Chapter 1

Multilinear Algebra

Unit Structure :

1.1 Objectives

After going through this chapter you will be able to:

1. Define a multilinear function, $k-$tensor, alternating tensor and wedge product.
2. Learn algebraic properties of alternating tensor and wedge product.
3. Identify basis and dimension of subspace of tensor.
4. Learn the concept of volume element.

1.2 $k-$tensor

Multilinear Function: If V is a vector space over $\mathbb{R}$, we will denote the $k-$fold product $V \times V \times ... \times V$ by V^k. A function $T : V^k \to \mathbb{R}$ is called multilinear if for each i with $1 \leq i \leq k$ we have

$$T(v_1, v_2, \cdots, v_i + v_i^{'} \ldots, v_k) = T(v_1, v_2, \cdots, v_i, \cdots, v_k) + T(v_1, v_2, \cdots, v_i^{'}, \cdots, v_k),$$

$$T(v_1, v_2, \cdots, av_i, \cdots, v_k) = aT(v_1, v_2, \cdots, v_i, \cdots, v_k).$$

Example: Consider the function $f : \mathbb{R}^3 \to \mathbb{R}$ defined as, $f(x, y, z) = xyz$. Show that f is $3-$linear.

Solution: We begin by fixing x and z and treat f as a function of one variable y.
Consider $f(x, \alpha y_1 + \beta y_2, z) = x(\alpha y_1 + \beta y_2)z$
$= x(\alpha y_1)z + x(\beta y_2)z$
$= \alpha x y_1 z + \beta x y_2 z$
$= \alpha f(x, y_1, z) + \beta f(x, y_2, z)$.
shows that f is linear in y.
Similarly we can show that f is linear in x and z variables.

$k-$**tensor:** A multilinear function $T : V^k \to \mathbb{R}$ is called a $k-$tensor on V and the set of all $k-$tensors denoted by $\Im^k(V)$, becomes a vector space over $\mathbb{R}$ if for $S, T \in \Im^k(V)$ and $a \in \mathbb{R}$ we define

$$(S+T)(v_1, v_2, \cdots, v_i, \cdots, v_k) = S(v_1, v_2, \cdots, v_i, \cdots, v_k) + T(v_1, v_2, \cdots, v_i, \cdots, v_k),$$

$$(aS)(v_1, v_2, \cdots, v_i, \cdots, v_k) = aS(v_1, v_2, \cdots, v_i, \cdots, v_k).$$

Tensor Product: There is an operation connecting the various spaces $\Im^k(V)$. If $S \in \Im^k(V)$ and $T \in \Im^l(V)$, we define the tensor product $S \otimes T \in \Im^{k+l}(V)$ by

$$S \otimes T(v_1, v_2, \cdots, v_k, v_{k+1}, \cdots, v_{k+l}) = S(v_1, v_2, \cdots, v_k) \cdot T(v_{k+1}, \cdots, v_{k+l}).$$

Note: The order of the factors S and T is crucial here since $S \otimes T$ and $T \otimes S$ are far from equal.

$$T \otimes S(v_1, v_2, \cdots, v_l, v_{l+1}, \cdots, v_{l+k}) = T(v_1, v_2, \cdots, v_l) \cdot S(v_{l+1}, \cdots, v_{l+k}).$$

Example: If $S_1, S_2 \in \Im^k(V)$,$T \in \Im^l(V)$, $U \in \Im^m(V)$ and $a \in \mathbb{R}$ then Show that
(1) $(S_1 + S_2) \otimes T = S_1 \otimes T + S_2 \otimes T$,
(2) $S \otimes (T_1 + T_2) = S \otimes T_1 + S \otimes T_2$,
(3) $(aS) \otimes T = S \otimes (aT) = a(S \otimes T)$,
(4) $(S \otimes T) \otimes U = S \otimes (T \otimes U)$.

Notes:
(1) Both $(S \otimes T) \otimes U$ and $S \otimes (T \otimes U)$ are usually denoted simply $S \otimes T \otimes U$.
(2) higher-order products $T_1 \otimes T_2 \otimes, \cdots \otimes T_r$ are defined similarly.
(3) The $\Im^1(V)$ is just the dual space V^*.

Note: Any vector space has a corresponding dual vector space (or dual space) consisting of all linear forms on. , together with the vector space structure of pointwise addition and scalar multiplication by constants.

Theorem-01: Let $v_1, \cdots, v_n$ be a basis for V, and let $\varphi_1, \varphi_2, \cdots \varphi_n$ be the dual basis, $\varphi_i(v_j) = \delta_{ij}$. Then the set of all $k-$fold tensor products

$$\varphi_{i_1} \otimes \varphi_{i_2} \otimes \cdots \otimes \varphi_{i_k}, \;\; 1 \leq i_1, \cdots, i_k \leq n$$

is a basis for $\Im^k(V)$, which therefore has dimension n^k.

Proof Note that

$$\varphi_{i_1} \otimes \varphi_{i_2} \otimes \cdots \otimes \varphi_{i_k}(v_{j_1}, v_{j_2}, \cdots, v_{j_k}) = \delta_{i_1,j_1} \cdot \delta_{i_2,j_2} \cdots \delta_{i_k,j_k}$$

$$= \begin{cases} 1 & \text{if } j_1 = i_1; \cdots; j_k = i_k, \\ 0 & \text{otherwise.} \end{cases}$$

Step I: Claim: $\varphi_{i_1} \otimes \varphi_{i_2} \otimes \cdots \otimes \varphi_{i_k}$ span $\Im^k(V)$.

If $w_1, w_2, \cdots, w_k$ are k vectors with $w_i = \sum_{j=1}^{n} a_{ij} v_j$ and T is in $\Im^k(V)$, then

$$T(w_1, w_2, \cdots, w_k) = \sum_{j_1, j_2, \cdots, j_k = 1}^{n} a_{1,j_1} \cdots \cdot a_{k,j_k} T(v_{j_1}, v_{j_2}, \cdots v_{j_k})$$

and

$$\varphi_{i_1} \otimes \varphi_{i_2} \otimes \cdots \otimes \varphi_{i_k}(w_1, , w_2, \cdots, w_k) = a_{1,j_1} \cdots \cdot a_{k,j_k} \varphi_{i_1} \otimes \varphi_{i_2} \otimes \cdots \otimes \varphi_{i_k}(v_{j_1}, v_{j_2}, \cdots v_{j_k})$$

$$\varphi_{i_1} \otimes \varphi_{i_2} \otimes \cdots \otimes \varphi_{i_k}(v_{j_1}, v_{j_2}, \cdots v_{j_k}) = \begin{cases} 1 & \text{if } j_1 = i_1; \cdots; j_k = i_k, \\ 0 & \text{otherwise.} \end{cases}$$

$$\Rightarrow \varphi_{i_1} \otimes \varphi_{i_2} \otimes \cdots \otimes \varphi_{i_k}(w_1, , w_2, \cdots, w_k) = a_{1,j_1} \cdots \cdot a_{k,j_k} \text{ if } j_1 = i_1; \cdots; j_k = i_k$$

This gives us

$$T(w_1, w_2, \cdots, w_k) = \sum_{i_1, i_2, \cdots, i_k = 1}^{n} T(v_{i_1}, v_{i_2}, \cdots v_{i_k}) \cdot \varphi_{i_1} \otimes \varphi_{i_2} \otimes \cdots \otimes \varphi_{i_k}(w_1, , w_2, \cdots, w_k).$$

Thus $T = \sum_{i_1, i_2, \cdots, i_k = 1}^{n} T(v_{i_1}, v_{i_2}, \cdots v_{i_k}) \cdot \varphi_{i_1} \otimes \varphi_{i_2} \otimes \cdots \otimes \varphi_{i_k}$.

Consequently the $\varphi_{i_1} \otimes \varphi_{i_2} \otimes \cdots \otimes \varphi_{i_k}$ span $\Im^k(V)$.

Step II: Claim: $\varphi_{i_1} \otimes \varphi_{i_2} \otimes \cdots \otimes \varphi_{i_k}$ is linearly independent

Suppose now that there are numbers $a_{i_1,i_2 \cdots i_k}$ such that

$$\sum_{i_1,i_2 \cdots i_k}^{n} a_{i_1,i_2 \cdots i_k} \varphi_{i_1} \otimes \varphi_{i_2} \otimes \cdots \otimes \varphi_{i_k} = 0.$$

Applying both sides of this equation to $(v_{j_1}, v_{j_2}, \cdots v_{j_k})$

$$\sum_{i_1,i_2 \cdots i_k}^{n} a_{i_1,i_2 \cdots i_k} \varphi_{i_1} \otimes \varphi_{i_2} \otimes \cdots \otimes \varphi_{i_k}(v_{j_1}, v_{j_2}, \cdots v_{j_k}) = 0.$$

This yields $a_{i_1,i_2 \cdots i_k} = 0$. Thus the $\varphi_{i_1} \otimes \varphi_{i_2} \otimes \cdots \otimes \varphi_{i_k}$ are lineraly independent.

hence by step I and II, we conclude

$$\varphi_{i_1} \otimes \varphi_{i_2} \otimes \cdots \otimes \varphi_{i_k}, \;\; 1 \leq i_1, \cdots, i_k \leq n$$

is a basis for $\Im^k(V)$, which therefore has dimension n^k.

Example: Determine which of the following are tensors on $\mathbb{R}^4$ and express those in terms of elementary tensors.

$$f(x, y, z) = 3x_1y_2z_3 - x_3y_1z_4$$
$$g(x, y, z) = 2x_1x_2z_3 + x_3y_1z_4$$

Solution:
(a) f is a 3-tensor since it is linear with respect to each variable x, y, z. (Verify)
If ω^1, ω^2, ω^3, ω^4 is the dual basis of the standard basis e_1, . . . , e_4 in $\mathbb{R}^4$, then

$$f = 3\omega^1 \otimes \omega^2 \otimes \omega^3 - \omega^3 \otimes \omega^1 \otimes \omega^4.$$

(b) g is not a tensor since g is not linear as

$$g(ax, y, z) = 2ax_1ax_2z_3 + ax_3y_1z_4 = 2a^2x_1x_2z_3 + ax_3y_1z_4 \neq ag(x, y, z).$$

Example: Consider the following tensors on $\mathbb{R}^4$,

$$f(x, y, z) = 2x_1y_2z_2 \text{ - } x_2y_3z_1$$
$$g(x, y) = \omega^2 \otimes \omega^1 \text{ - } 2\omega^3 \otimes \omega^1$$

where $\{\omega^1, \omega^2, \omega^3, \omega^4\}$ is the dual basis of the standard basis $\{e_1, . . . , e_4\}$ for $\mathbb{R}^4$. Write $f \otimes g$ as a linear combination of elementary 5-tensors.

Solution: (b) Since $f = 2\omega^1 \otimes \omega^2 \otimes \omega^2$ - $\omega^2 \otimes \omega^3 \otimes \omega^1$.
$f \otimes g$
$= (2\omega^1 \otimes \omega^2 \otimes \omega^2$ - $\omega^2 \otimes \omega^3 \otimes \omega^1) \otimes (\omega^2 \otimes \omega^1$-$2\omega^3 \otimes \omega^1)$
$= 2\omega^1 \otimes \omega^2 \otimes \omega^2 \otimes \omega^2 \otimes \omega^1$ - $4\omega^1 \otimes \omega^2 \otimes \omega^2 \otimes \omega^3 \otimes \omega^1 + \omega^2 \otimes \omega^3 \otimes \omega^1 \otimes \omega^2 \otimes \omega^1$ - $2\omega^2 \otimes \omega^3 \otimes \omega^1 \otimes \omega^3 \otimes \omega^1$.

Dual Transformation: If $f : V \to W$ is a linear transformation, a linear transformation
$f^* : \Im^k(W) \to \Im^k(V)$ is defined by

$$f^*T(v_1, v_2, \cdots, v_k) = T(f(v_1), f(v_2), \cdots, f(v_k))$$

for $T \in \Im^k(W)$ and $v_1, v_2, \cdots, v_k \in V$.

Examples:

(1) Show that $f^*(S \otimes T) = f^*S \otimes f^*T$.

(2) Show that an inner product on V to be a 2-tensor or $\langle\ \rangle \in \Im^2(\mathbb{R}^n)$.

Definition: We define an inner product on V to be a 2-tensor T such that
T is symmetric, that is $T(v, w) = T(w, v)$ for $v, w \in V$ and
T is positive-definite, that is $T(u, v) > 0$ if $v \neq 0$.
We distinguish $\langle\, ,\, \rangle$ as the usual inner product on $\mathbb{R}^n$.

Theorem-02: If T is an inner product on V, there is a basis $v_1, v_2, \cdots, v_n$ for V such that $T(v_i, v_j) = \delta_{ij}$.(Such a basis is called orthonormal with respect to T.) Consequently there is an isomorphism $f : \mathbb{R}^n \to V$ such that $T(f(x), f(y)) = \langle x, y\rangle$ for $x, y \in \mathbb{R}^n$. In other words $f^*T = \langle\, ,\, \rangle$.

Proof Let $w_1, w_2, \cdots, w_n$ be any basis of V. Define

$$\begin{aligned}
w_1' &= w_1, \\
w_2' &= w_2 - \frac{T(w_1', w_2)}{T(w_1', w_1')} \cdot w_1', \\
w_3' &= w_3 - \frac{T(w_1', w_3)}{T(w_1', w_1')} \cdot w_1' - \frac{T(w_2', w_3)}{T(w_2', w_2')} \cdot w_2', \\
&\text{etc.}
\end{aligned}$$

It is easy to check that $T(w_i', w_j') = 0$ if $i \neq j$ and

$w_i' \neq 0$ so that $T(w_i', w_i') > 0$.

Now define $v_i = \dfrac{w_i^{'}}{\sqrt{T(w_i^{'}, w_i^{'})}}$.

The isomorphism f may be defined by $f(e_i) = v_i$.

Now Consider $f^*T(e_i, e_j) = T(f(e_i), f(e_i)) = T(v_i, v_j) = \delta_{ij} = \langle e_i, e_j \rangle$.

1.3 Alternating Tensor

Alternating Tensor: A $k-$tensor $\omega \in \Im^k(V)$ is called alternating if

$$\omega(v_1, v_2, \cdots, v_i, \cdots, v_j, \cdots, v_k) = -\omega(v_1, v_2, \cdots, v_j, \cdots, v_i, \cdots, v_k) \;\; \forall v_1, v_2, \cdots, v_k \in V.$$

(In this equation v_i and v_j are interchanged and all other v's are left fixed.) The set of all alternating $k-$ tensors is clearly a subspace $\Lambda^k(V)$ of $\Im^k(V)$.

