Name of the Course : **B.Sc. (Hons.) Mathematics CBCS**

Semester : **IV**

Unique Paper Code : **32351403**

Name of the Paper : **C10 - Ring Theory and Linear Algebra-I**

Duration: **2 Hours** Maximum Marks: **75**

Attempt any four questions. All questions carry equal marks. All symbols have usual meaning.

1. If $P_n(\mathbb{R})$ denotes the space of all polynomials of degree n or less with coefficients in $\mathbb{R}$ and if $T: P_1(\mathbb{R}) \to P_2(\mathbb{R})$ is a linear transformation such that $T(x+1) = x^2 - 1$ and $T(x-1) = x^2 + x$, what is $T(7x+2)$? Show that $\{x+1, x-1\}$ is a basis of a $P_1(\mathbb{R})$.

Let a linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ be defined as $T(a,b,c) = (a+c, a+b+2c, 2a+b+3c)$. Find a basis for the null space of T and a basis for the range space of T. Verify dimension theorem. Is T one-one? Is T onto?

2. Let a linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$ be defined as $T(x,y) = (x-2y, 2x+y, x+y)$. Let $\beta = \{(1,-1), (0,1)\}$ and $\gamma = \{(1,1,0), (0,1,1), (1,-1,1)\}$ be bases of $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. Find the matrix of T with respect to β and γ, that is $[T]_\beta^\gamma$. If β' and γ' denote the standard basis of $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively then find $[T]_{\beta'}^{\gamma'}$. What is the relation between the two matrices?

3. Can the polynomial $6x^3 - 3x^2 + x + 2$ be expressed as a linear combination of the polynomials $x^3 - x^2 + 2x + 3$ and $2x^3 - 3x + 1$ in $P_3(\mathbb{R})$? Justify.

Determine if the set

$$\left\{ \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 2 & -2 \end{pmatrix} \right\}$$

in $M_{2\times 2}(\mathbb{R})$ is linearly independent or linearly dependent, where $M_{2\times 2}(\mathbb{R})$ is the set of all 2×2 matrices over $\mathbb{R}$. Justify.

Let $S = \{(1,1,0), (1,0,1), (0,1,1)\}$ be a subset of the vector space F^3. Prove that if $F = \mathbb{R}$, then S is linearly independent and if F has characteristic 2, then S is linearly dependent.

4. Determine if the polynomials $x^2 + x + 1$, $2x^2 + 3x - 1$ and $-x^2 - x + 5$ generate $P_2(\mathbb{R})$. Justify.

What is the standard basis in $P_n(F)$? For a fixed $a \in \mathbb{R}$, determine the dimension of the subspaces of $P_n(\mathbb{R})$ defined by $\{f \in P_n(\mathbb{R}) : f(a) = 0\}$.

5. Show that $S = \{a + ib : a, b \in \mathbb{Z}, b \text{ is even}\}$ is a subring of $\mathbb{Z}[i]$ but not an ideal of $\mathbb{Z}[i]$. Further, with complete explanation, find the number of elements in $\mathbb{Z}[i]/\langle 3 + i\rangle$. Find all maximal ideals in $\mathbb{Z}_8$.

6. Show that the field $\mathbb{Z}_3[i]$ is ring isomorphic to the field $\frac{\mathbb{Z}_3[x]}{\langle x^2+1\rangle}$.

Determine all ring homomorphism from $\mathbb{Z}_{20}$ to $\mathbb{Z}_{30}$.