1

LINEAR PROGRAMMING

Unit Structure:

1.0 OBJECTIVES

The learning objectives of Linear Programming are as follows:

1. To develop the skillsets of learners in formulating and solving LPP.

2. To formulate the linear programming problem by considering different real life business constraints and learners can get solutions to LPP with alternative courses of action.

1.1 INTRODUCTION

1.1.1 Introduction to Linear Programming Problem

The linear programming problems are the mathematical methods by which scare or limited resources of the organization are used effectively and optimally so as to achieve the objective of the organization i.e. to earn and to maximize the profit. The resources of the organization such as manpower, money, method, material and machines are utilized effectively as and when required by the management using linear programming method. The linear programming problems act as optimization problems involving maximization and minimization problem. The maximization refers to the maximization of resources such as profit and sales whereas the minimization states to minimization of time and cost. These linear programming techniques are widely used in the field of Product Mix Problem Production Allocation Problem, Inspection Problem, Production Planning Problem, Advertising Media Selection Problem, Blending Problem, Investment Problem, Portfolio Selection problem, Capital Budgeting Problem, Trim Loss Problem, War Strategy Problem, Production Scheduling Problem and many other fields where optimizations are required.

Linear Programming Problem refers to a process of applying mathematical technique to use the resources of organization effectively and efficiently using optimization method.

1.1.2 Assumptions of Linear Programming Model

While developing the linear programming model, assumptions required to be considered are as follows:

a. Certainty
b. Additivity
c. Linearity
d. Continuity or Divisibility
e. Proportionality
f. Limited Choices

Let us discuss each assumption of Linear Programming Model one by one as follows:

a. Certainty

The certainty assumption states that the various elements of Linear Programming Model such as objective function, various constraints of the problem are arbitrary or constant and are determined using maximization or minimization method to know their precise or deterministic value. Hence the various factors of Linear Programming Model are precise, deterministic and certain in values.

b. Additivity

The Additivity assumption states that the objective function for the given values of decision variable as dependent variable is the sum of given many independent variables and their correspondent addition will lead to the maximization of variable or minimization of variable depending on the nature of objective function.

The Additivity assumption also states that the constraint functions for the given values of decision variable as dependent variable is the either greater than or equal to the sum of given many independent variables or less than or equal to sum of given many independent variables and their correspondent addition will lead to the search of values for the variables say x and y or any other defined variable.

c. Linearity

The objective function and the defined constraint equations are assumed to be linear so as to find out the values of variables leading to find out the values of assumed variables and their optimized value in the objective function. The linear function states that there exists simultaneous equation by which the unknown values can be found out.

d. Continuity or Divisibility

The Continuity assumption states that the values of decision variables are either positive values or zero values (non-negative values). The Divisibility or continuity function states that the whole numbers can be directly taken to solve linear programming problems whereas fractional numbers are rounded off to the nearest numbers to use and solve linear programming problems.

e. Proportionality

The Proportionality assumption states that the values of objective function are in the multiple of assumed variables (say objective function $z=3x+4y$, the objective function z is equal to the summation of thrice proportion of x variable and fourth multiple (proportion) of y variable).

The Proportionality assumption also states that the values of constraint functions are assumed to be linear and in the multiple of assumed variables (say constraint function $4x+5y \geq 20$, assuming linear function, the constraint function $4x+5y$ is equal to 20 and the variables x and y are in the multiples of 5 and 4 for x and y variables respectively.

f. Limited Choices

The limited choice assumption states the decision maker has no broad choice, he has to select the alternative from all available alternatives only which are limited in choice. The decision variables are finite and decision maker has to consider the inter-related and available conditions mentioned in the linear programming problems as constraints including the non-negativity constraint (representing the realistic conditions).

1.1.3 Advantages of Linear Programming

The linear programming models or problems have several advantages in the form of profit maximization, sales maximization, time minimization or cost minimization and many more. The linear programming has following advantages:

a. Optimum Utilization of Limited or Available Resources
b. Improvement in Efficiency and Productivity
c. Assistance for Decision Making
d. Objective oriented direction
e. Attainment of Feasible Solution to the Real Life Situation
f. Understanding the limitations of the project
g. Helps for Change Management
h. Profit and Sales Maximization
i. Time and Cost Minimization

Let us discuss the advantages of Linear programming problem in detail as follows:

a. Optimum Utilization of Limited or Available Resources

The resources are limited in number that's why the decision maker has to utilize these resources efficiently and effectively so that the management will achieve the objective of Organization. The constraint represents the conditions of limited resources and objective function is related to the optimum utilization of all these resources to maximize the profit or to optimize the objective function.

b. Improvement in Efficiency and Productivity

The linear programming problem improve the efficiency and productivity of available resources such as manpower Maine money method machine material and 20 improvement in available resources will ultimately lead to the improvement in efficiency and productivity of Manpower and that leads to the attainment of organisational objective hence the linear programming problem helps to improve the efficiency and productivity.

c. Assistance for Decision Making

The decision makers are confused for using scare resources. These ambiguity can be cleared by using linear programming problems as the optimum utilization of resources can be done with the help of this technique hence decision makers are determine to use proper decision making principles by using LPP method.

d. Objective oriented direction

The linear programming problems are supposed to achieve the objective of organization that is to earn and to maximize the profit by utilizing different constraints and achieving the objective function. Hence it provides the direction to achieve the objective of organization by utilization of given real-life situations or conditions in LPP model or in linear programming model.

e. Attainment of Feasible Solution to the Real Life Situation

The linear programming problem to attend the feasible solution to the real life situation in organization it provides possible solution to the existing problems. The appropriate solution to different business conditions are brought into reality with the help of linear programming problems. The linear programming problems are solved to get different feasible solutions and their output. The decision variables with the given conditions are solved to get the feasible solution to the real-life problem of the corporate world and hence the LPP models are considered as the best solution to complex business situations.

g. Helps for Change Management

Linear programming problems help to understand the dynamism in the system and corresponding change management can be implemented with the help of linear programming problems.

h. Profit and Sales Maximization

Objective function of linear programming problems helps to maximize the profit as well as the sales of the organisation. The Optimisation technique used here is maximization.

i. Time and Cost Minimization

The objective function of linear programming problem is to minimise the time or duration of completion of the project as well as reduces the cost of production of the project.

1.1.4 Limitations of Linear Programming:

Although there are many advantages of linear programming problems, there are few limitations of linear programming. The limitations of linear programming as follows:

a. Linearity

The relationships among different variables are considered as linear relationship but it may not be possible every time that the relationship among variables of decision making Are linear. The linearity function in business or in real life situation may or may not exist the assumption of linearity function in the case of linear programming problems is a question mark.

b. Integer value solutions:

The solution obtained for the linear programming problems in terms of objective function many may or may not be integer value functions. It may be the whole number or fractional number. Thus every time, an integer value solution may not be possible. The optimal solution may not be appropriate if they reach rounding off to the nearest integer value. This is one of the biggest limitations of linear programming problem.

c. Uncertainty:

The linear programming problem may not consider the change factors such as internal change factors as well as external change factors. The uncontrollable external factors may lead to change the equation and the optimal solution may be inappropriate. The uncertainty in different variables of linear programming may lead to biased linear programming solution.

d. Value of time:

The linear programming problem does not consider the value of time that is the time for completion of the activity or the time required to consider the conditions of linear programming problem. Schedule of linear programming problem are unknown and undetermined that leads to to fluctuate the solution of linear programming problems

e. Small problems:

The linear programming models are suitable for big problems of multinational organizations whereas small part of problems cannot be solved cannot be fragmented by using linear programming problem methods.

f. constant parameters:

The linear programming problems have considered constant parameters of decision variables but it is not possible every time that the parameters are constant. Hence, this is one of the limitations of linear programming problem.

g. Single objective

The single objective is considered in a linear programming problem as objective function. The multiple objectives of organizations are not considered in the linear programming models full stop it is not possible to attain multiple objectives of organization by using linear programming model.

1.1.5 Requirements for a Linear Programming Problem

The requirements for a linear programming problem mean the need for a linear programming problem in the business field. The basic objective of management is to achieve the objective of organization that is to earn profit by effective utilization of all available resources. The available resources are men, material, money, method and machine. These resources are limited in number. The use of these resources in the proper way is an essential part for the management. The optimum use of these resources for the accomplishment of the objective of organization is done by one of the methods called- linear programming problem.

Therefore, there is a need of defining linear programming problem and the requirements for linear programming problem.

The conditions of linear programming problems and its requirements are as follows:

a. well defined objective function

b. potential constraints

c. alternative course of action

d. interrelated variables

e. non negative be constraint

f. Limited supply

A. Well defined objective function

Linear programming problems should contain well defined objective functions that highlight the objective of organization. The well-defined objective function should be an optimization function leading to either maximization or minimization case. The objective function is defined in a mathematical equation starting with the letter Z equal to sum function.

B. Potential constraints

The potential constant refers to the conditions or limitations for things that have the capacity to achieve the objective of organisation and this constants must be capable of being expressed as a linear function as either greater than or less than are equal to as the variable may be.

C. Alternative course of action:

Linear programming problems must consist of different alternatives called alternative course of actions. For example a product may be processed on three different machines and how much quantity of product can be processed on what machine that it is known through linear programming problems.

D. Interrelated variables

The decision variables of linear programming problem are interrelated to each other they are dependent on each other hence by using simultaneous equation we can find out the value of solution of a linear programming problem.

E. Non negative constraint

The decision variables of linear programming problem are non-negative that is either Zero OR greater than zero and hence the non-negative the constant indicates that the constant are realistic in nature and that they exist in real life situation.

F. Limited supply

The resources may be having limited supply that's why there is a requirement of linear programming problem.

1.1.6 Applications of Linear Programming

Applications of linear programming method are in the resource allocations of Product Mix Problem, Production Allocation Problem; Inspection Problem; Production Planning Problem; Advertising Media Selection Problem; Blending Problem; Investment Problem; Portfolio Selection problem and many other areas.

The linear programming problems are used in assembly line balancing, make or buy decisions, Facilities location planning, agriculture resource allocation problems, flight and train scheduling problems, dietary problems, Environment protection problems, profit planning problems, transportation problems, assignment problems, man power scheduling cases, profit planning problems and many other areas of business field.

1.1.7 Areas of Applications of Linear Programming

Linear programming problems have wide application in the field of industry, management and/or other fields. The areas of application of linear programming problem are as follows:

a. Production scheduling problems

The production scheduling is done in order to match the demand of the product, inventory and manpower at minimum threshold level so as to minimize the total cost of production and so as to keep the minimum inventory.

b. Product mix problems

The product mix problem involves the mixture of different raw materials, their ingredients and production capacity and the available resources. The optimum allocation of resources such as men, machine, money, material, method, and market can be properly executed with the help of linear programing problems.

c. Assembly Line balancing Problems

The high brand value products are not manufactured by a single company. The multiple brands are involved in the production of different components of that product. These parts are brought together and assembled to form brand value product. For example, The Reynolds Automobile do not build whole car, the tires, seat covers, chromes and other parts are manufactured by other companies and they are brought together and assembled to form a new car. The decisions of assembly line balancing are taken with the help of LPP models.

d. Make or buy decisions Problems

The make or buy decisions may be generally taken with a linear programming problem as the given conditions are related to make or buy decisions. When quantities to be manufactured are small in number, then buy decision may be made and when quantities to be produced are larger in number then make decision may be made. The quantities as a constraint and other limitations of the company may decide to make or buy decisions.

e. Transportation problems

Transportation problems involve transfer of goods or services from source to destination. Transportation involves different constraints such as quantity to be transferred, distance to be travelled, manufacturing capacity of source, supply from the source and demand of the destination, unit transportation cost and other conditions. The combined constraints and objectives of transportation can be achieved with the help of linear programming problems.

f. Assignment problems

Assignment problem involves allocation of different facilities like manpower or machines to different jobs. Proper allocations of different facilities make timely arrangement to execute various jobs on different machines or different jobs by different employees.

g. Portfolio selection problems

The investment banks or other non-banking financial services like Mutual Fund associations and other share market brokers always face the problem of portfolio selection. The linear programming problem assists to select a portfolio based on minimum risk and maximum return. Different investment alternatives are provided and are selected based on given constraints in LPP.

h. Agricultural problems

The allocation of different agricultural inputs such as water, labour, fertilizers, pesticides and capital to different crops are properly combined with the help of linear programming problems. The proper allocation of Agricultural food in more agricultural output in the form of Agricultural producers and hence the linear programming problems are greatly helpful in the allocation of Agricultural inputs.

1.1.8 Structure of Linear Programming Problem

Structure of linear programming model consists of three components.

1. Variables and their relationship
2. Objective function
3. Constraint.

The components of linear programming models are described in detail as follows:

1. Variables and their relationship

Variables are decision variables and they are under the control of the decision maker. The relationships between the variables are generally linear relationship and are solved using simultaneous solutions. All decision variables are controllable, non-negative and continuous. They are represented as $x_1, x_2,.....x_n$.

2. Objective function

Objective function is the objective of creating and solving linear programming problems. Generally the objective function is represented in the form of profit, sales, revenue, time and cost. Objective function is represented in two forms that is either to maximize or to minimise with the help of letter Z as

$Z=c_1x_1+c_2x_2+...+c_nx_n$

Where,

Z is the objective function.

$c_1, c_2,....c_n$ are parameters of uncontrollable variable and

$x_1, x_2,.....x_n$ are parameters of controllable variable.

3. **Constraints**

Constraints are the limitations or given conditions described in linear programming problems. They are the representations on the use of limited resources. Constraint represents the mathematical equation where limited resources can be used effectively so as to maximise the objective function.

1.2 FORMULATION OF LINEAR PROGRAMMING PROBLEM:-

The linear programming problems are widely used in different sectors of business, management and administration such asProduct Mix Problem, Production Allocation Problem; Inspection Problem, Production Planning Problem; Advertising Media Selection Problem; Blending Problem; Investment Problem; Portfolio Selection problem and many other sectors.

1.2.1 Product Mix Problem

Example: A firm manufactures two items. It purchases castings which are then machined, bored and polished. Castings for items A and B cost Rs 2 and 3 respectively and are sold at Rs 5 and 6 respectively. Running costs of three machines are Rs. 20, Rs. 14 and Rs. 17.50 per hour respectively. Capacities of the machine are

	Part A	Part B
Machining Capacity	25/hr	40/hr
Boring Capacity	28/hr	35/hr
Polishing Capacity	35/hr	25/hr

Formulate the Linear Programming model to determine the product mix that maximises the profit.

Solution:

1. Variables under study:

As a firm manufactures two items A and B, let us consider x1 and X2 are the number of units to be manufactured per hour.

2. Objective function:

Here, the objective function is to maximise the profit by utilising proper product mix.

Determination of Profit for part A and part B

Particulars	Formula	Part A (Rs.)	Part B (Rs.)
Machining Cost	= Running cost / Capacity per hr	20/25= 0.8	20/40=0.5
Boring Cost	= Running cost / Capacity per hr	14/28=0.5	14/35=0.4
Polishing cost	= Running cost / Capacity per hr	17.5/35= 0.5	17.5/25=0.7
Total Assembling Cost	= Machining cost+ Boring Cost+ Polishing Cost	0.8+0.5+0.5= 1.8	0.5+0.4+0.7= 1.6
Casting cost	Given	2	3
Total cost	=Total assembling cost+ Casting cost	1.8+2= 3.8	1.6+3 = 4.6
Selling Price	Given	5	6
Profit	Selling Price- Total cost	5-3.8= 1.2	6-4.6=1.4

3. Constraints

Constraints are the conditions mentioned on the capacities of machines.

Running machines for one hour requires following conditions

Machine Constraint : $(1/25)\ x_1+(1/40)\ x_2 \leq 1.$

$8x_1+5x_2 \leq 200.$

Boring Constraint : $(1/28)\ x_1+(1/35)\ x_2 \leq 1.$

$5x_1+4x_2 \leq 140.$

Polishing constraint : $(1/28)\ x_1+(1/35)\ x_2 \leq 1.$

$5x_1+4x_2 \leq 140.$

Non-negativity constraint :$x_1 \geq 0; x_2 \geq 0$

1.2.2 Production Allocation Problem

Example: A firm produces three products. These products are processed on three different machines. The time required to manufacture one unit of each of the three products and the daily capacity of the three machines are given in the table below:

Machine	Time per unit (minutes)			Machine capacity (minutes/day)
	Product 1	Product 2	Product 3	
M_1	2	3	2	440
M_2	4	-	3	470
M_3	2	5	-	430

Solution:

1. **Variables under study:**

As a firm produces three products and are processed on three different machines, let us consider x_1, x_2 and x_3 are the number of units to be manufactured daily for products 1, 2 and 3 respectively.

2. **Objective function:**

Here, the objective function is to maximise the profit.

The objective function $Z= 4x_1+3x_2+6x_3$ is to maximize.

3. **Constraints**

Constraints are the conditions mentioned on the capacities of machines.

Machine M_1 Constraint: $2x_1+3x_2+2x_3 \leq 440.$

Machine M_2 Constraint: $4x_1+0x_2+3x_3 \leq 470.$

Machine M_3 constraint : $2x_1+5x_2+0x_3 \leq 430$.

Non-negativity constraint : $x_1 \geq 0$; $x_2 \geq 0$ and $x_3 \geq 0$

1.2.3 Inspection Problem

Example: A company has two grades of inspectors I and II to undertake quality control inspection. At least, 1500 pieces must be inspected in an 8 hour day. Grade I inspector can check 20 pieces in an hour with an accuracy of 96%. Grade II inspector checks 14 pieces an hour with an accuracy of 92%.

Wages of grade I inspector are Rs. 5 per hour while those of grade II inspector are Rs. 4 per hour. Any error made by an inspector costs Rs. 3 to the company, find the optimal assignment of inspectors that minimizes the daily inspection cost.

Solution:

1. **Variables under study:**

As a company has two grades of inspectors I and II and are assigned grades to undertake quality control inspection. Let us consider x_1, and x_2 are the number of grades of inspectors I and II that are assigned the job of quality control inspection respectively.

2. **Objective function:**

Here, the objective function is to minimize the daily cost of inspection.

The daily cost of inspection includes two costs namely wages paid to the inspectors and cost of inspection.

The cost of grade I inspection per hour is = Rs. (5+3x0.04x20) = Rs. 7.4

The cost of grade II inspection per hour is = Rs. (4+3x0.08x14) = Rs. 7.36

The objective function $Z= 8\ (7.4x_1+7.36x_2)$ is to maximize.

3. **Constraints**

Constraints are the conditions mentioned on the grades of inspections.

Grade I Constraint : $x_1 \leq 10$.

Grade I Constraint : $x_2 \leq 15$.

Inspection constraint : $20(8x_{1)}+14\ (8x_2) \leq 1500$.

Non-negativity constraint : $x_1 \geq 0$; and $x_2 \geq 0$

1.2.4 Production Planning Problem

A factory manufactures a product each unit of which consists of 5 units of part A and 4 units of part B. The two parts A and B require different raw materials of which 120 units and 240 units respectively are available.

These parts can be manufactured by three different methods. Raw material requirements per production run and the number of units for each part produced are given below.

Method	Input per run (units)		Output per run (units)	
	Raw material 1	Raw material 2	Part A	Part B
1	7	5	6	4
2	4	7	5	8
3	2	9	7	3

Formulate the Linear Programming Problem model to determine the number of production runs for each method so as to maximize the total number of complete units of the final product.

Solution:

1. Variables under study:

Let us consider x_1, x_2 and x_3 are the number of production runs for each method 1, 2 and 3 respectively.

2. Objective function:

Here, the objective function is to maximize total units of the final product.

The total units of part A produced by different machines are $6x_1+5x_2+7x_3$ and The total units of part B produced by different machines are $4x_1+8x_2+3x_3$.

A factory manufactures a product each unit of which consists of 5 units of part A and 4 units of part B. Hence upper limit of production cannot go beyond $(6x_1+5x_2+7x_3)/5$ and $(4x_1+8x_2+3x_3)/4$.

The objective function Z= Mimimum of $(6x_1+5x_2+7x_3)/5$ and $(4x_1+8x_2+3x_3)/4)$.

This equation may violates the norms of LPP. We may convert this non-linear function to linear function by considering the following equation

Let y= Min $((6x_1+5x_2+7x_3)/5$ and $(4x_1+8x_2+3x_3)/4))$

It implies that

$(6x_1+5x_2+7x_3)/5 \geq y$ and $(4x_1+8x_2+3x_3)/4) \geq y$.

$6x_1+5x_2+7x_3 \geq 5y$ $\quad$ $4x_1+8x_2+3x_3 \geq 4y$

$6x_1+5x_2+7x_3 - 5y \geq 0$ $\quad$ $4x_1+8x_2+3x_3 - 4y \geq 0$

Hence the objective function is to maximize Z =y

3. **Constraints**

Constraints are the conditions mentioned.

Raw Material 1 Constraint : $7x_1+4x_2+2x_3 \leq 120$.

Raw Material 2 Constraint : $5x_1+7x_2+9x_3 \leq 240$

Output Part A Constraint : $6x_1+5x_2+7x_3-5y \geq 0$

Output Part B Constraint : $4x_1+8x_2+3x_3-4y \geq 0$

Non-negativity constraint : $x_1; x_2; x_3, y \geq 0$;

1.2.5 Advertising Media Selection Problem

An advertising company wishes to plan its advertising strategy in 3 different media- television, radio and magazines. The purpose of advertising is to reach as large a number of potential customers as possible. Following data have been obtained from market survey:

	Television	Radio	Magazine I	Magazine II
Cost of an advertising unit	Rs. 30,000	Rs. 20,000	Rs.15,000	Rs. 10,000
No. of potential customers reached per unit	2,00,000	6,00,000	1,50,000	1,00,000
No. of female customers reached per unit	1,50,000	4,00,000	70,000	50,000

The Company wants to spend no more than Rs.4,50,000 on advertising. Following are the further requirements that must be met:

i. at least 1 million exposures take place among female customers.

ii. advertising on magazines be limited to Rs. 1,50,000.

iii. atleast 3 advertising units to be brought on magazine 1& 2 units on magazine II,

iv. the number of advertising units on television and radio should each be between 5 and 10.

Formulate an L.P. model for the problem.

Solution:

1. Variables under study:

As an advertising company wishes to plan its advertising strategy in 3 different media- television, radio and magazines, Let us consider x_1, x_2, x_3 and x_4 are the number of advertising units to be brought on television, radio, magazine I and magazine II respectively.

2. Objective function:

Here, the objective function is to maximize the total number of potential customers reached.

The objective function $Z= 10,00,000\ (2x_1+6x_2+1.5x_3+x_4)$ is to maximize.

3. **Constraints**

Constraints are the conditions mentioned as follows.

Advertising Budget Constraint :

$30,000x_1+20,000x_2+15,000x_3+10,000x_4 \leq 4,50,000$.

i.e. $30x_1+20x_2+15x_3+10x_4 \leq 450$

Number of female customers reached by advertising campaign Constraint

$1,50,000x_1+4,00,000x_2+70,000x_3+50,000x_4 \geq 10,00,000$.

i.e. $15x_1+40x_2+7x_3+5x_4 \leq 100$.

Expenses on magazine advertising constraint

$15,000x_3+10,000x_4 \geq 1,50,000$.

i.e. $15x_3+10x_4 \leq 150$.

No. of units on magazines constraints

$x_3 \geq 3$

$x_4 \geq 2$.

No. of units on television constraints

$5 \leq x_1 \leq 10$

No. of units on radio constraints

$5 \leq x_2 \leq 10$

Non-negativity constraint : $x_1, x_2, x_3, x_4 \geq 0$

1.3 GRAPHICAL METHOD OF SOLUTION

The graphical method of solution is used to solve the problems of linear programming module. The detail steps can be understood through the numerical as follows

Numerical 1. Use the graphical method of solution for the following linear programming problem. Maximize $Z= 2x_1+x_2$subject to the constraints i.) $x_1+2x_2 \leq 10$ ii.) $x_1+x_2 \leq 6$ iii.) $x_1-x_2 \leq 2$ iv.) $x_1-2x_2 \leq 1$

and $x_{1,}\ x_2 \geq 0$.

Solution:

1. At first we convert the first constraint equation $x_1+2x_2 \leq 10$ into

$x_1+2x_2=10$.

Now put $x_2=0$ and calculate the value of $x_1$as $x_1=10$

Point (x, y) = (10, 0).

Now put x_1 =0 and calculate the value of x_2 as $x_2=5$

Point (x, y) = (0, 5).

2. Now, we convert the second constraint equation $x_1+x_2 \leq 6$ into

$x_1+x_2=6$.

Now put $x_2=0$ and calculate the value of $x_1$as $x_1=6$

Point (x, y) = (6, 0).

Now put x_1 =0 and calculate the value of x_2 as $x_2=6$

Point (x, y) = (0, 6).

3. Now, we convert the third constraint equation $x_1-x_2 \leq 2$into

$x_1-x_2=2$.

Now put $x_2=0$ and calculate the value of $x_1$as $x_1=2$

Point (x, y) = (2, 0).

Now put x_1 =0 and calculate the value of x_2 as $x_2= -2$

Point (x, y) = (0,-2).

$x_1-2x_2 \leq 1$

4. Now, we convert the third constraint equation $x_1-x_2 \leq 2$ into

$x_1-2x_2= 1$.

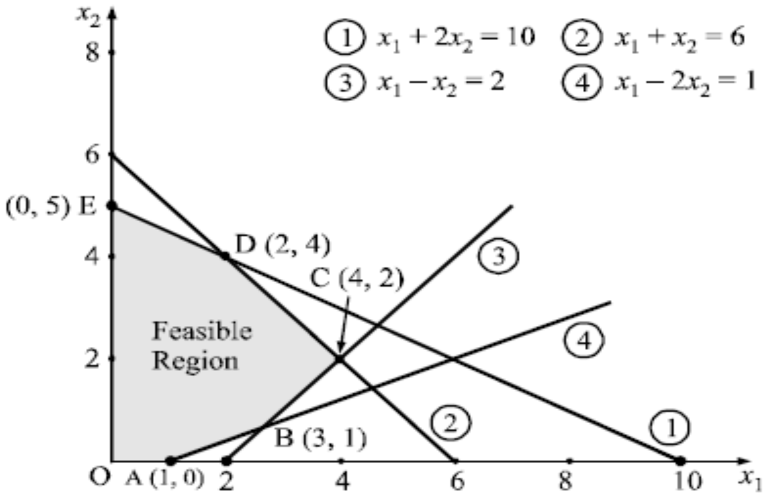
Now put $x_2=0$ and calculate the value of $x_1$as $x_1=2$

Point (x, y) = (2, 0).

Now put x_1 =0 and calculate the value of x_2 as $x_2= -1/2$

Point (x, y) = (0,-1/2).

Now using graphical points as follows



Source: Operations Research Book by J.K. Sharma

The point B is at the junction of two lines x_1-x_2= 2 and x_1-2x_2= 1

The coordinates of the junction point of intersecting lines can be obtained by solving the equation of two lines intersecting each other using simultaneous equation as follows

$x_1 - x_2 = 2$

$-x_1 + 2x_2 = -1$

As x_2= 1 and putting the value of x_2= 1 in above equation, we get x_1 as 3.

Point B (x,y)= B (3,1).

The point D is at the junction of two lines x_1+2x_2= 10 and . x_1+x_2= 6

The coordinates of the junction point of intersecting lines can be obtained by solving the equation of two lines intersecting each other using simultaneous equation as follows

$x_1 + 2x_2 = 10$

$-x_1 - x_2 = -6$

As x_2= 4 and putting the value of x_2= 4 in above equation, we get x_1 as 2.

Point D (x, y)= B (2,4).

Now as all equations are either less than or equal to some value, the feasible region lies below that all lines. Hence the solution of existing LPP lies in that feasible region.

Let us check all extreme values of feasible region as follows

Extreme point	Coordinates	Value (Objective function)
0	(0,0)	2(0)+1(0)=0
A	(1,0)	2(1)+1(0)=2
B	(3,1)	2(3)+1(1)=7
C	(4,2)	2(4)+1(2)=10
D	(2,4)	2(2)+1(4)=8
E	(0,5)	2(0)+1(5)=5

The maximum value of objective function lies at point C(4,2) as 10 and hence the optimal solution to the given LPP is : x1=4 and x2=2 and Max Z=10.

1.4 DUALITY

There is a set of unique linear programming problem for every linear programming problem. The other linear programming problem includes same data reflecting program and is called as dual Program. The dual program is obtained by transposing the rows and columns of linear programming problem. Dual variables are also called as shadow prices of various sources. Solution for dual program also exist just like that of original linear programming problem. Solutions of Primal program and dual program are very closely related to each other.

The features of duality in linear program is as follows

1. If the original linear programming problem has many rows and few columns then the calculation can be complex one. At that time, solution of linear programming problem can be obtained by converting rows and columns.

2. The duality in linear programming give the additional information as changes in options can change the optimal solution of linear programming problem. This change in coefficient leading to change in the solution of linear programming problem is reflex sensitivity analysis or post optimality analysis.

3. Generally, the duality in linear programming evaluates on different economic nature and hence, it helps to evaluate opportunity cost for different alternatives available to the solution and their relative values.

4. The accuracy of primal solution can be checked with the help of duality in linear programming,

5. The duality in linear programming is just same that of zero Sum game in Game Theory.

1.5 POST OPTIMALITY ANALYSIS AND SENSITIVITY ANALYSIS

The change in the values of variables of linear programming problem leads to changes in the optimal solution of a linear programming problem.

The sensitivity analysis is a study of sensitivity of optimal solution of linear programming problem by studying the discrete variation in the variables of linear programming problem.

The sensitivity can be measured from zero value to the corresponding value as per the corresponding changes in the optimal solution of linear programming. We first calculate the solution of Original linear programming problem and we consider the changes in the constraints of linear programming problem leading to the changes in the solution of linear programming problem. The variation can be noted and compared with original linear programming problem. The process of studying the sensitivity of optimal solution of linear programming problem is also called as post optimality analysis because it is done after an optimal solution.

The changes in one of the following parameters of original linear programming problem can make sensitivity analysis and alter the solution of linear programming problem as:

1. Changes in the objective function
2. Change in the value of constraints.
3. Changing the value of availability of resources.
4. Change in the consumption of resources.
5. Addition of new variable.
6. Addition of new constraints to the original linear programming problem

1.6 SELF ASSESSMENT QUESTIONS

Q1. Answer Multiple Choice Questions

1. The linear programming problems are the ……..by which scare or limited resources of the organization are used effectively and optimally so as to achieve the objective of the organization i.e. to earn and to maximize the profit.

   a. mathematical methods

   b. statistical methods

   c. quantitative methods

   d. numerical methods

2. ………refers to a process of applying mathematical technique to use the resources of organization effectively and efficiently using optimization method.

   a. Game Theory

   b. Linear Programming Problem

   c. Assignment Problem

   d. Transportation Problem.

3. Assumptions of Linear Programming Model do not include…

   a. Linearity

   b. Continuity or Divisibility

   c. Proportionality

   d. Unlimited Choices.

