5\. (a) Let $I \subseteq \mathbb{R}$ be an interval, let $c \in I$, and let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be functions that are differentiable at c. Prove that if $g(c) \neq 0$, the function f/g is differentiable at n, and

$$\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2}. \qquad (6)$$

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = |x| + |x + 1|$, $x \in \mathbb{R}$. Is f differentiable everywhere in $\mathbb{R}$? Find the derivative of f at the points where it is differentiable. (6)

(c) State Mean Value Theorem. If $f: [a, b] \to \mathbb{R}$ satisfies the conditions of Mean Value Theorem and $f'(x) = 0$ for all $x \in (a,b)$. Then prove that f is constant on [a, b]. (6)

6\. (a) Let I be an open interval and let $f : I \to \mathbb{R}$ have a second derivative on I. Then show that f is a convex function on I if and only if $f''(x) \geq 0$ for all $x \in I$. (6)

(b) Find the points of relative extrema of the functions $f(x) = |x^2 - 1|$, for $-4 \leq x \leq 4$. (6)

(c) Use Taylor's Theorem with n = 2 to approximate $\sqrt[3]{1+x}$, , $x > -1$. (6)

(3500)

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Your Roll No..............

Sr. No. of Question Paper : 1381 **C**

Unique Paper Code : 32351301 22 DEC 2022

Name of the Paper : BMATH 305 – Theory of Real Functions

Name of the Course : **CBCS (LOCF) B.Sc. (H) Mathematics**

Semester : III

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt any **two** parts from each question.
3. **All** questions are compulsory.

1\. (a) Let $A \subseteq \mathbb{R}$ and $c \in \mathbb{R}$ be a cluster point of A and $f : A \to \mathbb{R}$, then define limit of function f at c.

Use $\varepsilon - \delta$ definition to show that $\lim_{x \to 1} \frac{x}{x+1} = \frac{1}{2}$. (6)

P.T.O.

(b) Let $f: A \to \mathbb{R}$, $A \subseteq \mathbb{R}$ and $c \in \mathbb{R}$ be a cluster point of A. Then show that $\lim_{x \to c} f(x) = L$ if and only if for every sequence $\langle x_n \rangle$ in A that converges to c such that $x_n \neq c$, $\forall n \in \mathbb{R}$, the sequence $\langle f(x_n) \rangle$ converges to L. (6)

(c) Show that $\lim_{x \to 0} \sin\left(\frac{1}{x^2}\right)$ does not exist in $\mathbb{R}$ but $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x^2}\right) = 0$. (6)

2. (a) Let $A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$, $g: A \to \mathbb{R}$ and $c \in \mathbb{R}$ be a cluster point of A. Show that if f is bounded on a neighborhood of c and $\lim_{x \to c} g(x) = 0$, then $\lim_{x \to c} (fg)(x) = 0$. (6)

(b) Let $f(x) = e^{1/x}$ for $x \neq 0$, then find $\lim_{x \to 0} f(x)$ and $\lim_{x \to 0^+} f(x)$. (6)

(c) Let $f: \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} 2x & \text{if } x \text{ is rational} \\ x + 3 & \text{if } x \text{ is irrational} \end{cases}.$$

Find all the points at which f is continuous. (6)

3. (a) Let $A \subseteq \mathbb{R}$ and let f and g be real valued functions on A. Show that if f and g are continuous on A then their product fg is continuous on A. Also, give examples of two functions f and g such that both are discontinuous at a point $c \in A$ but their product is continuous at c. (7½)

(b) State and prove Boundedness Theorem for continuous functions on a closed and bounded interval. (7½)

(c) State Maximum-Minimum Theorem. Let $I = [a,b]$ and $f: I \to \mathbb{R}$ be a continuous function such that $f(x) > 0$ for each x in I. Prove that there exists a number $\alpha > 0$ such that $f(x) \geq \alpha$ for all x in I. (7½)

4. (a) Let $A \subseteq \mathbb{R}$ and $f: A \to \mathbb{R}$ such that $f(x) \geq 0$ for all $x \in A$. Show that if f is continuous at $c \in A$, then $\sqrt{f}$ is continuous at c. (6)

(b) Show that every uniformly continuous function on $A \subseteq R$ is continuous on A. Is the converse true? Justify your answer. (6)

(c) Show that the function $f(x) = \frac{1}{x^2}$, $x \neq 0$ is uniformly continuous on $[a, \infty)$, for $a > 0$ but not uniformly continuous on $(0, \infty)$. (6)

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