[This question paper contains 4 printed pages.]

Your Roll No...............

Sr. No. of Question Paper : 1505

Unique Paper Code : 42354401

Name of the Paper : Real Analysis

Name of the Course : **B.Sc. Mathematical Sciences/ B.Sc. (Prog.)**

Semester : IV

Duration : 3.5 Hours Maximum Marks : 75

Instructions for Candidates

1. Write your Roll No. on the top immediately on receipt of this question paper.
2. This question paper has **six** questions in all.
3. Attempt any **two** parts from each question.

1. (a) Show that a countable union of countable sets is countable. Deduce that the set $N \times N$ is countable.

   (b) State and prove the Archimedean property of real numbers.

   (c) Let S be a non-empty bounded set in R. Show that

$$Sup(a\,S) = a\,SupS \;\; if\; a > 0$$
$$\&\; \text{Inf}(a\,S) = a\,\text{Sup}\,S \;\; if\; a < 0$$

P.T.O.

(d) Find all $x \in R$ that satisfy the following inequalities :

(i) $|x-1| > |x+1|$

(ii) $|x| + |x+1| < 2$

(6, 6)

2. (a) Let A and B be two non-empty subsets of R that satisfy the property:

$a \le b \;\forall\, a \in A \;\&\; \forall\, b \in B$

show that $Sup\ A \le Inf\ B$.

(b)Use the definition of the limit of a sequence to establish the following limits

(i) $Lt\left(\frac{3n+1}{2n+5}\right) = \frac{3}{2}$

(ii) $Lt\left(\sqrt{n+1} - \sqrt{n}\right) = 0$

(c) If $Lt\,(x_n) = x > 0$, show that $\exists$ a natural no. k : if $n \ge k$, then

$\frac{1}{2}x < x_n < 2x$

(d) Let $< x_n >$ be a sequence of positive real numbers such that

$Lt\left({x_n}^{1/n}\right) = l < 1$, show that $Lt\,(x_n) = 0$.

(6, 6)

3. (a) State and prove Monotone Convergence theorem.

(b) Establish the convergence or the divergence of the sequence $< x_n >$, where

$x_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$, $n \in N$.

(c) State Cauchy Convergence Criteria for sequences.

Show that the sequence $< x_n >$, where

$x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{n}$ is a divergent sequence.

(d) State and prove the necessary condition for the convergence of an infinite series.

Is the condition sufficient ? Justify the answer.

(6.5, 6.5)

4. (a) Test for the convergence and absolute convergence of the series

$1 - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{7}} + \cdots$

(b) Show that the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is a divergent series when $p < 1$.

(c) Test for the convergence of the following series :

(i) $\sum_{n=1}^{\infty}\left(\sqrt{n^3+1} - n^{3/2}\right)$

(ii) $\sum_{n=1}^{\infty} \frac{1}{n^n}$

(d) Test the series

$1 + \frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + \cdots$ for convergence, for all positive values of x .

(6, 6)

5. (a) Prove that $\lim_{n\to\infty} \frac{nx}{1+n^2x^2} = 0 \;\forall\, x \in R$.

Also prove that $< \frac{nx}{1+n^2x^2} >$ is not uniformly convergent on $[0, \infty[$, but is uniformly convergent on $[a, \infty[$, $a > 0$.

(b) Discuss the convergence and uniform convergence of the series $\sum f_n(x)$ where $f_n(x)$ is given by $(x^2 + n^2)^{-1}$.

(c) Determine the radius of convergence and exact interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n$.

(d) State and prove Weierstrass M – test for uniform convergence of a series of functions $\sum f_n(x)$, also test the uniform convergence of $\sum \frac{\sin(x^2+n^2x)}{n(n+1)} \;\forall\, x \in R$.

(6.5, 6.5)

6. (a) Define exponential function in terms of power series. Prove that

(i) $E'(x) = E(x) \;\forall\, x \in R$

(ii) $E(x+y) = E(x)\,E(y) \;\forall\, x, y \in R$.

(iii) $E(r) = e^r \;\forall\, r \in Q$

where E denotes exponential function.

(b) Let f be a bounded real function defined on $[a, b]$. Let P be any partition of $[a, b]$. Define the upper and lower sums of f over P and show that

$m(b-a) \le L(P, f) \le U(P, f) \le M(b-a)$

(c) Show that the function $f(x) = x^2 + x$ is integrable on $[0, 1]$ and find $\int_0^1 f(x)$, using Sequential Criteria .

(d) Let f be a bounded real function defined on $[a,b]$. Let P be any partition of $[a,b]$. Define the upper and lower sums of f over P and show that

$$m(b-a) \leq L(P,f) \leq U(P,f) \leq M(b-a)$$

where m and M are bounds of f over $[a,b]$.

(6.5, 6.5)