4. The linear programming has following advantages

   a. Optimum Utilization of Limited or Available Resources

   b. Improvement in Efficiency and Productivity

   c. Assistance for Decision Making

   d. All of these

5. ….. states that the values of decision variables are either positive values or zero values (non-negative values).

   a. Continuity assumption

   b. Proportionality assumption

   c. Both a and b

   d. None of the above

6. The …….states that the values of objective function are in the multiple of assumed variables

   a. Continuity assumption

   b. Proportionality assumption

   c. Both a and b

   d. None of the above

Answers: 1: a ***2:b*** ***3:d***

4:d ***5:a*** ***6:b***

Q2. Write short notes on:

a. Assumptions of Linear Programming Model

b. Advantages of Linear Programming

c. Limitations of Linear Programming

d. Requirements for a Linear Programming Problem

e. Applications of Linear Programming

f. Areas of Applications of Linear Programming

g. Structure of Linear Programming Problem

h. Duality

i. Post Optimality Analysis

j. Sensitivity Analysis

❖❖❖❖

2

TRANSPORTATION

Unit Structure:

2.0 OBJECTIVES

The learning objectives of transportation are as follows:

1. To understand different concepts and terminologies used in the transportation problem.

2. To comprehend transportation model and to develop solution to the transportation model.

3. To develop logical thinking and analytical solutions to the transportation problem.

4. To use different methods of solving transportation model and to minimize the total transportation cost.

2.1 INTRODUCTION

Transportation is a typical type of operation research technique developed for transportation of goods from the sources to the destination. Transportation problem is a special type of linear programming problem which can be solved by using different methods of transportation. A single product can be transported from several sources to different destinations with the help of transportation problems.

2.1.1 Constraints or conditions of transportation problem:

The constraints of transportation problems involve different conditions under which transportation problem can be solved. The constraints of transportation problem are as follows:

1. Demand and supply relation

Necessary and sufficient condition for existence of a feasible solution to the transportation problem is that there should be total demand equal to total supply. (i.e. Total demand is equal to total supply).

2. Balanced Transportation problem

When total demand is equal to total supply then the transportation problem is said to be balanced.

3. Unbalanced transportation problem

If total demand is not equal to total supply then the transportation problem is an unbalanced transportation problem.

4. Occupied cells and unoccupied cells

The occupied cells are having positive allocation in a transportation problem and are denoted by a circular symbol. The cells are not having any allocation in a transportation problem.

5. Basic feasible solution

The basic feasible solution exists when number of occupied cells are equal to number of rows and number of columns minus one i.e. (No. of occupied cells = m+n-1). Where m is the number of rows and n is the number of columns.

6. Degenerate solution and non-degenerate solution

When the number of positive allocated cells are less than m+n-1 and a solution is degenerate solution and when the number of positive located cells are equal to m+n-1 then the solution is non-degenerate solution.

7. Prohibited routes

Due to unfavorable road conditions, bad weather conditions or any other condition etc., It may not be possible to transit products from the sources to the destination such routes are prohibited routes.

2.2 MATRIX TERMINOLOGY

Following matrix terminologies are often used in transportation model.

1 Cell

The smallest element of a matrix is called a cell. It is at the intersection of row and column. The cell in a transportation model represents the unit cost of transportation from source to the destination.

2 Row

The horizontal arrangement of cells in the matrix is termed as a row. Row heading may represent the source or origins or plants from where goods are manufactured and ready for transportation. It is related to the supply function.

3 Column

The vertical arrangement of cells in the matrix is termed as a column. The column heading may represent the destination center to which goods are ready for transportation. It is related to the demand function.

4 Source

Source is the origin or plant from where goods are manufactured and ready for transportation. Generally different sources are available in a transportation model.

5 Destination

Destination is the final place where goods are required to be transported. Transportation models with different destinations are available. Due to availability of different sources and destinations with different unit costs of transportation, a transportation model is designed.

2.3 SOLUTION OF TRANSPORTATION MODEL

The solution of transportation model is designed to minimize the total cost of transportation and to transit the goods from source to destination with minimum cost.

The solution of a transportation model has following steps:

a. Formulate the transportation model in matrix form
b. Obtain an initial feasible solution
c. Perform optimally test.
d. Find the transportation solution (after iteration if available)

Steps of solutions of transportation models are described in depth as follows:

a. Formulate the transportation model in matrix form

The objective function of transportation problems is to minimize transportation cost and the constraints are demands from the destination and supply from the sources and hence transportation problem is similar to that of linear programming problems. The formulation of transportation model requires objective function as total transportation cost to be minimized and demand and supply as constraints. Key decision is overall minimization of transportation cost. For example, the daily supplies of milk from different sources to the different destinations having milk demand are matched with transportation models. The total transportation cost can be minimized using the transportation model.

b. Obtain an initial feasible solution

An initial feasible solution is the first solution to the transportation problem. The transportation quantities and routes are decided at an initial stage.

This solution may be obtained using different methods such as

i. North West Corner Method
ii. Row Minima Method
iii. Column Minima Method
iv. Least Cost Method
v. Vogels' Approximation Method

c. Perform optimality test.

The initial feasible solution is tested for the optimality using the condition as when the number of occupied cells is equal to the sum of number of rows and number of columns -1.

a. Condition A

When the opposite condition is satisfied then the given initial feasible solution is optimal and this is the final solution where improvement for minimizing total transportation cost is not possible

b. Condition B

When the opposite condition is not satisfied then the given initial feasible solution is not optimal and this is not the final solution where improvement for minimizing total transportation cost is possible.

c. Find the transportation solution (after iteration if available)

The final minimized transportation solution (after iteration if available) is computed and this gives the most possible minimum total transportation cost subject to the availability of supply and demand constraints from the supply and demand respectively.

2.4 METHODS OF OBTAINING AN INITIAL FEASIBLE SOLUTION

An initial feasible solution is the first solution to the transportation problem. The transportation quantities and routes are decided at an initial stage.

This solution may be obtained using different methods such as:

2.4.1 NORTH WEST CORNER METHOD

North West Corner Method is the simple and easy method to compute initial feasible solution. This method considers neither the cost of transportation nor the path of transportation. North West Corner is the upper left corner of the matrix used for transportation. The process of transportation of goods starts from the North West Corner of the matrix.

There are several steps of calculating initial feasible solution using North West Corner Method as follows:

Step 1.
As North West Corner is the upper left corner of the matrix used for transportation. The process of allocation of goods in transit starts from the North West Corner of the matrix. Start the allocation of transportation units from the North West Corner (the upper left corner of the matrix) cell by considering minimum values from the corresponding first row and first column i.e. Min (First Row, First Column).

Step 2.
a. If the values of First Row and First Column are equal then the same units will be transferred from the first North West Corner of the matrix. This condition is termed as degeneracy.

b. If the values of First Row and First Column are not equal then the minimum units (i.e. Min.(First Row, First Column) will be transferred from the first North West Corner of the matrix. The balance value of either first row or first column is kept there and is to be considered for next North West Corner. The reduced first row is considered as the North West row for allocating transportation units to that cell.

Step 3.
Similar iteration is done for allocating transportation units until it reaches to the extreme South East Corner of the transportation matrix. At the last cell of the transportation matrix (extreme South East Corner), the remaining total demand and total supply would be equal (degeneracy condition). In this way balanced transportation problem will be solved and unbalanced transportation problem will be solved by making it balanced by adding dummy row or column making total demand=total supply.

Let us understand these steps by practical example.

1. Solve the following transportation problem using North West Corner method.

	D1	D2	D3	Supply
S1	3	2	1	20
S2	2	4	1	50
S3	3	5	2	30
S4	4	6	7	25
Demand	40	30	55	125

Solution:
As Total Supply= Total Demand (125 units), this is balanced transportation problem.

Step 1. The North West Corner is Upper Left Corner called S1D1 cell. Corresponding first row supply is 20 units and corresponding first column demand is 40 Units. Hence allocation of maximum 20 units is possible (Being the minimum of first row and first column) to the S1D1 cell.

After allocating these 20 units, supply side gets completely fulfilled and hence there is no requirement of supply in the first row and the demand side gets reduced from 40 units to 20 units (40-20=20) hence the reduced transportation matrix becomes

	D1	D2	D3	Supply
~~S1~~	~~3 (20)~~	~~2~~	~~1~~	~~20, 0~~
S2	2	4	1	50
S3	3	5	2	30
S4	4	6	7	25
Demand	~~40~~, 20	30	55	125

Step 2. Now, the next North West Corner is Upper Left Corner called S2D1 cell. Corresponding second row supply is 50 units and corresponding first column demand is 20 Units. Hence allocation of maximum 20 units is possible (Being the minimum of second row and first column) to the S2D1 cell.

After allocating these 20 units, demand side gets completely fulfilled and hence there is no requirement of demand in the first column and the supply side gets reduced from 50 units to 30 units (50-20=30) hence the reduced transportation matrix becomes

	D1	D2	D3	Supply
~~S1~~	~~3 (20)~~	~~2~~	~~1~~	~~20, 0~~
S2	~~2~~ (20)	4	1	~~50~~, 30
S3	~~3~~	5	2	30
S4	4	6	7	25
Demand	40, ~~20~~	30	55	125

Step 3. Now, the next North West Corner is Upper Left Corner called S2D2 cell. Corresponding second row supply is 30 units and corresponding second column demand is 30 Units. Hence allocation of maximum 30 units is possible (Being the minimum of second row and second column) to the S2D2 cell.

After allocating these 30 units, demand side gets completely fulfilled and hence there is no requirement of demand in the second column and the supply side gets fulfilled and hence the reduced transportation matrix becomes

	D1	D2	D3	Supply
S1	3 (20)	2	1	20, 0
S2	2 (20)	4 (30)	1	50, 30
S3	3	5	2	30
S4	4	6	7	25
Demand	40, 20	30	55	125

Step 4. Now, the next North West Corner is Upper Left Corner called S3D3 cell. Corresponding third row supply is 30 units and corresponding third column demand is 55 Units. Hence allocation of maximum 30 units is possible (Being the minimum of third row and third column) to the S3D3 cell.

After allocating these 30 units, supply side gets completely fulfilled and hence there is no requirement of supply in the third row and the demand side gets reduced from 55 to 25 units (55-30=25) and hence the reduced transportation matrix becomes

	D1	D2	D3	Supply
S1	3 (20)	2	1	20, 0
S2	2 (20)	4 (30)	1	50, 30
S3	3	5	2 (30)	30
S4	4	6	7	25
Demand	40, 20	30	55, 25	125

Step 5. Now, the next North West Corner is Upper Left Corner called S4D3 cell. Corresponding fourth row supply is 25 units and corresponding third column demand is 25 Units. Hence allocation of maximum 25 units is possible (Being the minimum of forth row and third column) to the S4D3 cell.

After allocating these 25 units, supply side gets completely fulfilled and hence there is no requirement of supply in the fourth row and the demand side gets completed fulfilled and hence there is no requirement of demand in the fourth row and hence the reduced transportation matrix becomes

	D1	D2	D3	Supply
S1	3 (20)	2	1	20, 0
S2	2 (20)	4 (30)	1	50, 30
S3	3	5	2 (30)	30
S4	4	6	7 (25)	25
Demand	40, 20	30	55, 25	125

Step 6. Allocation of Transportation Units.

Total Transportation Cost = 3 (20) + 2 (20) + 4 (30) + 2 (30) + 7(25)

= 60+40+120+60+175

= 455.

Hence, Total Transportation Cost is Rs.455 using North West Corner Method.

2. Solve the following transportation problem using North West Corner method.

	D1	D2	D3	D4	Supply
S1	20	30	40	30	50
S2	10	20	30	10	60
S3	20	40	60	10	70
Demand	30	50	30	70	180

Solution:

As Total Supply = Total Demand (180 units), this is balanced transportation problem.

Step 1. The North West Corner is Upper Left Corner called S1D1 cell. Corresponding first row supply is 50 units and corresponding first column demand is 30 Units. Hence allocation of maximum 30 units is possible (Being the minimum of first row and first column) to the S1D1 cell.

After allocating these 30 units, demand side gets completely fulfilled and hence there is no requirement of demand in the first column and the supply side gets reduced from 50 units to 20 units (50-30=20) hence the reduced transportation matrix becomes

	D1	D2	D3	D4	Supply
S1	20 (30)	30	40	30	~~50~~, 20
S2	10	20	30	10	60
S3	20	40	60	10	70
Demand	30	50	30	70	180

Step 2. The North West Corner is Upper Left Corner called S1D2 cell. Corresponding first row supply is 20 units and corresponding second column demand is 50 Units. Hence allocation of maximum 20 units is possible (Being the minimum of first row and second column) to the S1D2 cell.

After allocating these 20 units, supply side gets completely fulfilled and hence there is no requirement of supply in the first row and the demand side gets reduced from 50 units to 30 units (50-20=30) hence the reduced transportation matrix becomes

	D1	D2	D3	D4	Supply
S1	20 (30)	~~30 (20)~~	~~40~~	~~30~~	~~50, 20~~
S2	10	20	30	10	60
S3	20	40	60	10	70
Demand	30	~~50~~, 30	30	70	180

Step 3. The North West Corner is Upper Left Corner called S2D2 cell. Corresponding second row supply is 60 units and corresponding second column demand is 30 Units. Hence allocation of maximum 30 units is possible (Being the minimum of second row and second column) to the S2D2 cell.

After allocating these 30 units, demand side gets completely fulfilled and hence there is no requirement of demand in the second column and the supply side gets reduced from 60 units to 30 units (60-30=30) hence the reduced transportation matrix becomes

	D1	D2	D3	D4	Supply
S1	20 (30)	~~30 (20)~~	~~40~~	~~30~~	~~50, 20~~
S2	10	20 (30)	30	10	~~60~~, 30
S3	20	40	60	10	70
Demand	30	50, ~~30~~	30	70	180

Step 4. The North West Corner is Upper Left Corner called S2D3cell. Corresponding second row supply is 30 units and corresponding third column demand is 30 Units. Hence allocation of maximum 30 units is possible (Being the minimum of second row and second column) to the S2D2 cell.

After allocating these 30 units, demand side gets completely fulfilled and hence there is no requirement of demand in the third column and the supply side gets completed fulfilled and hence there is no requirement of supply in the second row hence the reduced transportation matrix becomes

	D1	D2	D3	D4	Supply
S1	20 (30)	30 (20)	40	30	50, 20
S2	10	20 (30)	30 (30)	10	60, 30
S3	20	40	60	10	70
Demand	30	50, 30	30	70	180

Step 5. The North West Corner is Upper Left Corner called S3D4 cell. Corresponding third row supply is 770 units and corresponding fourth column demand is 70 Units. Hence allocation of maximum 70 units is possible (Being the equal units) to the S3D4 cell.

After allocating these 70 units, demand side gets completely fulfilled and hence there is no requirement of demand in the fourth column and the supply side gets completed fulfilled and hence there is no requirement of supply in the third row hence the reduced transportation matrix becomes

	D1	D2	D3	D4	Supply
S1	20 (30)	30 (20)	40	30	50, 20
S2	10	20 (30)	30 (30)	10	60, 30
S3	20	40	60	10 (70)	70
Demand	30	50, 30	30	70	180

Step 6: Computation of Total Transportation Cost

Total transportation cost can be calculated by summing the multiplication of unit cost of transportation with transportation units of occupied cells as follows

Total Transportation Cost= 20(30)+30(20)+20(30)+30(30)+10(70)
= 600+600+600+900+700
= 3400

Total Transportation Cost using NWCM is Rs. 3400

3. Solve the following transportation problem using North West Corner method.

	D1	D2	D3	D4	Supply
S1	40	50	60	50	50
S2	30	40	50	30	60
S3	40	60	80	30	70
Demand	30	50	30	70	180

Solution:

As Total Supply= Total Demand (180 units), this is balanced transportation problem.

Step 1. The North West Corner is Upper Left Corner called S1D1 cell. Corresponding first row supply is 50 units and corresponding first column demand is 30 Units. Hence allocation of maximum 30 units is possible (Being the minimum of first row and first column) to the S1D1 cell.

After allocating these 30 units, demand side gets completely fulfilled and hence there is no requirement of demand in the first column and the supply side gets reduced from 50 units to 20 units (50-30=20) hence the reduced transportation matrix becomes

	D1	D2	D3	D4	Supply
S1	40 (30)	50	60	50	~~50~~, 20
S2	30	40	50	30	60
S3	40	60	80	30	70
Demand	30	50	30	70	180

Step 2. The North West Corner is Upper Left Corner called S1D2 cell. Corresponding first row supply is 20 units and corresponding second column demand is 50 Units. Hence allocation of maximum 20 units is

possible (Being the minimum of first row and second column) to the S1D2 cell.

After allocating these 20 units, supply side gets completely fulfilled and hence there is no requirement of supply in the first row and the demand side gets reduced from 50 units to 30 units (50-20=30) hence the reduced transportation matrix becomes

	D1	D2	D3	D4	Supply
S1	40 (30)	~~50 (20)~~	~~60~~	~~50~~	~~50, 20~~
S2	30	40	50	30	60
S3	40	60	80	30	70
Demand	30	~~50~~, 30	30	70	180

Step 3. The North West Corner is Upper Left Corner called S2D2 cell. Corresponding second row supply is 60 units and corresponding second column demand is 30 Units. Hence allocation of maximum 30 units is possible (Being the minimum of second row and second column) to the S2D2 cell.

After allocating these 30 units, demand side gets completely fulfilled and hence there is no requirement of demand in the second column and the supply side gets reduced from 60 units to 30 units (60-30=30) hence the reduced transportation matrix becomes

	D1	D2	D3	D4	Supply
S1	40 (30)	~~50 (20)~~	~~60~~	~~50~~	~~50, 20~~
S2	30	40 (30)	50	30	~~60~~, 30
S3	40	60	80	30	70
Demand	30	50, ~~30~~	30	70	180

Step 4. The North West Corner is Upper Left Corner called S2D3 cell. Corresponding second row supply is 30 units and corresponding third column demand is 30 Units. Hence allocation of maximum 30 units is possible (Being the minimum of second row and second column) to the S2D2 cell.

After allocating these 30 units, demand side gets completely fulfilled and hence there is no requirement of demand in the third column and the supply side gets completed fulfilled and hence there is no requirement of supply in the second row hence the reduced transportation matrix becomes

	D1	D2	D3	D4	Supply
S1	40 (30)	50 (20)	60	50	50, 20
S2	30	40 (30)	50 (30)	30	60, 30
S3	40	60	80	30	70
Demand	30	50, 30	30	70	180

Step 5. The North West Corner is Upper Left Corner called S3D4 cell. Corresponding third row supply is 770 units and corresponding fourth column demand is 70 Units. Hence allocation of maximum 70 units is possible (Being the equal units) to the S3D4 cell.

After allocating these 70 units, demand side gets completely fulfilled and hence there is no requirement of demand in the fourth column and the supply side gets completed fulfilled and hence there is no requirement of supply in the third row hence the reduced transportation matrix becomes

	D1	D2	D3	D4	Supply
S1	40 (30)	50 (20)	60	50	50, 20
S2	30	40 (30)	50 (30)	30	60, 30
S3	40	60	80	30 (70)	70
Demand	30	50, 30	30	70	180

Step 6. Computation of Total Transportation Cost

Total transportation cost can be calculated by summing the multiplication of unit cost of transportation with transportation units of occupied cells as follows

Total Transportation Cost = 40(30)+50(20)+40(30)+50(30)+30(70)

= 1200+1000+1200+1500+2100

= 7000

Total Transportation Cost using NWCM is Rs. 7000

2.4.2 ROW MINIMA METHOD

Row Minima Method is the simple and easy method to compute initial feasible solution. This method considers the cost of transportation and the paths of transportation both. Row Minima is the considering minimum value in the given row in sequence from Row 1 onwards used for transportation. The process of allocation of goods in transit starts from the first row by considering the minimum value in the first row and similar calculations are carried forward.

There are several steps of calculating initial feasible solution using Row Minima Method as follows:

Step 1. Row Minima is the considering minimum value in the given row in sequence from Row 1 onwards used for transportation. The process of allocation of goods in transit starts from the first row by considering the minimum value in the first row and similar calculations are carried forward. Start the allocation of transportation units from the first row by considering minimum value from the corresponding first row.

Step 2. After allocation of transportation units to the cell having minimum value in the first row then if first row transportation units are remaining then corresponding allocation of transportation units to the next smaller transportation cost of first row can be done considering the demand and supply side of transportation problem.

Step 3. Similar iteration is done for allocating transportation units until it reaches to last row with complete allocation of transportation units matching demand and supply side.

Let us understand these steps by practical example.

4. Solve the following transportation problem using Row Minima Method.

	D1	D2	D3	Supply
S1	3	2	1	20
S2	2	4	1	50
S3	3	5	2	30
S4	4	6	7	25
Demand	40	30	55	125

Solution:
As Total Supply= Total Demand (125 units), this is balanced transportation problem.

Step 1. The First row in the transportation matrix is S1. Corresponding minimum cell value representing unit transportation cost is S1D3 in the first row. The allocation of transportation unit will be done firstly in the cell S1D3 by considering corresponding demand and supply for that cell.

The supply is of 20 units and demand is of 55 units. Minimum of these two values is 20 and hence the allocation of 20 units can be done in this cell as below.

After allocating these 20 units, supply side gets completely fulfilled and hence there is no requirement of supply in the first row and the demand side gets reduced from 55 units to 35 units (55-20=35) hence the reduced transportation matrix becomes

	D1	D2	D3	Supply
~~S1~~	~~3~~	~~2~~	~~1 (20)~~	~~20, 0~~
S2	2	4	1	50
S3	3	5	2	30
S4	4	6	7	25
Demand	40,	30	~~55~~, 35	125

There is no other cell remaining in the first row.

Step 2. The Second row in the transportation matrix is S2. Corresponding minimum cell value representing unit transportation cost is S2D3 in the second row. The allocation of transportation unit will be done firstly in the cell S2D3 by considering corresponding demand and supply for that cell.

The supply is of 50 units and demand is of 35 units. Minimum of these two values is 35 and hence the allocation of 35 units can be done in this cell as below.

After allocating these 35 units, demand side gets completely fulfilled and hence there is no requirement of demand in the third column and the supply side gets reduced from 50 units to 15 units (50-35=15) hence the reduced transportation matrix becomes

	D1	D2	D3	Supply
~~S1~~	~~3~~	~~2~~	~~1 (20)~~	~~20, 0~~
S2	2	4	~~1~~ (35)	~~50~~, 15
S3	3	5	~~2~~	30
S4	4	6	7	25
Demand	40	30	55, ~~35~~	125

There is no other cell remaining in the third column.

Step 3. The reduced second row S2 is remaining in the transportation matrix. Corresponding minimum cell value representing unit transportation cost is S2D1 in the second row. The allocation of transportation unit will be done now in the cell S2D1 by considering corresponding demand and supply for that cell.

The supply is of 15 units and demand is of 40 units. Minimum of these two values is 15 and hence the allocation of 15 units can be done in this cell as below.

After allocating these 15 units, supply side gets completely fulfilled and hence there is no requirement of supply in the second row and the demand side gets reduced from 40 units to 25 units (40-15=25) hence the reduced transportation matrix becomes

	D1	D2	D3	Supply
S1	3	2	1 (20)	20, 0
S2	2 (15)	4	1 (35)	50, 15
S3	3	5	2	30
S4	4	6	7	25
Demand	40, 25	30	55, 35	125

There is no other cell remaining in the second row.

Step 4. The next row is third row S3 in the transportation matrix. Corresponding minimum cell value representing unit transportation cost is S3D1 in the third row. The allocation of transportation unit will be done now in the cell S3D1 by considering corresponding demand and supply for that cell.

The supply is of 30 units and demand is of 25 units. Minimum of these two values is 25 and hence the allocation of 25 units can be done in this cell as below.

After allocating these 25 units, demand side gets completely fulfilled and hence there is no requirement of demand in the first column and the supply side gets reduced from 30 units to 5 units (30-25=5) hence the reduced transportation matrix becomes

	D1	D2	D3	Supply
S1	3	2	1 (20)	20, 0
S2	2 (15)	4	1 (35)	50, 15
S3	3 (25)	5	2	30, 5
S4	4	6	7	25
Demand	40, 25	30	55, 35	125

There is no other cell remaining in the first column.

Step 5. The next row is again reduced third row S3 in the transportation matrix. Corresponding minimum cell value representing unit transportation cost is S3D2 in the third row. The allocation of transportation unit will be done now in the cell S3D2 by considering corresponding demand and supply for that cell.

The supply is of 5 units and demand is of 30 units. Minimum of these two values is 5 and hence the allocation of 5 units can be done in this cell as below.

After allocating these 5 units, supply side gets completely fulfilled and hence there is no requirement of supply in the third row and the demand side gets reduced from 30 units to 25 units (30-5=25) hence the reduced transportation matrix becomes

	D1	D2	D3	Supply
S1	3	2	1 (20)	20, 0
S2	2 (15)	4	1 (35)	50, 15
S3	3 (25)	5 (5)	2	30, 5
S4	4	6	7	25
Demand	40, 25	30, 25	55, 35	125

There is no other cell remaining in the third row.

Step 6. The next row is reduced fourth row S4 in the transportation matrix. Corresponding minimum cell value representing unit transportation cost is S4D2 in the fourth row. The allocation of transportation unit will be done now in the cell S4D2 by considering corresponding demand and supply for that cell.

The supply is of 25 units and demand is of 25 units. Minimum of these two values is 25 and hence the allocation of 25 units can be done in this cell as below.

After allocating these 25 units, supply side gets completely fulfilled and hence there is no requirement of supply in the fourth row and the demand side gets completed fulfilled and hence there is no requirement of demand in the second column hence the reduced transportation matrix becomes

	D1	D2	D3	Supply
S1	3	2	1 (20)	20, 0
S2	2 (15)	4	1 (35)	50, 15
S3	3 (25)	5 (5)	2	30, 5
S4	4	6 (25)	7	25
Demand	40, 25	30, 25	55, 35	125

There is no other cell remaining in the fourth row.

Step 7. Allocation of Transportation Units to the occupied cells
Total Transportation Cost = 1 (20)+ 2 (15)+ 1 (35)+ 3 (25)+ 5(5)+ 6 (25)
= 20+30+35+75+25+ 150
= 335.

Hence, Total Transportation Cost is Rs.335 using Row Minima Method.

5. Solve the following transportation problem using Row Minima method.

	D1	D2	D3	D4	Supply
S1	20	30	40	30	50
S2	10	20	30	10	60
S3	20	40	60	10	70
Demand	30	50	30	70	180

Solution:
As Total Supply= Total Demand (180 units), this is balanced transportation problem.

Step 1. The First row in the transportation matrix is S1. Corresponding minimum cell value representing unit transportation cost is S1D1 in the first row. The allocation of transportation unit will be done firstly in the cell S1D1 by considering corresponding demand and supply for that cell. The supply is of 50 units and demand is of 30 units. Minimum of these two values is 30 and hence the allocation of 30 units can be done in this cell as below.

After allocating these 30 units, demand side gets completely fulfilled and hence there is no requirement of demand in the first column and the supply side gets reduced from 50 units to 20 units (50-30=20) hence the reduced transportation matrix becomes

	D1	D2	D3	D4	Supply
S1	20 (30)	30	40	30	50, 20
S2	10	20	30	10	60
S3	20	40	60	10	70
Demand	30	50	30	70	180

There is no other cell remaining in the first column.

Step 2. The reduced first row in the transportation matrix is S1. Corresponding minimum cell value representing unit transportation cost is S1D2 in the first row. The allocation of transportation unit will be done firstly in the cell S1D2 by considering corresponding demand and supply for that cell.

The supply is of 20 units and demand is of 50 units. Minimum of these two values is 20 and hence the allocation of 20 units can be done in this cell as below.

After allocating these 20 units, supply side gets completely fulfilled and hence there is no requirement of supply in the first row and the demand side gets reduced from 50 units to 30 units (50-20=30) hence the reduced transportation matrix becomes

	D1	D2	D3	D4	Supply
S1	20 (30)	30 (20)	40	30	50, 20
S2	10	20	30	10	60
S3	20	40	60	10	70
Demand	30	50, (30)	30	70	180

There is no other cell remaining in the first row.

Step 3. The next second row in the transportation matrix is S2. Corresponding minimum cell value representing unit transportation cost is S2D4 in the second row. The allocation of transportation unit will be done firstly in the cell S2D4 by considering corresponding demand and supply for that cell.

The supply is of 60 units and demand is of 70 units. Minimum of these two values is 60 and hence the allocation of 60 units can be done in this cell as below.

After allocating these 60 units, supply side gets completely fulfilled and hence there is no requirement of supply in the second row and the demand side gets reduced from 70 units to 10 units (70-60=10) hence the reduced transportation matrix becomes

	D1	D2	D3	D4	Supply
S1	20 (30)	30 (20)	40	30	50, 20
S2	10	20	30	10 (60)	60
S3	20	40	60	10	70
Demand	30	50, (30)	30	70, 10	180

There is no other cell remaining in the second row.

Step 4. The next third row in the transportation matrix is S3. Corresponding minimum cell value representing unit transportation cost is S3D4 in the third row. The allocation of transportation unit will be done firstly in the cell S3D4 by considering corresponding demand and supply for that cell.

The supply is of 70 units and demand is of 10 units. Minimum of these two values is 10 and hence the allocation of 10 units can be done in this cell as below

After allocating these 10 units, demand side gets completely fulfilled and hence there is no requirement of demand in the fourth column and the supply side gets reduced from 70 units to 60 units (70-10=60) hence the reduced transportation matrix becomes

	D1	D2	D3	D4	Supply
S1	20 (30)	30 (20)	40	30	50, 20
S2	10	20	30	10 (60)	60
S3	20	40	60	10 (10)	70, 60
Demand	30	50, 30	30	70, 10	180

There is no other cell remaining in the fourth column.

Step 5. The reduced third row in the transportation matrix is S3. Corresponding minimum cell value representing unit transportation cost is S3D2 in the third row. The allocation of transportation unit will be done firstly in the cell S3D2 by considering corresponding demand and supply for that cell.

The supply is of 60 units and demand is of 30 units. Minimum of these two values is 30 and hence the allocation of 30 units can be done in this cell as below

After allocating these 30 units, demand side gets completely fulfilled and hence there is no requirement of demand in the second column and the supply side gets reduced from 60 units to 30 units (60-30=30) hence the reduced transportation matrix becomes

	D1	D2	D3	D4	Supply
S1	20 (30)	30 (20)	40	30	50, 20
S2	10	20	30	10 (60)	60
S3	20	40 (30)	60	10 (10)	70, 60, 30
Demand	30	50, 30	30	70, 10	180

There is no other cell remaining in the second column.

Step 6. The reduced third row in the transportation matrix is S3. Corresponding minimum cell value representing unit transportation cost is S3D3 in the third row. The allocation of transportation unit will be done firstly in the cell S3D3 by considering corresponding demand and supply for that cell.

The supply is of 30 units and demand is of 30 units. Minimum of these two values is 30 and hence the allocation of 30 units can be done in this cell as below.

After allocating these 30 units, demand side gets completely fulfilled and hence there is no requirement of demand in the third column and the supply side completed fulfilled and hence there is no requirement of supply in the third row hence the reduced transportation matrix becomes

	D1	D2	D3	D4	Supply
S1	20 (30)	30 (20)	40	30	50, 20
S2	10	20	30	10 (60)	60
S3	20	40 (30)	60 (30)	10 (10)	70, 60, 30
Demand	30	50, 30	30	70, 10	180

There is no other cell remaining in the third column and third row also.