Note: A $k-$tensor $\omega \in \Im^k(V)$ is called symmetric if

$$\omega(v_1, v_2, \cdots, v_i, \cdots, v_j, \cdots, v_k) = \omega(v_1, v_2, \cdots, v_j, \cdots, v_i, \cdots, v_k) \;\; \forall v_1, v_2, \cdots, v_k \in V.$$

Definition: If $T \in \Im^k(V)$, we define $\mathrm{Alt}(T)$ by

$$\mathrm{Alt}(T)(v_1, v_2, \cdots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}\ \sigma \cdot T(v_{\sigma(1)}, v_{\sigma(2)}, \cdots, v_{\sigma(k)}),$$

where S_k is the set of all permutations of the numbers 1 to k.

Note: Recall that the sign of a permutation σ denoted sgn σ, is $+1$ if σ is even and -1 is σ is odd.

Theorem-03

(1) If $T \in \Im^k(V)$, then $\mathrm{Alt}(T) \in \Lambda^k(V)$.
(2) If $\omega \in \Lambda^k(V)$, then $\mathrm{Alt}(\omega) = \omega$.
(3) If $T \in \Im^k(V)$, then $\mathrm{Alt}(\mathrm{Alt}(T)) = \mathrm{Alt}(T)$.

Proof (1) Let (i, j) be the permutation that interchanges i and j and leaves all other numbers fixed. If $\sigma \in S_k$, let $\sigma^{'} = \sigma \cdot (i, j)$. Then

$$\mathrm{Alt}(T)(v_1, v_2, \cdots, v_j, \cdots, v_i, \cdots, v_k)$$

$$= \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}\ \sigma \cdot T(v_{\sigma(1)}, v_{\sigma(2)}, \cdots, v_{\sigma(j)}, \cdots, v_{\sigma(i)}, \cdots, v_{\sigma(k)}),$$

$$= \frac{1}{k!} \sum_{\sigma \in S_k} \text{sgn } \sigma \cdot T(v_{\sigma'(1)}, v_{\sigma'(2)}, \cdots, v_{\sigma'(i)}, \cdots, v_{\sigma'(j)}, \cdots, v_{\sigma'(k)}),$$

$$= \frac{1}{k!} \sum_{\sigma' \in S_k} -\text{sgn } \sigma' \cdot T(v_{\sigma'(1)}, v_{\sigma'(2)}, \cdots, v_{\sigma'(k)}),$$

$$= -\text{Alt}(T)(v_1, v_2, \cdots, v_k),$$

(2) If $\omega \in \Lambda^k(V)$ and $\sigma = (i, j)$, then

$$\omega(v_{\sigma(1)}, v_{\sigma(2)}, \cdots, v_{\sigma(k)}) = \text{sgn } \sigma \cdot \omega(v_1, v_2, \cdots, v_k).$$

Since every σ is a product of permutations of the form (i, j), this equation holds for all σ. Therefore

$$\text{Alt } \omega(v_1, v_2, \cdots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \text{sgn } \sigma \cdot \omega(v_{\sigma(1)}, v_{\sigma(2)}, \cdots, v_{\sigma(k)})$$

$$= \frac{1}{k!} \sum_{\sigma \in S_k} \text{sgn } \sigma \cdot \text{sgn } \sigma \cdot \omega(v_1, v_2, \cdots, v_k)$$

$$= \omega(v_1, v_2, \cdots, v_k).$$

(3) follows immediately from (1) and (2).(Exercise)

1.4 Wedge product

Wedge product: If $\omega \in \Lambda^k(V)$ and $\eta \in \Lambda^l(V)$, then $\omega \otimes \eta$ is usually not in $\Lambda^{k+l}(V)$. We will therefore define a new product, the wedge product $\omega \wedge \eta \in \Lambda^{k+l}(V)$ by

$$\omega \wedge \eta = \frac{(k+l)!}{k!l!} \text{Alt}(\omega \otimes \eta).$$

Example: Show that
(1) $(\omega_1 + \omega_2) \wedge \eta = \omega_1 \wedge \eta + \omega_2 \wedge \eta$,
(2) $\omega \wedge (\eta_1 + \eta_2) = \omega \wedge \eta_1 + \omega \wedge \eta_2$,
(3) $a\omega \wedge \eta = \omega \wedge a\eta = a(\omega \wedge \eta)$,
(4) $\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega$,
(5) $f^*(\omega \wedge \eta) = f^*(\omega) \wedge f^*(\eta)$,
(6) $(\omega \wedge \eta) \wedge \theta = \omega \wedge (\eta \wedge \theta)$.

Theorem-04

(1) If $S \in \Im^k(V)$ and $T \in \Im^l(V)$ and $\mathrm{Alt}(S) = 0$, then
$$\mathrm{Alt}(S \otimes T) = \mathrm{Alt}(T \otimes S) = 0.$$

(2) $\mathrm{Alt}(\mathrm{Alt}(\omega \otimes \eta) \otimes \theta) = \mathrm{Alt}(\omega \otimes \eta \otimes \theta) = \mathrm{Alt}(\omega \otimes \mathrm{Alt}(\eta \otimes \theta))$.

(3) If $\omega \in \Lambda^k(V),\ \eta \in \Lambda^l(V)$ and $\theta \in \Lambda^m(V)$, then
$$(\omega \wedge \eta) \wedge \theta = \omega \wedge (\eta \wedge \theta) = \frac{(k+l+m)!}{k!l!m!}\mathrm{Alt}(\omega \otimes \eta \otimes \theta).$$

Proof: (1) **Step I: Claim:** $\mathrm{Alt}(S \otimes T) = 0$

$$\mathrm{Alt}(S\otimes T)(v_1, v_2, \cdots, v_{k+l}) = \frac{1}{(k+l)!}\sum_{\sigma \in S_{k+l}} \mathrm{sgn}\sigma \cdot (S\otimes T)(v_{\sigma(1)}, v_{\sigma(2)}, \cdots, v_{\sigma(k+l)}).$$

$$\begin{aligned}&(k+l)!\mathrm{Alt}(S \otimes T)(v_1, v_2, \cdots, v_k + l)\\ &= \sum_{\sigma \in S_{k+l}} \mathrm{sgn}\sigma \cdot S(v_{\sigma(1)}, v_{\sigma(2)}, \cdots, v_{\sigma(k)}) \cdot T(v_{\sigma(k+1)}, v_{\sigma(k+2)}, \cdots, v_{\sigma(k+l)}).\end{aligned} \tag{1}$$

Case I: If $G \subset S_{k+l}$ consists of all σ whcih leave $k+1, k+2, \cdots, k+l$ fixed, then

$$\begin{aligned}&\sum_{\sigma \in G} \mathrm{sgn}\sigma \cdot S(v_{\sigma(1)}, v_{\sigma(2)}, \cdots, v_{\sigma(k)}) \cdot T(v_{\sigma(k+1)}, v_{\sigma(k+2)}, \cdots, v_{\sigma(k+l)})\\ &= \sum_{\sigma' \in S_k} \mathrm{sgn}\sigma' \cdot S(v_{\sigma'(1)}, v_{\sigma'(2)}, \cdots, v_{\sigma'(k)}) \cdot T(v_{(k+1)}, v_{(k+2)}, \cdots, v_{(k+l)})\\ &= 0. \qquad (\text{Since } \mathrm{Alt}(S) = 0)\end{aligned}$$

Hence by equation (1), $\mathrm{Alt}(S \otimes T) = 0$

Case II: Suppose $\sigma_0 \notin G$.
Let $G \cdot \sigma_0 = \{\sigma \cdot \sigma_0 : \sigma \in G\}$ and
let $v_{\sigma_0(1)}, v_{\sigma_0(2)}, \cdots, v_{\sigma_0(k+l)} = w_1, w_2 \cdots, w_{k+l}$. Then

$$\begin{aligned}&\sum_{\sigma \in G\cdot\sigma_0} \mathrm{sgn}\sigma \cdot S(v_{\sigma(1)}, v_{\sigma(2)}, \cdots, v_{\sigma(k)}) \cdot T(v_{\sigma(k+1)}, v_{\sigma(k+2)}, \cdots, v_{\sigma(k+l)})\\ &= \left[\mathrm{sgn}\ \sigma_0 \cdot \sum_{\sigma' \in G} \mathrm{sgn}\ \sigma' \cdot S(w_{\sigma'(1)}, w_{\sigma'(2)}, \cdots, w_{\sigma'(k)}) \cdot\right] \cdot T(w_{k+1}, w_{k+2}, \cdots, w_{k+l})\\ &= 0. \qquad (\text{Since } \mathrm{Alt}(S) = 0)\end{aligned}$$

Hence by equation (1), $\mathrm{Alt}(S \otimes T) = 0$

Notice that $G \cap G \cdot \sigma_0 = \Phi$.
In fact, if $\sigma \in G \cap G \cdot \sigma_0$, then $\sigma = \sigma' \cdot \sigma_0$ for some $\sigma' \in G$ and $\sigma_0 = \sigma \cdot (\sigma')^{-1} \in G$, a contradiction.

We can then continue in this way, breaking S_{k+l} up into disjoint subsets; the sum over each subset is 0, so that the sum over S_{k+l} is 0. Hence $\mathrm{Alt}(S \otimes T) = 0$.

Step II: Claim: $\mathrm{Alt}(T \otimes S) = 0$ Show similarly as step I. Combining step I and II, we obtain
$\mathrm{Alt}(S \otimes T) = \mathrm{Alt}(T \otimes S) = 0$.

(2) **Step I: Claim:** $\mathrm{Alt}(\omega \otimes \eta \otimes \theta) = \mathrm{Alt}(\omega \otimes \mathrm{Alt}(\eta \otimes \theta))$
Consider $\mathrm{Alt}(\mathrm{Alt}(\eta \otimes \theta) - \eta \otimes \theta) = \mathrm{Alt}\{\mathrm{Alt}(\eta \otimes \theta)\} - \mathrm{Alt}(\eta \otimes \theta)$.
By theorem (3(III)), we have $\mathrm{Alt}\{\mathrm{Alt}(\eta \otimes \theta)\} = \mathrm{Alt}(\eta \otimes \theta)$,
hence we have

$$\mathrm{Alt}(\mathrm{Alt}(\eta \otimes \theta) - \eta \otimes \theta) = \mathrm{Alt}(\eta \otimes \theta) - \mathrm{Alt}(\eta \otimes \theta) = 0.$$

Hence by (1) we have

$$\mathrm{Alt}(\omega \otimes [\mathrm{Alt}(\eta \otimes \theta) - \eta \otimes \theta]) = 0$$

$$\mathrm{Alt}(\omega \otimes \mathrm{Alt}(\eta \otimes \theta)) - Alt(\omega \otimes \eta \otimes \theta) = 0$$

$$\mathrm{Alt}(\omega \otimes \mathrm{Alt}(\eta \otimes \theta)) = Alt(\omega \otimes \eta \otimes \theta)$$

Step II: Claim: $\mathrm{Alt}(\mathrm{Alt}(\omega \otimes \eta) \otimes \theta) = \mathrm{Alt}(\omega \otimes \eta \otimes \theta)$
Similarly as per step I.

(3) **Step I: Claim:** $(\omega \wedge \eta) \wedge \theta = \dfrac{(k+l+m)!}{k!l!m!}\mathrm{Alt}(\omega \otimes \eta \otimes \theta)$.

By definition of wedge product have

$$(\omega \wedge \eta) \wedge \theta = \frac{(k+l+m)!}{(k+l)!m!}\mathrm{Alt}((\omega \wedge \eta) \otimes \theta)$$

again applying definition of wedge product have

$$(\omega \wedge \eta) \wedge \theta = \frac{(k+l+m)!}{(k+l)!m!}\mathrm{Alt}\{(\frac{(k+l)!}{k!l!}\mathrm{Alt}(\omega \otimes \eta)) \otimes \theta\}$$

$$(\omega \wedge \eta) \wedge \theta = \frac{(k+l+m)!}{(k+l)!m!}\frac{(k+l)!}{k!l!}\mathrm{Alt}\{\mathrm{Alt}(\omega \otimes \eta) \otimes \theta\}$$

By 2 above

$$(\omega \wedge \eta) \wedge \theta = \frac{(k+l+m)!}{k!l!m!}\mathrm{Alt}(\omega \otimes \eta \otimes \theta)$$

Step II: Claim: $\omega \wedge (\eta \wedge \theta) = \dfrac{(k+l+m)!}{k!l!m!}\mathrm{Alt}(\omega \otimes \eta \otimes \theta)$.

Similarly as per step I.

Note: (1) $\omega \wedge (\eta \wedge \theta) = (\omega \wedge \eta) \wedge \theta = \omega \wedge \eta \wedge \theta$
and higher-order products $\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_r$ are defined similarly.
(2) If an alternating tensor ω and η are of odd order then $\omega \wedge \eta = -\eta \wedge \omega$
(3) If an alternating tensor ω is of odd order then $\omega \wedge \omega = 0$

Example: Consider the following tensors on $\mathbb{R}^5$

$$f(x,y,z) = 3x_2y_2z_1 \text{ - } x_1y_5z_4$$
$$g(x) = 2x_1 + x_3$$

(a) Write Alt f as a linear combination of elementary alternating tensors.
(b) Write (Alt f) $\wedge\, g$ as a linear combination of elementary alternating tensors.

