Name of the Course : **B.Sc. (Hons.) Mathematics CBCS**

Semester : **VI**

Unique Paper Code : **32351601**

Name of the Paper : **C13 - Complex Analysis**

Duration: **2 Hours** Maximum Marks: **75**

Attempt any four questions. All questions carry equal marks. All symbols have usual meaning.

1. Let $S = \{z \in \mathbb{C}: |z| < 2\}$ and let T denotes the boundary of S. Find interior points, exterior points, boundary points and accumulation points of T. Does there exists a sequence (z_n) in T such that the series

$$\sum_{n=1}^{\infty} z_n$$

converges? Justify your answer. Expand the function $1/(2-z)$ into the Maclaurin series valid in the disk S. If $f: S \to T$ is a function such that f is analytic everywhere in S, prove that f is constant throughout S. If $g: \mathbb{C} \to T$ is an entire function, prove that g is constant throughout the complex plane.

2. Show that the function

$$f(z) = ze^{-z}$$

is entire by verifying that the real and imaginary parts of f satisfy the Cauchy–Riemann equations at each point of the complex plane. What is the anti-derivative of f? If C is any contour extending from $z = 0$ to $z = i\,\pi$, find the value of the integral

$$\int_C f(z)dz.$$

Also, use the ML-inequality to prove that

$$\left| \int_C \frac{f(z)}{z^2 - 1} dz \right| \leq \frac{2\pi\sqrt{e}}{3}$$

where C is the positively oriented circle $|z| = 1/2$.

3. Consider the function $f: \mathbb{C} \to \mathbb{C}$ defined by $f(z) = (Im\, z)^2$. Use the Cauchy–Reimann equations to determine the points where f is differentiable. Is f analytic at those points? Compute the integral

$$\int_C f(z)dz$$

where C is the boundary of the square $\{0 < x < 1 \,\&\, 0 < y < 1\}$ in the counter clockwise direction.

4. Let C be the positively oriented circle $|z| = 1$. Use the Cauchy Integral Formula to evaluate

$$\int_C \frac{\cos z}{z} dz.$$

Deduce that

$$\int_0^{2\pi} \cos(\cos t) \cosh(\sin t)\, dt = 2\pi.$$

Use the extension of Cauchy Integral Formula to find the value of the integral

$$\int_C \frac{e^z \cos z}{z^4} dz.$$

5. Find the pair of complex numbers z_1 and z_2 such that

$$Log\,(z_1 z_2) \neq Log\, z_1 + Log\, z_2$$

where $Log\, z$ represents the principal value of $\log z$. Is $Log(1+i)(1-i) = Log(1+i) + Log(1-i)$? Justify your answer. Expand the functions $z^3 - 6z^2 + 7z - 3$ into a Taylor series about the point $z_0 = 1$. Give two Laurent series expansions in powers of z for the function

$$f(z) = \frac{z}{(z-1)(z-2)}$$

and specify the regions in which those expansions are valid.

6. Determine whether $z_0 = 0$ is a pole, a removable singularity or an essential singular point of the function

$$f(z) = \frac{1}{1 - \cos z}$$

and

$$g(z) = z^3 e^{1/z^2}.$$

Also, determine the residue of f and g at z_0. Use the substitution $z = e^{it}$ and the Cauchy Residue Theorem to evaluate the integral

$$\int_0^{2\pi} \frac{4\cos x}{5 - 4\cos x} dx.$$