Step 7. Allocation of Transportation Units to the occupied cells

Total Transportation Cost = 20(30)+30(20)+ 10(60)+ 40(30)+ 60(30)+ 10(10)
= 600+600+600+1200+1800+ 100
= 4900.

Hence, Total Transportation Cost is Rs.4900 using Row Minima Method.

2.4.3 COLUMN MINIMA METHOD

Column Minima Method is the simple and easy method to compute initial feasible solution. This method considers the cost of transportation and the paths of transportation both. Column Minima is the considering minimum value in the given column in sequence from Column 1 onwards used for transportation. The process of allocation of goods in transit starts from the first column by considering the minimum value in the column row and similar calculations are carried forward.

There are several steps of calculating initial feasible solution using Column Minima Method as follows:

Step 1. Column Minima is considering the minimum value in the given column in sequence from column 1 onwards used for transportation. The process of allocation of goods in transit starts from the first column by considering the minimum value in the first column and similar calculations are carried forward. Start the allocation of transportation units from the first column by considering minimum value from the corresponding first column.

Step 2. After allocation of transportation units to the cell having minimum value in the first column then if first column transportation units are remaining then corresponding allocation of transportation units to the next smaller transportation cost of first column can be done considering the demand and supply side of transportation problem.

Step 3. Similar iteration is done for allocating transportation units until it reaches to last column with complete allocation of transportation units matching demand and supply side.

Let us understand these steps by practical example.

6. Solve the following transportation problem using column Minima Method.

	D1	D2	D3	Supply
S1	3	2	1	20
S2	2	4	1	50
S3	3	5	2	30
S4	4	6	7	25
Demand	40	30	55	125

Solution:
As Total Supply= Total Demand (125 units), this is balanced transportation problem.

Step 1. The First column in the transportation matrix is D1. Corresponding minimum cell value representing unit transportation cost is S2D1 in the first column. The allocation of transportation unit will be done firstly in the cell S2D1 by considering corresponding demand and supply for that cell.

The supply is of 50 units and demand is of 40 units. Minimum of these two values is 40 and hence the allocation of 40 units can be done in this cell as below.

After allocating these 40 units, demand side gets completely fulfilled and hence there is no requirement of demand in the first column and the supply side gets reduced from 50 units to 10 units (50-40=10) hence the reduced transportation matrix becomes

	D1	D2	D3	Supply
S1	3	2	1	20
S2	2 (40)	4	1	50, 10
S3	3	5	2	30
S4	4	6	7	25
Demand	40	30	55	125

There is no other cell remaining in the first column.

Step 2. The Second column in the transportation matrix is D2. Corresponding minimum cell value representing unit transportation cost is S1D2 in the second column. The allocation of transportation unit will be done firstly in the cell S1D2 by considering corresponding demand and supply for that cell.

The supply is of 20 units and demand is of 30 units. Minimum of these two values is 20 and hence the allocation of 20 units can be done in this cell as below.

After allocating these 20 units, supply side gets completely fulfilled and hence there is no requirement of supply in the first row and the demand side gets reduced from 30 units to 10 units (30-20=10) hence the reduced transportation matrix becomes

	D1	D2	D3	Supply
S1	3	2 (20)	1	20
S2	2 (40)	4	1	50, 10
S3	3	5	2	30
S4	4	6	7	25
Demand	40	30, 10	55	125

There is no other cell remaining in the first row.

Step 3. The reduced second columnD2 is remaining in the transportation matrix. Corresponding minimum cell value representing unit transportation cost is S2D2 in the second column. The allocation of transportation unit will be done now in the cell S2D2 by considering corresponding demand and supply for that cell.

The supply is of 10 units and demand is of 10 units. Minimum of these two values is 10 and hence the allocation of 10 units can be done in this cell as below.

After allocating these 10 units, supply side gets completely fulfilled and hence there is no requirement of supply in the second row and the demand side gets completed fulfilled and hence there is no requirement of demand in the second column, hence the reduced transportation matrix becomes

	D1	D2	D3	Supply
S1	3	2 (20)	1	20
S2	2 (40)	4 (10)	1	50, 10
S3	3	5	2	30
S4	4	6	7	25
Demand	40	30, 10	55	125

There is no other cell remaining in the second row.

There is no other cell remaining in the second column.

Step 4. The next column is third column S3 in the transportation matrix. Corresponding minimum cell value representing unit transportation cost is S3D3 in the third column. The allocation of transportation unit will be done now in the cell S3D3 by considering corresponding demand and supply for that cell.

The supply is of 30 units and demand is of 55 units. Minimum of these two values is 30 and hence the allocation of 30 units can be done in this cell as below.

After allocating these 30 units, supply side gets completely fulfilled and hence there is no requirement of supply in the third row and the demand side gets reduced from 55 units to 25 units (55-30=25) hence the reduced transportation matrix becomes

	D1	D2	D3	Supply
S1	3	2 (20)	1	20
S2	2 (40)	4 (10)	1	50, 10
S3	3	5	2 (30)	30
S4	4	6	7	25
Demand	40	30, 10	55, 25	125

There is no other cell remaining in the third row.

Step 5. The next column is again reduced third columnD3 in the transportation matrix. Corresponding minimum cell value representing unit transportation cost is S4D3 in the third column. The allocation of transportation unit will be done now in the cell S4D3 by considering corresponding demand and supply for that cell.

The supply is of 25 units and demand is of 25 units. Minimum of these two values is 25 and hence the allocation of 25 units can be done in this cell as below.

After allocating these 25 units, supply side gets completely fulfilled and hence there is no requirement of supply in the third row and the demand side gets completed fulfilled and hence there is no requirement of demand in the third column, hence the reduced transportation matrix becomes

	D1	D2	D3	Supply
S1	3	2 (20)	1	20
S2	2 (40)	4 (10)	1	50, 10
S3	3	5	2 (30)	30
S4	4	6	7 (25)	25
Demand	40	30, 10	55, 25	125

There is no other cell remaining in the third row.

Step 6. Allocation of Transportation Units to the occupied cells

Total Transportation Cost = 2(40)+2(20)+ 4(10)+ 20(30)+ 7(25)
= 80+40+40+60+175
= 395.

Hence, Total Transportation Cost is Rs.395 using Column Minima Method.

7. Solve the following transportation problem using Column Minima method.

	D1	D2	D3	D4	Supply
S1	20	30	40	30	50
S2	10	20	30	10	60
S3	20	40	60	10	70
Demand	30	50	30	70	180

Solution:
As Total Supply= Total Demand (180 units), this is balanced transportation problem.

Step 1. The First column in the transportation matrix is D1. Corresponding minimum cell value representing unit transportation cost is S2D1 in the first column. The allocation of transportation unit will be done firstly in the cell S2D1 by considering corresponding demand and supply for that cell.

The supply is of 60 units and demand is of 30 units. Minimum of these two values is 30 and hence the allocation of 30 units can be done in this cell as below.

After allocating these 30 units, demand side gets completely fulfilled and hence there is no requirement of demand in the first column and the supply side gets reduced from 60 units to 30 units (60-30=30) hence the reduced transportation matrix becomes

	D1	D2	D3	D4	Supply
S1	20	30	40	30	50
S2	10 (30)	20	30	10	60, 30
S3	20	40	60	10	70
Demand	30	50	30	70	180

There is no other cell remaining in the first column.

Step 2. The second column in the transportation matrix is D2. Corresponding minimum cell value representing unit transportation cost is S2D2 in the second column. The allocation of transportation unit will be done firstly in the cell S2D2 by considering corresponding demand and supply for that cell.

The supply is of 30 units and demand is of 50 units. Minimum of these two values is 30 and hence the allocation of 30 units can be done in this cell as below.

After allocating these 30 units, supply side gets completely fulfilled and hence there is no requirement of supply in the second row and the demand side gets reduced from 50 units to 20 units (50-30=20) hence the reduced transportation matrix becomes

	D1	D2	D3	D4	Supply
S1	20	30	40	30	50
S2	10 (30)	20 (30)	30	10	60, 30
S3	20	40	60	10	70
Demand	30	50, (20)	30	70	180

There is no other cell remaining in the first row.

Step 3. The reduced second column in the transportation matrix is D2. Corresponding minimum cell value representing unit transportation cost is S1D2 in the second column. The allocation of transportation unit will be done firstly in the cell S1D2 by considering corresponding demand and supply for that cell.

The supply is of 50 units and demand is of 20 units. Minimum of these two values is 20 and hence the allocation of 20 units can be done in this cell as below.

After allocating these 20 units, demand side gets completely fulfilled and hence there is no requirement of demand in the second column and the supply side gets reduced from 50 units to 30 units (50-20=30) hence the reduced transportation matrix becomes

	D1	D2	D3	D4	Supply
S1	20	30 (20)	40	30	50, 30
S2	10 (30)	20 (30)	30	10	60, 30
S3	20	40	60	10	70
Demand	30	50, 30	30	70	180

There is no other cell remaining in the second column.

Step 4. The next third column in the transportation matrix is D3. Corresponding minimum cell value representing unit transportation cost is S1D4 in the third column. The allocation of transportation unit will be done firstly in the cell S1D4 by considering corresponding demand and supply for that cell.

The supply is of 30 units and demand is of 30 units. Minimum of these two values is 30 and hence the allocation of 30 units can be done in this cell as below.

After allocating these 30 units, demand side gets completely fulfilled and hence there is no requirement of demand in the third column and the supply side gets completed fulfilled and hence there is no requirement of demand in the third column hence the reduced transportation matrix becomes

	D1	D2	D3	D4	Supply
S1	20	30 (20)	40 (30)	30	50, 30
S2	10 (30)	20 (30)	30	10	60, 30
S3	20	40	60	10	70
Demand	30	50, 30	30	70	180

There is no other cell remaining in the third column.

Step 5. The fourth column in the transportation matrix is D4. Corresponding minimum cell value representing unit transportation cost is S3D4 in the fourth column. The allocation of transportation unit will be done firstly in the cell S3D4 by considering corresponding demand and supply for that cell.

The supply is of 70 units and demand is of 70 units. Minimum of these two values is 70 and hence the allocation of 70 units can be done in this cell as below.

After allocating these 70 units, demand side gets completely fulfilled and hence there is no requirement of demand in the fourth column and the supply side gets completed fulfilled and hence there is no requirement of supply in the third row, hence the reduced transportation matrix becomes

	D1	D2	D3	D4	Supply
S1	20	30 (20)	40 (30)	30	50, 30
S2	10 (30)	20 (30)	30	10	60, 30
S3	20	40	60	10 (70)	70
Demand	30	50, 30	30	70	180

There is no other cell remaining in the fourth column and third row.

Step 6. Allocation of Transportation Units to the occupied cells

Total Transportation Cost = 10(30)+30(20)+ 20(30)+ 40(30)+ 10(70)
= 300+600+600+1200+1700
= 4400.

Hence, Total Transportation Cost is Rs.4400 using Column Minima Method.

2.4.4 LEAST COST METHOD

Least cost method is also known as matrix minima method because in this method, minimum value in all rows and all columns are searched out and corresponding allocation can be made in the transportation model.

Least Cost Method is the simple and easy method to compute initial feasible solution. This method considers the cost of transportation and the paths of transportation both. Matrix Minima is the considering minimum value in the given row and column in sequence from Column 1 onwards used for transportation. The process of allocation of goods in transit starts from the anywhere in the matrix by considering the minimum value in the column or row and similar calculations are carried forward.

There are several steps of calculating initial feasible solution using Lest Cost Method as follows:

Step 1. Matrix Minima is considering the minimum value in the given column or row in sequence from column 1 onwards or from row 1 onwards used for transportation. The process of allocation of goods in transit starts from either row or column by considering the minimum value in the matrix and similar calculations are carried forward. Start the allocation of transportation units from the row or column by considering minimum value from the corresponding transportation matrix.

Step 2. After allocation of transportation units to the cell having minimum value in the matrix then, find out the next minimum value in the matrix considering the demand and supply side of transportation problem.

Step 3. Similar iteration is done for allocating transportation units until it reaches to cell of the matrix with complete allocation of transportation units matching demand and supply side.

Let us understand these steps by practical example.

8. Solve the following transportation problem using Least Cost Method.

	D1	D2	D3	Supply
S1	3	2	1	20
S2	2	4	1	50
S3	3	5	2	30
S4	4	6	7	25
Demand	40	30	55	125

Solution:

As Total Supply= Total Demand (125 units), this is balanced transportation problem.

Step 1. The minimum value in the transportation matrix is 1 available at cell S1D3 and S2D3. Now there are two possibilities of allocating transportation units as follows:

i. If we consider, the minimum value in the transportation matrix is 1 available at cell S1D3 then maximum 20 units can be transported from S1 to D3.

ii. If we consider, the minimum value in the transportation matrix is 1 available at cell S2D3 then maximum 50 units can be transported from S2 to D3.

Here, the maximum units should be transported so as to minimize the total transportation cost.

Hence, the second possibility is better choice.

The minimum value in the transportation matrix is 1 available at cell S2D3 then maximum 50 units can be transported from S2 to D3.

The allocation of transportation unit will be done firstly in the cell S2D3 by considering corresponding demand and supply for that cell.

The supply is of 50 units and demand is of 55 units. Minimum of these two values is 50 and hence the allocation of 50 units can be done in this cell as below.

After allocating these 50 units, supply side gets completely fulfilled and hence there is no requirement of supply in the second row and the demand side gets reduced from 55 units to 5 units (55-50=5). Hence the reduced transportation matrix becomes

	D1	D2	D3	Supply
S1	3	2	1	20
S2	2	4	1 (50)	50
S3	3	5	2	30
S4	4	6	7	25
Demand	40	30	55, 5	125

There is no other cell remaining in the second row.

Step 2. The minimum value in the transportation matrix is 1 available at cell S1D3 then maximum 20 units can be transported from S1 to D3.

The allocation of transportation unit will be done firstly in the cell S1D3 by considering corresponding demand and supply for that cell.

The supply is of 20 units and demand is of 5 units. Minimum of these two values is 5 and hence the allocation of 5 units can be done in this cell as below.

After allocating these 5 units, demand side gets completely fulfilled and hence there is no requirement of demand in the third column and the supply side gets reduced from 55 units to 5 units (55-50=5). Hence the reduced transportation matrix becomes

	D1	D2	D3	Supply
S1	3	2	1 (5)	20, 15
S2	2	4	1 (50)	50
S3	3	5	2	30
S4	4	6	7	25
Demand	40	30	55, 5	125

There is no other cell remaining in the third column.

Step 3. The minimum value in the transportation matrix is 2 available at cell S1D2 then 15 units can be transported from S1 to D2.

The allocation of transportation unit will be done firstly in the cell S1D2 by considering corresponding demand and supply for that cell.

The supply is of 15 units and demand is of 30 units. Minimum of these two values is 15 and hence the allocation of 15 units can be done in this cell as below.

After allocating these 15 units, supply side gets completely fulfilled and hence there is no requirement of supply in the first row and the demand side gets reduced from 30 units to 15 units (30-15=15). Hence the reduced transportation matrix becomes

	D1	D2	D3	Supply
S1	3	2 (15)	1 (5)	20, 15
S2	2	4	1 (50)	50
S3	3	5	2	30
S4	4	6	7	25
Demand	40	30, 15	55, 5	125

There is no other cell remaining in the third column.

Step 4. The minimum value in the transportation matrix is 3 available at cell S3D1 then 30 units can be transported from S3 to D1.

The allocation of transportation unit will be done firstly in the cell S3D1 by considering corresponding demand and supply for that cell.

The supply is of 30 units and demand is of 40 units. Minimum of these two values is 30 and hence the allocation of 30 units can be done in this cell as below.

After allocating these 30 units, supply side gets completely fulfilled and hence there is no requirement of supply in the third row and the demand side gets reduced from 40 units to 10 units (40-30=10). Hence the reduced transportation matrix becomes

	D1	D2	D3	Supply
S1	3	2 (15)	1 (5)	20, 15
S2	2	4	1 (50)	50
S3	3 (30)	5	2	30
S4	4	6	7	25
Demand	40, 10	30, 15	55, 5	125

There is no other cell remaining in the third row.

Step 5. The minimum value in the transportation matrix is 4 available at cell S4D1 then 30 units can be transported from S4 to D1.

The allocation of transportation unit will be done firstly in the cell S4D1 by considering corresponding demand and supply for that cell.
The supply is of 25 units and demand is of 10 units. Minimum of these two values is 10 and hence the allocation of 10 units can be done in this cell as below.

After allocating these 10 units, demand side gets completely fulfilled and hence there is no requirement of demand in the first column and the supply side gets reduced from 25 units to 15 units (25-10=15). Hence the reduced transportation matrix becomes

	D1	D2	D3	Supply
S1	3	2 (15)	1 (5)	20, 15
S2	2	4	1 (50)	50
S3	3 (30)	5	2	30
S4	4 (10)	6	7	25, 15
Demand	40, 10	30, 15	55, 5	125

There is no other cell remaining in the first column.

Step 6. The minimum value in the transportation matrix is 6 available at cell S4D2 then 15 units can be transported from S4 to D2.

The allocation of transportation unit will be done firstly in the cell S4D2 by considering corresponding demand and supply for that cell.

The supply is of 15 units and demand is of 15 units. Minimum of these two values is 15 and hence the allocation of 15 units can be done in this cell as below.

After allocating these 15 units, demand side gets completely fulfilled and hence there is no requirement of demand in the second column and the supply side gets completed fulfilled and hence there is no requirement of supply in the fourth row. Hence the reduced transportation matrix becomes

	D1	D2	D3	Supply
S1	3	2 (15)	1 (5)	20, 15
S2	2	4	1 (50)	50
S3	3 (30)	5	2	30
S4	4 (10)	6 (15)	7	25, 15
Demand	40, 10	30, 15	55, 5	125

Step 7. Allocation of Transportation Units to the occupied cells

Total Transportation Cost = 2(15)+1(5)+ 1(50)+ 3(30)+ 4(10)+6 (15)
= 30+5+50+90+40+90
= 305.

Hence, Total Transportation Cost is Rs.305 using Least Cost Method.

9. Solve the following transportation problem using Matrix Minima method.

	D1	D2	D3	D4	Supply
S1	20	30	40	30	50
S2	10	20	30	10	60
S3	20	40	60	10	70
Demand	30	50	30	70	180

Solution:
As Total Supply= Total Demand (180 units), this is balanced transportation problem.

Step 1. The minimum value in the transportation matrix is 10 available at cell S2D1; S2D4 and S3D4. Now there are three possibilities of allocating transportation units as follows

a. If we consider, the minimum value in the transportation matrix is 10 available at cell S2D1 then maximum 30 units can be transported from S2 to D1.

b. If we consider, the minimum value in the transportation matrix is 10 available at cell S2D4 then maximum 60 units can be transported from S2 to D4.

c. If we consider, the minimum value in the transportation matrix is 10 available at cell S3D4 then maximum 70 units can be transported from S3 to D4.

Here the maximum units should be transported so as to minimize the total transportation cost.

Hence, the third possibility is better choice.

The minimum value in the transportation matrix is 1 available at cell S3D4 then maximum 70 units can be transported from S3 to D4.

The allocation of transportation unit will be done firstly in the cell S3D4 by considering corresponding demand and supply for that cell.
The supply is of 70 units and demand is of 70 units. Minimum of these two values is 70 and hence the allocation of 70 units can be done in this cell as below

After allocating these 70 units, supply and demand both side gets completed fulfilled and hence there is no requirement of demand and supply. Hence the reduced transportation matrix becomes

	D1	D2	D3	D4	Supply
S1	20	30	40	30	50
S2	10	20	30	10	60
S3	20	40	60	10 (70)	70
Demand	30	50	30	70	180

There is no other cell remaining in the third row and fourth column.

Step 2. The minimum value in the transportation matrix is 10 available at cell S2D1 then maximum 30 units can be transported from S2 to D1.
The allocation of transportation unit will be done firstly in the cell S2D1 by considering corresponding demand and supply for that cell.
The supply is of 60 units and demand is of 30 units. Minimum of these two values is 30 and hence the allocation of 30 units can be done in this cell as below.

After allocating these 30 units, demand side gets completely fulfilled and hence there is no requirement of demand in the first column and the supply side gets reduced from 60 units to 30 units (60-30=30). Hence the reduced transportation matrix becomes

	D1	D2	D3	D4	Supply
S1	20	30	40	30	50
S2	10 (30)	20	30	10	60, 30
S3	20	40	60	10 (70)	70
Demand	30	50	30	70	180

There is no other cell remaining in the third row and fourth column.

Step 3. The minimum value in the transportation matrix is 20 available at cell S2D2 then maximum 30 units can be transported from S2 to D2.

The allocation of transportation unit will be done firstly in the cell S2D2 by considering corresponding demand and supply for that cell.

The supply is of 30 units and demand is of 50 units. Minimum of these two values is 30 and hence the allocation of 30 units can be done in this cell as below.

After allocating these 30 units, supply side gets completely fulfilled and hence there is no requirement of supply in the second row and the demand side gets reduced from 50 units to 20 units (50-30=20). Hence the reduced transportation matrix becomes

	D1	D2	D3	D4	Supply
S1	20	30	40	30	50
S2	10 (30)	20 (30)	30	10	60, 30
S3	20	40	60	10 (70)	70
Demand	30	50, 20	30	70	180

There is no other cell remaining in the third row and fourth column.

Step 4. The minimum value in the transportation matrix is 30 available at cell S1D2 then maximum 20 units can be transported from S1 to D2.

The allocation of transportation unit will be done firstly in the cell S1D2 by considering corresponding demand and supply for that cell.

The supply is of 50 units and demand is of 20 units. Minimum of these two values is 20 and hence the allocation of 20 units can be done in this cell as below.

After allocating these 20 units, demand side gets completely fulfilled and hence there is no requirement of demand in the second column and the supply side gets reduced from 50 units to 20 units (50-30=20). Hence the reduced transportation matrix becomes

	D1	D2	D3	D4	Supply
S1	20	30 (20)	40	30	50, 30
S2	10 (30)	20 (30)	30	10	60, 30
S3	20	40	60	10 (70)	70
Demand	30	50, 20	30	70	180

There is no other cell remaining in the third row and fourth column.

Step 5. The minimum value in the transportation matrix is 40 available at cell S2D3 then maximum 30 units can be transported from S2 to D3.

The allocation of transportation unit will be done firstly in the cell S2D3 by considering corresponding demand and supply for that cell.

The supply is of 30 units and demand is of 30 units. Minimum of these two values is 30 and hence the allocation of 30 units can be done in this cell as below.

After allocating these 30 units, supply side and demand side both get completed fulfilled and there is no requirement of demand and supply. Hence the reduced transportation matrix becomes

	D1	D2	D3	D4	Supply
S1	20	30 (20)	40 (30)	30	50, 30
S2	10 (30)	20 (30)	30	10	60, 30
S3	20	40	60	10 (70)	70
Demand	30	50, 20	30	70	180

There is no other cell remaining in the first row and third column.

Step 7. Allocation of Transportation Units to the occupied cells

Total Transportation Cost = 30(20)+40(30)+ 10(30)+ 20(30)+ 10(70)

= 600+1200+300+600+700

= 3400.

Hence, Total Transportation Cost is Rs.3400 using Least Cost Method.

2.4.5 VOGELS' APPROXIMATION METHOD

Vogels' Approximation Method considers the highest penalty value for making allocation of transportation units. Penalty is the difference between two minimum cost values. The highest penalty value present in either row or column will be considered as the row or column for making allocation of transportation units.

The minimum cost value will be chosen from the selected row or column of the highest penalty value. The iteration of highest penalty values and corresponding allocations of transportation values are done until and unless all units are transported from the sources to the different destinations.

Let us understand these steps by practical example.

10. Solve following transportation problem using Vogels' Approximation Method.

	D1	D2	D3	Supply
S1	3	2	1	20
S2	2	4	1	50
S3	3	5	2	30
S4	4	6	7	25
Demand	40	30	55	125

Solution:
As Total Supply= Total Demand (125 units), this is balanced transportation problem.

Step 1. The maximum penalty value in the transportation matrix is 2 available at cell S4 row and D2 column. Now there are two possibilities of allocating transportation units as follows:

A. If we consider, the maximum penalty value in the transportation matrix is 2 available at S4 row then for minimum unit of transportation 4 (S4D1) maximum 25 units can be transported from S4 to D1.

B. If we consider, the maximum penalty value in the transportation matrix is 2 available at D2 column then then for minimum unit of transportation 2, maximum 20 units can be transported from S1 to D2.

Here, the maximum units should be transported so as to minimize the total transportation cost.

Hence, the first possibility is better choice.

The maximum penalty value in the transportation matrix is 2 available at S4 row then for minimum unit of transportation 4 (S4D1) maximum 25 units can be transported from S4 to D1.

The allocation of transportation unit will be done firstly in the cell S4D1 by considering corresponding demand and supply for that cell.

The supply is of 25 units and demand is of 40 units. Minimum of these two values is 25 and hence the allocation of 25 units can be done in this cell as below.

After allocating these 25 units, supply side gets completely fulfilled and hence there is no requirement of supply in the fourth row and the demand side gets reduced from 40 units to 15 units (40-25=15). Hence the reduced transportation matrix becomes

	D1	D2	D3	Supply	Penalty1
S1	3	2	1	20	2-1=1
S2	2	4	1	50	2-1=1
S3	3	5	2	30	3-2=1
S4	4 (25)	6	7	25	6-4=2
Demand	40, 15	30	55	125	
Penalty1	3-2=1	4-2=2	1-1=0		

There is no other cell remaining in the first column.

Step 2. Repeat the process or steps
The maximum penalty value for second time in the transportation matrix is 2 available at S4 row then for minimum unit of transportation 4 (S4D1) maximum 25 units can be transported from S4 to D1.

The allocation of transportation unit will be done firstly in the cell S4D1 by considering corresponding demand and supply for that cell.

The supply is of 25 units and demand is of 40 units. Minimum of these two values is 25 and hence the allocation of 25 units can be done in this cell as below.

After allocating these 25 units, supply side gets completely fulfilled and hence there is no requirement of supply in the fourth row and the demand side gets reduced from 40 units to 15 units (40-25=15). Hence the reduced transportation matrix becomes

	D1	D2	D3	Supply	Penalty1	Penalty2
S1	3	2	1	20	2-1=1	2-1=1
S2	2	4	1	50	2-1=1	2-1=1
S3	3	5	2	30	3-2=1	3-2=1
S4	4 (25)	6	7	25	6-4=2	
Demand	40, 15	30	55	125		
Penalty1	3-2=1	4-2=2	1-1=0			
Penalty2	3-2=1	4-2=2	1-1=0			

There is no other cell remaining in the fourth row.

Step 3. Repeat the process or steps

The maximum penalty value for second time in the transportation matrix is 2 available at D2column then for minimum unit of transportation 2 (S1D2) maximum 20 units can be transported from S1 to D2.

The allocation of transportation unit will be done firstly in the cell S1D2 by considering corresponding demand and supply for that cell.

The supply is of 20 units and demand is of 30 units. Minimum of these two values is 20 and hence the allocation of 20 units can be done in this cell as below.

After allocating these 20 units, supply side gets completely fulfilled and hence there is no requirement of supply in the first row and the demand side gets reduced from 300 units to 10 units (30-20=10). Hence the reduced transportation matrix becomes

	D1	D2	D3	Supply	Penalty1	Penalty2
S1	3	2 (20)	1	20	2-1=1	2-1=1
S2	2	4	1	50	2-1=1	2-1=1
S3	3	5	2	30	3-2=1	3-2=1
S4	4 (25)	6	7	25	6-4=2	
Demand	40, 15	30, 10	55	125		
Penalty1	3-2=1	4-2=2	1-1=0			
Penalty2	3-2=1	4-2=2	1-1=0			

There is no other cell remaining in the first row.

Step 4. Repeat the process or steps
The maximum penalty value for third time in the transportation matrix is 2 available at D2column then for minimum unit of transportation 2 (S1D2) maximum 20 units can be transported from S1 to D2.

The allocation of transportation unit will be done firstly in the cell S1D2 by considering corresponding demand and supply for that cell.

The supply is of 20 units and demand is of 30 units. Minimum of these two values is 20 and hence the allocation of 20 units can be done in this cell as below.

After allocating these 20 units, supply side gets completely fulfilled and hence there is no requirement of supply in the first row and the demand side gets reduced from 300 units to 10 units (30-20=10). Hence the reduced transportation matrix becomes

	D1	D2	D3	Supply	Penalty1	Penalty2	Penalty3
S1	3	2 (20)	1	20	2-1=1	2-1=1	
S2	2	4	1	50	2-1=1	2-1=1	2-1=1
S3	3	5	2	30	3-2=1	3-2=1	3-2=1
S4	4 (25)	6	7	25	6-4=2		
Demand	40, 15	30, 10	55	125			
Penalty1	3-2=1	4-2=2	1-1=0				
Penalty2	3-2=1	4-2=2	1-1=0				
Penalty3	3-2=1	5-4=1	2-1=1				

There is no other cell remaining in the first row.

The maximum penalty value for this time in the transportation matrix is 1 available at S2; S3 rows and D1; D2 and D3columns then for minimum unit of transportation 1 (S2D3) maximum 50 units can be transported from S2 to D3.

The allocation of transportation unit will be done firstly in the cell S2D3 by considering corresponding demand and supply for that cell.

The supply is of 50 units and demand is of 55 units. Minimum of these two values is 50 and hence the allocation of 50 units can be done in this cell as below.

After allocating these 50 units, supply side gets completely fulfilled and hence there is no requirement of supply in the second row and the demand side gets reduced from 55 units to 5 units (55-50=5). Hence the reduced transportation matrix becomes

	D1	D2	D3	Supply	Penalty1	Penalty2	Penalty3
S1	3	2 (20)	1	20	2-1=1	2-1=1	
S2	2	4	1 (50)	50	2-1=1	2-1=1	2-1=1
S3	3	5	2	30	3-2=1	3-2=1	3-2=1
S4	4 (25)	6	7	25	6-4=2		
Demand	40, 15	30, 10	55, 5	125			
Penalty1	3-2=1	4-2=2	1-1=0				
Penalty2	3-2=1	4-2=2	1-1=0				
Penalty3	3-2=1	5-4=1	2-1=1				

There is no other cell remaining in the SECOND row.

Step 5. As single cells are remaining in the columns, direct allocation is possible as follows

	D1	D2	D3	Supply	Penalty1	Penalty2	Penalty3
S1	3	2 (20)	1	20	2-1=1	2-1=1	
S2	2	4	1 (50)	50	2-1=1	2-1=1	2-1=1
S3	3 (15)	5 (10)	2 (5)	30	3-2=1	3-2=1	3-2=1
S4	4 (25)	6	7	25	6-4=2		

Demand	40, 15	30, 10	55, 5	125			
Penalty1	3-2=1	4-2=2	1-1=0				
Penalty2	3-2=1	4-2=2	1-1=0				
Penalty3	3-2=1	5-4=1	2-1=1				

Step 6. Allocation of Transportation Units to the occupied cells

Total Transportation Cost = 2(20)+1(50)+ 3(15)+ 5(10)+ 2(5)+4 (25)
= 40+50+45+50+10+100
= 295.