Solution:
(a) Recall that if $I = (i_1,..., i_k)$ is an multi-index and

$$\omega^{i_1} \wedge \cdots \wedge \omega^{i_k} = \omega^I := k!Alt(\omega^{i_1} \otimes \cdots \otimes \omega^{i_k}) \tag{1.1}$$

Hence write f as a linear combination of elementary tensors,

$$f = 3\omega^2 \otimes \omega^2 \otimes \omega^1 - \omega^1 \otimes \omega^5 \otimes \omega^4$$

Then by equation (2),

Alt f $= 3\mathrm{Alt}(\omega^2 \otimes \omega^2 \otimes \omega^1) - Alt(\omega^1 \otimes \omega^5 \otimes \omega^4)$
$= \frac{3}{3!}\omega^2 \wedge \omega^2 \wedge \omega^1 - \frac{1}{3!}\omega^1 \wedge \omega^5 \wedge \omega^4$
$= -\frac{1}{3!}\omega^1 \wedge \omega^5 \wedge \omega^4$
$= \frac{1}{3!}\omega^1 \wedge \omega^4 \wedge \omega^5$

(b) Since $g = 2\omega^1 + \omega^3$ so that
(Alt f) $\wedge\, g = \frac{1}{3!}\omega^1 \wedge \omega^4 \wedge \omega^5 \wedge (2\omega^1 + \omega^3)$
$= \frac{1}{3!}\omega^1 \wedge \omega^4 \wedge \omega^5 \wedge \omega^3$
$= \text{-}\frac{1}{3!}\omega^1 \wedge \omega^4 \wedge \omega^3 \wedge \omega^5$
$= \frac{1}{3!}\omega^1 \wedge \omega^3 \wedge \omega^4 \wedge \omega^5$

Example 2: Let X_1, X_2, ... , $X_k \in$ V and let φ^1, ... ,$\varphi^k \in V^*$. Show that $\varphi^1\wedge$ ... $\wedge\varphi^k(X_1$, X_2, ... , $X_k) = \det[\varphi^i(X_j)]$

Solution:
By definition,

$\varphi^1 \wedge \ldots \wedge \varphi^k(X_1, X_2, \ldots , X_k) = \frac{(1+\cdots+1)!}{1!\cdots1!} \text{Alt}(\varphi^1 \otimes \cdots \otimes \varphi^k)(X_1, X_2, \ldots , X_k)$

$= k! \text{ Alt}(\varphi^1 \otimes \cdots \otimes \varphi^k)(X_1, X_2, \ldots , X_k)$

$= \frac{k!}{k!} \sum_{\sigma \in S_k} (\text{sign } \sigma)\, \varphi^1(X_{\sigma(1)})\, \varphi^2(X_{\sigma(2)}) \cdots \varphi^k(X_{\sigma(k)})$

$$= \det \begin{bmatrix} \varphi^1(X_1) & \ldots & \varphi^1(X_k) \\ . & & \\ . & & \\ . & & \\ \varphi^k(X_1) & \ldots & \varphi^k(X_k) \end{bmatrix}$$

1.5 Basis for $\Lambda^k(V)$

Theorem-05: The set of all

$$\varphi_{i_1} \wedge \varphi_{i_2} \wedge \cdots \wedge \varphi_{i_k}, \;\; 1 \le i_1, i_2, \cdots, i_k \le n$$

is a basis for $\Lambda^k(V)$, which therefore has dimension

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Proof: Step I: Claim: $\varphi_{i_1} \wedge \varphi_{i_2} \wedge \cdots \wedge \varphi_{i_k}, \;\; 1 \le i_1, i_2, \cdots, i_k \le n$ spans $\Lambda^k(V)$.
Let $v_1, v_2, \cdots v_n$ be a basis for V and let $\varphi_1, \varphi_2, \cdots \varphi_n$ be the dual basis. If $\omega \in \Lambda^k(V) \subset \Im^k(V)$, then we can write

$$\omega = \sum_{i_1, i_2, \cdots i_k} a_{i_1, i_2, \cdots i_k} \varphi_{i_1} \otimes \varphi_{i_2} \otimes \cdots \otimes \varphi_{i_k}.$$

Thus by theorem $3(II)$, we have

$$\omega = \text{Alt}(\omega) = \sum_{i_1, i_2, \cdots i_k} a_{i_1, i_2, \cdots i_k} \text{Alt}(\varphi_{i_1} \otimes \varphi_{i_2} \otimes \cdots \otimes \varphi_{i_k}).$$

Since by definition of wedge product, each $\text{Alt}(\varphi_{i_1} \otimes \varphi_{i_2} \otimes \cdots \otimes \varphi_{i_k})$ is a constant times one of the $(\varphi_{i_1} \wedge \varphi_{i_2} \wedge \cdots \wedge \varphi_{i_k})$, these elements span $\Lambda^k(V)$.

Step II: Claim: $\varphi_{i_1} \wedge \varphi_{i_2} \wedge \cdots \wedge \varphi_{i_k}, \;\; 1 \le i_1, i_2, \cdots, i_k \le n$ is linearly independent.
Linear independence is proved as in Theorem-01.

Step III: Claim: Dimension of $\Lambda^k(V)$ is $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$.
As $\Lambda^k(V)$ is set of all alternating $k-$ tensors which is subspace of $\Im^k(V)$, clearly Dimension of $\Lambda^k(V)$ is $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$

Note: If V has dimension n, it follows from Theorem-05 that $\Lambda^n(V)$ has dimension 1.

Example: Let V be a vector space of dimension $n = 3$. The space of alternating 2-tensors $\Lambda^2(V^*)$ has the dimension

$$\dim \Lambda^2(V^*) = \binom{3}{2} = \frac{3!}{2!(3-2)!} = 3$$

Theorem-06: Let $v_1, v_2, \cdots v_n$ be a basis for V and let $\omega \in \Lambda^n(V)$. If $\omega_i = \sum_{j=1}^{n} a_{ij} v_j$ are n vectors in V then

$$\omega(w_1, w_2, \cdots, w_n) = \det(a_{ij}) \cdot \omega(v_1, v_2, \cdots, v_n).$$

Proof: Define $\eta \in \Im^n(\mathbb{R}^n)$ by
$\eta((a_{11}, a_{12}, \cdots, a_{1n}), (a_{21}, a_{22}, \cdots, a_{2n}), \cdots, (a_{n1}, a_{n2}, \cdots, a_{nn})$
$= \omega\left(\sum a_{1_j} v_j, \sum a_{2_j} v_j, \cdots, \sum a_{n_j} v_j\right)$ As $\omega \in \Lambda^n(V)$ clearly $\eta \in \Lambda^n(\mathbb{R}^n)$ so $\eta = \lambda \cdot \det(a_{ij})$ for some $\lambda \in \mathbb{R}$ and

$$\lambda = \eta(e_1, e_2, \cdots, e_n) = \omega(v_1, v_2, \cdots, v_n).$$

$$\omega(w_1, w_2, \cdots, w_n) = \det(a_{ij}) \cdot \omega(v_1, v_2, \cdots, v_n).$$

1.6 Volume Element of V

Orientation: Theorem-06 shows that a non zero $\omega \in \Lambda^n(V)$ splits the bases of V into two disjoint groups, those with $\omega(v_1, v_2, \cdots, v_n) > 0$ and those for which $\omega(v_1, v_2, \cdots, v_n) < 0$; if $v_1, v_2, \cdots, v_n$ and $w_1, w_2, \cdots, w_n$ are two bases and $A = (a_{ij})$ is defined by $w_i = \sum a_{ij} v_j$ then $v_1, v_2, \cdots, v_n$ and $w_1, w_2, \cdots, w_n$ are in the same group if and only if $\det A > 0$.

This criterion is independent of ω and can always be used to divide the bases of V into two disjoint groups. Either of these two groups is called an orientation for V. The orientation to which a basis $v_1, v_2, \cdots, v_n$ belongs is denoted by $[v_1, v_2, \cdots, v_n]$ and the other orientation is denoted $-[v_1, v_2, \cdots, v_n]$.

Note: In $\mathbb{R}^n$ we define the usual orientation as $[e_1, e_2, \cdots, e_n]$.

Volume Element: The fact that $\dim \Lambda^n(\mathbb{R}^n) = 1$ is obvious since det is often defined as the unique element $\omega \in \Lambda^n(\mathbb{R}^n)$ such that $\omega(e_1, e_2, \cdots, e_n) = 1$. By theorem 6

$$\omega(w_1, w_2, \cdots, w_n) = \det(a_{ij}) \cdot \omega(e_1, e_2, \cdots, e_n).$$

$$\omega(w_1, w_2, \cdots, w_n) = \det(a_{ij})$$

Suppose that an inner product T for V is given. If $v_1, v_2, \cdots, v_n$ and $w_1, w_2, \cdots, w_n$ are two bases which are orthonormal with respect to T, and the matrix $A = (a_{ij})$ is defined by $w_i = \sum_{j=1}^{n} a_{ij} v_j$, then

$$\begin{aligned}
\delta_{ij} &= T(w_i, w_j) \\
&= T(\sum_{k=1}^{n} a_{ik} v_k, \sum_{l=1}^{n} a_{il} v_l) \\
&= \sum_{k,l=1}^{n} a_{ik} a_{jl} T(v_k, v_l) \\
&= \sum_{k,l=1}^{n} a_{ik} a_{jl} \delta_{kl} \\
&= \sum_{k=1}^{n} a_{ik} a_{jk}.
\end{aligned}$$

In other words, if A^T denotes the transpose of the matirix A, then we have $A \cdot A^T = I$, so $\det(A) = \pm 1$.
It follows from Theorem-06 that if $\omega \in \Lambda^n(V)$ satisfies $\omega(v_1, v_2, \cdots, v_n) = \pm 1$, then $\omega(w_1, w_2, \cdots, w_n) = \pm 1$. If an orientation μ for V has also been given, it follows that there is a unique $\omega \in \Lambda^n(V)$ such that $\omega(v_1, v_2, \cdots, v_n) = 1$ whenever $v_1, v_2, \cdots, v_n$ is an orthonormal basis such that $[v_1, v_2, \cdots, v_n] = \mu$.
Note that det is the volume element of $\mathbb{R}^n$ determined by the usual inner product and usual orientation and that $|\det(v_1, v_2, \cdots, v_n)|$ is the volume of the paralleopiped spanned by the line segments from 0 to each of $v_1, v_2, \cdots, v_n$.

Volume Element of $\mathbb{R}^n$: If $v_1, v_2, \cdots, v_{n-1} \in \mathbb{R}^n$ and φ is defined by

$$\varphi(w) = \det \begin{pmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ \cdot \\ v_{n-1} \\ w \end{pmatrix},$$

Then $\varphi \in \Lambda^1(V)$. Therefore there is a unique element $z \in \mathbb{R}^n$ such that

$$\langle w, z\rangle = \varphi(w) = \det \begin{pmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ \cdot \\ v_{n-1} \\ w \end{pmatrix},$$

This z is the denoted $v_1 \times v_2 \times \cdots \times v_{n-1}$ and called the cross product of $v_1, v_2, \cdots, v_{n-1}$.

The following properties are immediate from the definition:
(1) $v_{\sigma(1)} \times v_{\sigma(2)} \times \cdots \times v_{\sigma(n-1)} = \operatorname{sgn}\sigma \cdot v_1 \times v_2 \times \cdots \times v_{n-1}$,
(2) $v_1 \times v_2 \times \cdots \times av_i \times \cdots \times v_{n-1} = a \cdot (v_1 \times v_2 \times \cdots \times v_{n-1})$,
(3) $v_1 \times v_2 \times \cdots \times (v_i + v_i^{'}) \times \cdots \times v_{n-1} = (v_1 \times v_2 \times \cdots \times v_i \times \cdots \times v_{n-1}) + (v_1 \times v_2 \times \cdots \times v_i^{'} \times \cdots \times v_{n-1})$.

1.7 Chapter End Exercise

1. Let $T \in \Im^k(W)$ and $S \in \Im^l(W)$. Show that $f^*(S \otimes T) = f^*S \otimes f^*T$ where f^* is a dual transformation of a linear transformation $f : V \to W$.

2. Let V be a vector space of dimension 5. Find the dimension of the space of alternating $3-$tensor $\Lambda^3(V)$. Justify your answer.

3. Let $\omega \in \Lambda^2(V)$, $\eta \in \Lambda^3(V)$ and $\theta \in \Lambda^4(V)$. Find the wedge product $(\omega \wedge \eta) \wedge \theta$ in terms of alternating tensor of tensor product of ω, η and θ.

4. Let $S \in \Lambda^k(V)$ and $T \in \Lambda^l(V)$ and $\operatorname{Alt}(T) = 0$ then compute $T \wedge S$.

5. Let V be a vector space of dimension 3. Find the dimension of the space of alternating $2-$tensor $\Lambda^2(V)$. Justify your answer.

6. Let $\omega \in \Lambda^1(V)$, $\eta \in \Lambda^2(V)$ and $\theta \in \Lambda^3(V)$. Find the wedge product $(\omega \wedge \eta) \wedge \theta$ in terms of alternating tensor of tensor product of ω, η and θ.

7. Prove or disprove: An inner product on vector space V to be a 2-tensor.

8. If $T \in \Im^k(V)$, then show that $\mathrm{Alt}(\mathrm{Alt}(T)) = \mathrm{Alt}(T)$.

9. If $\omega \in \Lambda^k(V),\ \eta \in \Lambda^l(V)$ and $\theta \in \Lambda^m(V)$, then show that

$$(\omega \wedge \eta) \wedge \theta = \frac{(k+l+m)!}{k!l!m!}\mathrm{Alt}(\omega \otimes \eta \otimes \theta).$$

Chapter 2

Differential Forms

Unit Structure :

2.1 Objectives

After going through this chapter you will be able to:

1. Learn the concept of tangent space.
2. Define Differential Forms and Pullback Forms.
3. Learn properties of Pullback Forms.

2.2 Basic Preliminaries

1. The Del operator:

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right).$$

2. Gradient:
Suppose f is a function. ∇f is the gradient of f, sometimes denoted grad f.

$$\text{grad } f = \nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}.$$

Example: Compute the gradient of $f(x,y,z) = xye^{y^2z}$
Solution: ∇f = $\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$ = y$e^{y^2z}\hat{i}$+(xe^{y^2z}+ 2x$y^2e^{y^2z}$)$\hat{j}$+$\hat{k}$(x$y^3e^{y^2z}$).

3. Directional derivative
Definition: The directional derivative of f in the direction $\vec{u}$, denoted by $D_{\vec{u}}f$, is defined to be,

$$D_{\vec{u}}f = \frac{\nabla f \cdot \vec{u}}{|\vec{u}|}$$

Example: What is the directional derivative of $f(x,y) = x^2 + xy$, in the direction of $\vec{i}$+2$\vec{j}$ at the point (1, 1)?
Solution: Now we first find ∇f.
∇f = $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$ =$(2x + y, x)$
=(3,1)
Let $\vec{u} = \vec{i} + 2\vec{j}$
$|\vec{u}| = \sqrt{1^2 + 2^2} = \sqrt{5}$.
$D_{\vec{u}}f = \dfrac{\nabla f \cdot \vec{u}}{|\vec{u}|} = \dfrac{(3,1)\cdot(1,2)}{\sqrt{5}} = \sqrt{5}$.

• Properties of the gradient deduced from the formula of Directional derivatives

$$D_{\vec{u}}f = \frac{\nabla f \cdot \vec{u}}{|\vec{u}|} = \frac{|\nabla f||\vec{u}|cos(\theta)}{|\vec{u}|} = |\nabla f|cos(\theta)$$

1. If $\theta = 0$, i.e. $\vec{u}$ points in the same direction as ∇ f, then $D_{\vec{u}}f$ is maximum. Therefore we may conclude that,
(i) ∇f points in the steepest direction.
(ii) The magnitude of ∇f gives the slope in the steepest direction.

2. At any point P, $\nabla f(P)$ is perpendiular to level set through that point.

4. Divergence:
Definition: The Divergence is given by,

$$\text{div } \vec{F} = \nabla \cdot \vec{F}$$

where $\vec{F}$ should be vector field.

Example. Compute the divergence of $\vec{F}$ = (x^2+y)$\hat{i}$ + (y^2-z)$\hat{j}$ + (z^2+x)$\hat{k}$

Solution: div $\vec{F} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \cdot ((x^2\text{+y})\hat{i} + (y^2\text{-z})\hat{j} + (z^2\text{+x})\hat{k})$
$= 2x + 2y + 2z$.

5. Curl:
Definition: The curl is given by,

$$\text{curl } \vec{F} = \nabla \times \vec{F}$$

More specifically, suppose $\vec{F} = (F_1, F_2, F_3)$. Then

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

The cross product of two vectors is a vector, so curl takes a vector field to another vector field.

