Name of the Course : **B.Sc. (Hons.) Mathematics CBCS**

Semester : **IV**

Unique Paper Code : **32351402**

Name of the Paper : **C9 - Riemann Integration and Series of Functions**

Duration: **2 Hours** Maximum Marks: **75**

Attempt any four questions. All questions carry equal marks. All symbols have usual meaning.

1. Find the upper and lower Darboux integrals for $f(x) = x^2 + 2$ on the interval $[0,1]$. Is f integrable on $[0,1]$? Justify.

Let $g: [1,5] \to R$ be defined as

$$g(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 3, & \text{if } x \text{ is irrational.} \end{cases}$$

Find the upper and lower sums, $U(g)$ and $L(g)$ respectively, on $[1,5]$. Is g integrable on $[1,5]$?

Prove that $\left|\int_{-2\pi/3}^{2\pi/3} \left(\frac{3x}{4}\right)^2 \cos^3(x^5 + 3e^x)dx\right| \leq \frac{\pi^3}{9}$ clearly stating the results used.

2. Show that the $Sgn\ x$, the Signum function, is integrable on $[-a, a]$ for any $a > 0$.

Using Fundamental theorem of Calculus I, show that

$$\frac{1}{2}\int_2^3 x^{\frac{n}{2}}\, dx = \frac{3^{\frac{n+2}{2}} - 2^{\frac{n+2}{2}}}{n+2}.$$

Calculate $\lim_{h\to 0} \frac{1}{h}\int_1^{1+h} e^{(3t^2+t)}dt$. State the theorem used.

3. Define improper integral of type II. When does it converge? When is it said to diverge?

Examine the convergence of $\int_2^{+\infty} \frac{dx}{(x-1)^2}$ and $\int_{-\infty}^0 e^{-x}\, dx.$

Show that the improper integral $\int_0^1 x^{a+1}\, (1-x)^b dx$ converges if and only if $a > -2$ and $b > -1$.

4. Define $f_n : [0,1] \to \mathbb{R}$ as

$$f_n(x) = \begin{cases} 2n - 2n^2 x & \text{if } \ 0 \le x \le 1/n \\ 0 & \text{if } \ 1/n < x \le 1. \end{cases}$$

Show that the pointwise limit does not exist on $[0,1]$. Further if we modify and define $f_n : [0,1] \to \mathbb{R}$ as

$$f_{n(x)} = \begin{cases} 0 & \text{if} \quad x = 0 \\ -n^2(2x - 2/n) & \text{if} \quad 0 < x \le 1/n \\ 0 & \text{if} \quad 1/n \ \le x \le 1 \end{cases}$$

then find the pointwise limit f of (f_n) on $[0,1]$. Show that f_n and f are integrable on $[0,1]$ but $lim_{n\to\infty} \int_0^1 f_n \neq \int_0^1 f$. Hence discuss the uniform convergence of (f_n) on $[0,1]$.

Let $f_n(x) = \frac{nx}{1+n^2x^2}$. Compute the pointwise limt $f(x) = lim_{n\to\infty} f_n(x)$. Show that the sequence (f_n) does not converge uniformly to f on $[-1,1]$ but does converge uniformly on $[1,\infty)$.

5. State a sufficient condition for the uniform convergence of a series of functions. Using this prove that $\sum_{n=1}^{\infty} \frac{e^{-2x}}{3n^2+2n}$ is uniformly convergent on $[0,\infty)$. Also show that it is a continuous function on $[0,\infty)$.

Examine the pointwise convergence of the series $\sum f_n$, where $f_n(x) = \frac{x^n}{2x^n+5}$, $x \ge 0$.

Show that $\sum_{n=1}^{\infty} \sin\left(\frac{x^2}{n^3}\right)$ is uniformly convergent on $(-1,1)$ but it is not uniformly convergent on $\mathbb{R}$.

6. Find the radius of convergence and determine the exact interval of convergence of the power series

$$\sum_{k=1}^{\infty} \frac{2^k}{k^2} x^k, \quad \sum_{k=0}^{\infty} 2^{-k} x^{4k}.$$

Given

$$\frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k \quad \text{for } \ |x| < 1,$$

show that $\frac{1}{(1+x)^2} = \sum_{k=1}^{\infty} (-1)^{k+1} k x^{k-1}$ and hence $\frac{x}{(1+x)^2} = \sum_{k=1}^{\infty} (-1)^{k+1} k x^k$. Also evaluate $\sum_{k=1}^{\infty} (-1)^{k+1} k \left(\frac{3}{4}\right)^k$.