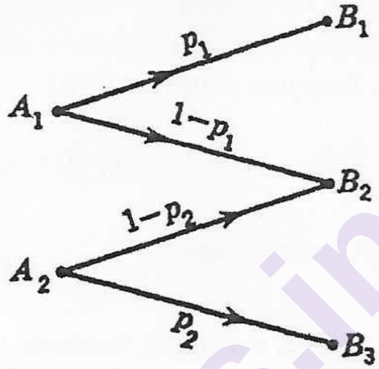
(i) $I(X;Y) = H(X) - H(X|Y)$

(ii) $I(X;Y) = H(X) + H(Y) - H(X, Y)$

(iii) $I(X;Y) = I(Y;X)$

(iv) $I(X;X) = H(X)$

(c) Compute the trans-information I(X,Y) in the channel, when symbols A_1 and A_2 are transmitted with probabilities $p(A_1) = 1/2$ and $p(A_2) = 1/2$, $p_1 = 2/3$ and $p_2 = 5/6$.



(d) If (X, Y) be the joint random variable of the random variables X and Y with $(X, Y) \sim p(x, y)$. The mutual information $I(X; Y)$ is a concave function of $p(x)$ for fixed $p(y|x)$ and a convex function of $p(y|x)$ for fixed $p(x)$.

4. (a) A codeword of the code {01010, 10101} is transmitted through a BSC with crossover probability $p = 0.1$, and a nearest-codeword decoder D is applied to the received word. Compute the decoding error probability Perr of D.

(b) Suppose d is odd. Then, prove that a binary (n, M, d) code exists iff a binary (n+1, M, d+1) code exists.

(c) Let $a(x) = x^4 + x^2 + x + 1$ and $b(x) = x^3 + 1$. Find $\gcd(a(x), b(x))$ over GF(2).

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(d) Construct the $GF(2^3)$ as a ring of residues over GF(2) modulo the polynomial $P(x) = x^3 + x^2 + 1$.

5. (a) Let F be a ternary field GF(3) and C be a linear code over F that is generated by

$$G = \begin{bmatrix} 2 & 2 & 2 & 0 \\ 1 & 2 & 1 & 1 \end{bmatrix}$$

(i) List all the codewords of C

(ii) Find the parameters n, k and d of the linear code C.

(b) Find the extended Hamming code for the linear [7,4,3] Hamming code over GF[2] with parity check matrix :

$$H = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Show that Extended Hamming code is a self-dual code.

(c) True or False. Also, correct the false statements.

(i) Every set of n-k columns of Parity Check matrix H of a linear code C is linearly independent.

(ii) Every set of k columns of Generator matrix G of a linear code C is linearly dependent.

(iii) The dual code of MDs code is not MDS.

(iv) The systematic generator matrix is of the (I|A), where I is an identity matrix of order (n-k) and A is k x k matrix.

(d) Let C be a linear [5,2,3] code over GF(2) with the generator matrix

$$G = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

Use syndrome decoding to decode the receiving vector v = (01111).

6. (a) Define Reed Solomon code. Show that it is a MDS code.

(b) State MacWilliams' Theorem. Let C be the Simplex Code, find $W_{c^{\perp}}(z)$ for q=2.

(c) State and prove Gilbert-Varshamov bound.

(d) Define perfect code and show that [k,1,k] repetition code over GF(2) and [n, n-m, 3] hamming code over GF(q) where m>0, $n = (q^m - 1)/(q - 1)$ are perfect codes.

[This question paper contains 4 printed pages.]

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E

Sr. No. of Question Paper : 4792

Unique Paper Code : 32351602

Name of the Paper : Ring Theory and Linear Algebra – II

Name of the Course : B.Sc. (H) Mathematics (CBCS – LOCF)

Semester : VI

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. All the questions are compulsory.
3. Attempt any **two** parts from each question.
4. Marks of each part are indicated

1. (a) (i) Prove that If F is a field, then F[x] is a Principal Ideal Domain.