Example. Compute the curl of $\vec{F} = (x^2+y)\hat{i} + (y^2\text{-}z)\hat{j} + (z^2\text{+x})\hat{k}$

Solution: curl $\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$

$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 + y & y^2 - z & z^2 + x \end{vmatrix}$

$= \hat{i}\text{-}\hat{j}\text{+}\hat{k} = (1, \text{-}1, 1)$.

Example. Show that curl grad $f = \vec{0}$
Solution: curl grad $f = \nabla \times \nabla\ f$

$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \end{vmatrix}$

$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \end{vmatrix}(f)$.

But the determinant of a matrix with two equal rows is 0, so the result is $\vec{0}$.

Example. div(curl $\vec{F}$) = 0
Solution: div(curl $\vec{F}$) = $\nabla \cdot (\nabla \times f)$

$$= \nabla \cdot \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$
$$= \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$
$= 0.$

Example. Find Curl(∇f) and Div(∇f)
Solution: Curl(∇f) $= \nabla \times \nabla f$
$= (f_{yz}$ - $f_{zy})\ \hat{i} + (f_{zx}$ - $f_{xz})\ \hat{j} + (f_{xy}$ - $f_{yx})\ \hat{k}$
$= 0$

Div(∇f) $= \nabla \cdot \nabla f$
$$= (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \cdot (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$$
$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

2.3 Fields and Forms

If $p \in \mathbb{R}^n$, the set of all pairs (p, v), for $v \in \mathbb{R}^n$, is denoted $\mathbb{R}^n_p$, and called the tangent space of $\mathbb{R}^n$ at p. This set is made into a vector space in the most obvious way, by defining

$$(p, v) + (p, w) = (p, v + w),$$
$$a \cdot (p, v) = (p, av).$$

Vector Field: A vector field is a function F such that $F(p) \in \mathbb{R}^n_p$, for each $p \in \mathbb{R}^n$. For each p there are numbers $F^1(p), F^2(p), \cdots, F^n(p)$ such that

$$F(p) = F^1(p) \cdot (e_1)_p + F^2(p) \cdot (e_2)_p + \cdots, F^n(p) \cdot (e_n)_p.$$

We thus obtain n component functions $F^i : \mathbb{R}^n \to \mathbb{R}$.

Note: (1) The vector field F is called continuous, differentiable etc., if the functions F^i are.
(2) A vector field defined only on an open subset of $\mathbb{R}^n$.
(3) Operations on vectors yield operations on vector field when applied

at each point separately. For example if F and G are vector fields and f is a function, we define

$$\begin{aligned}
&(F+G)(p) = F(p) + G(p),\\
&\langle F, G\rangle(p) = \langle F(p), G(p)\rangle,\\
&(f \cdot F)(p) = f(p)F(p).
\end{aligned}$$

If $F_1, F_2, \cdots, F_{n-1}$ are vector fields on $\mathbb{R}^n$, then we can similarly define

$$(F_1 \times F_2 \times \cdots \times F_{n-1})(p) = F_1(p) \times F_2(p) \times \cdots \times F_{n-1}(p).$$

Gradient, Divergence and Curl: Introduce the formal symbolism

$$\nabla = \sum_{i=1}^{n} D_i \cdot e_i.$$

The gradient of a scalar field f is defined as $\text{Grad} f = \nabla f$.
The divergence of a vector field F is defined as $\text{Div} F = \sum\limits_{i=1}^{n} D_i F^i$.
we can write, symbolically, $\text{Div} F = \langle \nabla, F\rangle$.
The curl of a vector field F is defined as $\text{Curl} F = \nabla \times F$.
If $n = 3$ we write, in conformity with this symbolism,

$$(\nabla \times F)(p) = (D_2F^3 - D_3F^2)(e_1)_p + (D_3F^1 - D_1F^3)(e_2)_p + (D_1F^2 - D_2F^1)(e_3)_p.$$

2.4 Differential Forms

Differential Forms or $k-$Forms: A function ω with $\omega(p) \in \Lambda^k(\mathbb{R}^n_p)$ is called a $k-$form on $\mathbb{R}^n$, or simply a differential form where $\Lambda^k(\mathbb{R}^n_p)$ be the set of all alternating $k-$ tensors which is a subspace of $\Im^k(\mathbb{R}^n_p)$ and $\mathbb{R}^n_p$ tangent space of $\mathbb{R}^n$ at p.
If $\varphi_1(p), \varphi_2(p), \cdots, \varphi_n(p)$ is the dual basis to $(e_1)_p, (e_2)_p, \cdots, (e_n)_p$, then

$$\omega(p) = \sum_{i_1 < i_2 < \cdots < i_k} \omega_{i_1, i_2, \cdots, i_k} \cdot \left[\varphi_{i_1}(p) \wedge \varphi_{i_2}(p) \wedge \cdots \wedge \varphi_{i_k}(p)\right],$$

for certain functions $\omega_{i_1}, \omega_{i_2}, \cdots, \omega_{i_k}$.

Note:
1. The form ω is continuous, differentiable, etc. if these functions $\omega_{i_1}, \omega_{i_2}, \cdots, \omega_{i_k}$ are continuous, differentiable, etc.
2. Let ω and η be two $k-$ forms then the sum $(\omega + \eta)(p) = \omega(p) + \eta(p)$.
3. The product $(f \cdot \omega)(p) = f \cdot \omega(p)$ and $f \cdot \omega$ is also written as $f \wedge \omega$.
4. Let ω be $k-$ form and and η be $l-$ forms then wedge product $\omega \wedge \eta$ is $(k + l)-$ form given by $(\omega \wedge \eta)(p) = \omega(p) \wedge \eta(p)$.

5. A arbitrary real valued function f is considered to be a $0-$form.

Differential Forms or $k-$Forms for a function $f : \mathbb{R}^n \to \mathbb{R}$: If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable, then $Df(p) \in \Lambda^1(\mathbb{R}^n)$ i.e. $Df(p)$ is $1-$form. A $1-$form df, defined by

$$df(p)(v_p) = Df(p)(v) \tag{2.1}$$

Let us consider in particular the $1-$forms $d\pi^i$.
Let x^i denote the function π^i.
Since

$$dx^i(p)(v_p) = d\pi^i(p)(v_p) = D\pi^i(p)(v) = v^i \tag{2.2}$$

We see that $dx^1(p), dx^2(p), \cdots, dx^n(p)$ is just the dual basis to $(e_1)_p, (e_2)_p, \cdots, (e_n)_p$.
Thus every $k-$form ω can be written

$$\omega = \sum_{i_1 < i_2 < \cdots i_k} \omega_{i_1 i_2 \cdots i_k} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k} \tag{2.3}$$

Note: Thus $\omega = \sum\limits_{i_1} \omega_{i_1} dx^{i_1}$ is $1-$form.
$\omega = \sum\limits_{i_1 < i_2} \omega_{i_1 i_2} dx^{i_1} \wedge dx^{i_2}$ is $2-$form.
$\omega = \sum\limits_{i_1 < i_2 < i_3} \omega_{i_1 i_2 i_3} dx^{i_1} \wedge dx^{i_2} \wedge dx^{i_3}$ is $3-$form and etc.

Theorem-07: If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable , then

$$df = D_1 f \cdot dx^1 + D_2 f \cdot dx^2 + \cdots D_n f \cdot dx^n.$$

In classical notation, $df = \dfrac{\partial f}{\partial x^1} dx^1 + \dfrac{\partial f}{\partial x^2} x^2 + \cdots + \dfrac{\partial f}{\partial x^n} dx^n$
Proof:

$$df(p)(v_p) = Df(p)(v_p) = \sum_{i=1}^{n} D_i f(p) \cdot v^i \quad \text{by equation 1}$$

$$df(p)(v_p) = \sum_{i=1}^{n} D_i f(p) \cdot dx^i(p)(v_p) \qquad \text{by equation 2}$$

This gives

$$df = D_1 f \cdot dx^1 + D_2 f \cdot dx^2 + \cdots D_n f \cdot dx^n \tag{2.4}$$

2.5 Pullback Forms

Differential Forms or $k-$Forms for a function $f : \mathbb{R}^n \to \mathbb{R}^m$: Pullback Forms : Consider a differentiable function $f : \mathbb{R}^n \to \mathbb{R}^m$ we have a linear transformation $Df(p) : \mathbb{R}^n \to \mathbb{R}^m$. Another minor modification therfore produces a linear transformation $f_* : \mathbb{R}^n_p \to \mathbb{R}^m_{f(p)}$ defined by

$$f_*(v_p) = (Df(p)(v))_{f(p)} \tag{2.5}$$

This linear transformation induces a linear transformation $f^* : \Lambda^k(\mathbb{R}^m_{f(p)}) \to \Lambda^k(\mathbb{R}^n_p)$. If ω is a $k-$form on $\mathbb{R}^m$ we can therefore define a $k-$form $f^*\omega$ on $\mathbb{R}^n$ by

$$(f^*\omega)(p) = f^*(\omega(f(p))) \tag{2.6}$$

i.e. if $v_1, v_2, \cdots, v_k \in \mathbb{R}^n_p$ then

$$f^*\omega(p)(v_1, v_2, \cdots, v_k) = \omega(f(p)(f_*(v_1), \cdots, f_*(v_k)) \tag{2.7}$$

Thus if ω is a $k-$form on $\mathbb{R}^m$, it can be pullback to $\mathbb{R}^n$ by $f^*\omega$ then $f^*\omega$ is an alternating $k-$tensor on $\mathbb{R}^n_p$ and hence $f^*\omega$ is $k-$form on $\mathbb{R}^n$ and is known as pullback form of ω by f

Theorem-08: If $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable, then

(1) $f^*(dx^i) = \sum\limits_{j=1}^{n} D_j f^i \cdot dx^j = \sum\limits_{j=1}^{n} \dfrac{\partial f^i}{\partial x^j} dx^j$.

(2) $f^*(\omega_1 + \omega_2) = f^*(\omega_1) + f^*(\omega_2)$.

(3) $f^*(g \cdot \omega) = (g \circ f) \cdot f^*\omega$.

(4) $f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta$.

Proof(1)

$f^*(dx^i)(p)(v_p) = (dx^i)(f(p))(f_*v_p)$ by equation 7

$= (dx^i)(f(p))(Df(p)(v))_{f(p)}$ by equation 5

$$= (dx^i)(f(p)) \left[\sum_{j=1}^{n} v^j \cdot D_j f^1(p), \sum_{j=1}^{n} v^j \cdot D_j f^2(p), \cdots, \sum_{j=1}^{n} v^j \cdot D_j f^m(p) \right]_{f(p)}$$

$$= \sum_{j=1}^{n} v^j \cdot D_j f^i(p)$$

$$= \sum_{j=1}^{n} D_j f^i(p) \cdot dx^j(p)(v_p) \quad \text{by equation 2}$$

Thus

$$f^*(dx^i) = \sum_{j=1}^{n} D_j f^i \cdot dx^j = \sum_{j=1}^{n} \frac{\partial f^i}{\partial x^j} dx^j \tag{2.8}$$

(2) Let ω_1 and ω_2 be $k-$forms. Consider

$$f^*(\omega_1+\omega_2)(p)(v_1,v_2,\cdots,v_k) = (\omega_1+\omega_2)(f(p))(f_*(v_1),\cdots,f_*(v_k)) \quad \text{by equation 7}$$
$$= \omega_1(f(p))(f_*(v_1),\cdots,f_*(v_k)) + \omega_2(f(p))(f_*(v_1),\cdots,f_*(v_k))$$
$$= f^*(\omega_1) + f^*(\omega_2)$$

(3) Consider

$$f^*(g\cdot\omega)(p)(v_1,v_2,\cdots,v_k) = (g\cdot\omega)(f(p))(f_*(v_1),\cdots,f_*(v_k)) \quad \text{by equation 7}$$
$$= \omega[g(f(p))](f_*(v_1),\cdots,f_*(v_k)) \quad \text{since g is 0-form}$$
$$= \omega[g\circ f(p)](f_*(v_1),\cdots,f_*(v_k))$$
$$= (g\circ f)\cdot f^*\omega$$

(4) Let ω be $k-$ form and and η be $l-$ forms then wedge product $\omega\wedge\eta$ is $(k+l)-$ form given by $(\omega\wedge\eta)(p) = \omega(p)\wedge\eta(p)$.
Consider

$$f^*(\omega\wedge\eta)(p)(v_1,\cdots,v_k,v_{k+1},\cdots,v_{k+l})$$
$$= (\omega\wedge\eta)(f(p))(f_*(v_1),\cdots,f_*(v_k),f_*(v_{k+1}),\cdots,f_*(v_{k+l})) \quad \text{by equation 7}$$
$$= \omega(f(p))(f_*(v_1),\cdots,f_*(v_k))\wedge\eta(f(p))(f_*(v_{k+1}),\cdots,f_*(v_{k+l}))$$
$$= f^*\omega\wedge f^*\eta$$

Theorem-09: If $f:\mathbb{R}^n\to\mathbb{R}^n$ is differentiable, then

$$f^*(hdx^1\wedge dx^2\wedge\cdots\wedge dx^n) = (h\circ f)(\det f^{'})(dx^1\wedge dx^2\wedge\cdots dx^n).$$

Proof: By theorm $8(III)$, we can write,

$$f^*(hdx^1\wedge dx^2\wedge\cdots\wedge dx^n) = (h\circ f)f^*(dx^1\wedge dx^2\wedge\cdots dx^n).$$

then it suffices to show that

$$f^*(dx^1\wedge dx^2\wedge\cdots\wedge dx^n) = (\det f^{'})dx^1\wedge dx^2\wedge\cdots dx^n.$$

Let $p\in\mathbb{R}^n$ and let $A=(a_{ij})$ be the matrix of $f^{'}(p)$. For convenience we shall omit "p". Then

$$f^*(dx^1\wedge dx^2\wedge\cdots\wedge dx^n)(e_1,e_2,\cdots,e_n)$$
$$= dx^1\wedge dx^2\wedge\cdots\wedge dx^n(f_*e_1,f_*e_2,\cdots,f_*e_n) \quad \text{by equation 7}$$
$$= dx^1\wedge dx^2\wedge\cdots\wedge dx^n(Df_1(e_i),Df_2(e_i),\cdots,Df_n(e_i)) \quad \text{by equation 5}$$
$$= dx^1\wedge dx^2\wedge\cdots\wedge dx^n\left(\sum_{i=1}^n a_{i1}e_i,\sum_{i=1}^n a_{i2}e_i,\cdots,\sum_{i=1}^n a_{in}e_i\right)$$

$$= \det(a_{ij})\cdot dx^1\wedge dx^2\wedge\cdots\wedge dx^n(e_1,e_2,\cdots,e_n) \quad \text{by theorem 6}$$
$$= \det(f^{'})\cdot dx^1\wedge dx^2\wedge\cdots\wedge dx^n(e_1,e_2,\cdots,e_n)$$

Example 1: Let $\omega = xydx + 2zdy - ydz \in \Omega^k(\mathbb{R}^3)$ and α: $\mathbb{R}^2 \to \mathbb{R}^3$ is defined as $\alpha(u, v) = (uv, u^2, 3u + v)$. Calculate $\alpha^*\omega$.