Hence, Total Transportation Cost is Rs.295 using Vogels' Approximation Method.

2.5 SELF ASSESSMENT TEST

Q1. Solve following Multiple Choice Questions.

1. Transportation is a typical type of operation research technique developed for ……… from the sources to the destination.

a. transportation of goods

b. assignment of job

c. transfer of employees

d. None of the above.

2. Transportation problem is a special type of ………which can be solved by using different methods of transportation.

a. Operations Method

b. linear programming problem

c. Research model

d. Business model.

3. The vertical arrangement of cells in the matrix is termed as…..

a. row b. column

c. table d. none of the above.

4. When total demand is equal to total supply then the transportation problem is said to be…

a. Complete b. incomplete

c. balanced d. Unbalanced.

5. The basic feasible solution exists when …….

a. number of occupied cells are equal to number of rows and number of columns minus one.

b. number of occupied cells are less than number of rows and number of columns minus one.

c. number of occupied cells are greater than number of rows and number of columns minus one.

d. number of occupied cells are either greater than or less than number of rows and number of columns minus one.

6. The objective function of transportation problems is…..

A. to minimize transportation cost

b. to maximize transportation cost

c. to either minimize or to maximize transportation cost

d. to neither minimize nor to maximize transportation cost

7. ……..is the first solution to the transportation problem.

a. An initial feasible solution

b. An initial infeasible solution

c. An initial feasible or infeasible solution

d. VAM

8. In the North West Corner Method, the process of transportation of goods starts from the……. Of the matrix.

a. Upper right corner

b. Lower right corner

a. lower left corner

d. upper left corner

Answers

1.) a 2.) b 3.) b 4.) c 5.) a 6.) a 7.) a 8.) d

Q2. Identify statements as true or false statement.

1. Necessary and sufficient condition for existence of a feasible solution to the transportation problem is that there should be total demand equal to total supply. (i.e. Total demand is equal to total supply). (True or False)
2. When total demand is equal to total supply then the transportation problem is said to be unbalanced. (True or False)
3. If total demand is not equal to total supply then the transportation problem is an unbalanced transportation problem. (True or False).
4. The occupied cells are having positive allocation in a transportation problem and are denoted by a circular symbol. (True or False).
5. The basic feasible solution exists when number of occupied cells are not equal to number of rows and number of columns minus one i.e. (No. of occupied cells = m+n-1). Where m is the number of rows and n is the number of columns. (True or False).
6. When the number of positive allocated cells are less than m+n-1 and a solution is degenerate solution and when the number of positive located cells are equal to m+n-1 then the solution is non-degenerate solution. (True or False)
7. Due to unfavorable road conditions, bad weather conditions or any other condition etc., It may not be possible to transit products from the sources to the destination such routes are prohibited routes. (True or False)

Answers:

1. True ***2. False.*** ***3. True***

4. True ***5. False*** ***6. True***

7. True

Q3. Write short notes on

1. Describe various terminologies used in the transportation method
2. Explain methods of finding initial feasible solution.
3. Describe steps of finding solutions to the transportation prolem.
4. Explain the steps of solving
   a. North West Corner Method.
   b. Row Minima Method
   c. Column Minima Method
   d. Least Cost Method
   e. Vogels' Approximation Method.

Q4. Determine the initial feasible solution to the following transportation problem using

a. North West Corner Method. (Answer: 116)

b. Least Cost Method (Answer: 112)

c. Vogels' Approximation Method. (Answer: 102)

Source/ Destination	D1	D2	D3	D4	Supply
S1	2	3	11	7	6
S2	1	0	6	1	1
S3	5	8	15	9	10
Demand	7	5	3	2	17

❖❖❖❖

3

ASSIGNMENT PROBLEMS

Unit Structure:

3.0 OBJECTIVES

The learning objectives of Game Theory are as follows:

1. To understand the concepts of assignment problems.
2. To allocate different jobs to different machines, different jobs to different employees on one to one basis.
3. To utilize maximum efficiency of employees and machine to get maximum output and with minimum duration of completion of tasks.
4. To study different methods of assignments in detail.

3.1 INTRODUCTION

Assignment is the particular type of optimization technique where object is to allocate limited resources to maximize the profit. In general, assignment is the method allocating limited organizational resources such as manpower, machine to different jobs. Hence assignment problems are the optimization problem to allocate jobs to different machines or to workers, salesmen to the sales zone or to allocate vehicles to different routes. Assignment works on one to one basis i.e. one resource to the one unit likewise.

All assignments are done in order to allocate scare resources to machines or jobs so as to maximize the efficiency or output of the machine or worker. Assignment problems have same number of rows and columns for allocating resources properly on one to one basis. If this situation is not satisfied, this may become unbalanced problem and it can be converted to the balanced assignment problems by adding dummy row or column to make it balanced problem and solution can be found out by using Hungarian Assignment Method or special case solutions. Let us study these concepts thoroughly one by one as follows

3.2 HUNGARIAN METHOD OF SOLUTION

The Statistician Hungarian put forward the method of finding solution to the assignment problems called as Hungarian Assignment Method (HAM). This method is used for minimization cases of optimization problems. This method is faster, easier and more efficient to find out the solutions to the Assignment Problems. The steps of finding solutions using Hungarian Assignment Method (HAM) are as follows:

Step 1. Verifying Balanced or Unbalanced assignment problem
The first step is to find out the balanced or unbalanced assignment problem. This can be done by considering equality of rows and columns. If the number of rows is equal to the number of columns then this is balanced assignment problem. If the number of rows is not equal to the number of columns then this is unbalanced assignment problem.

The balanced assignment problem is required for next step.

Step 2. Row Subtraction
In the balanced assignment problem, we have to find the smallest element from each row. Then subtract that smallest element of row from every element of corresponding row. This step is row subtraction.

Step 3. Column Subtraction
In the reduced assignment problem, we have to find the smallest element from each column. Then subtract that smallest element of column from every element of corresponding column. This step is column subtraction.

Step 4. Line Formation
Then form the lines using maximum number of zeros either in row or in column as first line and next lines are formed using the same principle of maximum number of zeros either in row or in column as second line and so on.

Step 5. Comparing Number of lines with Number of rows and Number of Columns
In the assignment problems, as jobs are assigned to machines or men on one to one basis, the number of rows and number of columns are already same. Here we have to verify the number of lines formed with number of rows and number of columns, if they are same then direct allocation is possible otherwise improvement has to be done. Let us discuss these facts in detail as follows

i. If number of lines formed are same as number of rows and number of columns then direct allocation is possible as follows

The allocation is one by considering the only one zero available in the row or column. Then second time same thing is observed (only one zero available) and remaining zero in the related column or row are cancelled accordingly as the allocations are done on one to one basis. The procedure is repeated until and unless all allocations or assignments are done.

ii. If number of lines formed are not same as number of rows and number of columns then direct allocation is possible as follows

Then Hungerian Assignment Method (HAM) is used.

Step 6. Hungerian Assignment Method (HAM)
If number of lines formed are not same as number of rows and number of columns then Hungerian Assignment Method (HAM) is used

As the lines are formed, few rows and columns are cancelled. The matrix is turned into reduced matrix with remaining elements of matrix that are not cancelled. Few lines cross each other and crossing points are called as junction point.

The smallest element in the remaining reduced matrix is subtracted from all elements of reduced matrix so that one additional zero is obtained at the location of smallest element of that remaining reduced matrix and that leads to the chance of formation of new line. At the junction point, that smallest number is to be added.

The allocation is one by considering the only one zero available in the row or column. Then second time same thing is observed (only one zero available) and remaining zero in the related column or row are cancelled accordingly as the allocations are done on one to one basis. The procedure is repeated until and unless all allocations or assignments are done.

Step 7. Assignments
The allocations or assignments are done on one to one basis so that every row and column as one allocation. It means that each worker or machine has one job to be done in minimum time.

Let us study these all steps by practical examples.

Example 1. A job requires four different activities- Sorting, Washing, Finishing and Assembling. Four workers are assigned all these activities. The time required by each worker to complete four different activities-Sorting, Washing, Finishing and Assembling are given below

	Worker 1	Worker 2	Worker 3	Worker 4
Sorting	31	25	33	25
Washing	25	24	23	21
Finishing	19	21	23	24
Assembling	38	36	34	40

How should these activities be arranged to the workers so that the job is completed in minimum time?

Solution:

This is an assignment problem as 4 activities are assigned to 4 workers on one to one basis. Our objective is to minimize time of completion of work by assigning proper activities to workers.

Step 1. Verifying Balanced or Unbalanced assignment problem

The first step is to find out the balanced or unbalanced assignment problem. This can be done by considering equality of rows and columns. If the number of rows is equal to the number of columns then this is balanced assignment problem. If the number of rows is not equal to the number of columns then this is unbalanced assignment problem.

Here the number of rows r=4 and number of columns c=4 and hence this is balanced assignment problem (r=c=4).

The balanced assignment problem is required for next step.

Step 2. Row Subtraction

In the balanced assignment problem, we have to find the smallest element from each row. Then subtract that smallest element of row from every element of corresponding row. This step is row subtraction.

Here, sorting activity is given in the first row and smallest element in the first row is 25. Subtract 25 from each element of first row. Similarly for the second row, washing activity is given in the second row and smallest element in the second row is 21. Subtract 21 from each element of second row. For the third row, finishing activity is given in the third row and smallest element in the third row is 19. Subtract 19 from each element of third row. For the fourth row, assembling activity is given in the fourth row and smallest element in the fourth row is 34. Subtract 34 from each element of fourth row. We get

	Worker 1	Worker 2	Worker 3	Worker 4
Sorting	6	0	8	0
Washing	4	3	2	0
Finishing	0	2	4	5
Assembling	4	2	0	6

Step 3. Column Subtraction

In the reduced assignment problem, we have to find the smallest element from each column. Then subtract that smallest element of column from every element of corresponding column. This step is column subtraction.

Here, worker 1 is given in the first column and smallest element in the first column is 0. Subtract 0 from each element of first column. Similarly for the second column, worker 2 is given in the second column and smallest element in the second column is 0. Subtract 0 from each element of second column. For the third column, worker 3 is given in the third column and smallest element in the third column is 0. Subtract 0 from each element of third column. For the fourth column, worker 4 is given in the fourth column and smallest element in the fourth column is 0. Subtract 0from each element of fourth column. We get

	Worker 1	Worker 2	Worker 3	Worker 4
Sorting	6	0	8	0
Washing	4	3	2	0
Finishing	0	2	4	5
Assembling	4	2	0	6

Step 4. Line Formation

Then form the lines using maximum number of zeros either in row or in column as first line and next lines are formed using the same principle of maximum number of zeros either in row or in column as second line and so on.

First row Sorting and fourth column Worker 4 has two zeroes. So, we may form first line in the first row or in the fourth column and other rows and columns have only one zero in their pattern. So, let us try with first row having two zeroes as maximum number of zeroes as follows

	Worker 1	Worker 2	Worker 3	Worker 4
Sorting	~~6~~	~~0~~	~~8~~	~~0~~
Washing	4	3	2	0
Finishing	0	2	4	5
Assembling	4	2	0	6

Next see that second, third and fourth rows have only one zero similarly first, second, third and fourth columns have only one zero and we may proceed either row wise or column wise. For the purpose of convenience, consider second, third and fourth rows have only one zero for line formation as follows

	Worker 1	Worker 2	Worker 3	Worker 4
Sorting	~~6~~	~~0~~	~~8~~	~~0~~
Washing	~~4~~	~~3~~	~~2~~	~~0~~
Finishing	~~0~~	~~2~~	~~4~~	~~5~~
Assembling	~~4~~	~~2~~	~~0~~	~~6~~

Step 5. Comparing Number of lines with Number of rows and Number of Columns

Here we have to verify the number of lines formed with number of rows and number of columns, if they are same then direct allocation is possible otherwise improvement has to be done. Let us discuss these facts in detail as follows:

The number of lines formed (4) are same as number of rows (4) and number of columns (4) then direct allocation is possible as follows
The allocation is done by considering the only one zero available in the row or column. Then second time same thing is observed (only one zero available) and remaining zero in the related column or row are cancelled accordingly as the allocations are done on one to one basis. The procedure is repeated until and unless all allocations or assignments are done.

	Worker 1	Worker 2	Worker 3	Worker 4
Sorting	6	0	8	0
Washing	4	3	2	0
Finishing	0	2	4	5
Assembling	4	2	0	6

In the second row, only one zero is available which is at the junction of worker 4 and washing. It means that washing can be done by worker 4. If worker 4 is busy with washing then he can't do other task sorting. Hence sorting can be done by worker 2 (zero availability) as follows

	Worker 1	**Worker 2**	**Worker 3**	**Worker 4**
Sorting	**6**	**0✔**	**8**	**0✖**
Washing	**4**	**3**	**2**	**0✔**
Finishing	**0✔**	**2**	**4**	**5**
Assembling	**4**	**2**	**0✔**	**6**

Step 7. Assignments

The allocations or assignments are done on one to one basis so that every row and column as one allocation. It means that each worker or machine has one job to be done in minimum time. The single zero in row or column indicates relative cell position to be selected for assignment purpose as follows

Sr. No.	Activities	Worker	Time (in minutes)
1	Sorting	2	25
2	Washing	4	21
3	Finishing	1	19
4	Assembling	3	34
		Total time	99 minutes

These activities should be arranged to the workers in above assignment so that the job is completed in minimum 99 minutes.

HAM for improvement type assignment problems

Example 2: The five different machines A, B, C, D and E perform five different tasks 1, 2, 3, 4 and 5. The costs for performing each task (in lack rupees) are given below

	1	**2**	**3**	**4**	**5**
A	**8**	**8**	**8**	**11**	**12**
B	**4**	**5**	**6**	**3**	**4**
C	**12**	**11**	**10**	**9**	**8**
D	**18**	**21**	**18**	**17**	**15**
E	**10**	**11**	**10**	**8**	**12**

How you will assign different tasks to different machines one to one basis so that the total cost of production should be minimum?

Solution:

This is an assignment problem as 5 tasks are assigned to 5 machines on one to one basis. Our objective is to minimize total cost of production by assigning proper tasks to machines.

The steps for assignment are as follows:

Step 1. Verifying Balanced or Unbalanced assignment problem

The first step is to find out the balanced or unbalanced assignment problem. This can be done by considering equality of rows and columns. If the number of rows is equal to the number of columns then this is balanced assignment problem. If the number of rows is not equal to the number of columns then this is unbalanced assignment problem.

Here, the number of tasks and machines are 5 and 5 respectively. Hence this is balanced assignment problem.

The balanced assignment problem is required for next step.

Step 2. Row Subtraction

In the balanced assignment problem, we have to find the smallest element from each row. Then subtract that smallest element of row from every element of corresponding row. This step is row subtraction.

Here machine A is at first row and the smallest element in the first row is 8. Subtract 8 from every element of first row. Machine B is at second row and the smallest element in the second row is 3. Subtract 3 from every element of second row. Similarly, for third, fourth and fifth row, activities are carried out as follows

	1	**2**	**3**	**4**	**5**
A	**0**	**0**	**0**	**3**	**4**
B	**1**	**2**	**3**	**0**	**1**
C	**4**	**3**	**2**	**1**	**0**
D	**3**	**6**	**3**	**2**	**0**
E	**2**	**3**	**2**	**0**	**4**

Step 3. Column Subtraction

In the reduced assignment problem, we have to find the smallest element from each column. Then subtract that smallest element of column from every element of corresponding column. This step is column subtraction.

Every column has smallest element 0 and hence subtracting smallest element from every element of first column will lead to same reduced assignment table.

	1	2	3	4	5
A	0	0	0	3	4
B	1	2	3	0	1
C	4	3	2	1	0
D	3	6	3	2	0
E	2	3	2	0	4

Step 4. Line Formation

Then form the lines using maximum number of zeros either in row or in column as first line and next lines are formed using the same principle of maximum number of zeros either in row or in column as second line and so on.

	1	2	3	4	5
A	0	0	0	3	4
B	1	2	3	0	1
C	4	3	2	1	0
D	3	6	3	2	0
E	2	3	2	0	4

By observation, it is found that the first row has maximum (3) zeroes and hence first line formed is in first row. Fourth and fifth columns have (2) maximum zeroes for the next time. Hence second and third lines are formed at Fourth and fifth columns. There are no other zeroes available and hence line formation is as follows

	1	2	3	4	5
A	0	0	0	3	4
B	1	2	3	0	1
C	4	3	2	1	0
D	3	6	3	2	0
E	2	3	2	0	4

Step 5. Comparing Number of lines with Number of rows and Number of Columns

In the assignment problems, as tasks are assigned to machines or men on one to one basis, the number of rows and number of columns are already same. Here we have to verify the number of lines formed with number of rows and number of columns, if they are same then direct allocation is possible otherwise improvement has to be done. Let us discuss these facts in detail as follows:

The number of lines formed (3) are not same as number of rows (5) and number of columns (5) then direct allocation is possible as follows

Then Hungerian Assignment Method (HAM) is used.

Step 6. Hungerian Assignment Method (HAM)

If number of lines formed are not same as number of rows and number of columns then Hungerian Assignment Method (HAM) is used

The smallest element in the remaining reduced matrix is subtracted from all elements of reduced matrix so that one additional zero is obtained at the location of smallest element of that remaining reduced matrix and that leads to the chance of formation of new line. At the junction point, that smallest number is to be added.

The allocation is one by considering the only one zero available in the row or column. Then second time same thing is observed (only one zero available) and remaining zero in the related column or row are cancelled accordingly as the allocations are done on one to one basis. The procedure is repeated until and unless all allocations or assignments are done.

	1	2	3	4	5
A	0	0	0	3	4
B	1	2	3	0	1
C	4	3	2	1	0
D	3	6	3	2	0
E	2	3	2	0	4

The smallest element in the reduced assignment matrix is 1. Subtract that smallest element 1 from all elements of reduced assignment matrix. Add that smallest element at two junctions point A4 and A5 as follows:

The number of lines formed (4) are not same as number of rows (5) and number of columns (5) then direct allocation is not possible as follows:

Line formation: As second row has zero, this is cancelled as below

	1	2	3	4	5
A	0	0	0	4	5
B	0	1	2	0	1
C	3	2	1	1	0
D	2	5	2	2	0
E	1	2	1	0	4

Here the smallest element in the remaining reduced matrix is 1, subtracted from all elements of reduced matrix so that one additional zero is obtained at the location of smallest element of that remaining reduced matrix and that leads to the chance of formation of new line. At the junction point, that smallest number (1) is to be added.

Step 7. Line formation:
As third column has 2 zeroes and hence this is cancelled as follows

	1	2	3	4	5
A	0	0	0	5	6
B	0	1	2	1	2
C	2	1	0	1	0
D	1	4	1	2	0
E	0	1	0	0	4

Step 8. Comparing number of lines formed and with number of rows and columns
The number of lines formed (5) are same as number of rows (5) and number of columns (5) then direct allocation is possible as follows

Step 9. Assignments
The allocations or assignments are done on one to one basis so that every row and column as one allocation. It means that each worker or machine has one job to be done in minimum time.

The allocation is done by considering the only one zero available in the row or column. Then second time same thing is observed (only one zero available) and remaining zero in the related column or row are cancelled accordingly as the allocations are done on one to one basis. The procedure is repeated until and unless all allocations or assignments are done.

	1	2	3	4	5
A	**0**	**0**	**0**	**5**	**6**
B	**0**	**1**	**2**	**1**	**2**
C	**2**	**1**	**0**	**1**	**0**
D	**1**	**4**	**1**	**2**	**0**
E	**0**	**1**	**0**	**0**	**4**

Now the second column has only one zero and hence it will be selected at first.

	1	2	3	4	5
A	**0✖**	**0✔**	**0✖**	**5**	**6**
B	**0**	**1**	**2**	**1**	**2**
C	**2**	**1**	**0**	**1**	**0**
D	**1**	**4**	**1**	**2**	**0**
E	**0**	**1**	**0**	**0**	**4**

As machine A is busy with task 2, it can't perform any other tasks like 1 and 3 (having zero values)

Now the machine D has only one zero and hence it can perform only 5th task.

	1	2	3	4	5
A	**0✖**	**0✔**	**0✖**	**5**	**6**
B	**0**	**1**	**2**	**1**	**2**
C	**2**	**1**	**0**	**1**	**0✖**
D	**1**	**4**	**1**	**2**	**0✔**
E	**0**	**1**	**0**	**0**	**4**

Now machine C is capable of performing task 3 as follows

	1	2	3	4	5
A	**0✖**	**0✔**	**0✖**	**5**	**6**
B	**0**	**1**	**2**	**1**	**2**
C	**2**	**1**	**0✔**	**1**	**0✖**
D	**1**	**4**	**1**	**2**	**0✔**
E	**0**	**1**	**0**	**0**	**4**

As task 3 is already taken up by machine C then machine E cant' do that task and machine E performs 4th task as follows

	1	2	3	4	5
A	**0✖**	**0✔**	**0✖**	**5**	**6**
B	**0**	**1**	**2**	**1**	**2**
C	**2**	**1**	**0✔**	**1**	**0✖**
D	**1**	**4**	**1**	**2**	**0✔**
E	**0✖**	**1**	**0✖**	**0✔**	**4**

Now task 1 can be performed by machine B as follows

	1	2	3	4	5
A	**0✖**	**0✔**	**0✖**	**5**	**6**
B	**0✔**	**1**	**2**	**1**	**2**
C	**2**	**1**	**0✔**	**1**	**0✖**
D	**1**	**4**	**1**	**2**	**0✔**
E	**0✖**	**1**	**0✖**	**0✔**	**4**

The allocation or assignment of different tasks to different machines are as follows

Sr. No.	Machine	Task	Cost of task
1	A	2	8
2	B	1	4
3	C	3	10
4	D	5	15
5	E	4	8
Total cost of production			**(in lack rupees)** 45

3.3 SPECIAL CASES IN ASSIGNMENT PROBLEMS

The special cases in assignment problems are unbalanced problems, multiple optimum solution, maximization problems and prohibited assignments as follows:

3.3.1 Unbalanced Problems

When the number of rows is not equal to the number of columns then this is unbalanced assignment problem. While solving it, we have to convert it to balanced problem either by adding dummy row or column as per the requirement and rest steps are similar.

Example 3: The 3 seminar slots are available for 4 MMS students. Now Professor wants to conduct seminars within minimum time slots available to students. The slots and students names are given below within cells representing time required for completing seminar.

Students	Slot1	Slot2	Slot3
Aaradhya	40	29	32
Aadvet	42	25	35
Vedika	35	22	38
Shrived	36	26	33

How will you assign slots to students so that minimum time slots are available to students?

Solution: As 3 slots are available to 4 students, one student is remaining with the slot. Hence dummy slot may be added to make it equal (number of rows and number of columns will be made same by adding dummy column) as follows

Students	Slot1	Slot2	Slot3	Dummy Slot
Aaradhya	40	29	32	0
Aadvet	42	25	35	0
Vedika	35	22	38	0
Shrived	36	26	33	0

Step 1. Verifying Balanced or Unbalanced assignment problem

The number of rows (4) is equal to the number of columns (4) hence, this is balanced assignment problem. The balanced assignment problem is required for next step.

Step 2. Row Subtraction

In the balanced assignment problem, we have to find the smallest element from each row. Then subtract that smallest element of row from every element of corresponding row. This step is row subtraction.

In all rows, smallest element is zero and hence assignment problem remains unchanged.

Students	Slot1	Slot2	Slot3	Dummy Slot
Aaradhya	40	29	32	0
Aadvet	42	25	35	0
Vedika	35	22	38	0
Shrived	36	26	33	0

Step 3. Column Subtraction
In the reduced assignment problem, we have to find the smallest element from each column. Then subtract that smallest element of column from every element of corresponding column. This step is column subtraction.

The smallest element in the first column is 35 and subtracts 35 from all elements of first column and so on

Students	Slot1	Slot2	Slot3	Dummy Slot
Aaradhya	5	7	0	0
Aadvet	7	3	3	0
Vedika	0	0	6	0
Shrived	1	4	1	0

Step 4. Line Formation
Then form the lines using maximum number of zeros either in row or in column as first line and next lines are formed using the same principle of maximum number of zeros either in row or in column as second line and so on.

Students	Slot1	Slot2	Slot3	Dummy Slot
Aaradhya	5	7	0	0
Aadvet	7	3	3	0
Vedika	0	0	6	0
Shrived	1	4	1	0

First line is formed by four zeroes available in the fourth column and second line is formed by two zeroes in the third row and third column contains only one zero available forming third line as follows

Students	Slot1	Slot2	Slot3	Dummy Slot
Aaradhya	5	7	0	0
Aadvet	7	3	3	0
Vedika	0	0	6	0
Shrived	1	4	1	0

Step 5. Comparing Number of lines with Number of rows and Number of Columns

In the assignment problems, as jobs are assigned to machines or men on one to one basis, the number of rows and number of columns are already same. Here we have to verify the number of lines formed with number of rows and number of columns, if they are same then direct allocation is possible otherwise improvement has to be done. Let us discuss these facts in detail as follows

The number of lines formed (3) are not same as number of rows (4) and number of columns (4)then direct allocation is possible as follows
Then Hungerian Assignment Method (HAM) is used.

Step 6. Hungerian Assignment Method (HAM)

If number of lines formed (3) are not same as number of rows (4) and number of columns (4) then Hungerian Assignment Method (HAM) is used

As the lines are formed, few rows and columns are cancelled. The matrix is turned into reduced matrix with remaining elements of matrix that are not cancelled. Few lines cross each other and crossing points are called as junction point.

The smallest element in the remaining reduced matrix is subtracted from all elements of reduced matrix so that one additional zero is obtained at the location of smallest element of that remaining reduced matrix and that leads to the chance of formation of new line. At the junction point, that smallest number is to be added.

Students	Slot1	Slot2	Slot3	Dummy Slot
Aaradhya	5	7	0	0
Aadvet	7	3	3	0
Vedika	0	0	6	0
Shrived	1	4	1	0

The smallest element (1) in the remaining reduced matrix is subtracted from all elements of reduced matrix so that one additional zero is obtained at the location of smallest element of that remaining reduced matrix and that leads to the chance of formation of new line. At the junction points 6 and 0, that smallest number (1) is to be added.

Students	Slot1	Slot2	Slot3	Dummy Slot
Aaradhya	4	6	0	0
Aadvet	6	2	3	0
Vedika	0	0	7	1
Shrived	0	3	1	0

Step 7. One additional line is formed and hence total 4 lines are formed and are equal to the number of rows (4) and number of columns (4). Hence Assignment is done as follows

Students	Slot1	Slot2	Slot3	Dummy Slot
Aaradhya	4	6	0	0
Aadvet	6	2	3	0
Vedika	0	0	7	1
Shrived	0	3	1	0

Step 8. Assignments

The allocations or assignments are done on one to one basis so that every row and column as one allocation. It means that each worker or machine has one job to be done in minimum time.

Students	Slot1	Slot2	Slot3	Dummy Slot
Aaradhya	4	6	0	0
Aadvet	6	2	3	0
Vedika	0	0	7	1
Shrived	0	3	1	0

The second row is having only one zero and hence allocation to Aadvet is dummy slot. As dummy slot is taken by Aadvet and hence dummy slot can't be given to any other student. For the slot2, only one zero is available at third row. Similarly for the slot3, only one zero is available at first row and the assignment problem takes the following form.

Students	Slot1	Slot2	Slot3	Dummy Slot
Aaradhya	**4**	**6**	**0✔**	**0✖**
Aadvet	**6**	**2**	**3**	**0✔**
Vedika	**0✖**	**0✔**	**7**	**1**
Shrived	**0✔**	**3**	**1**	**0✖**

The Fourth row Shrived is having zero at slot 1 and hence slot 1 cannot be taken by Vedika as follows

Sr. No.	Name of the student	Time slot	Time (in minutes)
1	**Aaradhya**	Slot3	32
2	**Aadvet**	Slot4	00* Dummy
3	**Vedika**	Slot2	22
4	**Shrived**	Slot1	36

* Dummy values are not known and are considered for the purpose of making allocations or assignments.

3.3.2 Multiple Optimum Solutions

When the assignment problem has more than one solution, this is termed as multiple optimum solutions to the assignment problem. The minimum two solutions exist for multiple optimum solutions. Hence there are possibilities of getting different solutions. We can compare solutions and get the optimum solution required.

Example 4. The railway ticket window has 4 reservation counters. Four employees are assigned these tasks of reservation counters. The cost of assigning each person to each counter is as follows

Employee	Reservation counter (Rc)			
	Rc1	Rc2	Rc3	Rc4
Aaradhya	1	8	15	22
Aadvet	13	18	23	28
Vedika	13	18	23	28
Shrived	19	23	27	31

Solution:

Step 1. Verifying Balanced or Unbalanced assignment problem
The number of rows (4) is equal to the number of columns (4) hence, this is balanced assignment problem. The balanced assignment problem is required for next step.

Step 2. Row Subtraction
In the balanced assignment problem, we have to find the smallest element from each row. Then subtract that smallest element of row from every element of corresponding row. This step is row subtraction.

In first row, the smallest element is 1 and In second and third rows, the smallest elements are 13 and in the fourth row, the smallest element is 19. Subtract these smallest elements from all corresponding rows and then we get

Employee	Reservation counter (Rc)			
	Rc1	Rc2	Rc3	Rc4
Aaradhya	0	7	14	21
Aadvet	0	5	10	15
Vedika	0	5	10	15
Shrived	0	4	27	12

Step 3. Column Subtraction
In the reduced assignment problem, we have to find the smallest element from each column. Then subtract that smallest element of column from every element of corresponding column. This step is column subtraction.
The smallest element in the first column is 0 and subtracts 0 from all elements of first column and so on

Employee	Reservation counter (Rc)			
	Rc1	Rc2	Rc3	Rc4
Aaradhya	0	3	4	9
Aadvet	0	1	0	3
Vedika	0	1	0	3
Shrived	0	0	17	0

Step 4. Line Formation
Then form the lines using maximum number of zeros either in row or in column as first line and next lines are formed using the same principle of maximum number of zeros either in row or in column as second line and so on.

First line is formed by four zeroes available in the first column and second line is formed by two zeroes in the third column and second and fourth column contains only one zero available forming third and fourth lines as follows

Employee	Reservation counter (Rc)			
	Rc1	Rc2	Rc3	Rc4
Aaradhya	0	3	4	9
Aadvet	0	1	0	3
Vedika	0	1	0	3
Shrived	0	0	17	0

Step 5. Comparing Number of lines with Number of rows and Number of Columns

In the assignment problems, as jobs are assigned to machines or men on one to one basis, the number of rows and number of columns are already same. Here we have to verify the number of lines formed with number of rows and number of columns, if they are same then direct allocation is possible otherwise improvement has to be done. Let us discuss these facts in detail as follows

The number of lines formed (4) are same as number of rows (4) and number of columns (4) then direct allocation is possible as follows

Employee	Reservation counter (Rc)			
	Rc1	Rc2	Rc3	Rc4
Aaradhya	0	3	4	9
Aadvet	0	1	0	3
Vedika	0	1	0	3
Shrived	0	0	17	0

Step 8. Assignments

The allocations or assignments are done on one to one basis so that every row and column as one allocation. It means that each worker or machine has one job to be done in minimum time.

