This question paper contains 4+2 printed pages]

Roll No. | | | | | | | | | | | |

10/5/19

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S. No. of Question Paper : **2253**

Unique Paper Code : **32351601** **IC**

Name of the Paper : **Complex Analysis**

Name of the Course : **B.Sc. (Hons.) Mathematics**

Semester : **VI**

Duration : **3 Hours** Maximum Marks : **75**

(Write your Roll No. on the top immediately on receipt of this question paper.)

Attempt all parts from Question No. **1**.

Each part carries **1½** marks.

Attempt any *two* parts from question Nos. **2** to **6**

Each part carries *six* marks.

1. State True *or* False. Justify your answer in brief :

(*a*) A point z_0 of a domain need not be an accumulation point of that domain.

(*b*) $\lim_{z \to 0} \frac{\overline{z}^2}{z} = 0$

P.T.O.

(c) The function $f(z) = e^z$ is periodic with period 2π.

(d) $\log(-ei) = 1 - \frac{\pi}{2}i$.

(e) The function $f(z) = |z|^2$ is analytic at $z = 0$.

(f) Let C denote the boundary of the triangle with vertices at the point 0, $3i$, and -4, oriented in the counterclockwise direction. Then $\left|\int_C (e^z - \bar{z})dz\right| \le 60$.

(g) If C is any simple closed contour, in either direction, then $\int_C \exp(z^3)dz = 0$.

(h) If C is the positively oriented unit circle $|z| = 1$, then $\int_C \frac{\exp(2z)}{z^4}dz = \frac{8\pi i}{3}$.

(i) $\text{Res}_{z=0} f(z) = -\frac{1}{3!}$, where $f(z) = z^2 \sin\left(\frac{1}{z}\right)$, $0 < |z| < \infty$.

(j) The function $f(z) = \frac{1}{\sin\left(\frac{\pi}{z}\right)}$ has no isolated singular point.

2. (a) Prove that a finite set of points cannot have any accumulation point.

(b) Suppose that $f(z) = u(x, y) + iv(x, y)$ $(z = x + iy)$ and $z_0 = x_0 + iy_0$, $w_0 = u_0 + iv_0$. Then $\lim_{z \to z_0} f(z) = w_0$ if and only if $\lim_{(x,y) \to (x_0,y_0)} u(x, y) = u_0$ and $\lim_{(x,y) \to (x_0,y_0)} v(x, y) = v_0$.

(c) Define neighbourhood of the point at infinity. Show that a set S is unbounded if and only if every neighbourhood of the point at infinity contains at least one point in S.

3. (a) Use Cauchy-Riemann equations to show that $f'(z)$ does not exist at any point if $f(z) = \exp(\bar{z})$. State sufficient conditions for differentiability of a function $f(z)$ at any point $z_0 = x_0 + iy_0 \in \mathbb{C}$.

(b) Suppose that a function $f(z) = u(x, y) + iv(x, y)$ and its conjugate $\overline{f(z)} = u(x, y) - iv(x, y)$ are both analytic in a given domain D. Show that $f(z)$ must be constant throughout D.

(c) Explain why $f(x) = \sin(x)$ is a bounded function on **R**, whereas $f(z) = \sin(z)$ is not a bounded function on the complex plane **C**, although $\sin^2(z) + \cos^2(z) = 1$ for all $z \in \mathbf{C}$.

4. (a) State and prove Cauchy Integral formula.

(b) State Liouville's theorem and use it to prove the fundamental theorem of algebra.

(c) Let C denote a contour of length L, and suppose that a function $f(z)$ is piecewise continuous on C. Show that $\left|\int_C f(z)dz\right| \le ML$, where M is a non-negative constant such that $|f(z)| \le M$ for all points z on C at which $f(z)$ is defined. Hence, show that

$$\left|\int_{C_R} \frac{2z^2 - 1}{z^4 + 5z^2 + 4} dz\right| \le \frac{\pi R(2R^2 + 1)}{(R^2 - 1)(R^2 - 4)},$$

where C_R denote the upper half of the circle $|Z| = R (R > 2)$, taken in the counterclockwise direction.

5. (a) Derive the expansions :

(i) $$\frac{\sinh z}{z^2} = \frac{1}{z} + \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+3)!} \quad (0 < |z| < \infty);$$

(ii) $$z^3 \cosh\left(\frac{1}{z}\right) = \frac{z}{2} + z^3 + \sum_{n=1}^{\infty} \frac{1}{(2n+2)!} \frac{1}{z^{2n-1}} \quad (0 < |z| < \infty)$$

(b) Give all the Laurent series expansions in powers of z for the function $f(z) = \dfrac{-1}{(z-1)(z-2)}$ and specify the domains in which those expansions are valid.

(c) If a power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges when $z = z_1 (z_1 \ne z_0)$, then show that it is absolutely convergent at each point z in the open disk $|z - z_0| < R_1$ where $R_1 = |z_1 - z_0|$.

6. (a) Show that the singular point of each of the following functions is a pole. Determine the order m of that pole and the corresponding residue B.

(i) $$f(z) = \frac{z^2 - 2z + 3}{z - 2}$$

(ii) $$f(z) = \frac{\sinh z}{z^4}$$

(iii) $$f(z) = \frac{\exp(2z)}{(z-1)^2}.$$

(b) Use residues to evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{5 + 4\sin\theta}$.

(*c*) If a function f is analytic everywhere in the finite plane except for a finite number of singular points interior to a positively oriented simple closed contour C, then show that $\int_C f(z)\,dz = 2\pi i\, \text{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]$. Use it to show that $\int_C \frac{5z-2}{z(z-1)}\,dz = 10\pi i,$ where C is the circle $|z| = 2$, described counterclockwise.