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| Name of Course | : **CBCS B.Sc. (H) Mathematics** |
| Unique Paper Code | : **32351602** |
| Name of Paper | : **C14-Ring Theory and Linear Algebra-II** |
| Semester | : **VI** |
| Duration | : **3 hours** |
| Maximum Marks | : **75 Marks** |

*Attempt any four questions. All questions carry equal marks.*

**1.** Prove that $\mathbb{Z}[x]$ is not a principal ideal domain. Also show that $2x^2 + x + 1$ is irreducible over $\mathbb{Z}_3$. Construct a field of order 16.

**2.** Prove that in a unique factorization domain, an element is irreducible if and only if it is prime. Prove or disprove that a subdomain of a principal ideal domain is a principal ideal domain. Show that $x^4 + 1$ is irreducible over $\mathbb{Q}$ but reducible over $\mathbb{R}$.

**3.** Let $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be a linear operator such that

$$T\big(f(x)\big) = f'(x) + f''(x).$$

Find the eigenvalues of $T$ and their corresponding eigenspaces. Is $T$ a diagonalizable linear operator? Find the minimal polynomial of $T$. Now suppose that $V = \mathbb{R}^3$ and$\beta = \{(1,0,2), (0,1,1), (1,1,0)\}$ be an ordered basis for $V$.Find an ordered basis $\beta^*$of $V^*$ which is the dual basis corresponding to $\beta$.

**4.** Find an ordered basis for the $T$-cyclic subspace $W$ of $\mathbb{R}^4$ generated by the vector $z$ where $T: \mathbb{R}^4 \to \mathbb{R}^4$ is a linear operator such that

$$T(a, b, c, d) = (c + d, -b, a + b, 2a + b)$$

and $z = e_1$.Is $W$ a $T$-invariant subspace of $\mathbb{R}^4$?Find the characteristic polynomial of $T_W$.Show that the characteristic polynomial of $T_W$ obtained above divides the characteristic polynomial of $T$.Verify Cayley-Hamilton Theorem for $T_W$.

**5.** Apply the Gram-Schmidt process to the subset

$$S = \{f_1, f_2, f_3\}$$

of the inner product space $V = C[-\pi, \pi]$ with the inner product given by

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(t)g(t)\,dt$$

to obtain an orthogonal basis forspan$(S)$, i.e., the subspace of $V$ spanned by the functions in $S$, where $f_1(x) = 1$, $f_2(x) = \sin x$ and $f_3(x) = \cos x$. Then normalize the vectors in this basis to obtain an orthonormal basis $\beta$ for span$(S)$.

**6.** Use the least squares approximation to find the best fit quadratic function for the set

$$\{(-1,5),(1,1),(2,1),(3,-3)\}.$$

Also compute the corresponding error $E$. Also find the minimal solution to the following system of linear equations:

$$x+y+z-w=1$$
$$2x-y+w=1$$