Show that G/N is isomorphic to the group of all the positive real numbers under multiplication.

(2×6.5=13)

[This question paper contains 8 printed pages.]

Your Roll No..............

Sr. No. of Question Paper : 1409 C

Unique Paper Code : 32351302

Name of the Paper : BMATH306 – Group Theory-I

Name of the Course : **B.Sc. (Hons) Mathematics**

Semester : III

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. **All** questions are compulsory.
3. Attempt any **two** parts from each question from **Q2** to **Q6**.
4. In the question paper, given notations have their usual meaning unless until stated otherwise.

P.T.O.

1. Give short answers to the following questions. Attempt any **six**.

(i) What is the total no of rotations and total no of reflections in the dihedral group D_3? Describe them (rotations and reflections) in pictures or words. What can you say about the total no of rotations and total no of reflections in the dihedral group D_n?

(ii) Give one non- trivial, proper subgroup of GL(2, R). Is GL(2, R) a group under addition of matrices? Answer in few lines.

(iii) Let G be a group with the property that for any a, b, c in G,

ab = ca implies b = c. Prove that G is Abelian.

(iv) Give an example of a cyclic group of order 5. Show that a group of order 5 is cyclic.

then $\phi^{-1}(\bar{K}) = [k \in G: \phi(k) \in \bar{K}\}$ is a subgroup of G.

(c) If H and K are two normal subgroups of a group G such that $H \subseteq X$, then prove that

$$G/K \approx \frac{G/H}{K/H}.$$

(2×6=12)

6. (a) Show that the mapping ϕ from $\mathbb{C}^*$ to $\mathbb{C}^*$ given by $\phi(z) = z^4$ is a homomorphism. Also find the set of all the elements that are mapped to 2.

(b) Prove that every group is isomorphic to a group of permutations.

(c) Let G be the group of non-zero complex numbers under multiplication and N be the set of complex numbers of absolute value 1.

(ii) Let $|G| = pq$, p and q are primes. Prove that $|Z(G)| = 1$ or pq. (4+2.5=6.5)

(c) (i) Prove that a subgroup of index 2 is normal.

(ii) Let $G = U(32)$, $H = U_8(32)$. Write all the elements of the factor group G/H. Also find order of 3H in G/H. (3+3.5=6.5)

5. (a) Show that the mapping from $\mathbb{R}$ under addition to $GL(2, \mathbb{R})$ that takes x to $\begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}$ is a group homomorphism. Also, find the kernel of the homomorphism.

(b) Let ϕ be a homomorphism from a group G to a group $\bar{G}$. Show that if $\bar{K}$ is a subgroup of $\bar{G}$,

(v) Prove that a cyclic group is Abelian. Is the converse true?

(vi) Find all subgroups of Z_{15}.

(vii) Prove that 1 and -1 are the only two generators of (Z,+). Give short answer in few lines.

(viii) "Z_n. $n \in N$, is always cyclic whereas U(n), $n \in N$; $n \geq 2$ may or may not be cyclic". Prove or disprove the statement in a few lines. (6×2=12)

2. (a) Let $G = \{a + b\sqrt{2} \mid a$ and b are rational nos not both zero$\}$

Prove that G is a group under ordinary multiplication. Is it Abelian or Non-Abelian? Justify your answer.

(b) Prove that a group of composite order has a non-trivial, proper subgroup.

(c) Prove that order of a cyclic group is equal to the order of its generator. (2×6.5=13)

3. (a) Prove that every permutation of a finite set can be written as a cycle or as a product of distinct cycles. (6)

(b) (i) In S_4, write a cyclic subgroup of order 4 and a non-cyclic subgroup of order 4.

(ii) Let $\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 4 & 8 & 6 & 7 & 5 & 1 & 3 \end{bmatrix}$ and

$$\beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 4 & 5 & 1 & 8 & 3 & 2 & 6 \end{bmatrix}$$

Write α, β and $\alpha\beta$ as product of 2-cycles. (3+3=6)

(c) (i) Let $|a| = 24$. How many left cosets of $H = \langle a^4 \rangle$ in $G = \langle a \rangle$ are there? Write each of them.

(ii) State Fermat's Little theorem. Also compute $5^{25} \bmod 7$ and $11^{17} \bmod 7$. (3+3=6)

4. (a) (i) Let H and K be two subgroups of a finite group. Prove that

$HK \leq G$ if G is Abelian.

(ii) Give an example of a group G and its two subgroups H and K ($H \neq K$) such that HK is not a subgroup of G. (3+3.5=6.5)

(b) (i) Let G be a group and let Z (G) be the centre of G. If G/Z (G) is cyclic, prove that G is Abelian.