Name of the Course : CBCS B.Sc. (H) Mathematics

Unique Paper Code : 32357607

Name of the Paper : DSE - III Probability Theory and Statistics

Semester : VI

Duration : 3 hours

Maximum Marks : 75

Attempt any four questions. All questions carry equal marks.

1. If the random variable T is the time to failure of a commercial product and the values of its probability density and distribution function at time t are $f(t)$ and $F(t)$, then its failure rate at time t is given by $\frac{f(t)}{1-F(t)}$. Thus, the failure rate at time t is the probability density of failure at time t given that failure does not occur prior to time t.

   Show that if T has the exponential distribution, the failure rate is constant.
   Show the random variable $\boldsymbol{X}$ has probability density function $f(x)$ if it is defined by

$$f(x) = \begin{cases} \frac{\alpha}{x^{\alpha+1}}, & for\ x > 1 \\ 0, & elsewhere \end{cases}$$

   where $\alpha > 0$. Also show that μ_r' exists only if $r < \alpha$.

2. Let $\boldsymbol{X}$ be binomially distributed with parameters $\boldsymbol{n}$ and $\boldsymbol{\theta}$. Show that as $\boldsymbol{k}$ goes from $\mathbf{0}$ to $\boldsymbol{n}$, $P(\boldsymbol{X} = \boldsymbol{k})$ increases monotonically, then decreases monotonically reaching its largest value in the case that $(\boldsymbol{n} + \mathbf{1})\, \boldsymbol{\theta}$ is an integer, when k equals either $(\boldsymbol{n} + \mathbf{1})\, \boldsymbol{\theta} - \mathbf{1}$ or $(\boldsymbol{n} + \mathbf{1})\, \boldsymbol{\theta}$.

   An airline knows that 5 percent of the people making reservations on a certain flight will not show up. Consequently, their policy is to sell 52 tickets for a flight that can hold only 50 passengers. What is the probability that there will be a seat available for every passenger who shows up?

3. The joint probability density function of X & Y is:

$$f(x,y) = \begin{cases} \frac{2}{3}\,(x+y) & ,0 < x < 1\,, 0 < y < 1 \\ 0 & ,\text{otherwise} \end{cases}$$

   Find **(a)** the marginal density functions of X and Y **(b)** conditional density of X given y **(c)** evaluate $P(X \le 1/2 \mid Y = 1/2\ )$ (d) conditional mean and variance of X given $Y = \frac{1}{2}$.

4. The joint probability density function of (X, Y) is given to be

$$f(x,y) = \begin{cases} k(y-x)e^{-y} & , \quad -y < x < y \\ 0 & , \quad 0 < y < \infty \end{cases}$$

Find **(a)** the constant k **(b)** mean of X **(c)** mean of Y **(d)** Covariance (X,Y)

5. Variates X and Y have zero means and standard deviations σ_1, σ_2 are normally correlated with correlation coefficient ρ. Show that

$$U = \frac{X}{\sigma_1} + \frac{Y}{\sigma_2}, \qquad V = \frac{X}{\sigma_1} - \frac{Y}{\sigma_2}$$

are independent random variables and follow the normal distribution.

Let the Markov chain consisting of the states 1, 2, 3, 4, 5, 6 and have the transition probability matrix

$$P = \begin{bmatrix} 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.4 & 0.6 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.3 & 0.0 & 0.4 & 0.2 & 0.1 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.3 & 0.7 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.5 & 0.0 & 0.5 \\ 0.0 & 0.0 & 0.0 & 0.3 & 0.0 & 0.7 \end{bmatrix}$$

Determine which states are transient and which are recurrent.

6. Let X has the probability density function

$$f(x) = \begin{cases} \frac{1}{2\sqrt{5}}, & -\sqrt{5} < x < \sqrt{5} \\ 0, & elsewhere \end{cases}$$

Find the actual probability $P\left[|X - E[X]| \geq \frac{3}{2}\sigma\right]$ and compare it with the upper bound obtained by Chebyshev's inequality. Further, if the variate X has the probability density function $f(x) = e^{-x}, \quad x \geq 0$. Use Chebyshev's inequality to show that

$$P[|X - 1| > 2] < \frac{1}{4}$$

and show that the actual probability is e^{-3} .