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[This question paper contains 4 printed pages]

Your Roll No. : ....

Sl. No. of Q. Paper : **7464** **J**

Unique Paper Code : 32351302

Name of the Course : **B.Sc.(Hons.) Mathematics**

Name of the Paper : Group Theory - I

Semester : III

Time : 3 Hours ***Maximum Marks : 75***

Instructions for Candidates :

(a) Write your Roll No. on the top immediately on receipt of this question paper.

(b) Attempt any **two** parts from each question.

(c) All questions carry equal marks.

1. (a) Let $G=\left\{\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} : a \in R, a \neq 0\right\}$. Show that G is a group under matrix multiplication.

(b) Let G be a group and H be a subset of G. Prove that H is a subgroup of G if $a, b \in H \Rightarrow ab^{-1} \in H$. Hence prove that H = {$A \in G$: det A is a power of 3} is a subgroup of GL (2, R).

(c) (i) Suppose G is a group that has exactly eight elements of order 3. How many subgroups of order 3 does G have ?

(ii) If $|a| = n$ and k divides n, prove that $|a^{n/k}|=k$.

$6 \times 2 = 12$

2. (a) Let G = <a> be a cyclic group of order n. Prove that G = <a^k> if and only if gcd (k, n) =1. List all the generators of Z_{20}.

(b) (i) If a cyclic group has an element of infinite order, how many elements of finite order does it have.

(ii) List all the elements of order 6 and 8 in Z_{30}.

(c) Suppose that a and b are group elements that commute and have orders m and n. If <a> $\cap$ <b> = {e}, Prove that the group contains an element whose order is the least common multiple of m and n. Show that this need not be true if a and b do not commute.

$6.5 \times 2 = 13$

3. (a) Let G be a group. Is $H = \{x^2 : x \in G\}$ a subgroup of G ? Justify.

(b) Prove that any two left cosets of a subgroup H in a group G are either equal or disjoint.

(c) Show that (Q ,+) has no proper subgroup of finite index. $6 \times 2 = 12$

4. (a) Prove that every subgroup of index 2 is normal. Show that A_5 is normal subgroup of S_5.

(b) Let G be a group and H be a normal subgroup of G. Prove that the set of all left cosets of H in G forms a group under the operation aH.bH = abH where $a, b \in G$.

(c) If H is a normal subgroup of G with $|H| = 2$, prove that $H \subseteq Z(G)$. Hence or otherwise show that A_5 cannot have a normal subgroup of order 2. $6.5 \times 2 = 13$

5. (a) Let C be the set of complex numbers and

$$M = \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} : a, b \in R \right\}.$$

Prove that C and M are isomorphic under addition and $C^* = C \setminus \{0\}$ and $M^* = M \setminus \{0\}$ are isomorphic under multiplication.

(b) Prove that a finite cyclic group of order n is isomorphic to the group $Z_n = \{0, 1, 2, \ldots, n-1\}$ under addition modulo *n*.

(c) (i) Suppose that φ is an isomorphism from a group G onto a group G*. Prove that G is cyclic if and only if G* is cyclic.

 P.T.O.

(ii) Show that Z, the group of integers under addition is not isomorphic to Q, the group of rationals under addition. $6 \times 2 = 12$

6. (a) Let φ be a group homomorphism from a group G to a group G^* then prove that :

(i) $|\varphi(x)|$ divides $|x|$, for all x in G.

(ii) φ is one-one if and only if $|\varphi(x)| = |x|$, for all x in G.

(b) State and prove the Third Isomorphism Theorem.

(c) (i) Let G be a group. Prove that the mapping $\phi(g) = g^{-1}$, for all $g \in G$, is an isomorphism on G if and only if G is Abelian.

(ii) Determine all homomorphisms from Z_n to itself. $6.5 \times 2 = 13$