This question paper contains 4 printed pages.

Your Roll No. ....

S. No. of Paper	: 93	I
Unique Paper Code	: 32351501	
Name of the Paper	: **Metric Spaces**	
Name of the Course	: **B.Sc. (Hons.) Mathematics**	
Semester	: **V**	
Duration	: **3 hours**	
Maximum Marks	: 75	

(Write your Roll No. on the top immediately on receipt of this question paper.)

*Attempt any **two** parts from each question.*
***All** questions are compulsory*

1. (*a*) (i) Let X= **R** ∪ {∞ }∪ {−∞ }. Define the metric d on X by :

$$d(x, y) = \tan^{-1} x - \tan^{-1} y|, x, y \in X,$$

where $\tan^{-1}(\infty) = \pi/2$ and $\tan^{-1}(-\infty) = -\pi/2$. Show that (X, d) is a metric space.

(ii) Let X denote the set of all Riemann integrable functions on [a, b]. For f, g in X, define:

$$d(f, g) = \int_a^b |f(x) - g(x)| \, dx.$$

Show that d is not a metric on X. 3+3=6

(*b*) Prove that a sequence in $\mathbf{R}^n$ is Cauchy in the Euclidean metric d_2 if and only if it is Cauchy in the maximum metric $d\infty$. 6

P. T. O.

(c) (i) Show that the metric space (X, d) of rational numbers is an incomplete metric space.

(ii) Let X be any nonempty set and d be the discrete metric defined on X. Prove that the metric space (X, d) is a complete metric space. 3+3=6

2. (a) Let (X, d) be a metric space. Prove that the intersection of any finite family of open sets in X is an open set in X. Is it true for the intersection of an arbitrary family of open sets? Justify your answer. 6

(b) Prove that if A is a subset of the metric space (X, d), then $d(A) = d(\overline{A})$. 6

(c) Let F be a subset of a metric space (X, d). Prove that the following are equivalent:

(i) $x \in \bar{F}$

(ii) $S(x, \epsilon) \cap F \neq \emptyset$ for every open ball $S(x, \epsilon)$ centered at x;

(iii) There exists an infinite sequence $\{x_n\}$, $n \geq 1$ of points (not necessarily distinct) of F such that $x_n \to x$. 6

3. (a) Let (X, d) be a metric space and $Z \subseteq Y \subseteq X$. If $cl_X(Z)$ and $cl_Y(Z)$ denote, respectively, the closures of Z in the metric spaces X and Y, then show that:

$$cl_Y(Z) = Y \cap cl_X(Z).$$ 6

(b) (i) Let Y be a nonempty subset of a metric space (X, d_X), and (Y, d_Y) is complete. Show that Y is closed in X.

(ii) Is the converse of part (i) true? Justify your answer. 4+2=6

(c) Let d_p ($p \geq 1$) on the set $\mathbf{R}^n$ be given by:

$$d_p(x, Iy) = \left(\sum_{j=1}^{n} |x_j - y_j|^p\right)^{1/p},$$

for all $x=(x_1, x_2, \ldots, x_n)$, $y=(y_1, y_2, \ldots, y_n)$ in $\mathbf{R}^n$. Show that ($\mathbf{R}^n$, d_p) is a separable metric space. 6

4. (a) Prove that a mapping f: (X, d_X) → (Y, d_Y) is continuous on X if and only if $f^{-1}(F)$ is closed in X for all closed subsets F of Y. 6 ½

(b) (i) Define an isometry between the metric spaces (X, d_X) and (Y, d_Y), and show that it is a homeomorphism.

(ii) Is the completeness of a metric space preserved under homeomorphism? Justify your answer.

4+2½=6½

(c) State and prove the Contraction Mapping Principle.

1½+5=6½

5. (a) Let f be a mapping of (X, d_X) into (Y, d_Y). Prove that f is continuous on X if and only if for every subset F of Y:

$$f^{-1}(F^o) \subseteq (f^{-1}(F))^o$$ 6½

P. T. O.

(b) Prove that the metrics d_1, d_2 and d_∞ defined on $\mathbf{R}^n$ by:

$$d_1(x, y) = \sum_{j=1}^{n} \left|x_j - y_j\right|;$$

$$d_2(x, y) = \left(\sum_{j=1}^{n} \left|x_j - y_j\right|^2\right)^{1/2}; \text{ and}$$

$$d_\infty(x, y) = \max\{ \left|x_j - y_j\right| : j = 1,2,\ldots,n\}$$

for $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n) \in \mathbf{R}^n$ are equivalent. 6½

(c) Prove that a metric space (X, d) is disconnected if and only if there exists a continuous mapping of (X, d) onto the discrete two element space (X_0, d_0). 6½

6. (a) If every two points in a metric space X are contained in some connected subset of X, prove that X is connected. 6½

(b) Let (X, d) be a metric space and Y a subset of X. Prove that if Y is compact subset of (X, d), then Y is bounded. Is the converse true? Justify your answer. 6½

(c) If f is a one-to-one continuous mapping of a compact metric space (X, d_X) onto a metric space (Y, d_Y), then prove that f is a homeomorphism. 6½