(ii) Is $\mathbb{Z}[x]$, a Principal Ideal Domain? Justify your answer.

(b) Prove that $<x^2 + 1>$ is not a maximal ideal in $\mathbb{Z}[x]$.

(c) State and prove the reducibility test for polynomials of degree 2 or 3. Does it fail in higher order polynomials? Justify. (4+2,6,6)

2. (a) (i) State and prove Gauss's Lemma.

(ii) Is every irreducible polynomial over $\mathbb{Z}$ primitive? Justify.

(b) Construct a field of order 25.

(c) In $\mathbb{Z}\left[\sqrt{(-5)}\right]$, prove that $1+3\sqrt{(-5)}$ is irreducible but not prime. (4+2.5,6.5,6.5)

3. (a) Let $V = \mathbb{R}^3$ and define $f_1, f_2, f_3 \in V^*$ as follows:

$f_1(x, y, z) = x - 2y,$
$f_2(x, y, z) = x + y + z,$
$f_3(x, y, z) = y - 3z.$

Prove that $\{f_1, f_2, f_3\}$ is a basis for V^* and then find a basis for V for which it is the dual basis.

(b) Test the linear operator $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$, $T(f(x)) = f(0) + f(1)(x + x^2)$ for diagonalizability and if diagonalizable, find a basis β for V such that $[T]_\beta$ is a diagonal matrix.

(c) Let $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \in M_{2\times2}(\mathbb{R})$. Find an expression for A^n where n is an arbitrary natural number. (6,6,6)

4. (a) For a linear operator $T: \mathbb{R}^3 \to \mathbb{R}^3$, $T(a, b, c) = (-b + c, a + c, 3c)$, determine the T-cyclic subspace W of $\mathbb{R}^3$ generated by $e_1 = (1, 0, 0)$. Also find the characteristic polynomial of the operator T_W.

(b) State Cayley-Hamilton theorem and verify it for the linear operator $T:P_2(\mathbb{R}) \to P_2(\mathbb{R})$, $T(f(x)) = f'(x)$.

(c) Show that the vector space $\mathbb{R}^4 = W_1 \oplus W_2 \oplus W_3$ where $W_1 = \{(a, b, 0, 0): a, b \in \mathbb{R}\}$, $W_2 = \{(0,0, c, 0): c \in \mathbb{R}\}$ and $W_3 = \{(0,0,0, d): d \in \mathbb{R}\}$. (6.5,6.5,6.5)

5. (a) Consider the vector space $\mathbb{C}$ over $\mathbb{R}$ with an inner product $\langle .,. \rangle$. Let $\bar{z}$ denote the conjugate of z. Show that $\langle .,. \rangle'$ defined by $\langle z, w\rangle' = \langle \bar{z}, \bar{w}\rangle$ for all $z, w \in \mathbb{C}$ is also an inner product on $\mathbb{C}$. Is $\langle .,. \rangle''$ defined by $\langle z, w\rangle'' = \langle z + \bar{z}, w + \bar{w}\rangle$ for all $z, w \in \mathbb{C}$ an inner product on $\mathbb{C}$? Justify your answer.

(b) Let $V = P(\mathbb{R})$ with the inner product $\langle p(x), q(x)\rangle = \int_{-1}^{1} p(t)q(t)\,dt \quad \forall p(x), q(x) \in V$. Compute the orthogonal projection of the vector $p(x) = x^{2k-1}$ on $P_2(\mathbb{R})$, where $k \in \mathbb{N}$.

(c) (i) For the inner product space $V = P_1(R)$ with $\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)dt$ and the linear operator T on V defined by $T(f) = f' + 3f$, compute $T^*(4 - 2t)$.

