(c) Let f be a continuous real-valued function on a compact metric space (X, d_X), then show that f is bounded and attains its bounds. Does the result hold when X is not compact? Justify.

(4+2.5)

[This question paper contains 8 printed pages.]

Your Roll No. ....

Sr. No. of Question Paper : 1013 C

Unique Paper Code : 32351501

Name of the Paper : BMATH511 – Metric Spaces

Name of the Course : **B.Sc. (Hons.) Mathematics CBCS (LOCF)**

Semester : V

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt any **two** parts from each question.

1. (a) Let (X, d) be a metric space. Define the mapping $d^*: X \times X \to \mathbb{R}$ by

$$d^*(x,y) = \frac{d(x,y)}{1+d(x,y)};\ \forall x, y \in X.$$

Show that (X, d*) is a metric space and $d^*(x, y) < 1$, for every $x, y \in X$. (6)

(b) Let $\langle x_n \rangle_{n\geq1}$ be a sequence of real numbers defined by $x_1 = a$, $x_2 = b$ and $x_{n+2} = \frac{1}{2}(x_{n+1} + x_n)$ for $n = 1, 2, \cdots$. Prove that $\langle x_n \rangle_{n\geq1}$ is a Cauchy sequence in $\mathbb{R}$ with usual metric. (6)

(c) Define a complete metric space. Is the metric space $(\mathbb{Z}, d)$ of integers, with usual metric d, a complete metric space? Justify. (6)

2. (a) (i) Let (X, d) be a metric space. Show that for every pair of distinct points x and y of X, there exist disjoint open sets U and V such that $x \in U$, $y \in V$. (2)

(c) Let (X, d) be a metric space. Then prove that (X, d) is disconnected if and only if there exists a continuous mapping of (X, d) onto the discrete two element space (X_0, d_0). (6.5)

6. (a) Prove that homeomorphism preserves compactness. Hence or otherwise show that

$$S(0,1) = \{z \in \mathbb{C} : |z| < 1\} \text{ and}$$
$$S[0,1] = \{z \in \mathbb{C} : |z| < 1\}$$

are not homeomorphic. (4+2.5)

(b) Let (X, d) be a metric space and $A \subseteq X$ such that every sequence in A has a subsequence converging in A. Show that for any $B \subseteq X$, there is a point $p \in A$ such that $d(p,B) = d(A,B)$. If B be a closed subset of X such that $A \cap B = \phi$, show that $d(A,B) > 0$. (4.5+2)

(c) (i) Let (X, d) be a complete metric space. Let $T: X \rightarrow X$ be a mapping such that $d(Tx,Ty) < d(x, y), \forall x, y \in X$. Does T always have a fixed point? Justify. (4)

(ii) Let X be any non-empty set and $T: X \rightarrow X$ be a mapping such that T^n (where n is a natural number, $n > 1$) has a unique fixed point $x_0 \in X$. Show that x_0 is also a unique fixed point of T. (2.5)

5. (a) Let $(\mathbb{R}, d)$ be the space of real numbers with usual metric. Prove that a connected subset of E must be an interval. Give an example of two connected subsets of E, such that their union is disconnected. (4+2.5)

(b) Let (X, d) be a metric space such that every two points of X are contained in some connected subset of X. Show that (X, d) is connected. (6.5)

(ii) Give an example of the following :

(a) A set in a metric space which is neither a closed ball nor an open set. (1)

(b) A metric space in which the interior of the intersection of an arbitrary family of the subsets may not be equal to the intersection of the interiors of the members of the family. (2)

(c) A metric space in which every singleton is an open set. (1)

(b) Let (X, d) be a metric space. Let A be a subset of X. Define closure of A and show that it is the smallest closed superset of A. (6)

(c) Let (X, d) be a complete metric space. Let $\langle F_n \rangle$ be a nested sequence of non-empty closed subsets of X such that $d(F_n) \to 0$. Show that $\bigcap_{n=1}^{\infty} F_n$ is a singleton. Does it hold if (X, d) is incomplete? Justify. (6)

3. (a) Let (X, d_X) and (Y, d_Y) be metric spaces and $f: X \to Y$ be a function. Prove that f is continuous on X if and only if $f(\overline{A}) \subseteq \overline{f(A)}$ for all subsets A of X. (6)

(b) Let A and B be non-empty disjoint closed subsets of a metric space (X, d). Show that there is a continuous real valued function f on X such that $f(x) = 0, \forall x \in A$, $f(x) = 1, \forall x \in B$ and $0 \le f(x) \le 1$, $\forall x \in X$. Further show that there exist disjoint open subsets G, H of X such that $A \subseteq G$ and $B \subseteq H$. (6)

(c) Define a dense subset of a metric space (X, d). Let $A \subseteq X$. Show that A is dense in X if and only if A^C has empty interior. Give an example of a metric space that has only one dense subset. (6)

4. (a) Show that the metrics d_1, d_2 and d_∞ defined on $\mathbb{R}^n$ by

$$d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i| ,$$

$$d_2(x,y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{1/2} \text{ and}$$

$$d_\infty(x,y) = \max\{|x_i - y_i| : 1 \le i \le n\}$$

are equivalent where $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$. (6.5)

(b) Show that the function $f: \mathbb{R} \to (-1, 1)$ defined by,

$f(x) = \frac{x}{1+|x|}$ is a homeomorphism but not an isometry. (6.5)