Employee	Reservation counter (Rc)			
	Rc1	Rc2	Rc3	Rc4
Aaradhya	0	3	4	9
Aadvet	0	1	0	3
Vedika	0	1	0	3
Shrived	0	0	17	0

As first row has only one zero. Aaradhya will serve to Reservation Counter 1 and As Reservation Counter 1 is served by Aaradhya and hence it can't be served by any other employee. Similarly, second and fourth columns have single zeroes. Hence allocation is direct there.

Employee	Reservation counter (Rc)			
	Rc1	Rc2	Rc3	Rc4
Aaradhya	0✔	3	4	9
Aadvet	0✖	1	0✔	3
Vedika	0✖	1	0✔	3
Shrived	0✖	0✔	17	0✔

As reservation counter 3 is occupied by the Aadvet and Vedika both so assignment solution is not optimum.

Hence again repeat the process.

Step 9. Row subtraction

As every row has zero element and then row subtraction is not possible.

Employee	Reservation counter (Rc)			
	Rc1	Rc2	Rc3	Rc4
Aaradhya	0✔	3	4	9
Aadvet	0✖	1	0✔	3
Vedika	0✖	1	0✔	3
Shrived	0✖	0✔	17	0✔

Step 10. Column subtraction

Employee	Reservation counter (Rc)			
	Rc1	Rc2	Rc3	Rc4
Aaradhya	0	3	4	9
Aadvet	0	1	0	3
Vedika	0	1	0	3
Shrived	0	0	17	0

Students	Slot1	Slot2	Slot3	Dummy Slot
Aaradhya	**4**	**6**	**0✔**	**0✖**
Aadvet	**6**	**2**	**3**	**0✔**
Vedika	**0✖**	**0✔**	**7**	**1**
Shrived	**0✔**	**3**	**1**	**0✖**

The Fourth row Shrived is having zero at slot 1 and hence slot 1 cannot be taken by Vedika as follows

Sr. No.	Name of the student	Time slot	Time (in minutes)
1	**Aaradhya**	Slot3	32
2	**Aadvet**	Slot4	00* Dummy
3	**Vedika**	Slot2	22
4	**Shrived**	Slot1	36

* Dummy values are not known and are considered for the purpose of making

3.3.3 Maximization Problems

The maximization problems are the assignment problems related to profit, sales or output maximization or any other units that are required to be maximized. The optimization techniques used here is to maximize the sales, profit or any other objective of assignment problem. The given maximization matrix is transformed into a relative loss matrix by subtracting every element from the largest element of the assignment matrix. The next steps are similar to the HAM assignment problems. All steps are designed to find out the relative position of allocation or selection of given alternatives from set of all combinations of alternatives as follow

Example 5. A marketing manager has five salesmen and five sales districts. Considering the capabilities of the salesmen and the nature of districts, the marketing manager estimates that sales per month (in hundred rupees) for each salesman in each district would be as follows

		Districts				
		A	B	C	D	E
Salesmen	1	32	38	40	28	40
	2	40	24	28	21	36
	3	41	27	33	30	37
	4	22	38	41	36	36
	5	29	33	40	35	39

Find the assignment of salesmen to districts that will result in maximum sales.

Solution:

Step 1. The given assignment problem is to maximize sales by proper assignment of salesmen to different districts. Hence this is maximization problem of assignment.

Step 2. Convert into minimization

This can be done by selecting the largest element (41) of the given problem and subtract it from all elements to form relative loss matrix called opportunity loss matrix as follows

		Districts				
		A	B	C	D	E
Salesmen	1	9	3	1	13	1
	2	1	17	13	20	5
	3	0	14	8	11	4
	4	19	3	0	5	5
	5	12	8	1	6	2

Step 3. Row subtraction

		Districts				
		A	B	C	D	E
Salesmen	1	8	0	0	7	0
	2	0	16	12	19	4
	3	0	14	8	11	4
	4	19	3	0	5	5
	5	11	7	0	5	1

Step 4. Column subtraction

		Districts				
		A	B	C	D	E
Salesmen	1	8	0	0	2	0
	2	0	16	12	14	4
	3	0	14	8	6	4
	4	19	3	0	0	5
	5	11	7	0	0	1

Step 5. Line formation

		Districts				
		A	B	C	D	E
Salesmen	1	8	0	0	2	0
	2	0	16	12	14	4
	3	0	14	8	6	4
	4	19	3	0	0	5
	5	11	7	0	0	1

The formed lines (4) are less than 5 rows and 5 columns. Hence improvement can be done by subtracting smallest element (4) from all elements of reduced matrix and adding this smallest element to the junction point so as to develop a chance of forming new line.

		Districts				
		A	B	C	D	E
Salesmen	1	12	0	0	2	0
	2	0	12	8	10	0
	3	0	10	4	2	0
	4	23	3	0	0	5
	5	15	7	0	0	1

Step 6. Line formation

		Districts				
		A	B	C	D	E
Salesmen	1	12	0	0	2	0
	2	0	12	8	10	0
	3	0	10	4	2	0
	4	23	3	0	0	5
	5	15	7	0	0	1

The formed lines (5) are equal to 5 rows and 5 columns. Hence direct allocation is possible as follows

		Districts				
		A	B	C	D	E
Salesmen	1	12	0✔	0✖	2	0✖
	2	0	12	8	10	0
	3	0	10	4	2	0
	4	23	3	0	0	5
	5	15	7	0	0	1

Now there are two alternatives for A district as

i. Salesman 2 is given an opportunity to serve in A district.
ii. Salesman 3 is given an opportunity to serve in A district.

		Districts				
		A	B	C	D	E
Salesmen	1	12	0✔	0✖	2	0✖
	2	0✔	12	8✖	10	0✖
	3	0✖	10	4	2	0✔
	4	23	3	0	0	5
	5	15	7	0	0	1

There are two additional conditions in first alternative as

a. Salesman 4 is given an opportunity to serve in C district.
b. Salesman 5 is given an opportunity to serve in C district.

Let us study (a) option from first alternative as follows

		Districts				
		A	B	C	D	E
Salesmen	1	12	0✔	0✖	2	0✖
	2	0✔	12	8✖	10	0✖
	3	0✖	10	4	2	0✔
	4	23	3	0✔	0✖	5
	5	15	7	0✖	0✔	1

Allocations are made as follows

Sr. No.	Salesman	District	Sales (in Hundred rupees)
1	1	B	38
2	2	A	40
3	3	E	37
4	4	C	41
5	5	D	35
		Total sales	191

Second option: Salesman 5 is given an opportunity to serve in C district.

Let us study (a) option from first alternative as follows

		Districts				
		A	B	C	D	E
Salesmen	1	12	0✔	0✖	2	0✖
	2	0✔	12	8	10	0✖
	3	0✖	10	4	2	0✔
	4	23	3	0✖	0✔	5
	5	15	7	0✔	0✖	1

Allocations are made as follows

Sr. No.	Salesman	District	Sales (in Hundred rupees)
1	1	B	38
2	2	A	40
3	3	E	37
4	4	D	36
5	5	C	40
		Total sales	191

3.4 SUMMARY

Assignment problems are the optimization problem to allocate jobs to different machines or to workers, salesmen to the sales zone or to allocate vehicles to different routes. Assignment works on one to one basis i.e. one resource to the one unit likewise.

Hungarian Assignment Method (HAM) is used for minimization cases of optimization problems. This method is faster, easier and more efficient to find out the solutions to the Assignment Problems. When the assignment problem has more than one solution, this is termed as multiple optimum solutions to the assignment problem. The minimum two solutions exist for multiple optimum solutions. Hence there are possibilities of getting different solutions. We can compare solutions and get the optimum solution required. The maximization problems are the assignment problems related to profit, sales or output maximization or any other units that are required to be maximized. The optimization techniques used here is to maximize the sales, profit or any other objective of assignment problem. The given maximization matrix is transformed into a relative loss matrix by subtracting every element from the largest element of the assignment matrix. The next steps are similar to the HAM assignment problems.

3.5 SELF-ASSIGNMENT QUESTIONS.

1. What is an assignment problem? What are the types of assignment problems?

2. What are the steps of solving assignment problem?

3. Five jobs are assigned to five persons, each person will do one job only. The expected time (in hours) required for each job for each person have been estimated and are shown in the following table. Determine the optimal assignments.

Job	Person				
	1	2	3	4	5
A	12	15	13	14	15
B	16	18	15	14	16
C	18	16	15	18	20
D	15	20	18	17	19
E	16	15	18	14	15

4. A company has a team of four salesmen and there are four districts where the company wants to start its business. The following is the profit per day in rupees for each salesman in each district. Find the assignment of salesmen to districts which will yield maximum profit.

Salesman	District			
	D1	D2	D3	D4
A	16	10	14	11
B	14	11	15	15
C	15	15	13	12
D	13	12	14	15

❖❖❖❖

4

GAME THEORY

Unit Structure:

4.0 OBJECTIVES

The learning objectives of Game Theory are as follows:

1. To understand and decide winning strategies in a competitive business world.
2. To study the concepts and terminologies of Game Theory.
3. To solve complex business situation and to assist in decision making by using game strategies.
4. To analyse the business game and solve it by using pure strategy game, mixed strategy game and principle of dominance.

4.1 INTRODUCTION

Today, business world is complex, dynamic, and competitive in nature. The monopoly is quite observed. There are many rivals in the business and environ is competitive in the industry. Every business firm tries to achieve success (win) by maximizing profit by defeating other rivalry players in the business world. Hence every business firm tries to win the game by defeating other players in the game using different strategies.

4.2 MEANING AND CONCEPT OF GAME THEORY

Game Theory is the competitive situation of business players in the business competitive environment. Game theory is the study of operations research technique involving appropriate use of strategies to win the game. Thus Game theory is the branch of Operations Research dealing with quantitative decision making when at least two players are involved in the conflicting and competitive environment, resulting in success and failure of players.

4.3 TERMINOLOGIES IN GAME THEORY

4.3.1 Strategy: The possible alternatives or available course of action that may be adopted to win the game. For example, In order to earn more profit, either selling price should be decreased to have more sales or cost of production should be lowered.

4.3.2 Player: Each participant of game is called Player. There are at least two players. One is considered as maximizing player (winner of the game) and other is assumed as minimizing player (looser of the game). But practically, such ideal situation may not come in reality.

4.3.3 Play: A play happens when each player select a particular strategy from all available strategies.

4.3.4 Pay off: The outcome of playing a game is known as Pay off. The Pay off is the net gain or profit earned by using appropriate strategy by the winning player.

4.3.5 Pay off Matrix: Pay off matrix is the table of game theory representing the two players out of which One is considered as maximizing player (winner of the game) and other is assumed as minimizing player (looser of the game).

4.3.6 Optimum Strategy: The optimum strategy is the strategy used by the maximizing player against the strategies used by the competing minimizing players.

4.3.7 Value of the Game: It is the expected outcome of the game theory when all players use their optimum strategy. If the value of the game is zero then game is said to be the fair game otherwise if the value of the game is not zero then game is said to be the unfair game.

4.4 TYPES OF GAMES

Game is a particular business situation where at least two players are involved to overcome the other player using optimum strategy and to get maximum output using different strategies. The types of games are as follows

a. Two Person Game
b. Multiple Person Game
c. Zero Sum Game

d. Non-Zero Sum Game
e. Two Person Zero Sum Game

These are explained as follows

a. **Two Person Game:** When there are two players involved in the game, then this is termed as two person game. One is considered as maximizing player and other is considered as minimizing player.

b. **Multiple Person Game:** When there are more than two players involved in the game, then this is termed as multiple person game. This is also termed as n-person game.

c. **Zero Sum Game:** Zero sum means addition becomes equal to zero. When the sum of gains of all winners is equal to the sum of losses of all the losers, this condition of game is called zero sum game.

d. **Non-Zero Sum Game:** When the sum of gains of all winners is not equal to the sum of losses of all the losers, this condition of game is called non-zero sum game.

e. **Two Person Zero Sum Game:** In the two person game, when the gain of one player is equal to the loss of other player, such type of game is termed as Two Person Zero Sum Game.

Two person game having two strategies for each player is called as a two person 2x2 game involving two rows and two columns in the payoff matrix.

Based on the type of strategy, Games are again classified as follow

4.4.1 Pure Strategy Game:

In the pure strategy game, players use only one strategy throughout the game. The pure strategy indicates that there is no other combination of strategy used by the game player.

4.4.1.1 Steps for solving Pure Strategy Game:

1. Maximizing and Minimizing Player: In the payoff matrix, maximizing players' strategy should be written in the row and minimizing players, strategy should be written in the column.

2. Row Maxi-min Strategy: In the row, maximum values from the minimum values of the given rows are taken. In simple words, at first row minima values should be taken for all rows and then maximum value from all row minima values are taken called as Row Maxi-min value.

3. Column Mini-max Strategy: In the column, minimum values from the maximum values of the given columns are taken. In simple words, at first column maxima values should be taken for all columns and then minimum value from all column maxima values are taken called as Column Mini-max value.

4. Comparing Row Maxi-min and Column Mini-max values: The Row Maxi-min and Column Mini-max values are compared and if these values are same then there exists a saddle point representing the value of the game.

5. Saddle Point: The saddle point is the point of intersection of row and column of Row Maxi-min and Column Mini-max values. This saddle point represents the value of the game.

6. One Saddle point and multiple saddle points: If there is only one saddle point then this represents the value of the game alone. If there is more than one saddle point, then this represents multiple solutions or multiple values of the game showing combination of different strategies to be used in the game strategy.

7. **Comparing Row Maxi-min and Column Mini-max values:** The Row Maxi-min and Column Mini-max values are compared and if these values are not same then there does not exist a saddle point and the game is mixed strategy game

.

Example 1: Two competing firms A and B are in the local market. They use different strategies to maximize the profit. Firm A uses two strategies A1 and A2. Firm B uses three strategies B1, B2 and B3. Their corresponding pay off matrix is given in the following table as

Firm B

		B1	B2	B3
Firm A	**A1**	**2**	**8**	**4**
	A2	**7**	**10**	**6**

Solution:

The steps to solve game theory problem is as follows

1. **Maximizing and Minimizing Player:** Firm A is considered as maximizing player and firm B is assumed as minimizing player. Maximizing players' (Firm A) strategy should be written in the row and minimizing players' (Firm B) strategy should be written in the column.

2. Row Maxi-min Strategy: In the row, maximum values from the minimum values of the given rows are taken. In simple words, at first row minima values should be taken for all rows

	B1	B2	B3	Row Minima
Firm A A1	2	8	4	2
A2	7	10	6	6

And then maximum value (6) from all row minima values (2 and 6) are taken called as Row Maxi-min value.

3. Column Mini-max Strategy: In the column, minimum values from the maximum values of the given columns are taken. In simple words, at first column maxima values should be taken for all columns

		B1	B2	B3
Firm A	A1	2	8	4
	A2	7	10	6
Column Maxima		7	10	6

and then minimum value (6) from all column maxima values (7, 10 and 6) are taken called as Column Mini-max value.

4. Comparing Row Maxi-min and Column Mini-max values: The Row Maxi-min (6) and Column Mini-max values (6) are compared and if these values are same then there exists a saddle point (6) representing the value of the game (6).

5. Saddle Point: The saddle point (A2B3) is the point of intersection of second row and third column of Row Maxi-min and Column Mini-max values. This saddle point represents the value of the game.

6. Optimum Strategy and Value of the Game:
The optimum strategy corresponding to the saddle point is A2B3 strategies. The value of the game is 6 and Thus for the firm A, strategy A2 is the optimal strategy and it is represented as A (0,1) i.e. probability of using A1 strategy is 0 and probability of using A2 strategy is 100%. Similarly for the firm B, strategy B3 is the optimal strategy and it is represented as B (0,0,1) i.e. probability of using B1 and B2 strategies are 0 and probability of using B3 strategy is 100%.

Example 2. A Company management and the labour union are negotiating a new 3 years settlement. Each player has 4 strategies namely

a. Hard and Aggressive Bargaining

b. Reasoning and Logical Approach

c. Legalistic Approach

d. Conciliatory Approach

The cost to the company (in the form of average wage rise in Rs.) are given for every pair of strategy choices

		Union Strategies			
		A	B	C	D
Company Strategies	A	20	25	40	-5
	B	15	14	2	4
	C	12	8	10	11
	D	35	10	5	0

Which strategy will the two sides adopt? Also determine the value of game.

Solution:
Solution: The steps to solve game theory problem is as follows

1. **Maximizing and Minimizing Player:** Company is considered as maximizing player and Union is assumed as minimizing player. But Union thinks to be the maximizing player and hence the transformation of row and column into column and row takes place.

Maximizing players' (Union) strategy should be written in the row and minimizing players' (Company) strategy should be written in the column.

		Company Strategies			
		A	B	C	D
Union Strategies	A	20	15	12	35
	B	25	14	8	10
	C	40	2	10	5
	D	-5	4	11	0

2. Row Maxi-min Strategy: In the row, maximum values from the minimum values of the given rows are taken. In simple words, at first row minima values should be taken for all rows

		Company Strategies				
		A	B	C	D	Row Minima
Union Strategies	A	20	15	12	35	12
	B	25	14	8	10	8
	C	40	2	10	5	2
	D	-5	4	11	0	-5

And then maximum value (12) from all row minima values (12, 8, 2 and -5) are taken called as Row Maxi-min value.

3. Column Mini-max Strategy: In the column, minimum values from the maximum values of the given columns are taken. In simple words, at first column maxima values should be taken for all columns

		Company Strategies			
		A	**B**	**C**	**D**
Union Strategies	**A**	**20**	**15**	**12**	**35**
	B	**25**	**14**	**8**	**10**
	C	**40**	**2**	**10**	**5**
	D	**-5**	**4**	**11**	**0**
Column Maxima		**40**	**15**	**12**	**35**

and then minimum value (12) from all column maxima values (40, 15,12 and 35) are taken called as Column Mini-max value.

4. Comparing Row Maxi-min and Column Mini-max values: The Row Maxi-min (12) and Column Mini-max values (12) are compared and if these values are same then there exists a saddle point (12) representing the value of the game (12).

5. Saddle Point: The saddle point (uAcC) is the point of intersection of first row and third column of Row Maxi-min and Column Mini-max values. The uA strategy or cell corresponds to the strategy A used by Union and cC strategy represents to the strategy C used by Company. This saddle point represents the value of the game.

6. Optimum Strategy and Value of the Game:
The optimum strategy corresponding to the saddle point is uAcC strategies. The value of the game is 12 and Thus for the Union, strategy A is the optimal strategy and it is represented as (1,0,0,0) i.e. probability of using A strategy is 10%0 and probability of using B,C and D strategies are 0%. Similarly for the Company, strategy C is the optimal strategy and it is represented as (0,0,1,0) i.e. probability of using A,B and D strategies are 0 and probability of using C strategy is 100%.

4.4.2 Mixed Strategy Game:
In the mixed strategy game, players use different strategies throughout the game. The mixed strategy indicates that there is other combination of strategies used by the game player.

Mixed strategy games are solved by

1. 2x2 matrix game solution if pay off matrix is having 2 rows and 2 columns otherwise

2. Principle of Dominance strategy is used to solve the problem.

Let us study these two methods as follows:

1. 2x2 matrix game solution

If pay off matrix is having 2 rows and 2 columns then the solution of game theory and their probabilities are obtained as follows

Let us consider,
p as the probability that A uses strategy A1 and
1-p is the probability that A uses strategy A2.

Similarly,
q as the probability that B uses strategy B1 and
1-q is the probability that B uses strategy B2.
Consider the pay off matrix

		B	
		B1 (q)	**B2(1-q)**
A	**A1(p)**	**a11**	**a12**
	A2(1-p)	**a21**	**a22**

Where,
a11 represents the cell position of row 1 and column 1.
a12 represents the cell position of row 1 and column 2.
a21 represents the cell position of row 2 and column 1.
a22 represents the cell position of row 2 and column 2.

p as the probability that A uses strategy A1 is obtained by the formula
p= (a22- a21)/D
q as the probability that B uses strategy B1 is obtained by the formula
q= (a22- a12)/D
and the value of the game is obtained by the formula
v= (a11a22-a21a12)/D
where, D= denominator= a11+a22-a21-a12

Example 3. Two business firms A and B are competing in the local market. Player A is having the choice of two strategies A1 and A2. Player B is having the choice of two strategies B1 and B2. Determine the optimum strategy and value of the game using pure strategy and then by using 2x2 matrix game solution method.

		Player B	
		B1	**B2**
Player A	**A1**	**3**	**5**
	A2	**4**	**1**

Solution:

1. Maximizing and Minimizing Player: In the payoff matrix, maximizing players' (A) strategy should be written in the row and minimizing players' (B) strategy should be written in the column.

2. Row Maxi-min Strategy: In the row, maximum values from the minimum values of the given rows are taken. In simple words, at first row minima values should be taken for all rows

		Player B		
		B1	**B2**	**Row Minima**
Player A	**A1**	**3**	**5**	**3**
	A2	**4**	**1**	**1**

Then maximum value (1) from all row minima values (3,1) are taken called as Row Maxi-min value.

3. Column Mini-max Strategy: In the column, minimum values from the maximum values of the given columns are taken. In simple words, at first column maxima values should be taken for all columns

		Player B	
		B1	**B2**
Player A	**A1**	**3**	**5**
	A2	**4**	**1**
Column Maxima		**4**	**5**

then minimum value (4) from all column maxima values (4 and 5) are taken called as Column Mini-max value.

4. Comparing Row Maxi-min and Column Mini-max values: The Row Maxi-min (1) and Column Mini-max (4) values are compared and these values are not same then there does not exist a saddle point representing the value of the game.

5. Mixed Strategy Game: 2x2 matrix two person mixed strategy game can be solved by using the formula

p as the probability that A uses strategy A1 is obtained by the formula
p= (a22- a21)/D
q as the probability that B uses strategy B1 is obtained by the formula
q= (a22- a12)/D
and the value of the game is obtained by the formula
v= (a11a22-a21a12)/D
where D= denominator= a11+a22-a21-a12

		Player B	
		B1	**B2**
Player A	**A1**	**3**	**5**
	A2	**4**	**1**

Here,
a11= 3; a12=5; a21=4; a22=1
and D= denominator= a11+a22-a21-a12= 3+1-4-5= -5
p as the probability that A uses strategy A1 is obtained by the formula
p= (a22- a21)/D = (1-4)/(-5)= (-3)/-5= 0.6 and
and the probability that A uses strategy A2 is 1-p=1-0.6=0.4
q as the probability that B uses strategy B1 is obtained by the formula
q= (a22- a12)/D= (1-5)/-5= (-4)/-5= 0.8
and the probability that B uses strategy B2 is 1-q=1-0.8=0.2
and the value of the game is obtained by the formula
v= (a11a22-a21a12)/ D = (3x1-4x5)/-5=(3-20)/-5=-17/-5=3.4

Therefore, the value of the game is 3.4 and the optimum strategy for A is A1 and for B is B1.

4.5 PRINCIPLE OF DOMINANCE

The Principle of Dominance states that, if a strategy of a player dominates over another strategy in all conditions (that is for all counter strategies by other player), then the later strategy (being dominated) can be ignored. A strategy dominates over the other strategy, only if it is preferable over the other, under all conditions.

This Principle of Dominance is difficult to understand and hence simple and logical trick has been developed by the content writer Dr. Gajanan Mudholkar to remember this principle as follows

The trick to remember and apply Principle of Dominance is ‘RC’. The first capital letter R stands for Capital R means Big R (R for row and R for retain, it means that Bigger Row should be retained and Smaller Row should be cancelled. The second capital letter **C** means Big C (C for Column and C for cancellation) Greater **C**olumn should be **C**ancelled and smaller column should be retained.

This principle can be applied in all type of Game Theory problem. This principle can be used for pure strategy game and mixed strategy game also.

Example 4. Solve the following game directly and by using Principle of Dominance

		Player Y				
		1	**2**	**3**	**4**	**5**
Player X	**I**	**1**	**3**	**2**	**7**	**4**
	II	**3**	**4**	**1**	**5**	**6**
	III	**6**	**5**	**7**	**6**	**5**
	IV	**2**	**0**	**6**	**3**	**1**

Solution:
Solve the following game directly means by using pure strategy game as follows:

1. Maximizing and Minimizing Player: In the payoff matrix, maximizing players' strategy (Player X) should be written in the row and minimizing players' strategy (Player Y) should be written in the column.

2. Row Maxi-min Strategy: In the row, maximum values from the minimum values of the given rows are taken. In simple words, at first row minima values should be taken for all rows

		Player Y					
		1	**2**	**3**	**4**	**5**	**Row Minima**
Player X	**I**	**1**	**3**	**2**	**7**	**4**	**1**
	II	**3**	**4**	**1**	**5**	**6**	**1**
	III	**6**	**5**	**7**	**6**	**5**	**5**
	IV	**2**	**0**	**6**	**3**	**1**	**0**

and then maximum value (5) from all row minima values (1,1,5,0) are taken called as Row Maxi-min value (5).

3. Column Mini-max Strategy: In the column, minimum values from the maximum values of the given columns are taken. In simple words, at first column maxima values should be taken for all columns

		Player Y				
		1	2	3	4	5
Player X	**I**	**1**	**3**	**2**	**7**	**4**
	II	**3**	**4**	**1**	**5**	**6**
	III	**6**	**5**	**7**	**6**	**5**
	IV	**2**	**0**	**6**	**3**	**1**
Column Maxima		**6**	**5**	**7**	**7**	

and then minimum value (5) from all column maxima values (6,5,7,7) are taken called as Column Mini-max value (5).

4. Comparing Row Maxi-min and Column Mini-max values: The Row Maxi-min (5) and Column Mini-max values (5) are compared and as these values are same then there exists a saddle point representing the value of the game.

5. Saddle Point: The saddle point (5) is the point of intersection of row and column of Row Maxi-min and Column Mini-max values. This saddle point represents the value of the game.

V= Value of the Game=5 and the corresponding optimum strategy for player X is III strategy and for Player Y is 2nd strategy.

6. By using Principle of Dominance

		Player Y				
		1	2	3	4	5
Player X	**I**	**1**	**3**	**2**	**7**	**4**
	II	**3**	**4**	**1**	**5**	**6**
	III	**6**	**5**	**7**	**6**	**5**
	IV	**2**	**0**	**6**	**3**	**1**

As we compare first and second row, we find that 1<3; 3<4; 2>1; 7>5 and 4<6 and there is no comparison possible. Similarly first row and third row can't be compared. Similarly first row and fourth row can't be compared. Now we compare second row with third and fourth row as follows.

Similarly, second row and third row can't be compared. Similarly second row and fourth row can't be compared.

Now we compare third row with fourth row as follows
We get 6>1; 5>0; 7>6; 6>3 and 5>1. Hence third row is greater than fourth row.

According to the Principle of Dominance, RC means R for Greater Row and R for Retain. It means that the greater row should be retained and smaller row should be cancelled.

III>IV i.e. third row will be retained and fourth row will be cancelled as follows

		Player Y				
		1	2	3	4	5
Player X	I	1	3	2	7	4
	II	3	4	1	5	6
	III	6	5	7	6	5
	~~IV~~	~~2~~	~~0~~	~~6~~	~~3~~	~~1~~

Now, compare column wise
First column 1 with second column 2 as follows
1<3; 3<4; 2>1; 7>5 and 4<6 and hence comparison is not possible.
First column 1 with third column 3 as follows
1<2; 3>1 and 6<7 and hence comparison is not possible.
First column 1 with fourth column 4 as follows
1<7; 3<5 and 6=6 and hence First Column 1≤ fourth column.
According to the Principle of Dominance, RC means C for Greater Column and C for cancel. It means that the greater column should be cancelled and smaller column should be retained.

Fourth column being greater should be cancelled and smaller first column should be retained as follows

		Player Y				
		1	2	3	~~4~~	5
Player X	I	1	3	2	~~7~~	4
	II	3	4	1	~~5~~	6
	III	6	5	7	~~6~~	5
	~~IV~~	~~2~~	~~0~~	~~6~~	~~3~~	~~1~~

Now compare second column with fifth column we get
3<4; 4<6 and 5=5 and hence comparison is possible as follows
Column 3≤ Column 5

According to the Principle of Dominance, RC means C for Greater Column and C for cancel. It means that the greater column should be cancelled and smaller column should be retained.

Fourth column being greater should be cancelled and smaller first column should be retained as follows

		Player Y				
		1	**2**	**3**	**4**	**5**
Player X	**I**	**1**	**3**	**2**	**7**	**4**
	II	**3**	**4**	**1**	**5**	**6**
	III	**6**	**5**	**7**	**6**	**5**
	IV	**2**	**0**	**6**	**3**	**1**

Now compare first row with second row, here comparison is not possible. Compare first row with third row, we came to know First row I < Third row III.

According to the Principle of Dominance, RC means R for Greater Row and R for Retain. It means that the greater row should be retained and smaller row should be cancelled.

First row I < Third row III i.e. third row will be retained and first row will be cancelled as follows

		Player Y				
		1	**2**	**3**	**4**	**5**
Player X	**I**	**1**	**3**	**2**	**7**	**4**
	II	**3**	**4**	**1**	**5**	**6**
	III	**6**	**5**	**7**	**6**	**5**
	IV	**2**	**0**	**6**	**3**	**1**

Compare second row with third row, we came to know second row II < Third row III.

According to the Principle of Dominance, RC means R for Greater Row and R for Retain. It means that the greater row should be retained and smaller row should be cancelled.

Second row II < Third row III i.e. third row will be retained and second row will be cancelled as follows

		Player Y				
		1	2	3	4	5
Player X	I	1	3	2	7	4
	II	3	4	1	5	6
	III	6	5	7	6	5
	IV	2	0	6	3	1

Now compare column wise first column (value 6) is greater than second column (value 5).

According to the Principle of Dominance, RC means C for Greater Column and C for cancel. It means that the greater column should be cancelled and smaller column should be retained.
First column being greater should be cancelled and smaller second column should be retained as follows

		Player Y				
		1	2	3	4	5
Player X	I	1	3	2	7	4
	II	3	4	1	5	6
	III	6	5	7	6	5
	IV	2	0	6	3	1

Now only remaining comparison is in between second column and third column.
Second column 2< Third column 3.

According to the Principle of Dominance, RC means C for Greater Column and C for cancel. It means that the greater column should be cancelled and smaller column should be retained.

Third column being greater should be cancelled and smaller second column should be retained as follows

		Player Y				
		1	2	3	4	5
Player X	I	1	3	2	7	4
	II	3	4	1	5	6
	III	6	5	7	6	5
	IV	2	0	6	3	1

Hence only reminder cell in the given matrix is 5 representing the value of game. The corresponding optimum strategy for player X is third III strategy and for player Y is second strategy 2.

Example 5. Solve the following game directly.

		Player B			
		B1	B2	B3	B4
Player A	A1	7	6	8	9
	A2	-4	-3	9	10
	A3	3	0	4	2
	A4	10	5	-2	0

Solution:
Solve the following game directly means by using pure strategy game as follows

1. Maximizing and Minimizing Player: In the payoff matrix, maximizing players' strategy (Player A) should be written in the row and minimizing players' strategy (Player B) should be written in the column.