Solution: Instead of thinking of α as a map, think of it as a substition of varibles:
$x = uv, y = u^2, z = 3u + v$
Then,
$dx = \dfrac{\partial x}{\partial u}du + \dfrac{\partial x}{\partial v}dv = vdu + udv$ and similarly,
$dy = 2udu$ and $dz = 3du + dv$
Consider,
$\omega = xydx + 2zdy - ydz = (uv)(u^2)\ (vdu + udv) + 2(3u + v)2udu$ - $u^2(3du + dv)$
$= (u^3v^2 + 9u^2 + 4uv)\ du + (u^4v$ - $u^2)\ dv$
We conclude that,
$\alpha^*\omega = \alpha^*(xydx + 2zdy - ydz) = (u^3v^2 + 9u^2 + 4uv)du + (u^4$v - $u^2)$ dv.

Example 2: Consider a map F: $\mathbb{R}^3 \to \mathbb{R}^2$ given as,

$$F(x, y, z) = (x^2{+}yz,\ e^{xyz})$$

and 2 form $\omega = uv^3\ du \wedge dv$ on $\mathbb{R}^2$. Then calculate $F^*\omega$.

Solution: $F^*\omega = (x^2{+}yz)e^{3xyz}\ d(x^2{+}yz) \wedge de^{xyz}$
$= (x^2{+}yz)e^{3xyz}\ (2xdx + zdy + ydz) \wedge$ (yz$e^{xyz}dx{+}xz\ e^{xyz}dy + xye^{xyz}dz)$
$= (x^2{+}yz)e^{4xyz}(2x^2zdx \wedge dy{+}2x^2ydx \wedge dz{+}z^2ydy \wedge dx{+}xyzdy \wedge dz + y^2zdz \wedge dx + xyz\ dz \wedge dy)$
$= (x^2{+}yz)e^{4xyz}((2x^2z$-$yz^2)dx \wedge dy + (2x^2y$-$zy^2)dx \wedge dz)$.

2.6 Chapter End Exercise

1. In $\mathbb{R}^3$, let $\omega = xydx + 2zdy - ydz$ and $\alpha : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be given by $\alpha(u, v) = (uv, u^2, 3u + v)$. Calculate $\alpha^*(\omega)$.

2. If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable then show that $df = \dfrac{\partial f}{\partial x^1}dx^1 + \dfrac{\partial f}{\partial x^2}x^2 + \cdots + \dfrac{\partial f}{\partial x^n}dx^n$

Chapter 3

Exterior Derivatives

Unit Structure :
3.1 Objective
3.2 Exterior Derivative
3.3 Closed and Exact Forms
3.4 Chapter End Exercise

3.1 Objectives

After going through this chapter you will be able to:

1. Define and calculate Exterior Derivative.
2. Learn properties of Exterior Derivative.
3. Identify closed and exact forms.
4. Learn the concept of Star Shaped Set.

3.2 Exterior Derivatives

The operator d which changes $0-$forms into $1-$forms. If

$$\omega = \sum_{i_1<i_2<i_3\cdots i_k} \omega_{i_1,i_2,\cdots i_k} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}$$

be a given $k-$form, we define a $(k+1)-$form $d\omega$ which is the differential of ω, by

$$d\omega = \sum_{i_1<i_2<i_3\cdots i_k} d\omega_{i_1,i_2,\cdots i_k} \wedge dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}$$

$$d\omega = \sum_{i_1,i_2,\cdots i_k} \sum_{\alpha=1}^{n} D_\alpha(\omega_{i_1,i_2,\cdots i_k}) \cdot dx^\alpha \wedge dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k} \qquad (3.1)$$

Theroem-10

(1) $d(\omega + \eta) = d\omega + d\eta$.

(2) If ω is a $k-$form and η is a $l-$form, then $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$.

(3) Cocycle condition: $d(d\omega) = 0$. Briefly, $d^2 = 0$.

(4) If ω is a $k-$form on $\mathbb{R}^m$ and $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable, then $f^*(d\omega) = d(f^*\omega)$.

Proof: (1) Let ω and η are $k-$form. From equation (3), We have

$$\omega = \sum_{i_1<i_2<i_3\cdots<i_k} \omega_{i_1,i_2,\cdots i_k} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}$$

and

$$\eta = \sum_{i_1<i_2<i_3\cdots i_k} \eta_{i_1,i_2,\cdots<i_k} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}$$

From equation (9), We have

$$d\omega = \sum_{i_1<i_2<\cdots<i_k} \sum_{\alpha=1}^{n} D_\alpha(\omega_{i_1,i_2,\cdots i_k}) \cdot dx^\alpha \wedge dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}$$

$$d\eta = \sum_{i_1<i_2<\cdots<i_k} \sum_{\alpha=1}^{n} D_\alpha(\eta_{i_1,i_2,\cdots i_k}) \cdot dx^\alpha \wedge dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}$$

$\Rightarrow$

$$d(\omega+\eta) = \sum_{i_1<i_2<\cdots<i_k} \sum_{\alpha=1}^{n} D_\alpha(\omega_{i_1,i_2,\cdots i_k} + \eta_{i_1<i_2<\cdots<i_k}) \cdot dx^\alpha \wedge dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}$$

$$d(\omega + \eta) = \sum_{i_1<i_2<\cdots<i_k} \sum_{\alpha=1}^{n} D_\alpha(\omega_{i_1,i_2,\cdots i_k}) \cdot dx^\alpha \wedge dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}$$

$$+ \sum_{i_1<i_2<\cdots<i_k} \sum_{\alpha=1}^{n} D_\alpha(\eta_{i_1,i_2,\cdots i_k}) \cdot dx^\alpha \wedge dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}$$

$$d(\omega + \eta) = d(\omega) + d(\eta)$$

(2) Let ω is a $k-$form and η is a $l-$form.
Claim: $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$.

Case I: Let ω and η both are $0-$form. Then $\omega = f$ and $\eta = g$ for some scalar field f and g. Consider

$$\begin{aligned} d(\omega \wedge \eta) = d(f \wedge g) &= \sum_{i=1}^{n} D_i(f \cdot g)dx^i \\ &= \sum_{i=1}^{n}(D_i f) \cdot g dx^i + \sum_{i=1}^{n} f \cdot (D_i g)dx^i \\ &= (df) \wedge g + f \wedge (dg) \\ &= (df) \wedge g + (-1)^0 f \wedge (dg) \end{aligned}$$

Case II: If $\omega = dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}$ and
$\eta = dx^{j_1} \wedge dx^{j_2} \wedge \cdots \wedge dx^{j_l}$ then
since $D(1) = 0$ all terms vanish, formula is true.

Case III: Let ω is a $0-$form and η is a $l-$form.
Since ω is a $0-$form, let $\omega = f$, for some scalar field f.
Since η is a $l-$form, we have

$$\eta = \sum_{j_1<j_2<j_3\cdots<j_l} \eta_{j_1,j_2,\cdots j_l} dx^{j_1} \wedge dx^{j_2} \wedge \cdots \wedge dx^{j_l}$$

$$\begin{aligned} & d(\omega \wedge \eta) = d(f \wedge \eta) = d(f \cdot \eta) \\ &= \sum_{j_1<j_2<j_3\cdots<j_l} \sum_{\beta=1}^{n} D_\beta(f \cdot \eta_{j_1,j_2,\cdots j_l})dx^\beta \wedge dx^{j_1} \wedge dx^{j_2} \wedge \cdots \wedge dx^{j_l} \\ &= \sum_{j_1<j_2<j_3\cdots<j_l} \sum_{\beta=1}^{n}[(D_\beta f) \cdot \eta_{j_1,j_2,\cdots j_l} + f \cdot (D_\beta \eta_{j_1,j_2,\cdots j_l})]dx^\beta \wedge dx^{j_1} \wedge dx^{j_2} \wedge \cdots \wedge dx^{j_l} \end{aligned}$$

$$\begin{aligned} &= \sum_{j_1<j_2<j_3\cdots<j_l} \sum_{\beta=1}^{n}[(D_\beta f) \cdot \eta_{j_1,j_2,\cdots j_l} dx^\beta \wedge dx^{j_1} \wedge dx^{j_2} \wedge \cdots \wedge dx^{j_l} \\ &\quad + f \cdot (D_\beta \eta_{j_1,j_2,\cdots j_l})dx^\beta \wedge dx^{j_1} \wedge dx^{j_2} \wedge \cdots \wedge dx^{j_l}] \\ &= df \wedge \eta + f \wedge d\eta \\ &= df \wedge \eta + (-1)^0 f \wedge d\eta \end{aligned}$$

Case IV: Let ω is a $k-$form and η is a $l-$form. Let ω is $k-$form, We have

$$\omega = \sum_{i_1<i_2<i_3\cdots<i_k} \omega_{i_1,i_2,\cdots i_k} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}$$

Since η is a $l-$form, we have

$$\eta = \sum_{j_1<j_2<j_3\cdots<j_l} \eta_{j_1,j_2,\cdots j_l} dx^{j_1} \wedge dx^{j_2} \wedge \cdots \wedge dx^{j_l}$$

$\Rightarrow$

$$\omega \wedge \eta = (\sum_{i_1<i_2<i_3\cdots<i_k} \omega_{i_1,i_2,\cdots i_k} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k})$$
$$\wedge(\sum_{j_1<j_2<j_3\cdots<j_l} \eta_{j_1,j_2,\cdots j_l} dx^{j_1} \wedge dx^{j_2} \wedge \cdots \wedge dx^{j_l})$$

$\Rightarrow$

$$\omega\wedge\eta = \sum_{i_1<i_2<i_3\cdots<i_k} \sum_{j_1<j_2<j_3\cdots<j_l} (\omega_{i_1,i_2,\cdots i_k} \cdot \eta_{j_1,j_2,\cdots j_l}) dx^{i_1}\wedge dx^{i_2}\wedge\cdots\wedge dx^{i_k}\wedge dx^{j_1}\wedge dx^{j_2}\wedge\cdots\wedge dx^{j_l}$$

$$d(\omega \wedge \eta) = \sum_{i_1<i_2<\cdots<i_k} \sum_{j_1<j_2<\cdots<j_l} \sum_{\alpha=1}^{n} D_\alpha(\omega_{i_1,i_2,\cdots i_k} \cdot \eta_{j_1,j_2,\cdots j_l})$$
$$dx^\alpha \wedge dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k} \wedge dx^{j_1} \wedge dx^{j_2} \wedge \cdots \wedge dx^{j_l}$$
$$= \sum_{i_1<i_2<\cdots<i_k} \sum_{j_1<j_2<\cdots<j_l} \sum_{\alpha=1}^{n} [D_\alpha(\omega_{i_1,i_2,\cdots i_k}) \wedge (\eta_{j_1,j_2,\cdots j_l}) + (\omega_{i_1,i_2,\cdots i_k}) \wedge D_\alpha(\eta_{j_1,j_2,\cdots j_l})]$$
$$dx^\alpha \wedge dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k} \wedge dx^{j_1} \wedge dx^{j_2} \wedge \cdots \wedge dx^{j_l}$$
$$= \sum_{i_1<i_2<\cdots<i_k} \sum_{j_1<j_2<\cdots<j_l} \sum_{\alpha=1}^{n} [D_\alpha(\omega_{i_1,i_2,\cdots i_k}) \wedge (\eta_{j_1,j_2,\cdots j_l})$$
$$dx^\alpha \wedge dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k} \wedge dx^{j_1} \wedge dx^{j_2} \wedge \cdots \wedge dx^{j_l}$$
$$+ (\omega_{i_1,i_2,\cdots i_k}) \wedge D_\alpha(\eta_{j_1,j_2,\cdots j_l}) dx^\alpha \wedge dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k} \wedge dx^{j_1} \wedge dx^{j_2} \wedge \cdots \wedge dx^{j_l}]$$

$$= \sum_{i_1<i_2<\cdots i_k} \sum_{j_1<j_2<\cdots j_l} \sum_{\alpha=1}^{n} [D_\alpha(\omega_{i_1,i_2,\cdots i_k}) dx^\alpha \wedge dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}]$$
$$\wedge [(\eta_{j_1,j_2,\cdots j_l}) dx^{j_1} \wedge dx^{j_2} \wedge \cdots \wedge dx^{j_l}]$$
$$+ (-1)^k [(\omega_{i_1,i_2,\cdots i_k}) dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}] \wedge [D_\alpha(\eta_{j_1,j_2,\cdots j_l}) dx^\alpha \wedge dx^{j_1} \wedge dx^{j_2} \wedge \cdots \wedge dx^{j_l}]$$
$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$$

The sign $(-1)^k$ added since $dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}$ is $k-$form and $D_\alpha(\eta_{j_1,j_2,\cdots j_l})$ is $1-$form.

(3) Let ω is $k-$form. From equation (3), We have

$$\omega = \sum_{i_1<i_2<i_3\cdots i_k} \omega_{i_1,i_2,\cdots i_k} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}$$

From equation (9), We have

$$d\omega = \sum_{i_1,i_2,\cdots i_k} \sum_{\alpha=1}^{n} D_\alpha(\omega_{i_1,i_2,\cdots i_k}) \cdot dx^\alpha \wedge dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}$$

Operating d again on $d\omega$ we have

$$d(d\omega) = \sum_{i_1<i_2<\cdots i_k} \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} D_{\alpha,\beta}(\omega_{i_1 i_2 \cdots i_k}) dx^{\beta} \wedge dx^{\alpha} \wedge dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}.$$

In this sum the terms
$D_{\alpha,\beta}(\omega_{i_1 i_2 \cdots i_k}) dx^{\beta} \wedge dx^{\alpha} \wedge dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}$ and
$D_{\beta,\alpha}(\omega_{i_1 i_2 \cdots i_k}) dx^{\alpha} \wedge dx^{\beta} \wedge dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}$ cancel in pairs since

$$\begin{aligned} &D_{\alpha,\beta}(\omega_{i_1 i_2 \cdots i_k}) dx^{\beta} \wedge dx^{\alpha} \wedge dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k} \\ &= -D_{\beta,\alpha}(\omega_{i_1 i_2 \cdots i_k}) dx^{\alpha} \wedge dx^{\beta} \wedge dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k} \end{aligned}$$

and hence

$$d(d\omega) = 0$$

(4) **Claim:** If ω is a $k-$form on $\mathbb{R}^m$ and $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable, then
$f^*(d\omega) = d(f^*\omega)$.
To prove this result let's apply induction on k.

Step I: Subclaim: Result is true when $k = 0$, i.e. if ω is a $0-$ form.
Since ω is a $0-$ form, $\omega = f$ for some scalar field f.
Consider $f^*(d\omega) = f^*(df) = d(f^*(f)) == d(f^*\omega)$.