(ii) For the standard inner product space $V = \mathbb{R}^3$ and a linear transformation $g: V \to \mathbb{R}$ given by $g(a_1, a_2, a_3) = a_1 - 2a_2 + 4a_3$, find a vector $y \in V$ such that $g(x) = \langle x, y\rangle$ for all $x \in V$. (6,6,2+4)

6. (a) Prove that a normal operator T on a finite-dimensional complex inner product space V yields an orthonormal basis for V consisting of eigenvectors of T. Justify the validity of the conclusion of this result if V is a finite-dimensional real inner product space.

(b) Let $V = M_{2\times2}(\mathbb{R})$ and $T: V \to V$ be a linear operator given by $T(A) = A^T$. Determine whether T is normal, self-adjoint, or neither. If possible, produce an orthonormal basis of eigenvectors of T for V and list the corresponding eigenvalues.

(c) For the matrix $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ find an orthogonal matrix P and a diagonal matrix D such that $P^*AP = D$. (6.5,6.5,6.5)

[This question paper contains 4 printed pages.]

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Sr. No. of Question Paper : 4797 E

Unique Paper Code : 32231602

Name of the Paper : Evolutionary Biology

Name of the Course : B.Sc. (Hons.) Zoology (LOCF)

Semester : VI

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Draw well-labeled diagrams wherever necessary.
3. Attempt **five** questions in all. Question No. 1 is compulsory.

1. (a) Define the following terms : (5)

   (i) Adaptation

   (ii) Cline

   (iii) Kin selection

(iv) Stromatolites

(v) Pseudogene

(b) Differentiate between the following (10)

(i) Coacervates and Microspheres

(ii) Allopatric speciation and Sympatric speciation

(iii) Rooted and Unrooted tree

(iv) Stabilizing and Disruptive selection

(v) Micro and Macro Evolution

(c) State the contribution of the following Scientists (5)

(i) Stanley Miller

(ii) Jean Baptiste de Lamarck

(iii) Motoo Kimura

(iv) Alfred Russell Wallace

(v) Raymond Dart

(d) Fill in the blanks : (3)

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(i) Morphologically similar but reproductively isolated species are called ____________ species.

(ii) ____________ is the process by which organic material becomes a fossil through the replacement of the original material and the filling of the original pore spaces with minerals.

(iii) The most ancestral stage of *Equus* was. ____________

(e) Justify the following statements. (4)

(i) Mutation proposes, Selection disposes.

(ii) Mesozoic Era is the Age of Reptiles.

2. (a) What is endosymbiotic theory and how can it explain the origin of eukaryotic cells? (6)

(b) Explain K-T mass extinction and its biological significance. (6)

3. (a) What do you understand by isolating mechanisms? Discuss the role of reproductive isolating mechanisms leading to speciation. (8)

(b) Describe the major changes undergone during the course of evolution of horse. (4)

4. (a) Define 'fossil'. State the process of fossillization, and the importance of fossils in the evolutionary studies. (6)

(b) How do organic variations contribute to the process of evolution? (6)

5. (a) Explain the pre-requisites for the Hardy-Weinberg equilibrium to operate in a Population. (6)

(b) Compare and contrast the different concepts of the species proposed in evolution. (6)

6. Give an account of the Darwin's observations on the Galapagos islands which led him to describe the origin of species. (12)

7. Write short notes on any **three** of the following :

(a) Australopithecines

(b) Neo-Darwinism

(c) Genetic drift

(d) Globin gene family

(e) Chemogeny (12)

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[This question paper contains 8 printed pages.]

29 MAY 2023

Your Roll No. ....

E

Sr. No. of Question Paper : 4810

Unique Paper Code : 32351403

Name of the Paper : Ring Theory & Linear Algebra – I

Name of the Course : **B.Sc. [Hons.] Mathematics CBCS (LOCF)**

Semester : IV

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. **All** questions are compulsory.
3. Attempt any **two** parts from each question.

1. (a) Find all the zero divisors and units in $\mathbb{Z}_3 \oplus \mathbb{Z}_6$

(6)

(b) Prove that characteristic of an integral domain is 0 or prime number p.

