[This question paper contains 6 printed pages.]

Your Roll No. ....

Sr. No. of Question Paper : 1359

Unique Paper Code : 32351602

Name of the Paper : BMATH614: Ring Theory and Linear Algebra II

Name of the Course : **B.Sc. (Hons.) Mathematics**

Semester : VI

Duration : 3 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. Attempt any **two** parts from each question.

1. (a) (i) If D is an Integral domain, prove that D[x] is an integral domain.

(ii) If R is a commutative ring, prove that the characteristic of R[x] is same as the characteristic of R.

(b) Let $f(x) = 5x^4 + 3x^3 + 1$ and $g(x) = 3x^2 + 2x + 1$ in $Z_7[x]$. Compute the product f(x)g(x). Determine the quotient and the remainder upon dividing f(x) by g(x).

P.T.O.

(c) Let F be a field and let $I = \{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0 | a_i \in F$ and $f(1) = a_n + \ldots a_0 = 0\}$. Prove that I is an Ideal of F[x] and find a generator of I.

(d) Let R[x] denote the ring of polynomials with real coefficients. Then prove that $\frac{R[x]}{\langle x^2+1 \rangle}$ is isomorphic to the ring of complex numbers.

(3+3.5,6.5,6.5,6.5)

2. (a) (i) Let F be a field and $p(x) \in F[x]$ be irreducible over F. Prove that $\langle p(x) \rangle$ is a maximal ideal in F[x].

(ii) Show that, $\frac{Z_2[x]}{\langle x^3+x+1 \rangle}$ is a field with 8 elements.

(b) Determine which of the polynomials below are irreducible over Q.

(i) $3x^5 + 15x^4 - 20x^3 + 10x + 20$

(ii) $x^4 + x + 1$

(c) In integral domain $Z[\sqrt{-3}]$, prove that $1+\sqrt{-3}$ is irreducible but not prime.

(d) Define Euclidean domain. Prove that every Euclidean domain is a principal ideal domain.

(3+3,3+3,6,6)

3. (a) Let $V = P_1(R)$ and V^* denote the dual space of V. For $p(x) \in V$, define $f_1, f_2 \in V^*$ by $f_1(p(x)) = \int_0^1 p(t)dt$ and $f_2(p(x)) = \int_0^2 p(t)dt$. Prove that $\{f_1, f_2\}$ is a basis for V^* and find a basis for V for which it is the dual basis.

(b) Let W be a subspace of finite dimensional vector space V. Prove that $\dim(W) + \dim(W^\circ) = \dim(V)$, where W° is annihilator of W.

(c) Let T be a linear operator on $M_{n\times n}(R)$ defined by $T(A) = A^t$. Show that ± 1 are the only eigenvalues of T. Find the eigenvectors corresponding to each eigenvalue. Also find bases for $M_{2\times 2}(R)$ consisting of eigenvectors of T.

(d) Let T be a linear operator on R^3 defined by $T(a, b, c) = (3a + b, 3b + 4c, 4c)$. Show that T is digonalizable by finding a basis for R^3 consisting of eigen vectors of T. (6.5,6.5,6.5,6.5)

4. (a) Let T be a linear operator on finite dimensional vector space V and let W be the T-cyclic subspace of V generated by a non-zero vector $v \in V$. Let k = dim (W). Then prove that $\{v, T(v), \ldots\ldots, T^{k-1}(v)\}$ is basis for W.

(b) State Cayley Hamilton Theorem. Verify the theorem for linear operator $T: R^2 \to R^2$ defined by $T(a, b) = (a + 2b, -2a + b)$.

(c) Let T be a linear operator on R^3 defined by $T(a, b, c) = (3a - b, 2b, a - b + 2c)$. Find the characteristic polynomial and minimal polynomial of T.

(d) (i) Let T be an invertible linear operator. Prove that a scalar λ is an eigen value of T if and only if λ^{-1} is an eigenvalue of T^{-1}.

(ii) Prove that similar matrices have the same characteristic polynomial. (6,6,6,3+3)

5. (a) Show that in a complex inner product space V over field F. For $x, y \in V$, prove the following identities

(i) $\langle x, y\rangle = \frac{1}{4}\|x + y\|^2 - \frac{1}{4}\|x - y\|^2$ if $F = R$

(ii) $\langle x, y\rangle = \frac{1}{4}\sum_{k=1}^{4} i^k \left\|x + i^k y\right\|^2$ if F = C, where $i^2 = -1$.

(b) Let V be an inner product space, and let $S = \{v_1, v_2, \ldots, v_n\}$ be an orthonormal subset of V. Prove the Bessel's Inequality :

$$\|x\|^2 \geq \sum_{i=1}^{n} \left|\langle x, v_i\rangle\right|^2 \quad \text{for any } x \in V.$$

Further prove that Bessel's Inequality is an equality if and only if $x \in \text{span}(S)$.

(c) Let $V = P_2(R)$, with the inner product

$$\langle f(x), g(x)\rangle = \int_0^1 f(t)g(t)dt$$

and with the standard basis $\{1, x, x^2\}$. Use Gram-Scmidth process to obtain an orthonormal basis β of $P_2(R)$. Also, compute the Fourier coefficients of $h(x) = 1 + x$ relative to β.

P.T.O.

(d) Find the minimal solution to the following system of linear equations

$x + 2y - z = 1$

$2x + 3y + z = 2$

$4x + 7y - z = 4$ (3+3.5,6.5,6.5,6.5)

6. (a) For the data {(–3, 9), (–2, 6), (0, 2), (1, 1)}, use the least squares approximation to find the best fit with a linear function and compute the error E.

(b) Let T be a linear operator on a finite dimensional inner product space V. Suppose that the characteristic polynomial of T splits. Then prove that there exists an orthonormal basis β for V such that the matrix $[T]_\beta$ is upper triangular.

(c) (i) Let T be a linear operator on $\mathbb{C}^2$ defined by $T(a, b) = (2a + ib, a + 2b)$. Determine whether T is normal, self-adjoint, or neither.

(ii) For $z \in \mathbb{C}$, define $T_z: \mathbb{C} \to \mathbb{C}$ by $T_z(u) = zu$. Characterize those z for which T_z is normal, self adjoint, or unitary.

(d) Let U be a Unitary operator on an inner product space V and let W be a finite dimensional U–invariant subspace of V. Then, prove that

(i) $U(W) = W$

(ii) $W^\perp$ is U-invariant (6,6,3+3,3+3)

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