Step II: Suppose result is true when ω is a $k-$form.
i.e. if ω is a $k-$form on $\mathbb{R}^m$ then $f^*(d\omega) = d(f^*\omega)$.

Subclaim: Result is true when ω is $(k+1)-$form of the type $\omega \wedge dx^i$.
Consider

$$\begin{aligned} &f^*(d(\omega \wedge dx^i)) = f^*(d\omega \wedge dx^i + (-1)^k \omega \wedge d(dx^i)) \quad \text{by theorm 10(II)} \\ &= f^*(d\omega \wedge dx^i) \quad \text{by theorm 10(III)} \\ &= f^*(d\omega) \wedge f^*(dx^i) \quad \text{by theorm 8(IV)} \\ &= d(f^*\omega) \wedge f^*(dx^i)) \quad \text{result is true for k-form} \\ &= d(f^*(\omega \wedge dx^i)) \end{aligned}$$

Example I: Calculate exterior derivatives of the $1-$ forms $z^2 dx \wedge dy + (z^2 + 2y) dx \wedge dz$ in $\mathbb{R}^3$.

Solution: Consider $\omega = z^2 dx \wedge dy + (z^2 + 2y) dx \wedge dz$ be given $2-$forms.

Consider

$$d\omega = 2zdz \wedge dx \wedge dy + (2zdz + 2dy) \wedge dx \wedge dz$$
$$d\omega = -2zdx \wedge dz \wedge dy + 2zdz \wedge dx \wedge dz + 2dy \wedge dx \wedge dz$$
$$d\omega = 2zdx \wedge dy \wedge dz - 2zdz \wedge dz \wedge dx - 2dx \wedge dy \wedge dz$$
$$d\omega = 2zdx \wedge dy \wedge dz - 0 - 2dx \wedge dy \wedge dz$$
$$d\omega = 2(z-1)dx \wedge dy \wedge dz$$

Example II: Calculate exterior derivatives of fdg where f and g are functions.

Solution: Let $f = f(x, y, z)$ and $g = g(x, y, z)$
$\Rightarrow dg = g_x dx + g_y dy + g_z dz$
Thus we have $fdg = f(x, y, z) \cdot (g_x dx + g_y dy + g_z dz)$
Consider

$d(f \cdot dg) = df \wedge dg + f \wedge d(dg) \quad f$ is $0-$ form
$= df \wedge dg + f \wedge d(dg) \quad$ since $d(dg) = 0$
$= (f_x dx + f_y dy + f_z dz) \wedge (g_x dx + g_y dy + g_z dz)$
$= f_x dx \wedge (g_x dx + g_y dy + g_z dz) + f_y dy \wedge (g_x dx + g_y dy + g_z dz)$
$+ f_z dz \wedge (g_x dx + g_y dy + g_z dz)$
$= f_x \cdot g_x dx \wedge dx + f_x \cdot g_y dx \wedge dy + f_x \cdot g_z dx \wedge dz + f_y \cdot g_x dy \wedge dx$
$+ f_y \cdot g_y dy \wedge dy + f_y \cdot g_z dy \wedge dz + f_z \cdot g_x dz \wedge dx + f_z \cdot g_y dz \wedge dy + f_z \cdot g_z dz \wedge dz$
$= 0 + f_x \cdot g_y dx \wedge dy + f_x \cdot g_z dx \wedge dz - f_y \cdot g_x dx \wedge dy + 0$
$+ f_y \cdot g_z dy \wedge dz - f_z \cdot g_x dx \wedge dz - f_z \cdot g_y dy \wedge dz + 0$
$= (f_x \cdot g_y - f_y \cdot g_x)dx \wedge dy + (f_x \cdot g_z - f_z \cdot g_x)dx \wedge dz + (f_y \cdot g_z - f_z \cdot g_y)dy \wedge dz$

Example III: If F is a vector field on $\mathbb{R}^3$, define the forms

$$\omega_F^1 = F^1 dx + F^2 dy + F^3 dz$$
$$\omega_F^2 = F^1 dy \wedge dz + F^2 dz \wedge dx + F^3 dx \wedge dy$$

Prove that
(1) $df = \omega^1_{grad\ f}$ where f is a scalar field in $\mathbb{R}^3$
(2) $d(\omega_F^1) = \omega^2_{curl\ F}$
(3) $d(\omega_F^2) = (div\ F)dx \wedge dy \wedge dz$
(4) $curl\ grad\ f = 0$
(5) $div\ curl\ F = 0$

Solution:

(1) Let $f = f(x, y, z)$ be a scalar field in $\mathbb{R}^3$.
$\Rightarrow$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

where $(\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial f}{\partial z}) = grad\ f$
by definition of ω^1_F, we can write df as $df = \omega^1_{grad\ f}$.

(2) Let $\omega^1_F = F^1dx + F^2dy + F^3dz$ be a 1−form. Consider

$$\begin{aligned}
d(\omega^1_F) &= F^1_x dx \wedge dx + F^1_y dy \wedge dx + F^1_z dz \wedge dx \\
&+ F^2_x dx \wedge dy + F^2_y dy \wedge dy + F^2_z dz \wedge dy \\
&+ F^3_x dx \wedge dz + F^3_y dy \wedge dz + F^3_z dz \wedge dz \\
&= 0 - F^1_y dx \wedge dy + F^1_z dz \wedge dx \\
&+ F^2_x dx \wedge dy + 0 - F^2_z dy \wedge dz \\
&- F^3_x dz \wedge dx + F^3_y dy \wedge dz + +0 \\
&= (F^2_x - F^1_y) dx \wedge dy + (F^3_y - F^2_z) dy \wedge dz + (F^1_z - F^3_x) dz \wedge dx
\end{aligned}$$

where $((F^2_x - F^1_y), (F^3_y - F^2_z), (F^1_z - F^3_x)) = curl\ F$
by definition of ω^2_F, we can write $d(\omega^1_F)$ as $d(\omega^1_F) = \omega^2_{curl\ F}$.

(3) Let $\omega^2_F = F^1dy \wedge dz + F^2dz \wedge dx + F^3dx \wedge dy$ be given 2−form. Consider

$$\begin{aligned}
d(\omega^2_F) &= dF^1 \wedge dx \wedge dy \wedge dz + dF^2 \wedge dy \wedge dz \wedge dx + dF^3 \wedge dz \wedge dx \wedge dy \\
&= dF^1 \wedge dx \wedge dy \wedge dz + dF^2 \wedge dx \wedge dy \wedge dz + dF^3 \wedge dx \wedge dy \wedge dz \\
&= (dF^1 + dF^2 + dF^3) \wedge dx \wedge dy \wedge dz \\
&= (div\ F) dx \wedge dy \wedge dz
\end{aligned}$$

(4) By (2), we have $\omega^2_{curl\ F} = d(\omega^1_F)$
Replace F by $grad\ f$, we obtain
$\omega^2_{curl\ grad\ f} = d(\omega^1_{grad\ f})$
By (1), we have $\omega^2_{curl\ grad\ f} = d(d(f)) = 0$
$\Rightarrow curl\ grad\ f = 0$.

(5) By (3), we have $(div\ F)dx \wedge dy \wedge dz = d(\omega^2_F)$
Replace F by $curl\ F$, we obtain
$(div\ curl\ F)dx \wedge dy \wedge dz = d(\omega^2_{curl\ F})$
By (2), we have $(div\ curl\ F)dx \wedge dy \wedge dz = d(d(\omega^1_F)) = 0$
$\Rightarrow div\ curl\ F = 0$.

Example 1: Let $\alpha = xdx + ydy + zdz$, $\beta = zdx + xdy + ydz$ and $\gamma = xydz$ in the following problems.

1. Calculate
(a) $\alpha \wedge \beta$
(b) $\alpha \wedge \gamma$
(c) $\beta \wedge \gamma$
(d) $(\alpha+\gamma) \wedge (\alpha+\gamma)$

2. Calculate
(a) $d\alpha$
(b) $d\beta$
(c) $d(\alpha + \gamma)$
(d) $d(x\alpha)$

Example 2: Consider the forms,
$\omega = xydx + 3dy - yzdz$,
$\eta = xdx$ - yz^2 dy + $2xdz$ in $\mathbb{R}^3$.
Verify by direct computation that
$d(d\omega) = 0$ and $d(\omega \wedge \eta) = (d\omega) \wedge \eta$ - $\omega \wedge d\eta$.

Example 3: In $\mathbb{R}^3$, let $\omega = xydx + 2zdy - ydz$
Let α: $\mathbb{R}^2 \to \mathbb{R}^3$ be given by the equation,
$\alpha(u, v) = (uv, u^2, 3u + v)$
Calculate $d\omega$, $\alpha^*\omega$, $\alpha^*(d\omega)$ and $d(\alpha^*\omega)$ directly.

3.3 Closed and Exact Form

Closed Form: A form ω is called closed if $d\omega = 0$.

Exact Form: A form ω is called exact if $\omega = d\eta$, for some η.

Note: Theorem $10(III)$ shows that every exact form is closed since $d\omega = d(d\eta) = 0$.

Note: Is every closed form is exact?
In general every closed form is not exact.
If ω is the $1-$form $Pdx + Qdy$ on $\mathbb{R}^2$ and is closed, then

$$d\omega = (D_1Pdx + D_2Pdy) \wedge dx + (D_1Qdx + D_2Qdy) \wedge dy$$

$$d\omega = D_1Pdx \wedge dx + D_2Pdy \wedge dx + D_1Qdx \wedge dy + D_2Qdy \wedge dy$$

$$d\omega = 0 - D_2Pdx \wedge dy + D_1Qdx \wedge dy + 0$$

$$d\omega = (D_1Q - D_2P)dx \wedge dy$$

Thus since ω is closed $d\omega = 0 \Rightarrow 0 = (D_1Q - D_2P)dx \wedge dy$ then $D_1Q = D_2P$ Thus we have $\omega = Pdx + Qdy$ is exact if $D_1Q = D_2P$ i.e. $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$.

Example II: Let $A = \mathbb{R}^2 - 0$ and

$$\omega = \frac{-y}{x^2 + y^2}dx + \frac{x}{x^2 + y^2}dy$$

in A. Show that, ω is closed but not exact.

Star Shaped Set: Suppose that $\omega = \sum\limits_{i=1}^{n} \omega_i dx^i$ is a 1− form on $\mathbb{R}^n$. If ω is exact then $\omega = df = \sum\limits_{i=1}^{n} D_i f dx^i$ with assumption $f(0) = 0$. We have

$$\begin{aligned} f(x) &= \int_0^1 \frac{d}{dt} f(tx) dt \\ &= \int_0^1 \sum_{i=1}^{n} D_i f(tx) x^i dt \\ &= \int_0^1 \sum_{i=1}^{n} \omega_i(tx) x^i dt \end{aligned}$$

$\Rightarrow$ To find f, for a given ω such that $\omega = df$, we consider the function $I\omega$, defined by

$$I_\omega(x) = \int_0^1 \sum_{i=1}^{n} \omega_i(tx) \cdot x^i dt,$$

Note that the I_ω is well defined if ω is defined only on an open set $A \subset \mathbb{R}^n$ with the property that whenever$x \in A$, the line segment from 0 to x is contained in A. Such an open set is called star shaped with respect to 0.

Theorem-11 : Poincaré Lemma If $A \subset \mathbb{R}^n$ is an open set star-shaped with respect to 0, then every closed form on A is exact.

Proof: Let ω be $l-$form

$$\omega = \sum_{i_1 < i_2 < \cdots i_l} \omega_{i_1 i_2 \cdots i_l} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_l}.$$

Define a function $(l-1)-$forms I from $l-$forms ω (for each l), such that $I(0) = 0$ and $\omega = I(d\omega) + d(I\omega)$ for any form ω.
Since A is star-shaped we can define

$$I\omega(x) = \sum_{i_1 < i_2 < \cdots i_l} \sum_{\alpha=1}^{l} (-1)^{\alpha-1} \left(\int_0^1 t^{l-1} \omega_{i_1 i_2 \cdots i_l}(tx) dt \right) x^{i_\alpha} dx^{i_1} \cdots \wedge \widehat{dx^{i_\alpha}} \wedge \cdots \wedge dx^{i_l} \tag{3.2}$$

Note that the symbol $\widehat{dx^{i_\alpha}}$ indicates that it is omitted. Now let's consider $d(I\omega(x))$, note that

$$
\begin{aligned}
& d[(\omega_{i_1 i_2 \cdots i_l}(tx))x^{i_\alpha} dx^{i_1} \cdots \wedge \widehat{dx^{i_\alpha}} \wedge \cdots \wedge dx^{i_l}] \\
& = (\omega_{i_1 i_2 \cdots i_l}(tx)) d[x^{i_\alpha}] \wedge dx^{i_1} \cdots \wedge \widehat{dx^{i_\alpha}} \wedge \cdots \wedge dx^{i_l} \\
& + d(\omega_{i_1 i_2 \cdots i_l}(tx)) x^{i_\alpha} dx^{i_1} \cdots \wedge \widehat{dx^{i_\alpha}} \wedge \cdots \wedge dx^{i_l} \\
& = (-1)^{\alpha-1} \cdot l \cdot (\omega_{i_1 i_2 \cdots i_l}(tx)) dx^{i_1} \cdots \wedge dx^{i_\alpha} \wedge \cdots \wedge dx^{i_l} \\
& + \sum_{j=1}^{n} t \cdot D_j(\omega_{i_1 i_2 \cdots i_l}(tx)) x^{i_\alpha} dx^{i_1} \wedge \cdots \wedge \widehat{dx^{i_\alpha}} \wedge \cdots \wedge dx^{i_l} \\
& \text{ since } \alpha \text{ running from 1 to } l \text{ and} \\
& (-1)^{\alpha-1} \text{ added because of } (\alpha-1) \text{ permutations of } dx^{i_\alpha}
\end{aligned}
$$

hence $d(I\omega(x))$ becomes

$$
\begin{aligned}
d(I\omega(x)) &= l \cdot \sum_{i_1 < i_2 < \cdots i_l} \left(\int_0^1 t^{l-1} \omega_{i_1 i_2 \cdots i_l}(tx) dt \right) dx^{i_1} \cdots \wedge dx^{i_\alpha} \wedge \cdots \wedge dx^{i_l} \\
&+ \sum_{i_1 < i_2 < \cdots i_l} \sum_{\alpha=1}^{l} \sum_{j=1}^{n} (-1)^{\alpha-1} \left(\int_0^1 t^l D_j \omega_{i_1 i_2 \cdots i_l}(tx) dt \right) x^{i_\alpha} dx^{i_1} \cdots \wedge \widehat{dx^{i_\alpha}} \wedge \cdots \wedge dx^{i_l}
\end{aligned} \tag{11}
$$

Using equation (9), consider $d\omega$ as

$$
d\omega = \sum_{i_1 < i_2 < \cdots i_l} \sum_{j=1}^{n} D_j(\omega_{i_1 i_2 \cdots i_l}) dx^j \wedge dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_l}
$$

Applying I to the $(l+1)-$form $d\omega$, as per definition of I we obtain $l-$form as

$$
\begin{aligned}
I(d\omega) &= \sum_{i_1 < i_2 < \cdots i_l} \sum_{j=1}^{n} \left(\int_0^1 t^l x^j D_j(\omega_{i_1 i_2 \cdots i_l})(tx) dt \right) dx^{i_1} \wedge \cdots \wedge dx^{i_\alpha} \wedge \cdots \wedge dx^{i_l} \\
&- \sum_{i_1 < \cdots i_l} \sum_{j=1}^{n} \sum_{\alpha=1}^{l} (-1)^{\alpha-1} \left(\int_0^1 t^l D_j(\omega_{i_1 i_2 \cdots i_l})(tx) dt \right) x^{i_\alpha} dx^j \wedge dx^{i_1} \wedge \cdots \wedge \widehat{dx^{i_\alpha}} \wedge \cdots \wedge dx^{i_l}
\end{aligned} \tag{12}
$$

Adding equations (11) and (12), the triple sums cancel, and we obtain

$$d(I\omega) + d(d\omega) = \sum_{i_1<i_2<\cdots i_l} l \cdot \left(\int_0^1 t^{l-1} (\omega_{i_1 i_2 \cdots i_l})(tx) dt \right) dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_l}$$

$$+ \sum_{i_1<i_2<\cdots i_l} \sum_{j=1}^{n} \left(\int_0^1 t^l x^j D_j (\omega_{i_1 i_2 \cdots i_l})(tx) dt \right) dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_l}$$

$$= \sum_{i_1<i_2<\cdots i_l} \left(\int_0^1 \frac{d}{dt} [t^l (\omega_{i_1 i_2 \cdots i_l})(tx)] dt \right) dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_l}$$

$$= \sum_{i_1<i_2<\cdots i_l} (\omega_{i_1 i_2 \cdots i_l}) \, dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_l}$$

$$= \omega.$$

Thus we have $\omega = d(I\omega) + d(d\omega)$ since ω is closed $d\omega = 0$.
Thus $\omega = d(I\omega)$ hence ω is exact.

