12.12.18 (M)

This question paper contains **4+2** printed pages]

Roll No. | | | | | | | | | | | |

S. No. of Question Paper : **94**

Unique Paper Code : **32351502** **I**

Name of the Paper : **Group Theory-II**

Name of the Course : **B.Sc. (H) Mathematics**

Semester : **V**

Duration : **3 Hours** Maximum Marks : **75**

(Write your Roll No. on the top immediately on receipt of this question paper.)

All questions are compulsory.

Question No. **1** has been divided in 10 parts

and each part is of **1½** marks.

Each question from **2** to **6** has 3 parts and each part is of

6 marks. Attempt any *two* parts from each question.

1. State true (T) or false (F). Justify your answer in brief :

(*a*) $\mathbf{Z}_2 \oplus \mathbf{Z}_3$ is isomorphic to $\mathbf{Z}_6$ where $\mathbf{Z}_n$ is used for group $\{0, 1, 2, ..., n - 1\}$ under addition modulo n.

(*b*) The largest possible order of any element of external direct product $\mathbf{Z}_3 \oplus \mathbf{Z}_6 \oplus \mathbf{Z}_2$ is 36.

(*c*) If H, K and L are normal subgroups of a group G. Then G is internal direct product of H, K and L if G = HKL and $H \cap K \cap L = \{e\}$ where e is identity of G.

(*d*) The order of the group of inner automorphisms of additive group of integers is greater than 1.

(*e*) The dihedral group D_8 of order 8 is a subgroup of the symmetric group S_4.

(*f*) For any two groups G_1 and G_2, $G_1 \oplus G_2$ is isomorphic to $G_2 \oplus G_1$.

(*g*) Let G be a non-abelian group. A map $G \times G \to G$ is given by $(g, a) \mapsto g.a = ag$ for all g and a in G. This is an action of G on itself.

(*h*) Every subgroup H of a group G of index 2 is normal in G.

(*i*) If order of a group G is greater than 1, then the conjugacy action of G on itself is transitive.

(*j*) In S_3 the all conjugacy classes are {(1 2), (1 3), (2 3)} and {(1 2 3), (1 3 2)}.

2. (*a*) Prove that for any positive integer n, $Aut(\mathbf{Z}_n)$ is isomorphic to $U(n)$, where $\mathbf{Z}_n$ is the group {0, 1, 2 ......, $n - 1$} under addition modulo n and $U(n)$ the group of units under multiplication modulo n and $Aut(\mathbf{Z}_n)$ denotes the group of automorphisms of $\mathbf{Z}_n$.

(*b*) Define the commutator subgroup G′ of a group G. Prove that G/G′ is abelian and if G/N is abelian then G′ is subgroup of N.

(*c*) Prove that the order of an element of a direct product of fnite number of finite groups is the least common multiple of the orders of the components of the element.

3. (*a*) Prove that if a group G is the internal direct product of a fnite number of subgroups $H_1, H_2, ..., H_n$, then G is isomorphic to the external direct product of $H_1, H_2, ..., H_n$.

(*b*) Find all subgroups of order 4 in $\mathbf{Z}_4 \oplus \mathbf{Z}_4$.

(*c*) Let G = {1, 7, 17, 23, 49, 55, 65, 71} be the group under multiplication modulo 96. Express G as an internal direct product of cyclic groups.

4. (*a*) Let G be an abelian group of order 120 and G has exactly three elements of order 2. Determine the isomorphism class of G.

(*b*) (*i*) Let G be a group acting on a non-empty set A. Define kernel of action of G on A and explain when this action will be called faithful.

(*ii*) Consider the action of the dihedral group D_8 of order 8 on the set $A = \{\{1, 3\}, \{2, 4\}\}$ of the unordered pair of opposite vertices of a square. Show that this action is not faithful. Further, show that for either $a \in A$ ($a = \{1, 3\}$ or $\{2, 4\}$), the stabilizer of a in D_8 equals the kernel of the action.

(*c*) Let G be a group and A be any subset of G. Define centralizer $C_G(A)$ and normalizer $N_G(A)$ of A in G. Further, for the symmetric group S_3 and a subgroup $A = \{I, (1, 2)\}$ of S_3, find centralizer and normalizer of A in S_3 where I denotes identity of S_3.

5. (*a*) Let G be a group, H be a subgroup of G and let G act by left multiplication on the set A of left cosets of H in G. Let π_H be the associated permutation representation afforded by this action. Then, show that the following hold :

(*i*) G acts transitively on A.

(*ii*) The stabilizer in G of $1H \in A$ is a subgroup of H where 1 is identity of G.

(*iii*) Kernel of π_H is equal to $\bigcap_{x \in G} xHx^{-1}$ and the kernel of π_H is the largest normal subgroup of G contained in H.

(*b*) Let G be a group acting on a non-empty set A given by $g.a$ for all $g \in G$ and for all $a \in A$. If $a, b \in A$ and $b = g.a$, for $g \in G$, then show that $G_b = gG_ag^{-1}$. Deduce that, if G acts transitively on A, then kernel of the action is $\bigcap_{g \in G} gG_ag^{-1}$ where G_x denotes stabilizer of x in G.

(*c*) (*i*) State the class equation for a finite group G. Find all conjugacy classes and their sizes in the alternating group A_4.

(*ii*) Let G be a group of order p^2 for some prime p. Show that it is isomorphic to either $\mathbf{Z}_{p^2}$ *or* $\mathbf{Z}_p \times \mathbf{Z}_p$.

6. (*a*) Show that for any positive integer n greater than or equal to 5, the alternating group A_n of degree n does not have a proper subgroup of index less than n.

(*b*) Prove that if order of a group G is 105, then it has normal Sylow 5-subgroup and normal Sylow 7-subgroup.

(*c*) State and prove the Index theorem. Hence or otherwise, show that there is no simple group of order 216.