2. Row Maxi-min Strategy: In the row, maximum values from the minimum values of the given rows are taken. In simple words, at first row minima values should be taken for all rows

		Player B				Row Minima
		B1	B2	B3	B4	
Player A	A1	7	6	8	9	6
	A2	-4	-3	9	10	-4
	A3	3	0	4	2	0
	A4	10	5	-2	0	-2

and then maximum value (6) from all row minima values (6,-4,0,-2) are taken called as Row Maxi-min value (6).

3. Column Mini-max Strategy: In the column, minimum values from the maximum values of the given columns are taken. In simple words, at first column maxima values should be taken for all columns

		Player B			
		B1	B2	B3	B4
Player A	A1	7	6	8	9
	A2	-4	-3	9	10
	A3	3	0	4	2
	A4	10	5	-2	0
Column Maxima		10	6	9	10

and then minimum value (6) from all column maxima values (10,6,9,10) are taken called as Column Mini-max value (6).

4. Comparing Row Maxi-min and Column Mini-max values: The Row Maxi-min (6) and Column Mini-max values (6) are compared and as these values are same then there exists a saddle point representing the value of the game.

5. Saddle Point: The saddle point (6) is the point of intersection of row and column of Row Maxi-min and Column Mini-max values. This saddle point represents the value of the game.

V= Value of the Game=6 and the corresponding optimum strategy for player A is A1 strategy and for Player B is B2 strategy.
Example 6.

4.6 LIMITATIONS OF GAME THEORY

1. Players of game are not competent to know their own and competitors pay off values. Hence the application of game theory is not realistic in nature.

2. The theory of game theory is restricted to two players' duopoly, but the situation of many players are complex and difficult to solve.

3. Zero sum game situations are not realistic in nature.

4. Players do not have complete knowledge of the strategies and hence the strategies of Maximin and Minimax is not applicable for conservative cases and risk averse cases of the players.

4.7 SUMMARY

Today, business world is complex, dynamic, and competitive in nature. The monopoly is quite observed. Every business firm tries to win the game by defeating other players in the game using different strategies. Game Theory is the competitive situation of business players in the business competitive environment. Game theory is the study of operations research technique involving appropriate use of strategies to win the game. Thus Game theory is the branch of Operations Research dealing with quantitative decision making when at least two players are involved in the conflicting and competitive environment, resulting in success and failure of players. Game is a particular business situation where at least two players are involved to overcome the other player using optimum strategy and to get maximum output using different strategies. The types of games are Two Person Game, Multiple Person Game, Zero Sum Game, Non-Zero Sum Game and Two Person Zero Sum Game. In the pure strategy game, players use only one strategy throughout the game. The pure strategy indicates that there is no other combination of strategy used by the game player. In the mixed strategy game, players use different strategies throughout the game. The mixed strategy indicates that there is other combination of strategies used by the game player. The Principle of Dominance states that, if a strategy of a player dominates over another strategy in all conditions (that is for all counter strategies by other player), then the later strategy (being dominated) can be ignored. A strategy dominates over the other strategy, only if it is preferable over the other, under all conditions.

4.8 SELF-ASSESSMENT QUESTIONS

Q1. Explain the Meaning and Concept of Game Theory.
Q2. Describe various Terminologies in Game Theory.
Q3. What are types of Game? Explain in depth.
Q4. Explain Principle of Dominance with practical example.
Q5. Describe Limitations of Game Theory.
Q6. Solve the following game directly and by using Principle of Dominance

		Player B				
		1	2	3	4	5
Player A	I	1	3	2	7	4
	II	3	4	1	5	6
	III	6	5	7	6	5
	IV	2	0	6	3	1

Q7. Solve the following game directly and by using Principle of Dominance

			Player B					
		1	2	3	4	5	6
Player A	I	4	2	0	2	1	1
	II	4	3	1	3	2	2
	III	4	3	7	-5	1	2
	IV	4	3	4	-1	2	2
	V	4	3	3	-2	2	2

❖❖❖❖

5

DECISION THEORY

Decision Theory - Under Risk, Uncertainty, and decision tree

Unit Structure

5.0 OBJECTIVES

After reading this unit, you should be able to:

- structure a decision problem involving various alternatives and uncertainties in outcomes
- apply marginal analysis for solving decision problems under uncertainty
- analyse sequential problems using decision tree approach
- appreciate the use of utility theory in decision-making
- analyse uncertain situations where probabilities of outcomes are not known.

5.1 INTRODUCTION

In every sphere of our life we need to take various kinds of decisions. The ubiquity of decision problems, together with the need to make good decisions, have led many people from different time and fields, to analyse the decision-making process. A growing body of literature on Decision Analysis is thus found today. The analysis varies with the nature of the decision problem, so that any classification base for decision problems provides us with a means to segregate the Decision Analysis literature. A necessary condition for the existence of a decision problem is the presence of alternative ways of action. Each action leads to a consequence through a possible set of outcome, the information on which might be known or unknown. One of the several ways of classifying decision problems has been based on this knowledge about the information on outcomes. Broadly, two classifications result:

a) The information on outcomes are deterministic and are known with certainty, and

b) The information on outcomes are probabilistic, with the probabilities known or unknown.

The former may be classified as Decision Making under certainty, while the latter is called Decision Making under uncertainty. The theory that has resulted from analysing decision problems in uncertain situations is commonly referred to as Decision Theory. With our background in the Probability Theory, we are in a position to undertake a study of Decision Theory in this unit. The objective of this unit is to study certain methods for solving decision problems under uncertainty. The methods are consequent to certain key issues of such problems. Accordingly, in the next section we discuss the issues and in subsequent sections we present the different methods for resolving them.

The success or failure that an individual or organization experiences, depends to a large extent, on the ability of making acceptable decisions on time. To arrive at such a decision, a decision-maker needs to enumerate feasible and viable courses of action (alternatives or strategies), the projection of consequences associated with each course of action, and a measure of effectiveness (or an objective) to identify the best course of action.

Decision theory is both descriptive and prescriptive business modeling approach to classify the degree of knowledge and compare expected outcomes due to several courses of action. The degree of knowledge is divided into four categories: complete knowledge (i.e. certainty), ignorance, risk and uncertainty.

5.2 CERTAIN KEY ISSUES IN DECISION THEORY

Different issues arise while analysing decision problems under uncertain conditions of outcomes. Firstly, decisions we take can be viewed either as independent decisions, or as decisions figuring in the whole sequence of decisions that are taken over a period of time. Thus, depending on the planning horizon under consideration, as also the nature of decisions, we have either a single stage decision problem, or a sequential decision problem. In real life, the decision maker provides the common thread, and perhaps all his decisions, past, present and future, can be considered to be sequential. The problem becomes combinatorial, and hence difficult to solve. Fortunately, valid assumptions in most of the cases help to reduce the number of stages, and make the problem tractable.

Irrespective of the type of decision model, following essential components are common to all:

Decision alternatives: There is a finite number of decision alternatives available to the decision-maker at each point in time when a decision is made. The number and type of such alternatives may depend on the previous decisions made and their outcomes. Decision alternatives may be described numerically, such as stocking 100 units of a particular item, or non-numerically, such as conducting a market survey to know the likely demand of an item.

States of nature: A state of nature is an event or scenario that is not under the control of decision makers. For instance, it may be the state of economy (e.g. inflation), a weather condition, a political development etc.

The states of nature may be identified through *Scenario Analysis* where a section of people are interviewed – stakeholders, long-time managers, etc., to understand states of nature that may have serious impact on a decision.

The states of nature are mutually exclusive and collectively exhaustive with respect to any decision problem. The states of nature may be described numerically such as, demand of 100 units of an item or non-numerically such as, employees strike, etc.

Payoff It is a numerical value (outcome) obtained due to the application of each possible combination of decision alternatives and states of nature. The payoff values are always conditional values because of unknown states of nature.

The payoff values are measured within a specified period (e.g. within one year, month, etc.) called the *decision horizon*. The payoffs in most decisions are monetary. Payoffs resulting from each possible combination of decision alternatives and states of natures are displayed in a matrix (also called *payoff matrix*) form.

		Courses of Action (Alternatives)			
Sates of Nature	**Probability**	S_1	S_2	…	S_n
N_1	P_1	P_{11}	P_{21}	…	P_{1n}
N_2	P_2	P_{12}	P_{22}	…	P_{2n}
...	...	...	...	…	...
Nm	P_n	P_{1n}	P_{2n}	…	P_{mn}

5.3 MARGINAL ANALYSIS

Expected value can be used while deciding on one alternative from among several alternative courses of actions, each of which is characterized by a set of uncertain outcomes. It is easy to see that the computations become tedious as the number of values, the random variable can take, increases.

Consider the example of the newspaper vendor. Instead of weekly seven values of the demand of the newspapers, if the demand could take, say, twenty values, with different chances of occurrence of each value, the computation would become very tedious. In such cases, marginal analysis is very helpful. In this section, we explain the concept behind this analysis.

Consider the demand of the newspaper vendor. Let us assume that the newspaper man has found from the past data that the demand can take values ranging from 31, 32... to 50. For easy representation, let us assume that each of these values has got an equal chance of occurrence, viz., 1/20. The problem is to decide on the number of copies to be ordered.

Marginal Analysis proceeds by examining whether ordering an additional unit is worthwhile or not. Thus, we will order X copies, provided ordering the X^{th} copy is worthwhile but ordering the $(X+1)^{th}$ copy is not. To find out whether ordering X copies are worthwhile, we note the following. Ordering of the X^{th} copy may meet with two consequences, depending on the occurrences of two events:

A. The copy can be sold.

B. The copy cannot be sold.

The X^{th} copy can be sold only if the demand exceeds or equals X, whereas, the copy cannot be sold if the demand turns out to be less than X. Also, if event A occurs, we will make a profit of 50 p. on the extra copy, and if even B occurs, there will be a loss of 30 p. As this profit and loss pertains to the additional or marginal unit, these are referred to as marginal profit or loss and the resulting analysis is called marginal analysis.

Using the following notations:

K_1 = Marginal profit = 50p.

K_2 = Marginal loss = 30p.

P(A) = Probability (Demand $\geq$ X) = 1-Probability (Demand $\leq$ X - 1).

P(B) = Probability (Demand < X) = Probability (Demand $\leq$ X - 1).

We can write down the expected marginal profit and expected marginal loss as:

Expected Marginal Profit = K_1 P(A)

Expected Marginal Loss = K_2 P(B)

Ordering the X^{th} copy is worthwhile only if the expected profit due to it is more than the expected loss, so that

$K_1 P(A) \geq K_2 P(B)$

Now, if F(D) denotes the c.d.f. of demand, then by definition,

Probability Demand $\leq$(X-1) ± F(X-1)

Hence, $K_1 [1-F(X-1)] \geq K_2 F(X-1)$

or;

$K_1 - K_1 F(X-1) - K_2 F(X-1) \geq 0$

Thus, if condition 1 holds well, it is worthwhile to order the X^{th} copy.

If the optimal decision is to order X copies, then ordering the $(X+1)^{th}$ copy will not be worthwhile, i.e. the expected marginal profit due to the $(X+1)^{th}$ copy should be less than the expected loss.

Proceeding with the analysis in the same way as above, we have:

Expected Marginal Profit = K_1 Probability (Demand $\geq X + 1$)

$= K_1 [1 - F(X)]$

Expected Marginal Loss = K_2 F(X)

∴For the $(X+1)^{th}$ copy: $K_1 [1-F(X)]\ K_2 \leq F(X)$

From conditions (1) and (2) and the definition of Fractile, it is clear that X will be the

$K = \frac{K_1}{K_1 + K_2}$ the fractile of the Demand distribution.

Thus, for our problem, given the above result, all that we have to do is to calculate

$$K = \frac{K_1}{K_1 + K_2}$$

and find the K^{th} fractile of the distribution, which will give us the required answer.

In our problem:

K = [0.5 / (0.5 + 0.3)] = 0.625 and the 0.625^{th} fractile is 43.

∴The optimal decision is to order 13 copies.

The above shows how marginal analysis helps us in arriving at the optimal decision with very little computation. This is especially useful when the random variable of interest takes a large number of values. Though we have demonstrated this for a discrete demand distribution the same logic can be shown to be applicable for continuous distributions also. Instead of the distribution we have taken, if we would have assumed that demand is normal with a specific μ and σ, then also the same K^{th} fractile of N (μ,σ) would have given us the optimal decision.

5.4 THE STEPS OF DECISION-MAKING PROCESS

The decision-making process involves the following steps:

1. Identify and define the problem.

2. List all possible future events (not under the control of decision-maker) that are likely to occur.

3. Identify all the *courses of action* available to the decision-maker.

4. Express the payoffs (*pij*) resulting from each combination of course of action and state of nature.

5. Apply an appropriate decision theory model to select the best course of action from the given list on the basis of a criterion (measure of effectiveness) to get optimal (desired) payoff.

Example 5.1 A firm manufactures three types of products. The fixed and variable costs are given below:

	Fixed Cost (Rs)	*Variable Cost per Unit (Rs)*
Product *A*	25,000	12
Product *B*	35,000	9
Product *C*	53,000	7

The likely demand (units) of the products is given below:

Poor demand : 3,000

Moderate demand : 7,000

High demand : 11,000

If the sale price of each type of product is Rs 25, then prepare the payoff matrix.

Solution Let *D*1, *D*2 and *D*3 be the poor, moderate and high demand, respectively. The payoff is determined as:

Payoff = Sales revenue – Cost

The calculations for payoff (in '000 Rs) for each pair of alternative demand (course of action) and the types of product (state of nature) are shown below:

$D_1A = 3 \times 25 - 25 - 3 \times 12 = 14$ $D_2A = 7 \times 25 - 25 - 7 \times 12 = 66$

$D_1B = 3 \times 25 - 35 - 3 \times 9 = 13$ $D_2B = 7 \times 25 - 35 - 7 \times 9 = 77$

$D_1C = 3 \times 25 - 53 - 3 \times 7 = 1$ $D_2C = 7 \times 25 - 53 - 7 \times 7 = 73$

$D_3A = 11 \times 25 - 25 - 11 \times 12 = 118$

$D_3B = 11 \times 25 - 35 - 11 \times 9 = 141$

$D_3C = 11 \times 25 - 53 - 11 \times 7 = 145$

The payoff values are shown in Table below:

Product Type	**Alternative demand in ('000) Rs.**		
	D_1	D_2	D_3
A	14	66	118
B	13	77	141
C	1	73	145

5.5 DECISION UNDER VARIOUS DECISION-MAKING ENVIRONMENTS

To arrive at an optimal decision it is essential to have an exhaustive list of decision-alternatives, knowledge of decision environment, and use of appropriate quantitative approach for decision-making.

In this section three types of decision-making environments: *certainty*, *uncertainty*, *and risk*, have been discussed. The knowledge of these environments helps in choosing the quantitative approach for decision-making.

Type 1: Decision-Making under Certainty

In this decision-making environment, decision-maker has complete knowledge (perfect information) of outcome due to each decision-alternative (course of action). In such a case he would select a decision alternative that yields the maximum return (payoff) under known state of nature. For example, the decision to invest in *National Saving Certificate*, *Indira Vikas Patra*, *Public Provident Fund,* etc., is where complete information about the future return due and the principal at maturity is known.

Type 2: Decision-Making under Risk

In this decision-environment, decision-maker does not have perfect knowledge about possible outcome of every decision alternative. It may be due to more than one states of nature. In such a case, he makes an assumption of the probability for occurrence of particular state of nature.

Type 3: Decision-Making under Uncertainty

In this decision environment, decision-maker is unable to specify the probability for occurrence of particular state of nature. However, this is not the case of decision-making under ignorance, because the possible states of nature are *known*. Thus, decisions under uncertainty are taken even with less information than decisions under risk. For example, the probability that Mr. X will be the prime minister of the country 15years from now is not known.

5.6 DECISION-MAKING UNDER UNCERTAINTY

When probability of any outcome cannot be quantified, the decision-maker must arrive at a decision only on the actual conditional payoff values, keeping in view the criterion of effectiveness (policy).

The following criteria of decision-making under uncertainty have been discussed in this section:

- Optimism (Maximax or Minimin) criterion
- Pessimism (Maximin or Minimax) criterion
- Equal probabilities (Laplace) criterion
- Coefficient of optimism (Hurwiez) criterion
- Regret (salvage) criterion.

i) Optimism (Maximax or Minimin) Criterion

In this criterion the decision-maker ensures that he should not miss the opportunity to achieve the largest possible profit (maximax) or the lowest possible cost (minimin). Thus, he selects the decision alternative that represents the maximum of the maxima (or minimum of the minima) payoffs (consequences or outcomes).

The working method is summarized as follows:

(a) Locate the maximum (or minimum) payoff values corresponding to each decision alternative.

(b) Select a decision alternative with best payoff value (maximum for profit and minimum for cost).

Since in this criterion the decision-maker selects a decision-alternative with largest (or lowest) possible payoff value, it is also called an *optimistic decision criterion.*

ii) Pessimism (Maximin or Minimax) Criterion

In this criterion the decision-maker ensures that he would earn no less (or pay no more) than some specified amount. Thus, he selects the decision alternative that represents the maximum of the minima (or minimum of the minima in case of loss) payoff in case of profits. The working method is summarized as follows:

(a) Locate the minimum (or maximum in case of profit) payoff value in case of loss (or cost) data corresponding to each decision alternative.

(b) Select a decision alternative with the best payoff value (maximum for profit and minimum for loss or cost).

Since in this criterion the decision-maker is conservative about the future and always anticipates the worst possible outcome (minimum for profit and maximum for cost or loss), it is called a *pessimistic decision criterion*. This criterion is also known as *Wald's criterion*.

iii) Equal Probabilities (Laplace) Criterion

Since the probabilities of states of nature are not known, it is assumed that all states of nature will occur with equal probability, i.e. each state of nature is assigned an equal probability. As states of nature are mutually exclusive and collectively exhaustive, so the probability of each of these must be: 1/(number of states of nature). The working method is summarized as follows:

(a) Assign equal probability value to each state of nature by using the formula:

1 ÷ (number of states of nature).

(b) Compute the expected (or average) payoff for each alternative (course of action) by adding all the payoffs and dividing by the number of possible states of nature, or by applying the formula:

(*Probability of state of nature j*) × (*Payoff value for the combination of alternative I and state of nature j*.)

(c) Select the best expected payoff value (maximum for profit and minimum for cost).

This criterion is also known as the criterion of insufficient reason. This is because except in a few cases, some information of the likelihood of occurrence of states of nature is available.

iv) Coefficient of Optimism (Hurwicz) Criterion

This criterion suggests that a decision-maker should be neither completely optimistic nor pessimistic and, therefore, must display a mixture of both. Hurwicz, who suggested this criterion, introduced the idea of a coefficient of optimism (denoted by α) to measure the decision-maker's degree of optimism. This coefficient lies between 0 and 1, where 0 represents a complete pessimistic attitude about the future and 1 a complete optimistic attitude about the future. Thus, if α is the coefficient of optimism, then (1 – α) will represent the coefficient of pessimism.

The Hurwicz approach suggests that the decision-maker must select an alternative that maximizes

H (Criterion of realism) = α (*Maximum in column*) + (*1* – α) (*Minimum in column*)

The working method is summarized as follows:

(a) Decide the coefficient of optimism α (alpha) and then coefficient of pessimism (1 – α).

(b) For each decision alternative select the largest and lowest payoff value and multiply these with α and (1 – α) values, respectively. Then calculate the weighted average, *H* by using above formula.

(c) Select an alternative with best weighted average payoff value.

v) Regret (Savage) Criterion

This criterion is also known as *opportunity loss decision criterion or minimax regret decision criterion* because decision-maker regrets for choosing wrong decision alternative resulting in an opportunity loss of payoff. Thus, he always intends to minimize this regret. The working method is summarized as follows:

(a) From the given payoff matrix, develop an opportunity-loss (or regret) matrix as follows:

(i) Find the best payoff corresponding to each state of nature

(ii) Subtract all other payoff values in that row from this value.

(b) For each decision alternative identify the worst (or maximum regret) payoff value. Record this value in the new row.

(c) Select a decision alternative resulting in a smallest anticipated opportunity-loss value.

Example 5.2 Given the following payoff tables where the table represents profits:

Alternatives	**States of Nature**			
	S1	**S2**	**S3**	**S4**
A1	3	5	8	-1
A2	6	5	2	0
A3	0	5	6	4

Calculate the payoff using:

(a) Maximax, (b) Maximin,

➢ Decision making under uncertainty.

a) Maximax :

Max(Max Ai) = Max (8, 6, 6) = 8

Decision : Select A1

b) Maxmin :

Max(Min Ai) =Max (-1, 0 ,0) =0

Decision : Select A2 or A3

Example 5.3 A food products' company is contemplating the introduction of a revolutionary new product with new packaging or replacing the existing product at much higher price (S_1). It may even make a moderate change in the composition of the existing product, with a new packaging at a small increase in price (S_2), or may mall a small change in the composition of the existing product, backing it with the word 'New' and a negligible increase in price (S_3).

The three possible states of nature or events are: (i) high increase in sales (N_1), (ii) no change in sales (N_2) and (iii) decrease in sales (N_3). The marketing department of the company worked out the payoffs in terms of yearly net profits for each of the strategies of three events (expected sales). This is represented in the following table:

Strategies	**States of Nature**		
	N_1	N_2	N_3
S_1	700	300	150
S_2	500	450	0
S_3	300	300	300

Which strategy should the concerned executive choose on the basis of

(a) Maximin criterion (b) Maximax criterion

(c) Minimax regret criterion (d) Laplace criterion?

Solution: The payoff matrix is rewritten as follows:

(a) *Maximin Criterion*

States of Nature	Strategies		
	S_1	S_2	S_3
N_1	700	500	300
N_2	300	450	300
N_3	150	0	300
Column Minimum	150	0	300
Maximum Payoff ->			**300**

The maximum of column minima is 300. Hence, the company should adopt strategy *S*3.

(b) *Maximax Criterion*

States of Nature	Strategies		
	S_1	S_2	S_3
N_1	700	500	300
N_2	300	450	300
N_3	150	0	300
Column Maximum	700	500	300
	700	**<- Maximum Payoff**	

The maximum of column maxima is 7,00,000. Hence, the company should adopt strategy *S*1.

(c) *Minimax Regret Criterion* Opportunity loss table is shown below:

States of Nature	Strategies		
	S_1	S_2	S_3
N_1	700-700=0	700-500=200	700-300=400
N_2	450-300=150	450-450=0	450-300=150
N_3	300-150=150	300-0=300	300-300=0
Column Maximum	150	300	400
	150	**<- Minimum Regret**	

Hence the company should adopt minimum opportunity loss strategy, $S1$.

(d) *Laplace Criterion* Assuming that each state of nature has a probability 1/3 of occurrence. Thus,

Strategies	Expected Return (Rs)
S_1	(700 + 300 + 150)/3 = 383.33 <-***Largest Payoff***
S_2	(500 + 450 + 0)/3 = 316.66
S_3	(300 + 300 + 300)/3 = 300

Since the largest expected return is from strategy $S1$, the executive must select strategy $S1$.

5.7 PROBABILITY ESTIMATES USING BAYESIAN ANALYSIS

In this decision-making environment, decision-maker has sufficient information to assign probability to the likely occurrence of each outcome (state of nature). Knowing the probability distribution of outcomes (states of nature), the decision-maker needs to select a course of action resulting a largest expected (average) payoff value. The expected payoff is the sum of all possible weighted payoffs resulting from choosing a decision alternative.

The widely used criterion for evaluating decision alternatives (courses of action) under risk is the *Expected Monetary Value* (EMV) or *Expected Utility.*

(a) *Expected Monetary Value (EMV)*

The expected monetary value (EMV) for a given course of action is obtained by adding payoff values multiplied by the probabilities associated with each state of nature. Mathematically, EMV is stated as follows:

$$\text{EMV } (\textit{Course of action, } S_j) = \sum_{i=1}^{m} p_{ij}\, p_i$$

Where, m = number of possible states of nature

p_i= probability of occurrence of state of nature, Ni

p_{ij}= payoff associated with state of nature Ni and course of action, Sj

The Procedure

1. Construct a payoff matrix listing all possible courses of action and states of nature. Enter the conditional payoff values associated with each possible combination of course of action and state of nature along with the probabilities of the occurrence of each state of nature.

2. Calculate the EMV for each course of action by multiplying the conditional payoffs by the associated probabilities and adding these weighted values for each course of action.

3. Select the course of action that yields the optimal EMV.

Example 5.4 Mr X flies quite often from town A to town B. He can use the airport bus which costs Rs 25 but if he takes it, there is a 0.08 chance that he will miss the flight. The stay in a hotel costs Rs 270 with a 0.96 chance of being on time for the flight. For Rs 350 he can use a taxi which will make 99 per cent chance of being on time for the flight. If Mr X catches the plane on time, he will conclude a business transaction that will produce a profit of Rs 10,000, otherwise he will lose it. Which mode of transport should Mr X use? Answer on the basis of the EMV criterion.

Solution: Computation of EMV associated with various courses of action is shown in Table.

States of Nature	**Courses of Action**								
	Bus			**Stay in Hotel**			**Taxi**		
	Cost	**Prob.**	**Expected Value**	**Cost**	**Prob.**	**Expected Value**	**Cost**	**Prob.**	**Expected Value**
Catches the Flight	1000 -25 = 9975	0.92	9177	10000 -270 = 9730	0.96	9340.80	10000 – 350 = 9650	0.99	9553.50
Miss the Flight	- 25	0.08	- 2.00	- 270	0.04	- 10.80	- 350	0.01	- 3.5
Expected Monetary Value (EMV)			**9175**			**9330**			**9550**

Since EMV associated with course of action 'Taxi' is largest (= Rs 9,550), it is the logical alternative.

5.8 DECISION TREES FOR MAKING DECISION

Decision-making problems discussed earlier were limited to arrive at a decision over a fixed period of time. That is, payoffs, states of nature, courses of action and probabilities associated with the occurrence of states of nature were not subject to change. However, situations may arise when a decision-maker needs to revise his previous decisions due to availability of additional information. Thus he intends to make a sequence of interrelated decisions over several future periods. Such a situation is called a *sequential or multiperiod decision process*. For example, in the process of marketing a new product, a company usually first go for 'Test Marketing' and other alternative courses of action might be either

'Intensive Testing' or 'Gradual Testing'. Given the various possible consequences – good, fair, or poor, the company may be required to decide between redesigning the product, an aggressive advertising campaign or complete withdrawal of product, etc. Based on this decision there might be an outcome that leads to another decision and so on.

A decision tree analysis involves the construction of a diagram that shows, at a glance, when decisions are expected to be made – in what sequence, their possible outcomes, and the corresponding payoffs. A decision tree consists of *nodes, branches, probability estimates,* and *payoffs*. There are two types of nodes:

- ***Decision (or act) node:*** A decision node is represented by a square and represents a point of time where a decision-maker must select one *alternative course of action* among the available. The courses of action are shown as *branches* or *arcs* emerging out of decision node.

- ***Chance (or event) node:*** Each course of action may result in a *chance node*. The chance node is represented by a circle and indicates a point of time where the decision-maker will discover the response to his decision.

Branches emerge from and connect various nodes and represent either decisions or states of nature. There are two types of branches:

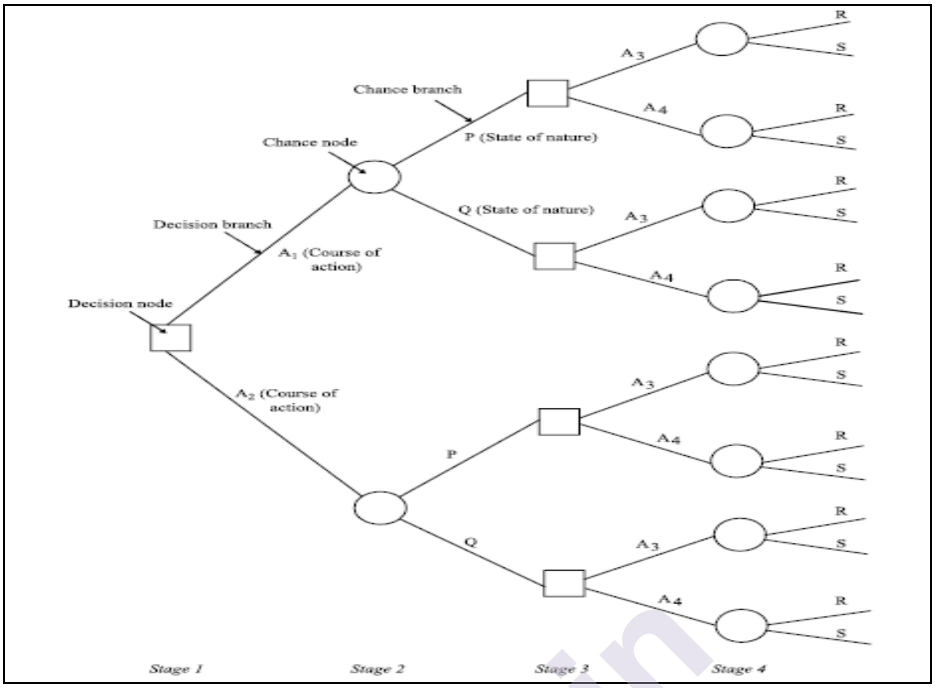
- ***Decision branch:*** It is the branch leading away from a decision node and represents a course of action that can be chosen at a decision point.

- ***Chance branch:*** It is the branch leading away from a chance node and represents the state of nature of a set of chance events. The assumed probabilities of the states of nature are written alongside their respective chance branch.

- ***Terminal branch:*** Any branch that makes the end of the decision tree (not followed by either a decision or chance node), is called a *terminal branch.* A terminal branch can represent either a course of action. The terminal points of a decision tree are supposed to be mutually exclusive points so that exactly one course of action will be chosen.

The *payoff* can be positive (i.e. revenue or sales) or negative (i.e. expenditure or cost) and it can be associated either with decision or chance branches.

An illustration of a decision tree is shown in Fig. below. It is possible for a decision tree to be deterministic or probabilistic. It can also further be divided in terms of stages – into single stage (a decision under condition of certainty) and multistage (a sequence of decisions).



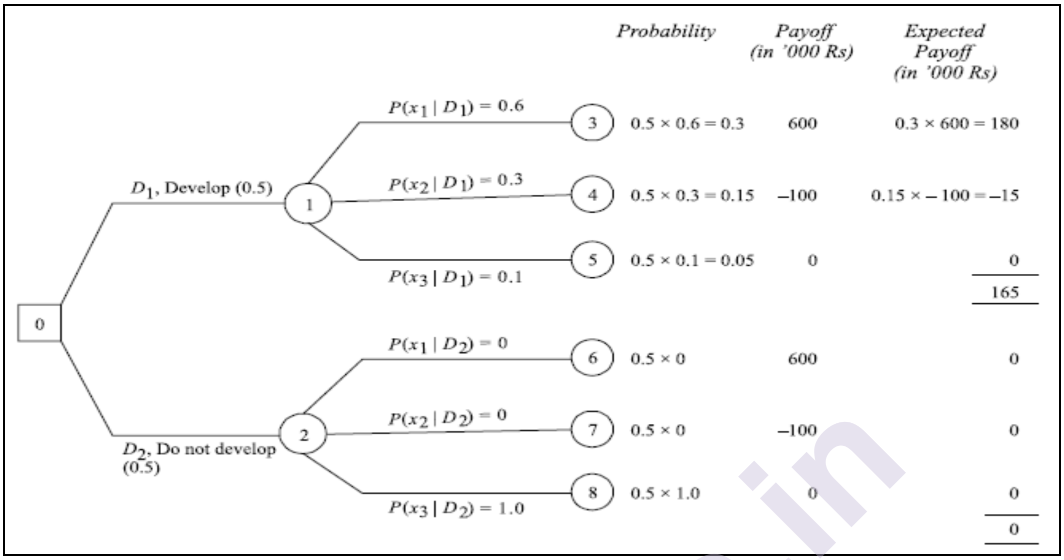
The optimal sequence of decisions in a tree is found by starting at the right-hand side and rolling backwards. At each node, an expected return is calculated (called position value). If the node is a chance node, then the position value is calculated as the sum of the products of the probabilities or the branches emanating from the chance node and their respective position values. If the node is a decision node, then the expected return is calculated for each of its branches and the highest return is selected. This procedure continues until the initial node is reached. The position values for this node correspond to the maximum expected return obtainable from the decision sequence.

