Name of the Course : **B.Sc. (Hons.) Mathematics CBCS**

Semester : **VI**

Unique Paper Code : **32351602**

Name of the Paper : **C14 - Ring Theory and Linear Algebra-II**

Duration: **2 Hours** Maximum Marks: **75**

Attempt any four questions. All questions carry equal marks. All symbols have usual meaning.

1. Suppose that $f(x) \in \mathbb{Z}_p[x]$ and is irreducible over $\mathbb{Z}_p$, where p is a prime. If $\deg f(x) = n$, find the number of elements in the field $\mathbb{Z}_p[x]/\langle f(x)\rangle$.

Show that the polynomial $x^2 + 2x + 3$ is irreducible over $\mathbb{Z}_5$ and use this to construct a field of order 25.

2. Let D be a Euclidean domain and d the associated function. If a and b are associates in D, then what is the relation between $d(a)$ and $d(b)$? Justify your answer.

Also prove that $2 + 3i$ and $2 - 3i$ are not associates in $\mathbb{Z}[i]$.

3. Let $V = \mathbb{R}^3$ and define $f_1, f_2, f_3 \in V^*$ as follows

$$f_1(x,y,z) = x + y, \quad f_2(x,y,z) = x - 2y + z, \quad f_3(x,y,z) = 3z.$$

Prove that $\{f_1, f_2, f_3\}$ is a basis for V^*, and then find a basis for V for which it is the dual basis.

4. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. Show that A is diagonalizable and find a 2×2 invertible martix Q such that $Q^{-1}AQ$ is a diagonal matrix. Also compute A^n where n is a positive integer.

5. Let $V = \mathbb{R}^3$, $u = (1,2,2)$ and $W = \{(x,y,z) : x + y - 3z = 0\}$. Find the orthogonal projection of the given vector u on the given subspace W of the inner product space V.

6. Show that every self-adjoint operator on a finite-dimensional inner product space is normal. Is the converse true? Justify your answer.

Let V be an inner product space over $\mathbb{R}$ and let T be a normal operator on V. Then show that $T - 3I$ is normal, where I is the identity operator on V.