3.4 Chapter End Exercise

1. Is the $1-$form $\omega = (x^2 + y^2)dx + 2xydy$ closed and exact? Justify your answer.

2. Let ω be a any $3-$form. Prove or disprove: $d(d\omega) = 0$.

3. Let $A = \mathbb{R}^2 - 0$ and $\omega = \dfrac{(-ydx + xdy)}{(x^2 + y^2)}$ in A. Prove or disprove: ω is closed and exact in A.

4. In $\mathbb{R}^3$, let $\omega = xydx + 2zdy - ydz$ and $\alpha : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be given by $\alpha(u, v) = (uv, u^2, 3u + v)$. Calculate $\alpha^*(d\omega)$.

5. State the necessary condition for every closed form on $A \subset \mathbb{R}^n$ to be exact. Is the $1-$form $\omega = (1 + e^x)dy + e^x(y - x)dy$ closed and exact? Justify your answer.

6. If ω is a $0-$form and η is a $l-$form, then show that $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$.

7. If F is a vector field on $\mathbb{R}^3$. Let $\omega_F^1 = F^1 dx + F^2 dy + F^3 dz$ and $\omega_F^2 = F^1 dy \wedge dz + F^2 dz \wedge dx + F^3 dx \wedge dy$ then show that $d(\omega_F^1) = \omega_{curl\ F}^2$.

8. Show that every exact form is closed. Is the converse true? Justify your answer.

Chapter 4

Basics of Submanifolds of $\mathbb{R}^n$

Unit Structure :

4.1 Objectives

After going through this chapter you will be able to:

1. Define a manifolds with and without boundary.
2. Learn the concepts of Coordinate system and M conditions.
3. Learn the properties of tangent space of manifolds and vector field on manifolds.
4. Identify orientation of Manifolds.

4.2 Basic Preliminaries

Smooth map: A mapping f of an open set $U \subset \mathbb{R}^n$ into $\mathbb{R}^m$ is called smooth if it has continuous partial derivatives of all orders.

Note: For partial derivatives domain of f is essentially required to be open.

Diffeomorphism: A smooth map $f : X \longrightarrow Y$ of subsets of two euclidean spaces is a diffeomorphism if it is bijective and if the inverse $f^{-1} : Y \longrightarrow X$ is also smooth. X and Y are diffeomorphic if such a map exists.
OR
If U and V are open sets in $\mathbb{R}^n$, a differentiable function $h : U \to V$ with a differntiable inverse $h^{-1} : V \to U$, will be called a diffeomorphism.

("Differntiable"hencefoth, means "$\mathbb{C}^{\infty}$".)

Exercise: Give an example of differomorphism.

4.3 Manifolds in $\mathbb{R}^n$

A subset M of $\mathbb{R}^n$ is called a $k-$dimensional manifold in $\mathbb{R}^n$ if for every point $x \in M$, the following condition is satisfied
Condition M: If there is an open set U containing x, an open set $V \subset \mathbb{R}^n$, and a diffeomorphism $h : U \to V$ such that

$$h(U \cap M) = V \cap (\mathbb{R}^k \times \{0\}) = \{y \in V : y^{k+1} = y^{k+2} = \cdots = y^n = 0\}.$$

i.e. $(y^1, \cdots, y^k, y^{k+1}, \cdots, y^n) \longrightarrow (y^1, \cdots, y^k, 0, \cdots, 0)$
OR
A subset M of a euclidean space $\mathbb{R}^n$ is known as a $k-$dimensional manifold if it is locally diffeomorphic to $\mathbb{R}^k$.
Note that, local referring to behaviour only in some neighborhood of a point.

Submanifolds: If M_1 and M_2 are both manifolds in $\mathbb{R}^n$ and $M_1 \subset M_2$ then M_1 is known as submanifold of M_2.

Note:
(1) M is itself submanifold of $\mathbb{R}^n$.
(2) Any open set of M is submanifold of M.
(3) A point in $\mathbb{R}^n$ is a $0-$dimensional manifolds.
(4) An open subset in $\mathbb{R}^n$ is an $n-$dimensional manifolds.

Theorem-01: Let $A \subset \mathbb{R}^n$ be open and let $g : A \to \mathbb{R}^p$ be a differentiable function such that $g^{'}(x)$ has rank p whenever $g(x) = 0$.
Then $g^{-1}(0)$ is an $(n-p)-$dimensional manifold in $\mathbb{R}^n$.

Proof: Step I: Consider following theorem from Real Analysis
Subclaim: Theorem: Let $f : \mathbb{R}^n \to \mathbb{R}^p$ be a continuously differentiable function in an open set containing a where $p \leq n$. If $f(a) = 0$

and the $p \times n$ matrix $D_j f^i(a)$ has rank p then there is an open set $A \subset \mathbb{R}^n$ containing a and a differentiable function $h : A \to \mathbb{R}^n$ with differentiable inverse such that

$$f oh(x^1, x^2, \cdots, x^n) = (x^{n-p+1}, x^{n-p+2}, \cdots, x^n).$$

Add proof of above theorem.

Step II: By applying above theorem and by definition of manifold we can conclude that $g^{-1}(0)$ is an $(n-p)-$dimensional manifold in $\mathbb{R}^n$.

Example: Show that the $n-$Sphere S^n, defined as $\{x \in \mathbb{R}^{n+1} : |x| = 1\}$ is $n-$dimensional manifold.

Solution: Apply above theorem (1) by considering $S^n = g^{-1}(0)$, where $g : \mathbb{R}^{n+1} \to \mathbb{R}$ is defined by $g(x) = |x|^2 - 1$.
Note that n is replaced by $n+1$,
$p = 1$,
$g(0) = 0$.
By theorem (1), Sphere S^n is $(n-p) = (n+1-1) = n$ dimensional manifold.

Theorem-02: A subset M of $\mathbb{R}^n$ is a $k-$dimensional manifold if and only if for each point $x \in M$ the following "coordinate condition"is satisfied:

Coordinate condition C: There is an open set U containing x, an open set $W \subset \mathbb{R}^k$, and a $1-1$ differentiable function $f : W \to \mathbb{R}^n$ such that

(1) $f(W) = M \cap U$,
(2) $f'(y)$ has rank k for each $y \in W$,
(3) $f^{-1} : f(W) \to W$ is continuous.
note that, such a function f is called a coordinate system around x.

Proof: Step I: Assume that M is a $k-$dimensional manifold in $\mathbb{R}^n$.
Claim: Each point $x \in M$ satisfies the coordinate condition.
Since M is $k-$dimensional manifold in $\mathbb{R}^n$ by definition each point $x \in M$ satisfies the following condition

If there is an open set U containing x, an open set $V \subset \mathbb{R}^n$, and a diffeomorphism $h : U \to V$ such that

$$h(U \cap M) = V \cap (\mathbb{R}^k \times \{0\}) = \{y \in V : y^{k+1} = y^{k+2} = \cdots = y^n = 0\}.$$

Let $W = \{a \in R^k : (a, 0) \in h(M)\}$.
Define $f : W \to \mathbb{R}^n$ by $f(a) = h^{-1}(a, 0)$.
Clearly

(1) Since $h : U \to V \Rightarrow h^{-1}(V) = U$ and
$(a, 0) \in h(M) \Rightarrow h^{-1}(a, 0) = M$
hence $f(W) = M \cap U$,
(2) Since h is diffomorphism, f^{-l} is continuous and
(3) If $H : U \to \mathbb{R}^k$ is defined by $H(z) = (h^1(z), \cdots, h^k(z))$,
then $H(f(y)) = y$ for all $y \in W$ ($\because$ Since $f = h^{-1}$)
Therefore on differentiating by using chain rule we obtain
$H^{'}(f(y)) \cdot f'(y) = I$ and $f^{'}(y)$ must have rank k.
Thus each point $x \in M$ satisfies the coordinate conditions.

Step II: Suppose that $f : W \to \mathbb{R}^n$ satisfies coordinate conditions.
Claim: M is a $k-$dimensional manifold in $\mathbb{R}^n$.
Let $f(y) = x$.
Assume that the matrix $(D_j f^i(y))$, $1 \leq i, j \leq k$ has a non-zero determinant.
Define $g : W \times \mathbb{R}^{n-k} \to \mathbb{R}^n$ by $g(a, b) = f(a) + f(0, b)$.
Then $\det g^{'}(a, b) = \det\ (D_j f^i(a))$,
so $\det\ g^{'}(y, 0) \neq 0$.
Now lets use Inverse Function Theorem as

Inverse Function Theorem: Suppose that $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is continuously differentiable in an open set containing a and $\det f'(a) \neq 0$. Then there is an open set V containing a and open set W containing $f(a)$ such that $f : V \longrightarrow W$ has a continuous inverse $f^{-1} : W \longrightarrow V$ which is differentiable and for all $y \in W$ satisfies $(f^{-1})'(y) = [f'(f^{-1}(y))]^{-1}$.

By Inverse Function Theorem
There is an open set $V_1^{'}$ containing $(y, 0)$ and an open set $V_2^{'}$ containing $g(y, 0) = x$ such that $g : V_1^{'} \to V_2^{'}$ has a differentiable inverse $h : V_2^{'} \to V_1^{'}$.
By third coordinate condition, f^{-1} is continuous,
$\{f(a) : (a, 0) \in V_1^{'}\} = U \cap f(W)$ for some open set U (By first coordinate condition).
Let $V_2 = V_2^{'} \cap U$ and $V_1 = g^{-1}(V2)$.
Then $V_2 \cap M$ is exactly $\{f(a) : (a, 0) \in V_1\} = \{g(a, 0) : (a, 0) \in V_1\}$, where $M \subset \mathbb{R}^n$ So

$$
\begin{aligned}
&h(V_2 \cap M) = g^{-1}(V_2 \cap M) \text{ since } h = g^{-1} \\
&= g^{-1}(\{g(a, 0) : (a, 0) \in V_1\}) = (\{(a, 0) : (a, 0) \in V_1\}) \\
&= V_1 \cap (\mathbb{R}^k \times \{0\}).
\end{aligned}
$$

hence by definition M is a $k-$dimensional manifold in $\mathbb{R}^n$.

Note: If $f_1 : W_1 \subset \mathbb{R}^k \longrightarrow \mathbb{R}^n$ and $f_2 : W_2 \subset \mathbb{R}^k \longrightarrow \mathbb{R}^n$ are two coordinate systems, then

$$f_2^{-1} \circ f_1 : f_1^{-1}(f_2(W_2)) \to \mathbb{R}^k$$

is differentiable with non-singular Jacobian. If fact, $f_2^{-1}(y)$ consists of the first k components of $h(y)$.

4.4 Manifolds of $\mathbb{R}^n$ without boundary

Manifolds in $\mathbb{R}^n$ without boundary: Let $k > 0$. Suppose that M is a subspace of $\mathbb{R}^n$ having the following property:
For each $p \in M$, there is an open set V containing p that is open in M, a set U that is open in $\mathbb{R}^k$, and a continuous map $f : U \to V$ carrying U onto V in a $1-1$ fashion such that

(1) f is of class $\mathbb{C}^r$
(2) $Df(x)$ has rank k for each $x \in U$,
(3) $f^{-1} : V \to U$ is continuous.

Then M is called a $k-$ manifold without boundary $\mathbb{R}^n$ of class $\mathbb{C}^r$. The map f is called a coordinate patch on M about p.

Example 1: Let $\alpha : \mathbb{R} \longrightarrow \mathbb{R}^2$ be given by $\alpha(t) = (t^3, t^2)$. Let M be image set of α. Is M $1-$manifold without boundary in $\mathbb{R}^2$? Justify your answer.

Solution: Let $\alpha : \mathbb{R} \longrightarrow \mathbb{R}^2$ be given by $\alpha(t) = (t^3, t^2)$ is a $1-1$ map. Clearly
(1) α is of class $\mathbb{C}^\infty$
(2) $\alpha^{-1} : V \to U$ is continuous where U is open in $\mathbb{R}$ and V is open in $\mathbb{R}^2$,
(3) $D\alpha(t) = (3t^2, 2t)$ has not rank 1 at $t = 0$.

hence M not $1-$manifold without boundary in $\mathbb{R}^2$.

Example 2: Let $\beta : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be given by $\beta(x, y) = (x(x^2 + y^2), y(x^2 + y^2), (x^2 + y^2),)$. Let M be image set of β. Is M $2-$manifold without boundary in $\mathbb{R}^3$? Justify your answer.

Solution: Let $\beta : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be given by $\beta(x, y) = (x(x^2 + y^2), y(x^2 + y^2), (x^2 + y^2),)$ is a $1-1$ map. Clearly
(1) β is of class $\mathbb{C}^\infty$
(2) $\beta^{-1} : V \to U$ is continuous where U is open in $\mathbb{R}$ and V is open in $\mathbb{R}^2$,
(3) $D\beta(t) = \begin{bmatrix} (x^2 + y^2) + 2x^2 & 2xy & 2x \\ 2xy & (x^2 + y^2) + 2y^2 & 2y \end{bmatrix}$
$D\beta(t)$ has not rank 2 at 0.

hence M not 2−manifold without boundary in $\mathbb{R}^3$.