(6)

(c) State and prove the Subring test

(6)

2. (a) Let R be a commutative ring with unity and let A be an ideal of R then prove that R/A is a field if and only if A is a maximal ideal of R.

(6)

(b) Let A and B are two ideals of a commutative ring R with unity and A+B=R then show that $A \cap B = AB$.

(6)

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(c) If an ideal I of a ring R contains a unit then show that I=R. Hence prove that the only ideals of a field F are {0} and F itself.

(6)

3. (a) Find all ring homomorphism from $\mathbb{Z}_6$ to $\mathbb{Z}_{15}$.

(6.5)

(b) Let $R = \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} \middle| a, b \in \mathbb{Z} \right\}$ and Φ be the mapping that takes $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$ to a-b. Show that

(i) Φ is a ring homomorphism.

(ii) Determine Ker Φ.

(iii) Show that R/Ker Φ is isomorphic to $\mathbb{Z}$.

(6.5)

(c) Using homomorphism, prove that an integer n with decimal representation $a_k a_{k-1} \dots \dots a_0$ is divisible by 9 iff $a_k + a_{k-1} + \dots + a_0$ is divisible by 9. (6.5)

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4. (a) Let V(F) be the vector space of all real valued function over $\mathbb{R}$.

Let $V_e = \{f \in V \mid f(x) = f(-x) \ \forall \ x \in \mathbb{R})$

and $V_o = \{f \in V \mid f(-x) = -f(x) \ \forall \ x \in \mathbb{R})$

Prove that V_e and V_0 are subspaces of V and $V = V_e \oplus V_o$. (6)

(b) Let V(F) be a vector space and let $S_1 \subseteq S_2 \subseteq V$. Prove that

(i) If S_1 is linearly dependent then S_2 is linearly dependent

(ii) If S_2 is linearly independent then S_1 is linearly independent (6)

(c) Show that $S = \left\{\begin{pmatrix}1 & 1\\1 & 0\end{pmatrix}, \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\1 & 1\end{pmatrix}, \begin{pmatrix}0 & 1\\1 & 1\end{pmatrix}\right\}$ forms a basis for $M_{2X2}(\mathbb{R})$. (6)

5. (a) Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by

$$T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3).$$

Find Null space and Range space of T and verify Dimension Theorem. (6.5)

(b) Define $T: M_{2X2}(\mathbb{R}) \rightarrow P_2(\mathbb{R})$ by $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a+b) + (2d)x + bx^2$

Let $\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ and $\gamma = \{1, x, x^2\}$ be basis of $M_{2X2}(\mathbb{R})$ and $P_2(\mathbb{R})$ respectively. Compute $[T]_\beta^\gamma$. (6.5)

(c) Let V and W be vector spaces over F, and suppose that $\{v_1, v_2, \ldots, v_n\}$ be a basis for V. For $w_1, w_2, \ldots, w_n$ in W. Prove that there exists exactly one linear transformation $T: V \rightarrow W$ such that $T(v_i) = w_i$ for $i = 1, 2, \ldots, n$. (6.5)

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6. (a) Let T be the linear operator on $\mathbb{R}^2$ define by

$$T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2a+b \\ a-3b \end{pmatrix}$$

Let β be the standard ordered basis for $\mathbb{R}^2$ and let

$$\beta' = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}.$$

Find $[T]_{\beta'}$. (6.5)

(b) Let V and W be finite dimensional vector spaces with ordered basis β and γ respectively. Let $T: V \rightarrow W$ be linear. Then T is invertible if and only if $[T]_\beta^\gamma$ is invertible.

Furthermore, $[T^{-1}]_\gamma^\beta = ([T]_\beta^\gamma)^{-1}$. (6.5)

(c) Let V, W and Z be finite dimensional vector spaces with ordered basis α, β, γ respectively. Let T: V → W and U: W → Z be linear transformations.

Then $[UT]_{\alpha}^{\gamma} = [U]_{\beta}^{\gamma}[T]_{\alpha}^{\beta}$. (6.5)

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