Example 5.5 You are given the following estimates concerning a Research and Development programme:

Decision (D_i)	**Probability of Decision Di Given Research R P(Di | R)**	**Outcome Number**	**Probability of Outcome xi Given Di) P(xi|Di)**	**Payoff Value of Outcome, xi (Rs '000)**
Develop	0.5	1	0.6	600
		2	0.3	-100
		3	0.1	0
	0.5	1	0.0	600
Do not develop		2	0.0	-100
		3	1.1	0

Construct and evaluate the decision tree diagram for the above data. Show your workings for evaluation.

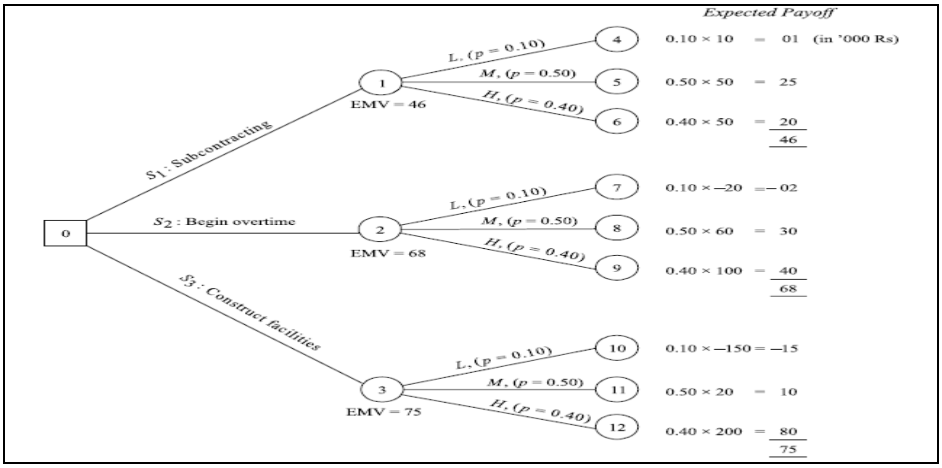
Solution: The decision tree of the given problem along with necessary calculations is shown in Fig.



Example 5.6 A glass factory that specializes in crystal is developing a substantial backlog and for this the firm's management is considering three courses of action: To arrange for subcontracting ($S1$), to begin overtime production ($S2$), and to construct new facilities ($S3$). The correct choice depends largely upon the future demand, which may be low, medium, or high. By consensus, management ranks the respective probabilities as 0.10, 0.50 and 0.40. A cost analysis reveals the effect upon the profits. This is shown in the table below:

Demand	**Probability**	**Course of Action**		
		S_1	**S_2**	**S_3**
		Subcontracting	**Begin Overtime**	**Construct facilities**
Low (L)	0.1	10	-20	-50
Medium (M)	0.5	50	60	20
High (H)	0.4	50	100	200

Solution: A decision tree that represents possible courses of action and states of nature is shown in Fig. In order to analyze the tree, we start working backwards from the end branches. The most preferred decision at the decision node 0 is found by calculating the expected value of each decision branch and selecting the path (course of action) that has the highest value.



Since node 3 has the highest EMV, therefore, the decision at node 0 will be to choose the course of action *S*3, i.e. construct new facilities.

5.9 SUMMARY

Decision Theory provides us with the framework and methods for analysing decision problems under uncertainty. A decision problem under uncertainty is characterized by different alternative courses of action and uncertain outcomes corresponding to each action. The problems can involve a single stage or a multi-stage decision process. Marginal Analysis is helpful in solving single stage problems, whereas the Decision Tree Approach is useful for solving multi-stage problems. In this unit we examined Optimism (Maximax or Minimin) criterion, Pessimism (Maximin or Minimax) criterion, Equal probabilities (Laplace) criterion, Coefficient of optimism (Hurwiez) criterion and Regret (salvage) criterion and the decision tree analysis.

5.10 EXERCISES

Exercise Example 1. The manager of a flower shop promises its customers delivery within four hours on all flower orders. All flowers are purchased on the previous day and delivered to Parker by 8.00 am the next morning. The daily demand for roses is as follows.

Dozens of roses:	70	80	90	100
Probability:	0.1	0.2	0.4	0.3

The manager purchases roses for Rs 10 per dozen and sells them for Rs 30. All unsold roses are donated to a local hospital. How many dozens of roses should Parker order each evening to maximize its profits? What is the optimum expected profit?

Solution: The quantity of roses to be purchased per day is considered as 'course of action' and the daily demand of the roses is considered as a 'state of nature' because demand is uncertain with known probability. From the data, it is clear that the flower shop must not purchase less than 7 or more than 10 dozen roses, per day. Also each dozen roses sold within a day yields a profit of Rs (30– 10) = Rs 20 and otherwise it is a loss of Rs 10.

Thus

Marginal profit (MP) = Selling price – Cost = 30 – 10 = Rs 20

Marginal loss (ML) = Loss on unsold roses = Rs 10

Using the information given in the problem, the various conditional profit (payoff) values for each combination of decision alternatives and state of nature are given by Conditional profit = MP × Roses sold – ML × Roses not sold

$$= \begin{cases} 20D, & \text{if } D \geq S \\ 20D - 10(S - D) = 30D - 10S, & \text{if } D < S \end{cases}$$

Where, D = number of roses sold within a day and S = number of roses stocked.

The resulting conditional profit values and corresponding expected payoffs are computed in the table below:

States if Nature (Demand per Day)	**Probability**	**Conditional profit (Rs.) due to courses of action (Purchase per Day)**				**Conditional profit (Rs.) due to courses of action (Purchase per Day)**			
		70	80	90	100	70	80	90	100
	(1)	(2)	(3)	(4)	(5)	(1)X(2)	(1)X(3)	(1)X(4)	(1)X(5)
70	0.1	140	130	120	110	14	13	12	11
80	0.2	140	160	150	140	28	32	30	28
90	0.4	140	160	180	170	56	64	72	68
100	0.3	140	160	180	200	42	48	54	60
Expected Monetary Value						140	157	168	167

Since the highest EMV of Rs 168 corresponds to the course of action 90, the flower shop should purchase nine dozen roses every day.

Exercise Example 2: The probability of demand for hiring cars on any day in a given city is as follows:

No. of cars demanded:	0	1	2	3	4
Probability:		0.1	0.2	0.3	0.2	0.2

Cars have a fixed cost of Rs 90 each day to keep the daily hire charges (variable costs of running) Rs. 200. If the car-hire company owns 4 cars, what is its daily expectation? If the company is about to go into business and currently has no car, how many cars should it buy?

Solution Given that Rs 90 is the fixed cost and Rs 200 is variable cost. The payoff values with 4 cars at the disposal of decision-maker are calculated as under:

No. of cars demanded:

0	1	2	3	4

Payoff (with 4 cars):

0 – 90 × 4	200 – 90 × 4	400 – 90 × 4	600 – 90 × 4	800 – 90 × 4
= – 360	= – 160	= 40	= 240	= 440

Thus, the daily expectation is obtained by multiplying the payoff values with the given corresponding probabilities of demand:

Daily Expectation =

(– 360)(0.1) + (– 160)(0.2) + (40)(0.3) + (240)(0.2) + (440)(0.2) = Rs 80

The conditional payoffs and expected payoffs for each course of action are shown in Tables.

Conditional Payoff Values

Demand of Cars	**Probability**	**Conditional Payoff (Rs.) due to Decision to Purchase Cars (Course of Action)**				
		0	1	2	3	4
0	0.1	0	-90	-180	-270	-360
1	0.2	0	110	20	-70	-160
2	0.3	0	110	220	130	40
3	0.2	0	110	220	330	240
4	0.2	0	110	220	330	440

Expected Payoffs and EMV

Demand of Cars	**Probability**	**Conditional Payoff (Rs.) due to Decision to Purchase Cars (Course of Action)**				
		0	1	2	3	4
0	0.1	0	-9	-18	-27	-36
1	0.2	0	22	4	-14	-32
2	0.3	0	33	66	39	12
3	0.2	0	22	44	66	48
4	0.2	0	22	44	66	88
EMV		**0**	**90**	**140**	**130**	**80**

Since the EMV of Rs 140 for the course of action 2 is the highest, the company should buy 2 cars.

❖❖❖❖

6

WAITING LINES MODEL

Unit Structure

6.0 OBJECTIVES

After studying this unit, you should be able to:

- Identify the occurrence of waiting line or queuing in real life situations.
- Describe the characteristics of a queuing problem.
- Use the statistical methods necessary to analyse queuing problems.
- Apply the common queuing models in suitable problems.
- Estimate the optimum parameters of a queuing model with respect to cost and service level for M/M/1 system.

6.1 INTRODUCTION

The waiting line models or the queuing problem is identified by the presence of a group of customers who arrive randomly to receive some service. The customer upon arrival may be attended to immediately or may have to wait until the server is free. The service time required to serve the customers is also a statistical variable. This methodology can be applied in the field of business (banks, booking counters), industries (servicing of machines), government (railway or post-office counters), transportation (airport, harbor) and everyday life (elevators, restaurants, and doctor's chamber).

The waiting line or queuing models are basically relevant to service oriented organisations and suggest ways and means to improve the efficiency of the service. An improvement of service level is always possible by increasing the number of employees. Apart from increasing the cost an immediate consequence of such a step is unutilized or idle time of the servers. In addition, it is unrealistic to assume that a large-scale increase in staff is possible in an organisation. Queuing methodology indicates the optimal usage of existing manpower and other resources to improve the service. It can also indicate the cost implications if the existing service facility has to be improved by adding more servers.

The relationship between queuing and service rates can be diagrammatically illustrated using the cost curves shown in Figure.

At a slow service rate, queues build up and the cost of queuing increases. An ideal service unit will minimise the operating cost of the entire system.

Many real-life situations in which study of queuing theory can provide solution to waiting line problems are listed below:

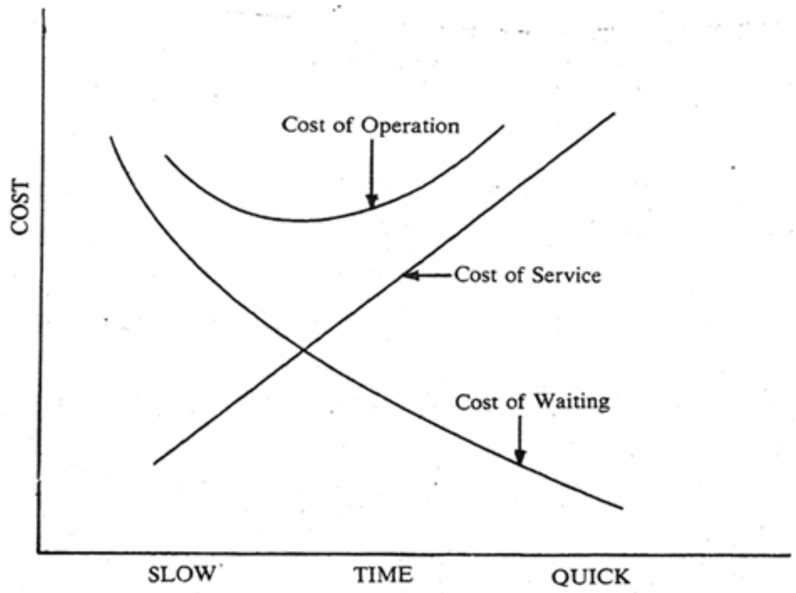
Situation	Customers	Service Facilities
Petrol pumps (stations)	Automobiles	Pumps
Hospital	Patients	Doctors/Nurses/Rooms
Airport	Aircraft	Runways
Post office	Letters	Sorting system
Job interviews	Applicants	Interviewers
Cargo	Trucks	Loader/unloaders
Workshop	Machines/Cars	Mechanics/Floor space
Factory	Employees	Cafeteria/Punching Machine

6.2 CHARACTERISTICS OF A WAITING LINE OR A QUEUEING MODEL

A **queuing system** can be described by the following components:

Arrival

The statistical pattern of the arrival can be indicated through the probability distribution of the number of arrivals in an interval. This is a discrete random variable. Alternatively, the probability distribution of the time between successive arrivals (known as **inter-arrival** time) can also be studied to ascertain the stochastic aspect of the problem. This variable is **continuous** in nature.



Cost Structure of a Queuing Problem

The probability distribution of the arrival pattern can be identified through analysis of past data. The discrete random variable indicating the number of arrivals in a time interval and the continuous random variable indicating the time between two successive arrivals **(interarrival time)** are obviously inter related. A remarkable result in this context is that if the number of arrivals follows a **Poisson distribution**, the corresponding interarrival time follows an **exponential distribution.** This property is frequently used to derive elegant results on queuing problems.

Service

The time taken by a server to complete service is known as service time. The service time is a statistical variable and can be studied either as the number of services completed in a given period of time or the completion period of a service. The data on actual service time should be analyzed to find out the probability distribution of service time.

Server

The service may be offered through a single server such as a ticket counter or through several channels such as a train arriving in a station with several platforms.

Sometimes the service is to be carried out sequentially through several phases known as multiphase service. In government, the papers move through a number of phases in terms of official hierarchy till they arrive at the appropriate level where a decision can be taken.

Time spent in the queuing system

The time spent by a customer in a queuing system is the sum of waiting time before service and the service time.

Queue discipline

The queue discipline indicates the order in which members of the queue are selected for service. It is most frequently assumed that the customers are served on a first' come first serve basis. This is commonly referred to as FIFO (first in, first out) system. Occasionally, a certain group of customers receive priority in service over others even if they arrive late. This is commonly referred to as priority queue; the queue discipline does not always take into account the order of arrival. The server chooses one of the customers to offer service at random. Such a system is known as service in random order (SIRO). While allotting an item with high demand and limited supply such as a test match cricketer share of a public limited company, SIRO is the only possible way of offering service when it is not possible to identify the order of arrival.

Size of a population

The collection of potential customers may be very large or of a moderate size. In a railway booking counter the total number of potential passengers is so large that although theoretically finite it can be regarded as infinity for all practical purposes. The assumption of infinite population is very convenient for analysing a queuing model. However, this assumption is not valid where the customer group is represented by a few looms in a spinning mill that require operator facility from time to time. If the population size is finite then the analysis of queuing model becomes more involved.

Maximum size of a queue

Sometimes only a finite number of customers are allowed to stay in the system although the total number of customers in the population may or may not be finite. For example, a doctor may make appointments with k patients in a day. If the number of patients asking for appointment exceeds k, they are not allowed to join the queue. Thus, although the size of the population is infinite, the maximum number permissible in the system is k.

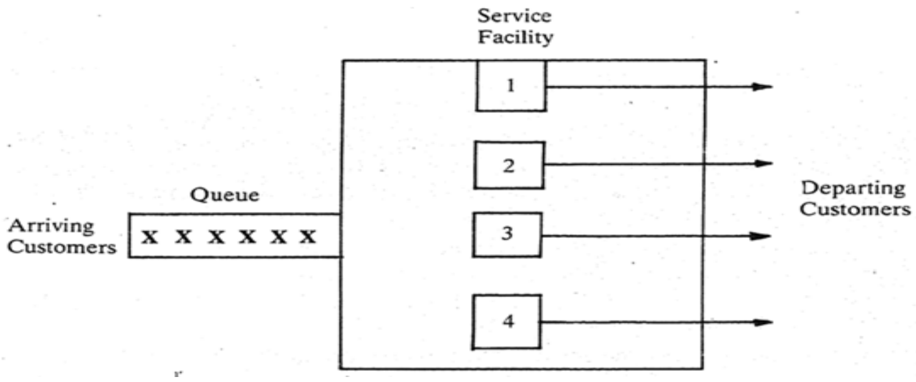


Fig.: Schematic Representation of a Queuing Problem

6.3 NOTATIONS AND SYMBOLS

Kendall (Kendall, 1951) has introduced set of notations which have become standard in the literature of queuing models. A general queuing system is denoted by (a/b/c): (d/e) where

a = probability distribution of the interarrival time.

b=probability distribution of the service time.

c=number of servers in the system.

d = maximum number of customers allowed in the system.

e= queue discipline.

In addition, the size of the population as mentioned in the previous section is important for certain types of queuing problem although not explicitly mentioned in the Kendall's notation. Traditionally, the exponential distribution in queuing problems is denoted by M. Thus (M/M/1) : (□ /FIFO) indicates queuing system when the interarrival times and service times are exponentially distributed having one server in the system with first in first out discipline and the number of customers allowed in the system can be infinite.

In general, the behaviour of a queuing system will depend upon time. Such a system is said to be in a **transient state.** This usually occurs at an early stage of formation of queues where its behavior is still dependent up on the initial conditions. When sufficient time has elapsed since the beginning of the operation (mathematically as the time approaches to infinity) the behaviour of the system may become **independent** of time. The system then is said to be in a steady state. Only steady state queuing problems will be analyzed in this unit.

The following symbols will be used while studying queuing models.

n=numberofunitsinthesystemwhoarewaitingforserviceandwhoarebeingserved.

$P_n(t)$=transientstateprobabilitiesofexactlyncustomersinthesystemattimetassumingthat the system has started its operation at time zero.

P_n=steady state probability of having n customers in the system.

£ =mean effective arrival rate (number of customers arriving per unit time).

μ=mean service rate per busy server (number of customers served per unit time)

C=number of parallel servers.

W(t)= density function of the waiting time.

W_s = expected waiting time per customer in the system including the service time. W_q=expected waiting time per customer in the queue excluding the service time.

L_s=expected number of customers in the system including those who are receiving service.

L_q=expected number of customers in the queue excluding those who are receiving service.

6.4 STATISTICAL METHODS IN QUEUEING

The arrival of customers, the departure of customers after service and the waiting time of a customer before being served in a queuing system are random phenomena. In order to develop appropriate statistical model of this problem the following assumptions are made:

1) If N(t) denotes the number of arrivals or departures after service in the timeinterval(0,t)thenN(t)has**independentincrements.**Thiscanbestatedstatistically as follows. If $t_1<t_2<t_3$ then $\{N(t_3) - N(t_2)\}$ and $\{N(t_2) - N(t_1)\}$ are **independent random variables**

2) If there are n units in the system, the probability of exactly one arrival from t to $t+\Delta t$ is $\lambda_n \Delta t + o(\Delta t)$. Likewise, the probability that exactly one departure after service will occur from t to $t+\Delta t$ is $\mu_n \Delta t + o(\Delta t)$. The quantity $o(\Delta t)$ represents a function of Δt such that

$$\lim_{t \to 0} \frac{o(\Delta t)}{t} = 0.$$

3) If there are n customers at time t, the probability that the number of arrivals and departures combined will exceed one during the time interval t to $t+\Delta t$ is $o(\Delta t)$.

Poisson Process

It can be shown (Taha, 1971) when $\lambda_n = \lambda$ and $\mu_n=0$ for all n that under conditions (1), (2) and (3), the probability $P_n(t)$ of having exactly n customers in the system (assuming that there is no customer in the system at time 0) is given by

$$P_n(t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}; \quad n = 0,1,2,..........$$

Thus, if the service component is ignored the number of arrivals in the interval (0,t) has the probability distribution of a Poisson variable with parameter λ t which is the product of the **arrival rate** and the length of the interval t. This probability law is known as the **Poisson Process.**

RelationshipbetweenPoissonProcessandExponentialProbabilityDistribution

a) Consider a queuing model in which the number of arrivals in an interval of length t follows a **Poisson Process** with **arrival rate** λ**.** If the interarrival times are **independent** random variables, they must follow an exponential distribution with density f(t) where

$$f(t)= \lambda e^{-\lambda t}\ ; t>0.$$

It can be readily shown by integration that E(t) = 1 / £. Thus if the arrival rate £=20/hour, the average time between two successive arrivals is 1/20 hour or 3 minutes.

b) If the interarrival times in a queuing system are independently, identically distributed **exponential** random variables then the number of arrivals in an interval follows a **Poisson Process** with arrival rate identical with parameter of the exponential distribution.

c) Consider a random variable X with an exponential probability distribution. Therefore, t>0

P(X>t+s | X>s) = P(X>t)

This property is important for solving queuing problems. Assuming that when a customer arrives the service is in progress for s units of time. The chance that the service will continue for at least an additional t unit is identical with the probability that the service will prolong for at least t units after a fresh start.

One way to look at this property is that the time already spent in the system has no relevance to the additional time the customer is likely to spend in the system further. Thus, this property is sometimes referred to **as lack of memory or forgetfulness** of the exponential distribution.

6.5 THE M/M/L SYSTEM

In this queuing model, it is assumed that the number of customers arriving in a time interval t follows a Poisson Process with parameter λ. Equivalently, the interval between any two successive arrivals is exponentially distributed with parameter A. The time taken to complete a single service is exponentially distributed with parameter μ. The number of servers is one. Although not explicitly stated both the population and the queue size can be infinity. The order of service is assumed to be FIFO.

Performance Measures of a Queuing System

The performance measures (operating characteristics) for the evaluation of the performance of an existing queuing system, and for designing a new system in terms of the level of service a customer receives as well as the proper utilization of the service facilities are listed as follows:

1. Average (or expected) time spent by a customer in the queue and system

Wq: Average time an arriving customer has to wait in a queue before being served,

Ws: Average time an arriving customer spends in the system, including waiting and service.

2. Average (expected) number of customers in the queue and system

Lq: Average number of customers waiting for service in the queue (queue length)

Ls: Average number of customers in the system (either waiting for services in the queue or being served).

3. Value of time both for customers and servers

Pw : Probability that an arriving customer has to wait before being served (also called *blocking probability*).

$\rho = \lambda / \mu$: Percentage of time a server is busy serving customers, i.e., the system utilization.

Pn: Probability of *n* customers waiting for service in the queuing system.

Pd: Probability that an arriving customer is not allowed to enter in the queuing i.e., system capacity is full.

4. Average cost required to operate the queuing system

• Average cost required to operate the system per unit of time?

• Number of servers (service centres) required to achieve cost effectiveness?

Transient-State and Steady-State

At the beginning of service operations, a queuing system is influenced by the initial conditions, such as number of customers waiting for service and percentage of time servers are busy serving customers, etc.

This initial period is termed as *transient-state.* However, after certain period of time, the system becomes independent of the initial conditions and enters into a *steady-state* condition.

To quantify various measures of system performance in each queuing model, it is assumed that the system has entered into a steady-state.

Let *Pn*(*t*) be the probability that there are *n* customers in the system at a particular time *t*. Any change in the value of *Pn*(*t*), with respect to time *t*, is denoted by *Pn'* (*t*). In the case of steady-state, we have:

$$\lim_{t \to \infty} P_n(t) = P_n \text{ (independent of time, } t)$$

or

$$\lim_{t \to \infty} \frac{d}{dt}\{P_n(t)\} = \frac{d}{dt}(P_n)$$

or

$$\lim_{t \to \infty} P_n'(t) = 0$$

If the arrival rate of customers at the system is more than the service rate, then a steady-state cannot be reached, regardless of the length of the elapsed time.

Queue size, also referred as *line length* represents average number of customers waiting in the system for service.

Queue length represents average number of customers waiting in the system and being served.

Notations The notations used for analyzing of a queuing system are as follows:

n = number of customers in the system (waiting and in service)

Pn= probability of n customers in the system

λ = average customer arrival rate or average number of arrivals per unit of time

in the queuing system

μ = average service rate or average number of customers served per unit time at the place of service

λ / μ = ρ= Average service completion time(1/ μ) / Average interarrival time(1/ λ)

= traffic intensity or server utilization factor

P_0 = probability of no customer in the system

s = number of service channels (service facilities or servers)

N = maximum number of customers allowed in the system

Ls= average number of customers in the system (waiting and in service)

Lq= average number of customers in the queue (queue length)

Ws= average waiting time in the system (waiting and in service)

Wq= average waiting time in the queue

Pw = probability that an arriving customer has to wait (system being busy),

$1 - P_0 = (\lambda/\mu)$

For achieving a steady-state condition, it is necessary that, $\lambda/\mu < 1$ (i.e. the arrival rate must be less than the service rate). Such a situation arises when the queue length is limited, generally because of space, capacity limitation or customers balk.

Relationships among Performance Measures

The following basic relationships hold for all infinite source queuing models.

$$L_s = \sum_{n=0}^{\infty} n P_n \quad \text{and} \quad L_q = \sum_{n=s}^{\infty} (n-s) P_n$$

The general relationship among various performance measures is as follows:

(i) Average number of customers in the system is equal to the average number of customers in queue(line) plus average number of customers being served per unit of time (system utilization).

Ls=*Lq* + Customer being served

= *Lq*+ λ / μ

The value of ρ =λ / μ is true for a single server finite-source queuing model.

(ii) Average waiting time for a customer in the queue (line)

Wq=*Lq* / λ

(iii) Average waiting time for a customer in the system including average service time

Ws= *Wq*+1 / μ

(iv) Probability of being in the system (waiting and being served) longer than time *t* is given by:

$$P(T > t) = e^{-(\mu - \lambda)\lambda t} \quad \text{and} \quad P(T \leq t) = 1 - P(T > t)$$

Where, *T* = time spent in the system

t = specified time period

e = 2.718

(v) Probability of only waiting for service longer than time *t* is given by:

$$P(T > t) = \frac{\lambda}{\mu} e^{-(\mu - \lambda)t}$$

(vi) Probability of exactly *n* customers in the system is given by:

$$P_n = P_0 \left(\frac{\lambda}{\mu}\right)^n = \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n$$

(vii) Probability that the number of customers in the system, *n* exceeds a given number, *r* is given by:

$$P(n > r) = \left(\frac{\lambda}{\mu}\right)^{r+1}$$

The general relationships among various performance measures are:

(a) $L_s = \lambda W_s$ (b) $W_s = W_q + \frac{1}{\mu} = \frac{1}{\lambda} L_s$

(c) $L_q = L_s - \frac{\lambda}{\mu} = \lambda W_q$ (d) $W_q = W_s - \frac{1}{\mu} = \frac{1}{\lambda} L_q$

(e) $L_s = \sum_{n=0}^{\infty} nP_n \rightarrow W_s = \frac{L_s}{\lambda} \rightarrow W_q = W_s - \frac{1}{\mu} \rightarrow L_q = \lambda W_q$

(M/M/1) System Examples

Example 1:

A petroleum company is considering expansion of its one unloading facility at its refinery. Due to random variations in weather, loading delays and other factors, ships arriving at the refinery to unload crude oil arrive at a rate of 5 ships per week. The service rate is 10 ships per week. Assume arrivals follow a Poisson Process and the service time is exponential.

a) Find the average time a ship must wait before beginning to deliver its cargo to the refinery.

b) If a second berth is rented what will be the average number of ships waiting before being unloaded?

c) What would be the average time a ship would wait before being unloaded with two berths?

d) What is the average number of idle berths atany specified time?

Solution:

a) The expected waiting time Wq before a ship begins to unload crude oil is:

$$W_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{5}{10\times5} = \frac{1}{10} \text{ week.}$$

b) With a second berth the queueing model is an M/M/2 system.

$$\rho = \frac{\lambda}{2\mu} = \frac{5}{20} = \frac{1}{4}.$$

$$\frac{1}{P_0} = 1 + \frac{\lambda}{\mu} + \frac{(\lambda/\mu)^2}{2} \times \left(\frac{1}{1-\frac{1}{4}}\right)$$

$$= 1 + \frac{1}{2} + \frac{1}{8} \times \frac{4}{3} = \frac{5}{3}$$

$$\therefore\ P_0 = \frac{3}{5}.$$

$$L_q = \frac{\frac{3}{5} \times (\frac{1}{2})^2 \times \frac{1}{4}}{2(1-\frac{1}{4})^2} = \frac{1}{30}$$

c) $W_q = \frac{L_q}{\lambda} = \frac{1}{150}$ week

d) If there is no ship in the system then both the berths are idle. If there is only one ship in the system one of the berths is empty. Hence the expected number of idle berths is

$$2P_0 + P_1 = 2 \times \frac{3}{5} + \frac{1}{2} \times \frac{3}{5} = 1.5.$$

Example 2: A television repairman finds that the time spent on his jobs has an exponential distribution with a mean of 30 minutes. If he repairs the sets in the order in which they came in, and if the arrival of sets follows a Poisson distribution with an approximate average rate of 10 per 8-hour day, what is the repairman's expected idle time each day? How many jobs are ahead of the average set just brought in? [*Rajasthan, MCom, 2000; Delhi Univ., MCom 2002*]

Solution: From the data of the problem, we have:

λ = 10/8 = 5/4 sets per hour; and μ = (1/30) 60 = 2 sets per hour

(a) Expected idle time of repairman each day

Since number of hours for which the repairman remains busy in an 8-hour day (traffic intensity) is given by:

(8) (λ/μ) = (8) (5/8) = 5 hours

Therefore, the idle time for a repairman in an 8-hour day will be: (8 – 5) = 3 hours

(b) Expected (or average) number of TV sets in the system

$$L_s = \frac{\lambda}{\mu-\lambda} = \frac{5/4}{2-(5/4)} = \frac{5}{3} = 2 \text{ (approx.) TV sets}$$

6.6 SUMMARY

A common situation that occurs in everyday life is that of waiting in a line either at bus stops, petrol pumps, restaurants, ticket booths, doctors' clinics, bank counters, traffic lights and so on. Queues (waiting lines) are also found in workshops where the machines wait to be repaired; at a tool crib where the mechanics wait to receive tools; in a warehouse where items wait to be used, incoming calls wait to mature in the telephone exchange, trucks wait to be unloaded, airplanes wait either to take off or land and so on.

In general, a queue is formed at a production/operation system when either customers (human beings or physical entities) requiring service wait because number of customers exceeds the number of service facilities, or service facilities do not work efficiently/take more time than prescribed to serve a customer.

A queuing problem is characterised by a flow of customers arriving randomly at one or more service facilities. The customers upon arriving at the facility may be served immediately or, if willing, may have to wait until the facility is available.

