Name of Course	: **CBCS B.Sc. Hons Mathematics**
Unique Paper Code	: **32351302**
Name of Paper	: **BMATH306-Group Theory-1**
Semester	: **III**
Duration	: **3 hours**
Maximum Marks	: **75 marks**

Attempt any four questions. All questions carry equal marks.

1. Let A be a non-empty set and $\langle G, . \rangle$ be a group. Let F be the set of all functions from A to G. Define an operation $*$ on F as follows:

For $f, g \in F$, let $f * g : A \to G$ as $(f * g)(x) = f(x). g(x)\ \forall\, x \in A$.

Prove that $\langle F, * \rangle$ is a group.

Find the inverse of $\begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$ in $GL(2, \mathbb{Z}_5)$, the group of 2×2 non-singular matrices over $\mathbb{Z}_5$. Verify the answer by direct calculation.

Describe the group of symmetries of a non-square rectangle and draw its Cayley's table.

2. Let a be an element of a group such that $|a| = 3$, prove that $C(a) = C(a^2)$. Give an example to show that the conclusion fails if $|a| = 4$.

Find the orders of each of the elements of U(14). Show that it is cyclic and find all its generators.

3. Define Centre $Z(G)$ of a group G and prove that $Z(S_4) = \{e\}$.

For $n > 2$, show that every even permutation in S_n is a product of 3-cycles.

Let σ = (1,5,7)(2,5,3)(1,6). Express σ^{17} as a cycle.

4. Prove or disprove any six, stating the results used

(i) $\langle \mathbb{R}, + \rangle \approx \langle \mathbb{Q}, + \rangle$, (ii) $\langle \mathbb{Q}, + \rangle \approx \langle \mathbb{Z}, + \rangle$, (iii) $\langle \mathbb{R}, + \rangle \approx \langle \mathbb{R}+, . \rangle$,

(iv) $D_4 \approx$ Group Q of Quaternions, (v) $U(20) \approx D_4$,

(vi) $U(8) \approx U(12)$, (vii) $U(10) \approx \mathbb{Z}_4$, (viii) $\frac{GL(2,\mathbb{R})}{SL(2,\mathbb{R})} \approx \mathbb{R}^*$.

5. Let H be a subgroup of a group G. Prove that $aH \mapsto Ha^{-1}$ is a bijective mapping from the set of all left cosets of H in G to the set of all right cosets of H in G. Can the same be said for $aH \mapsto Ha$?

If G is a non-abelian group of order 8 with $Z(G) \neq \{e\}$, prove that $|Z(G)| = 2$.

6. Let *N* be a normal subgroup of *G* and *M* be a normal subgroup of *N*. If *N* is cyclic, prove that *M* is a normal subgroup of *G*. Show by an example that the conclusion fails to hold if *N* is not cyclic.

If φ is a homomorphism from a finite group G to a finite group G', prove that $|\varphi(G)|$ divides the gcd of $|G|$ and $|G'|$.