The arrivals are characterised by either by **the number of arrivals time** in a specific period of time or by the time between two successive arrivals, known as **interarrival time**. The number of arrivals is a **discrete random variable** whereas the interarrival times are **continuous random variables.**

The service is characterised either by the **number of services completed** in a given time period or the **time taken to complete the service.** The number of services completed is a **discrete random variable** while the service time is **a continuous random variable.**

Other characteristics of a queuing model are the **number of servers, order of service, size of the population of the customers, maximum size of the queue.** The various **characteristics** of a queuing problem can be expressed in terms of a suitable notation known as **Kendall's notation.**

The time spent by an incoming customer in a queuing system is a random variable and is of interest to the decision maker. If the customer under consideration is a down machine it is inoperative during the period of service. There is, thus, a cost involved due to its lack of functioning. One of the objectives of studying a queuing problem is to find out the optimum service rate and the number of servers so that **the average cost of being in queuing system and the cost of service are minimized.** The time a customer spends in a system before the start of service is a random variable known as **waiting time.** The probability distribution of waiting time depends upon the probability distribution of interarrival time and service time.

Let $P_n(t)$ indicate the probability of having n customers in the system at time t. Then if P(t) depends upon t, the queuing system is said to be in the **transient** state. After the queuing system has become operative for a considerable period of time the probability $P_n(t)$ may become **independent of t.** The probabilities are then known as **steady state probabilities.**

When the number of arrivals in a time interval of length t follows a Poisson distribution with mean Jet where **A** is the rate of arrival, the arrivals are said to follow a **Poisson Process.** In this case, the interarrival times are independently, identically distributed random variables with an **exponential probability distribution with parameter X** and vice versa. If the service times are independently, identically distributed **exponential** random variables with **parameter** 11 and there is only one server in the queuing model, the resulting queuing model is denoted by M/M/1 according to Kendall's notation.

6.7 EXERCISES

Example 3: In a railway marshaling yard, goods trains arrive at a rate of 30 trains per day. Assuming that the interarrival time follows an exponential distribution and the service time (the time taken to hump a train) distribution is also exponential with an average of 36 minutes. Calculate:

(a) expected queue size (line length)

(b) probability that the queue size exceeds 10

If the input of trains increases to an average of 33 per day, what will be the change in (i) and (ii)?

Solution to Exercise 3: From the data of the problem, we have

£=30/(60 * 24)=1/48 trains per minute and μ=1/36 trains per minute.

The traffic intensity then is, ρ= £/μ = 36/48 = 0.75

(a) Expected queue size (line length):

$$L_s = \frac{\rho}{1-\rho} = \frac{0.75}{1-0.75} = 3 \text{ trains}$$

(b) Probability that the queue size exceeds 10:

$P\ (n \geq 10) = \rho^{10} = (0.75)^{10} = 0.06$

If the input increases to 33 trains per day, then we have £ = 33/(60* 24)= 11/480 trains per minute and μ = 1/36 trains per minute.

Thus, traffic intensity,

$$\rho = \frac{\lambda}{\mu} = \left(\frac{11}{480}\right)(36) = 0.83$$

Hence, recalculating the values for (a) and (bi)

$$L_s = \frac{\rho}{1-\rho} = \frac{0.83}{1-0.83} = 5 \text{ trains (approx.), and}$$

$$P(n \geq 10) = \rho^{10} = (0.83)^{10} = 0.2 \quad \text{(approx.)}$$

Example 4: Arrivals at telephone booth are considered to be Poisson with an average time of 10minutes between one arrival and the next. The length of phone calls is assumed to be distributed exponentially, with a mean of 3 minutes.

(a) What is the probability that a person arriving at the booth will have to wait?

(b) The telephone department will install a second booth when convinced that an arrival would expect waiting for at least 3 minutes for a phone call. By how much should the flow of arrivals increase in order to justify a second booth?

(c) What is the average length of the queue that forms from time to time?

(d) What is the probability that it will take a customer more than 10 minutes altogether to wait for the phone and complete his call?

Solution: From the data of the problem, we have

$\lambda = 1/10 = 0.10$ person per minute and $\mu = 1/3 = 0.33$ person per minute

(a) Probability that a person has to wait at the booth.

$P(n > 0) = 1 - P_0 = \lambda/\mu = 0.10 / 0.33 = 0.3$

(b) The installation of second booth will be justified only if the arrival rate is more than the waiting time.

Let λ' be the increased arrival rate. Then the expected waiting time in the queue will be

$$W_q = \frac{\lambda'}{\mu(\mu - \lambda')}$$

$$3 = \frac{\lambda'}{0.33(0.33 - \lambda')} \quad \text{or} \quad \lambda' = 0.16$$

Where, $Wq = 3$ (given) and $\lambda = \lambda'$ (say) for second booth. Hence, the increase in the arrival rate is0.16 – 0.10 = 0.06 arrivals per minute.

(c) Average length of non-empty queue:

$$L = \frac{\mu}{\mu - \lambda} = \frac{0.33}{0.23} = 2 \text{ customers (approx.)}$$

(d) Probability of waiting for 10 minutes or more is given by

$$P(t \geq 10) = \int_{10}^{\infty} \frac{\lambda}{\mu}(\mu - \lambda)\, e^{-(\mu-\lambda)t}\, dt$$

$$= \int_{10}^{\infty} (0.3)(0.23)\, e^{-0.23t}\, dt = 0.069\left[\frac{e^{-0.23t}}{-0.23}\right]_{10}^{\infty} = 0.03$$

This shows that on an average 3 per cent of the arrivals will have to wait for 10 minutes or more before they can use the phone.

❖❖❖❖

7

SIMULATION- QUEUE SYSTEM, INVENTORY AND DEMAND SIMULATION

Unit Structure

7.0 OBJECTIVES

After studying this unit, you will be able to:

- Discuss the need for simulation in management problems where it will not be possible to use precise mathematical techniques
- Explain that simulation may be the only method in situations where it will be extremely difficult to observe actual environment
- Describe the process of simulation based on a sound conceptual framework
- Apply simulation techniques in solving queuing and inventory control problems.

7.1 INTRODUCTION

Simulation is a quantitative procedure which describes a process by developing a model of that process and then conducting a series of organized experiments to predict the behaviour of the process over time. Observing the experiments is very much like observing the process in operation. To find out how the real process would react to certain changes, we can produce these changes in our model and simulate the reaction of the real process to them.

For instance, in designing an airplane, the designer can solve various equations describing the aerodynamics of the plane. Or, if these equations are very difficult and complex to solve, scale model can be built and its behaviour observed in a wind tunnel. In simulation, we build mathematical models which we cannot solve and run them on sample data to simulate the behaviour of the-system. Thus, simulation involves performing experiments on the model of a system.

Simulation is one of the widely used techniques by corporate managers as an aid for decision-making. This technique uses a computer to simulate (imitate) the operation of any system or process. It is also used to analyse systems that operate indefinitely. In such a case, the computer randomly generates and records the occurrence of the events that drive the system as if it was physically operating. Recording the performance of the simulated operation of the system for a number of alternative options of operating procedures enables us to evaluate and compare these alternatives to choose the most desired one.

7.2 REASONS FOR USING SIMULATION

In the case of number of problems, we have been able to find through straight forward techniques, mathematical solutions to the situation. The economic order quantity, the simplex solution to a linear programming problem, and a branch-and- bound solution to an integer programming problem are some of the typical examples we can cite. However, in each of those cases the problem was simplified by certain assumptions so that the appropriate mathematical techniques could be employed. It is not difficult to think of managerial situations so complex that mathematical solution is impossible given the current state of the art in mathematics. In these cases, simulation offers a good alternative, If we insist that all managerial problems have to be solved mathematically, then we may find ourselves simplifying the situation so that it can he solved; sacrificing realism to solve the problem can get us in real trouble. Whereas the assumption of normality-in dealing with a distribution of inventory demand may he reasonable, the assumption of linearity in a specific linear programming environment may be totally unrealistic.

While in some cases the solutions which result from simplifying assumptions are suitable for the decision-maker, in other cases, they simply are not. Simulation is an appropriate substitute for mathematical evaluation of a model in many situations.

Although it also involves assumptions, they are manageable. The use of simulation enables us to provide insight into certain management problems where mathematical evaluation of a model is not possible.

Among the reasons why management scientists would consider using simulation to solve management problems are the following:

- Simulation may be the only method available because it is difficult to observe the actual environment. (In space flight or the charting of satellite trajectories, it is widely used).
- It is not possible to develop a mathematical solution.
- Actual observation of a system may be too expensive. (The operation of a large computer centre under a number of different operating alternatives might be too expensive to be feasible).

There may not be sufficient time to allow the system to operate extensively. (If we were studying long-run trends in world population, for instance, we simply could not wait the required number of years to see results.)

Actual operation and observation of a system may be too disruptive. (If you are comparing two ways of providing food service in a hospital, the confusion that would result from operating two different systems for long enough to get valid observations might be too great).

7.3 LIMITATIONS OF SIMULATION

Use of simulation in place of other techniques, like everything else, involves a trade- off, and we should be mindful of the disadvantages involved in the simulation approach. These include the facts that

- Simulation is not precise. It is not optimization and does not yield an answer but merely provides a set of the system's responses to different operating conditions. In many cases, this lack of precision is difficult to measure.

- A good simulation model may-be very expensive. Often it takes years to develop a usable corporate planning model.

- Not all situations can be evaluated using simulation; only situations involving uncertainty are candidates, and without a random component, all simulated experiments would produce the same answer.

- Simulation generates a way of evaluating solutions but does not generate solutions themselves. Managers must still generate the solutions they want to test.

7.4 STEPS IN THE SIMULATION PROCESS

All effective simulations require a great deal of planning and organization. Although simulations vary in complexity from situation to situation, in general you will have to go through these steps:

1. Define the problem or system you intended to simulate.
2. Formulate the model you intend to use.
3. Test the model; compare its behaviour with the behaviour of the actual problem.

4. Identify and collect the data needed to test the model.
5. Run the simulation.
6. Analyze the results of the simulation and, if desired, change the solution you are evaluating.
7. Rerun the simulation to test the new solution.
8. Validate the simulation; this involves increasing the chances of the inferences you may draw about the real situation from running the simulation to become valid.

7.5 SIMULATION- QUEUE SYSTEM

Example: A dentist schedules all his patients for 30-minute appointments. Some of the patients take more 30 minutes some less, depending on the type of dental work to be done. The following summary shows the various categories of work, their probabilities and time actually needed to complete the work:

Category of Service	Time Required (minutes)	Probability of Category
Filling	45	0.40
Crown	60	0.15
Cleaning	15	0.15
Extraction	45	0.10
Checkup	15	0.20

Simulate the dentist's clinic for four hours and determine the average waiting time for the patients as well as the idleness of the doctor. Assume that all the patients show up at the clinic at exactly their scheduled arrival time starting at 8.00 a.m. Use the following random numbers for handling the above problem:

40 82 11 34 25 66 17 79 [AMIE, 2005]

Solution: The cumulative probability distribution and random number interval for service time are shown in Table:

Category of Service	**Time Required (minutes)**	**Probability of Category**	**Cumulative Probability**	**Random Number Interval**
Filling	45	0.40	0.40	00-39
Crown	60	0.15	0.55	40-54
Cleaning	15	0.15	0.70	55-69
Extraction	45	0.10	0.80	70-79
Checkup	15	0.20	1.00	80-99

The various parameters of a queuing system such as arrival pattern of customers, service time, waiting time, in the context of the given problem, are shown in below Tables:

Table: Arrival Pattern and Nature of Service

Patient Number	**Scheduled Arrival**	**Random Number**	**Category of Service**	**Service Time (Minute)**
1	8.00	40	Crown	60
2	8.30	82	Checkup	15
3	9.00	11	Filling	45
4	9.30	34	Filling	45
5	10.00	25	Filling	45
6	10.30	66	Cleaning	15
7	11.00	17	Filling	45
8	11.30	79	Extraction	45

Table: Computation of Arrivals, Departures and Waiting of Patients

Time	Event (Patient Number)	Patient Number (Time to Exit)	Waiting (Patient Number)
8.00	1 arrive	1 (60)	-
8.30	2 arrive	1 (30)	2
9.00	1 departs; 3 arrive	2 (15)	3
9.15	2 departs	3 (45)	-
9.30	4 arrive	3 (30)	4
10.00	3 departs; 3 arrive	4 (45)	5
10.30	6 arrive	4 (15)	5, 6
10.45	4 departs	5 (45)	6
11.00	7 arrive	5 (30)	6, 7
11.30	5 departs; 8 arrive	6 (15)	7, 8
11.45	6 departs	7 (45)	8
12.00	End	7 (30)	8

The dentist was not idle even once during the entire simulated period. The waiting times for the patients were as follows:

Table: Computation of Average Waiting Time

Patient Number	Arrival Time	Service Starts at	Waiting Time (Minute)
1	8.00	8.00	0
2	8.30	9.00	30
3	9.00	9.15	15
4	9.30	10.00	30
5	10.00	10.45	45
6	10.30	11.30	60
7	11.00	11.45	45
8	11.30	12.30	60
			280

Average Waiting Time = 280 / 8 = 25 minutes

7.6 SIMULATION- INVENTORY

Example: Using random numbers to simulate a sample, find the probability that a packet of 6products does not contain any defective product, when the production line produces 10 per cent defective products. Compare your answer with the expected probability.

Solution Given that 10 per cent of the total production is defective and 90 per cent is non-defective, if we have 100 random numbers (0 to 99), then 90 or 90 per cent of them represent non-defective products and the remaining 10 (or 10 per cent) of them represent defective products. Thus, the random numbers 00to 89 are assigned to variables that represent non-defective products and 90 to 100 are assigned to variables that represent defective products.

If we choose a set of 2-digit random numbers in the range 00 to 99 to represent a packet of 6 products as shown below, then we would expect that 90 per cent of the time they would fall in the range 00 to 89.

It may be noted that out of ten simulated samples 6 contain one or more defectives and 4 contain no defectives. Thus, the expected percentage of non-defective products is 40 per cent. However, theoretically the probability that a packet of 6 products containing no defective product, is $(0.9)^6 = 0.53144 = 53.14\%$.

Sample Number	**Random Number**					
A	83	02	22	57	51	68
B	39	77	32	77	09	79
C	28	06	24	25	93	22
D	97	66	63	99	61	80
E	69	30	16	09	05	53
F	33	63	99	19	87	26
G	87	14	77	43	96	43
H	99	53	93	61	28	52
I	93	86	52	77	65	15
J	18	46	23	34	25	85

Example: A book store wishes to carry a particular book in stock. The demand of the book is not certain and there is a lead time of 2 days for stock replenishment.

The probabilities of demand are given below:

Demand (units/day) :	0	1	2	3	4
Probability :	0.05	0.10	0.30	0.45	0.10

Each time an order is placed, the store incurs an ordering cost of Rs 10 per order. The store also incurs a carrying cost of Re 0.5 per book per day. The inventory carrying cost is calculated on the basis of stock at the end of each day. The manager of the book store wishes to compare two options for his inventory decision.

A : Order 5 books when the present inventory plus any outstanding order falls below 8 books.

B : Order 8 books when the present inventory plus any outstanding order falls below 8 books.

Currently (beginning of 1st day) the store has a stock of 8 books plus 6 books ordered two days ago and are expected to arrive the next day. Carryout simulation runs for 10 days to recommend an appropriate option. You may use random numbers in the sequences, using the first number for day one as given below:

89, 34, 78, 63, 61, 81, 39, 16, 13, 73 [*AMIE, 2005*]

Solution: Using the daily demand distribution, we obtain a probability distribution, as shown in Table of daily demand distribution.

Daily Demand	**Probability**	**Cumulative Probability**	**Random Number Intervals**
0	0.05	0.05	00-04
1	0.10	0.15	05-14
2	0.30	0.45	15-44
3	0.45	0.90	45-89
4	0.10	1.00	90-99

The stock in hand is of 8 books and stock on order is 5 books (expected next day).

Table: Optimal A

Random Number	Daily Demand	Opening Stock in Hand	Receipt	Closing Stock in Hand	Order Quantity	Closing Stock
89	3	8	-	8-3=5	5	5
34	2	5	5	5+5-2=8	-	8
78	3	8	-	8-3=5	5	5
63	3	5	5	10-3=7	-	7
61	3	7	-	7-3=4	5	4
81	3	4	5	9-3=6	-	6
39	2	6	-	6-2=4	5	4
16	2	4	5	9-2=7	-	7
13	1	7	-	7-1=6	-	6
73	3	6	-	6-3=3	-	3
						55

Since 5 books have been ordered four times as shown in Table above, therefore, the total ordering cost is Rs (4 × 10) = Rs 40.

Closing stock of 10 days is of 55 books. Therefore, the holding cost at the rate of Re 0.5 per book per day is Rs 55 × 0.5 = Rs 27.5

Total cost for 10 days = Ordering cost + Holding cost = Rs 40 + 27.5 = Rs 67.5

Table: Optimal B

Random Number	Daily Demand	Opening Stock in Hand	Receipt	Closing Stock in Hand	Order Quantity	Closing Stock
89	3	8	-	8-3=5	8	5
34	2	5	8	8+5-2=11	-	11
78	3	11	-	11-3=8	-	8
63	3	8	-	8-3=5	8	5
61	3	5	8	13-3=10	-	10
81	3	10	-	10-3=7	-	7
39	2	8	-	7-2=5	8	5
16	2	5	8	13-2=11	-	11
13	1	11	-	11-1=10	-	10
73	3	10	-	10-3=7	-	7
					71	

Eight books have been ordered three times, as shown in Table above, when the inventory of books at the beginning of the day plus outstanding orders is less than 8. Therefore, the total ordering cost is: Rs (3 × 10) = Rs 30.

Closing stock of 10 days is of 71 books. Therefore, the holding cost, Re 0.5 per book per day is Rs 71 × 0.5 = Rs 35.5

The total cost for 10 days = Rs (30 + 35.5) = Rs 65.5. Since option B has a lower total cost than option A, therefore, the manager should choose option B.

7.7 SIMULATION- DEMAND

Example: A bakery keeps stock of a popular brand of cake. Previous experience shows the daily demand pattern for the item with associated probabilities, as given below:

Daily demand (number) :	0	10	20	30	40	50
Probability:	0.01	0.20	0.15	0.50	0.12	0.02

Use the following sequence of random numbers to simulate the demand for next 10 days.

Random numbers: 25, 39, 65, 76, 12, 05, 73, 89, 19, 49.

Also estimate the daily average demand for the cakes on the basis of the simulated data.

Solution: Using the daily demand distribution, we first obtain a probability distribution as shown in Table below:

Table: Daily Demand Distribution

Daily Demand	Probability	Cumulative Probability	Random Number Intervals
0	0.01	0.01	00
10	0.20	0.21	01-20
20	0.15	0.36	21-35
30	0.50	0.86	36-85
40	0.12	0.98	86-97
50	0.02	1.00	98-99

Next to conduct the simulation experiment for demand take a sample of 10 random numbers from a table of random numbers, which represent the sequence of 10 samples. Each random sample number represents a sample of demand.

The simulation calculations for a period of 10 days are given in Table below:

Days	Random Number	Simulated Demand	
1	40	30	Because random number 40 falls in the interval 36-85
2	19	10	Because random number 19 falls in the interval 01-20 and so on
3	87	40	
4	83	30	
5	73	30	
6	84	30	
7	29	20	
8	09	10	
9	02	10	
10	20	10	
Expected Demand = 220/10 = 22 units per day			

7.8 SUMMARY

Simulation is a powerful and intuitive technique and uses computer to simulate the operation of an entire system or process. Random numbers are generated using a probability distribution to generate various outcomes over a period of time. These outcomes provide at a glance view of different configurations of the system at a least cost in comparison of actually operating the system. Hence, many alternative system configurations can be investigated and compared before selecting the most appropriate one to use.

Simulation approach has applications to a wide variety of areas such as queuing system, inventory system, and demand distribution, etc.

Spread sheet software is increasingly being used to perform basic computer simulations. The availability of such software enables decision makers to use simulation approach for solving real-life decision problems.

A simulation imitates the operation of real world processes or systems with the use of models. The model represents the key behaviours and characteristics of the selected process or system while the simulation represents how the model evolves under different conditions over time.

Simulations are usually computer-based, using a software-generated model to provide support for the decisions of managers and engineers as well as for training purposes. Simulation techniques aid understanding and experimentation, as the models are both visual and interactive.

Simulation systems include discrete event simulation, process simulation and dynamic simulation. Businesses may use all of these systems across different levels of the organisation.

7.9 EXERCISES

Example 1: A firm has a single channel service station with the following arrival and service time probability distributions:

Interarrival time in minutes	**Probability**	**Service time in minutes**	**Probability**
10	0.10	5	0.08
15	0.25	10	0.14
20	0.30	15	0.18
25	0.25	20	0.24
30	0.10	25	0.22
		30	0.14

The customer's arrival at the service station is a random phenomenon and the time between the arrivals varies from 10 to 30 minutes. The service time varies from 5 minutes to 30 minutes. The queuing process begins at 10 a.m. and proceeds for nearly 8 hours. An arrival immediately, goes to the service facility if it is free. Otherwise it waits in a queue. The queue discipline is first-come first-served. If the attendant's wages are Rs 10 per hour and the customer's waiting time costs Rs 15 per hour, then would it be an economical proposition to engage a second attendant? Answer using Monte Carlo simulation technique.

Solution: The cumulative probability distributions and random number interval, both for interarrival time and service time, are shown in the following Tables:

Table: Interarrival Time

Interarrival time in minutes	**Probability**	**Cumulative Probability**	**Random number interval**
10	0.10	0.10	00-09
15	0.25	0.35	10-34
20	0.30	0.65	35-64
25	0.25	0.90	65-89
30	0.10	1.00	90-99

Table: Service Time

Interarrival time in minutes	**Probability**	**Cumulative Probability**	**Random number interval**
5	0.08	0.08	00-07
10	0.14	0.22	08-21
15	0.18	0.40	22-39
20	0.24	0.64	40-63
25	0.22	0.86	64-85
30	0.14	1.00	86-99

Table: Single Server Queuing Simulation for 15 Arrivals

Arrival Number	Random Number	Interarrival Time (min)	Arrival Time (min)	Service Starts (min)	Waiting Time	Random Number	Service Time	Exit Time	Time in System
1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10= 6+8**
1	20	15	10.15	10.15	0	26	15	30	15
2	73	25	10.40	10.40	0	43	20	60	20
3	30	15	10.55	11.00	5	98	30	90	35
4	99	30	11.25	11.30	5	87	30	120	35
5	66	25	11.45	12.00	15	58	20	140	35
6	83	25	12.10	12.20	10	90	30	170	40
7	32	15	12.25	1.05	35	94	25	195	60
8	75	25	12.50	1.30	40	60	20	215	60
9	04	10	1.00	1.50	50	08	10	225	60
10	15	15	1.15	2.00	45	50	20	245	65
11	29	15	1.30	2.20	50	37	15	260	65
12	62	20	1.50	2.35	45	42	20	280	65
13	37	20	2.10	2.55	45	28	15	295	60
14	68	25	2.35	3.10	35	84	25	320	60
15	94	30	3.05	3.35	30	65	25	345	55

From the 15 samples of waiting time, 225 minutes, and the time spent, 545 minutes, by the customer in the system, we compute all average waiting time in the system and average service time as follows:

Average waiting time = 380/15 = 25.3 minutes.

Average service time = 545/15 = 36.33 minutes

Thus, the average cost of waiting and service is given by:

Cost of waiting = 15 × (25.30/60) = Rs 6.32 per hour

Cost of service = 10 × (36.33/60) = Rs 6.05 per hour

Since the average cost of service, per hour, is less than the average cost of waiting per hour, therefore second attendant may be hired.

Example 2: A company trading in motor vehicle spare parts wishes to determine the levels of stock it should carry for the items in its range. The demand is not certain and there is a lead time for stock replenishment. For an item A, the following information is obtained:

Demand (units/day) : 3 4 5 6 7

Probability : 0.10 0.20 0.30 0.30 0.10

Carrying cost (per unit/day) : Rs 2

Ordering cost (per order) : Rs 50

Lead time for replenishment : 3 days

Stock on hand at the beginning of the simulation exercise was 20 units.

Carry out a simulation run over a period of 10 days with the objective of evaluating the inventory rule:

Order 15 units when present inventory plus any outstanding order falls below 15 units.

You may use random numbers in the sequence of: 0, 9, 1, 1, 5, 1, 8, 6, 3, 5, 7, 1, 2, 9, using the first number for day one. Your calculation should include the total cost of operating this inventory rule for 10 days.
[*AMIE, 2004*]

Solution Let us begin simulation by assuming that:

(i) Orders are placed at the end of the day and received after 3 days, at the end of a day.

(ii) Back orders are accumulated in case of short supply and are supplied when stock is available.

The cumulative probability distribution and the random number range for daily demand is shown in Table below:

Table - Daily Demand Distribution

Daily Demand	**Probability**	**Cumulative Probability**	**Random Number Intervals**
3	0.10	0.10	00
4	0.20	0.30	01-02
5	0.30	0.60	03-05
6	0.30	0.90	06-08
7	0.10	1.00	09

The results of the simulation experiment conducted are shown in Table below:

Table - Simulation Experiments

Days	Opening Stock	Random Number	Resulting Demand	Closing Stock	Order Placed	Order Delivered	Average Stock in the Evening
1	20	0	3	17	-	-	18.5
2	17	9	7	10	15	-	13.5
3	10	1	4	6	-	-	8
4	6	1	4	2	-	-	4
5	2	5	5	0(-3)*	15	15	1
6	12	1	4	8	-	-	10
7	8	8	6	2	-	-	6
8	2	6	6	0(-4)*	15	15	1
9	11	3	5	6	-	-	8.5
10	6	5	5	1	-	-	3.5

* Negative figures indicate back orders.

Average ending stock = 78/10 = 7.8 units/day

Daily ordering cost = (Cost of placing one order) × (Number of orders placed per day) = 50 × 3 = Rs 150

Daily carrying cost = (Cost of carrying one unit for one day) × (Average ending stock) = 2 × 7.8 = Rs 15.60

Total daily inventory cost = Daily ordering cost + Daily carrying cost = 150 + 15.60 = Rs 165.60.

Example 3: The manager of a warehouse is interested in designing an inventory control system for one of the products in stock. The demand for the product comes from numerous retail outlets and the orders arrive on a weekly basis. The warehouse receives its stock from a factory but the lead time is not constant. The manager wants to determine the best time to release orders to the factory so that stockouts are minimized, yet the inventory holding costs are at acceptable levels. Any order from retailers, not supplied on a given day, constitute lost demand. Based on a sampling study, the following data are available:

Demand per week in thousand	Probability	Lead time	Probability
0	0.20	2	0.30
1	0.40	3	0.40
2	0.30	4	0.30
3	0.10		

The manager of the warehouse has determined the following cost parameters: Ordering cost (C_0) per order equals Rs 50, carrying cost (C_h) equals Rs 2 per thousand units per week, and shortage cost (C_s) equals Rs 10 per thousand units.

The objective of inventory analysis is to determine the optimal size of an order and the best time to place an order. The following ordering policy has been suggested.

Policy **:** Whenever the inventory level becomes less than or equal to 2,000 units (reorder level), an order equal to the difference between current inventory balance and the specified maximum replenishment level, is equal to 4,000 units, is placed.

Simulate the policy for a week's period assuming that the (i) the beginning inventory is 3,000 units, (ii) no back orders are permitted, (iii) each order is placed at the beginning of the week, as soon as the inventory level is less than or equal to the reorder level, and (iv) the replenishment orders are received at the beginning of the week.
[*AMIE, 2005*]

Solution:

Using weekly demand and lead time distributions, assign an appropriate set of random numbers to represent value (range) of variables as shown in below respective Tables:

Table: Probabilities and Random number Interval for Weekly Demand

Weekly demand (in thousand)	Probability	Cumulative probability	Random number interval
0	0.20	0.20	00-19
1	0.40	0.60	20-59
2	0.30	0.90	60-89
3	0.10	1.00	90-99

Table: Probabilities and Random number Interval for Lead Time

Lead time in weeks	Probability	Cumulative probability	Random number interval
2	0.30	0.30	00-29
3	0.40	0.70	30-69
4	0.30	1.00	70-99

The simulation experiment conducted for a 10 week period is shown in the next Table. The simulation process begins with an inventory level of 3,000 units. The following four steps occur in the simulation process:

1. Begin each simulation week by checking whether any order has just arrived. If it has, increase the beginning (current) stock (inventory) by the quantity received.

2. Generate a weekly demand from the demand probability distribution in weekly demand Table given above by selection of a random number. This random number is recorded in column 4. The demand simulated is recorded in column 5.The random number 31 generates a demand of 1,000 units when it is subtracted from the initial inventory level value of 3,000 units. It yields an ending inventory of 2,000 units at the end of the first week.

3. Compute the ending inventory every week and record it in column 7.

Ending inventory = Beginning inventory – Demand = 3,000 – 1,000 = 2,000

If on hand inventory is not sufficient to meet the week's demand, then record the number of units short in column 6.

4. Determine whether the week's ending inventory has reached the reorder level. If it has, and if thereis no outstanding order (back orders), then place an order.

Since the ending inventory of 2,000 units is equal to the reorder level, therefore, an order for4,000 – 2,000 = 2,000 units is placed.

5. The lead time for the new order is simulated by first choosing a random number and recording it in column 8. Finally, this random number is converted into a lead time (column 9) by using the lead time distribution in the earlier Table.

The random number 29 corresponds to a lead time of 2 weeks, with 2,000 units to be held (carried) in stock. Therefore, the holding cost of Rs 4 is paid and since there were no shortages, there is no shortage cost.

Summing these cost yields a total inventory cost (column 10) for week one of Rs 54.

The same step-by-step process is repeated for the remaining 10 weeks of the simulation experiment.

Analysis of Inventory Cost

Average ending inventory = (1,000 total unit / 10 weeks) = 100 units per week.

Average number of orders placed = (2 orders / 10 weeks) = 0.2 order per week.

Average number of lost sales = (7,000 / 1,000) = 7 units per week.

Total average inventory cost = Ordering cost + Holding cost + Shortage cost

= (Cost of placing one order) × (Number of orders placed per week)
+

(Cost of holding one unit for one week) × (Average ending inventory) + (Cost per lost sale) × (Average number of lost sales per week)

= (100 / 10) + (16 / 10) + (70 / 10)

= 10 + 1.6 + 7

= Rs 18.6

Maximum Inventory Level = 4,000 units Reorder Level = 2,000 units

Week	Order Receipt	Beginning Inventory	Random Number	Demand	Ending Inventory	Quantity Ordered	Random Number	Lead Time	Total Cost				
									C_0+	C_h+	C_s	=	TC (R.)
1	0	3000	31	1000	2000	2000	29	2	50	4	-	-	54
2	0	2000	73	2000	0	0	-	-	-	-	-	=	-
3	0	0	53	1000	(-1000)	0	-	-	0	0	10	=	10
4	2000	2000	86	2000	0	4000	83	4	50	-	-	=	50
5	0	0	32	1000	(-1000)	0					10	=	10
6	0	0	78	2000	(-2000)	0					20	=	20
7	0	0	26	1000	(-1000)	0					10	=	10
8	0	0	64	2000	(-2000)	0					20	=	20
9	4000	4000	45	1000	3000	0				6	-	=	6
10	0	3000	12	0	3000	0				6	-	=	6
			Total		1000				100	16	70		

The negative figures in Table above enclosed in brackets indicate loss of sales.

❖❖❖❖