Example 3: Let $\gamma : \mathbb{R} \longrightarrow \mathbb{R}^2$ be given by $\gamma(t) = (\sin 2t)(\mid \cos t \mid, \sin t)$ for $0 < t < \pi$. Let M be image set of γ. Is M 1−manifold without boundary in $\mathbb{R}^3$? Justify your answer.

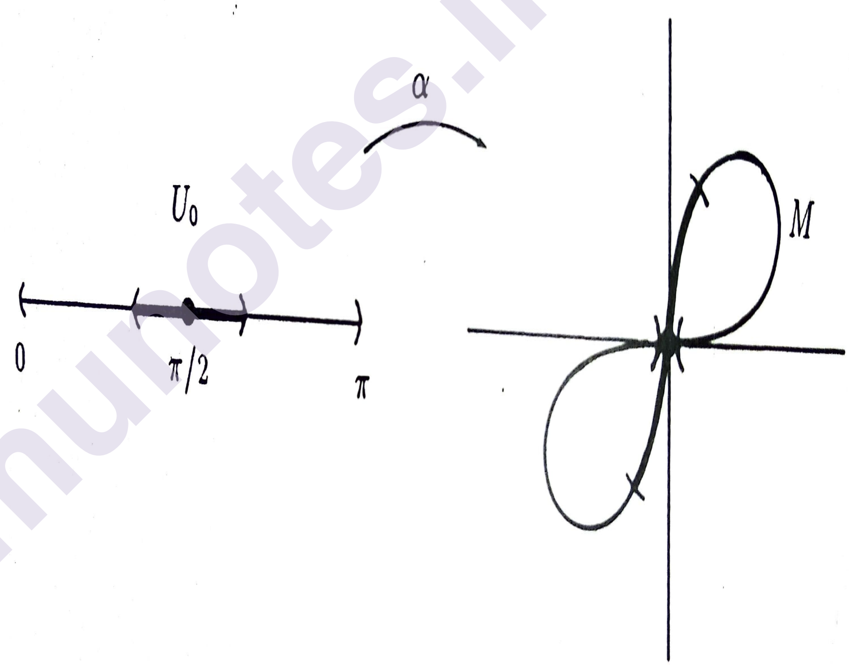
Solution: Let $\gamma : \mathbb{R} \longrightarrow \mathbb{R}^2$ be given by $\alpha(t) = (\sin 2t)(\mid \cos t \mid, \sin t)$ is a $1-1$ map for $0 < t < \pi\pi$. Clearly

(1) γ is of class $\mathbb{C}^1$

(2) $D\gamma(t) = (\sin 2t)(\mid \sin t \mid, \cos t) + (2\cos 2t)(\mid \cos t \mid, \sin t)$ has rank 1 for all t.

(3) Since image of smaller interval U which contains $\frac{\pi}{2}$ is not open in M hence $\gamma^{-1} : V \to U$ is not continuous where V is open in $\mathbb{R}^2$,

hence M not 1−manifold without boundary in $\mathbb{R}^3$.



4.5 Manifolds of $\mathbb{R}^n$ with boundary

Half Space: The half-space $H^k \subset R^k$ is defined as $\{x \in \mathbb{R}^k : x^k \geq 0\}$.

Manifold with Boundary: A subset M of $\mathbb{R}^n$ is a k−dimensional manifold-with boundary if for every point $x \in M$ either condition (M) or the following condition is satisfied:

Condition M': There is an open set U containing x, an open set $V \subset \mathbb{R}^n$, and a diffeomorphism $h : U \to V$ such that

$$h(U \cap M) = V \cap (H^k \times \{0\}) = \{y \in V : y^k \geq 0, \text{ and } y^{k+1} = y^{k+2} = \cdots = y^n = 0\}$$

and $h(x)$ has k^{th} component $= 0$.
The set of all points $x \in M$ for which condition M' is satisfied is called the boundary of M and denoted ∂M.

Note: Conditions (M) and (M') cannot both hold for the same x.

Examples: (1) Let $\alpha : \mathbb{R} \longrightarrow \mathbb{R}^2$ be the map $\alpha(t) = (t, t^2)$. Let M be image set of α. Show that M $1-$manifold in $\mathbb{R}^2$ covered by the single coordinate patch α.

(2) Let $\beta : H^1 \longrightarrow \mathbb{R}^2$ be the map $\beta(t) = (t, t^2)$. Let N be image set of β. Show that N is $1-$manifold in $\mathbb{R}^2$.

(3) Show that unit circle S^1 is a $1-$manifold in $\mathbb{R}^2$.

(4) Show that the function $\alpha : [0, 1] \longrightarrow S^1$ given by $\alpha(t) = (\cos 2\pi t, \sin 2\pi t)$ is not a coordinate patch on S^1.

4.6 Fields and Forms on Manifolds

Tangent Space of M: Let M be a $k-$dimensional manifold in $\mathbb{R}^n$ and let
$f : W \to \mathbb{R}^n$ be a coordinate system around $x = f(a)$.
Since $f'(a)$ has rank k, the linear transformation $f_* : \mathbb{R}^k_a \to \mathbb{R}^n_x$, is $1-1$, and $f_*(\mathbb{R}^k_a)$ is a $k-$dimensional subspace of $\mathbb{R}^n_x$.
If $g : V \to \mathbb{R}^n$ is another coordinate system, with $x = g(b)$, then

$$g_*(\mathbb{R}^k_b) = f_*(f^{-1} \circ g) * (\mathbb{R}^k_b) = f_*(\mathbb{R}^k_a)$$

Thus the $k-$dimensional subspace $f_*(\mathbb{R}^k_a)$ does not depend on the coordinate system f. This subspace is denoted M_x, and is called the tangent space of M at x.

Note: There is a natural inner product T_x, on M_x, induced by that on $\mathbb{R}^n_x$,
if $v,\, w \in M_x$, define $T_x(v,\, w) = \langle v,\, w\rangle_x$.

Vector field on M: Suppose that A is an open set containing M, and F is a differentiable vector field on A such that $F(x) \in M_x$, for each $x \in M$. If $f : W \to \mathbb{R}^n$ is a coordinate system, there is a unique differentiable vector field G on W such that $f_*(G(a)) = F(f(a))$ for each $a \in W$. such a function F is called a vector field on M.

Note: (1) we define F to be differentiable if G is differentiable.
(2) Note that our definition does not depend on the coordinate system chosen.
if $g : V \to \mathbb{R}^n$ and $g_*(H(b)) = F(g(b))$ for all $b \in V$, then the component functions of $H(b)$ must equal the component functions of $G(f^{-1}(g(b)))$, so H is differentiable if G is differentiable.

$p-$**form on** M**:** A function ω which assigns $\omega(x) \in \Lambda^p(M_x)$ for each $x \in M$ is called a $p-$form on M.
If $f : W \to \mathbb{R}^n$ is a coordinate system, then $f^*\omega$ is a $p-$form on W.

Note: (1) We define ω to be differentiable if $f^*\omega$ is differentiable.
(2) A $p-$form ω on M can be written as

$$\omega = \sum_{i_1 < i_2 < \cdots < i_p} \omega_{i_1 i_2 \cdots i_p} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_p}.$$

here the functions $\omega_{i_1 i_2 \cdots i_p}$ are defined only on M.

Theorem-03: There is a unique $(p+1)-$form $d\omega$ on M such that for every coordinate system $f : W \to \mathbb{R}^n$ we have $f^*(d\omega) = d(f^*\omega)$.

Proof: If $f : W \to \mathbb{R}^n$ is a coordinate system with $x = f(a)$ and $v_1, v_2, \cdots, v_{p+1} \in M_x$, there are unique $\omega_1, \omega_2, \cdots, \omega_{p+1}$ in $\mathbb{R}^k_a$ such that $f * (\omega_i) = v_i$.
Define $d\omega(x)(v_1, v_2, \cdots, v_{p+1}) = df^*(\omega)(a)(\omega_1, \omega_2, \cdots, \omega_{p+1})$.
One can check that this definition of $d\omega(x)$ does not depend on the coordinate system f, so that $d\omega$ is well-defined.
Moreover, it is clear that $d\omega$ has to be defined this way, so $d\omega$ is unique.

4.7 Orientable Manifolds

Consistent: For each tangent space M_x of a manifold M, it is necessary to choose an orientation μ_x. Such choices are called consistent provided that for every coordinate systems $f : W \to \mathbb{R}^n$ and $a, b \in W$ the relation

$$[f_*((e_1)_a), f_*((e_2)_a), \cdots, f_*((e_k)_a) = \mu_{f(a)}$$

holds if and only if

$$[f_*((e_1)_b), f_*((e_2)_b), \cdots, f_*((e_k)_b) = \mu_{f(b)}.$$

Orientation Preserving: Suppose orientations μ_x have been chosen consistently. If $f : W \to \mathbb{R}^n$ is a coordinate system such that

$$[f_*((e_1)_a), f_*((e_2)_a), \cdots, f_*((e_k)_a) = \mu_{f(a)}$$

for one, and hence for every $a \in W$, then f is called orientation-preserving.

Note: (1) If f is not orientation-preserving and $T : \mathbb{R}^k \to \mathbb{R}^k$ is a linear transformation with $\det T = -1$, then $f \circ T$ is orientation-preserving.
(2) Therefore there is an orientation-preserving coordinate system around each point.
(3) If f and g are orientation-preserving and $x = f(a) = g(b)$, then the relation

$$[f_*((e_1)a), f_*((e_2)a), \cdots, f_*((e_k)a)] = \mu_x = [g_*((e_1)b), g_*((e_2)b), \cdots, g_*((e_k)b)]$$

implies that

$$[(g^{-1} \circ f)_*((e_1)a), (g^{-1} \circ f)_*((e_2)a), \cdots, (g^{-1} \circ f)_*((e_k)a)] = [(e_1)b, (e_2)b, \cdots, (e_k)b],$$

so that $\det\,(g^{-1} \circ f)' > 0$.

Orientable Manifold: A manifold for which orientations μ_x can be chosen consistently is called orientable, and a particular choice of the μ_x is called an orientation μ of M. A manifold together with an orientation μ is called an oriented manifold.

Outward Unit Normal: If M is a $k-$dimensional manifold-with-boundary and $x \in \partial M$, then $(\partial M)_x$, is a $(k-1)-$dimensional subspace of the $k-$dimensional vector space M_x. Thus there are exactly two unit vectors in M, which are perpendicular to $(\partial M)_x$. They can be

distinguished as follows.
If $f : W \to \mathbb{R}^n$ is a coordinate system with $W \subset H^k$ and $f(0) = x$, then only one of these unit vectors is $f_*(v_0)$ for some v_0 with $v^k < 0$. This unit vector is called the outward unit normal $n(x)$.
Note: Outward unit normal does not depend on the coordinate system f.

Induced Orientation: Suppose that μ is an orientation of a $k-$dimensional manifold with-boundary M. If $x \in \partial M$, choose $v_1, v_2, \cdots, v_{k-1} \in (\partial M)_x$, so that $[(n(x), \omega_1, \omega_1, \cdots, \omega_{k-1}] = \mu_x$. If it is also true that $[(n(x), \omega_1, \omega_1, \cdots, \omega_{k-1}] = \mu_x$, then both $[v_1, v_2, \cdots, v_{k-1}]$ and $[(\omega_1, \omega_1, \cdots, \omega_{k-1}]$ are the same orientation for $(\partial M)_x$. This orientation is denoted $(\partial\mu)_x$. The orientations $(\partial\mu)_x$, for $x \in \partial M$, are consistent on ∂M. Thus if M is orientable, ∂M is also orientable, and an orientation μ for M determines an orientation $\partial\mu$ for ∂M, called the induced orientation.

Note: If we apply these definitions to H^k with the usual orientation, we find that the induced orientation on $\mathbb{R}^{k-1} = \{(x \in H^k : x^k = 0\}$ is $(-1)^k$ times the usual orientation.

Example: Show that the Möbius strip is a non-orientable manifold.

4.8 Chapter End Exercise

1. Define diffeomorphism and give an example of diffeomorphism. Justify your answer.

2. Show that unit circle S^1 is a $1-$manifold in $\mathbb{R}^2$.

3. Let $\gamma : \mathbb{R} \longrightarrow \mathbb{R}^2$ be given by $\gamma(t) = (\sin 2t)(|\cos t|, \sin t)$ for $0 < t < \pi$. Let M be image set of γ. Is M $1-$manifold without boundary in $\mathbb{R}^3$? Justify your answer.

4. Let $f : \mathbb{R}^1 \longrightarrow \mathbb{R}^1$ is given by

$$f(x) = \begin{cases} e^{\frac{-1}{x^2}}, & \text{x} > 0, \\ 0, & \text{x} \leq 0 \end{cases}$$

   Prove or disprove: f is diffeomorphism.

5. Let $\beta : H^1 \longrightarrow \mathbb{R}^2$ be the map $\beta(t) = (t, t^2)$. Let N be image set of β. Show that N is $1-$manifold in $\mathbb{R}^2$.

6. Prove or disprove: the Möbius strip is a orientable manifold.

7. Is the $n-$Sphere S^n, defined by $\{x \in \mathbb{R}^{n+1} : |x| = 1\}$ a $n-$dimensional manifold? Justify your answer.

8. Let $\gamma : \mathbb{R} \longrightarrow \mathbb{R}^2$ be given by $\gamma(t) = (\sin 2t)(|\cos t|, \sin t)$ for $0 < t < \pi$. Let M be image set of γ. Is M $1-$manifold without boundary in $\mathbb{R}^3$? Justify your answer.

9. Show that there is a unique $(p+1)-$form $d\omega$ on M such that for every coordinate system $f : W \to \mathbb{R}^n$ we have $f^*(d\omega) = d(f^*\omega)$.

Chapter 5

Stokes's Theorem

Unit Structure :

5.1 Objectives

After going through this chapter you will be able to:

1. Define a integral of $k-$forms.
2. Learn the concepts of line integral, surface integral and volume integral.
3. Learn the properties of the volume element.

5.2 Basic Preliminaries

$n-$**fold product:** $[0,1]^n$ denotes the $n-$fold product and is given by

$$[0,1]^n = [0,1] \times [0,1] \times \cdots \times [0,1]$$

Singular $n-$**cube:** A singular $n-$cube in $A \subset \mathbb{R}^n$ is a continuous function $C : [0,1]^n \longrightarrow A$.

Note: Let $\mathbb{R}^0$ and $[0,1]^0$ both denote $\{0\}$.

Standard $n-$cube: The standard $n-$cube $I^n : [0,1]^n \longrightarrow \mathbb{R}^n$ defined by $I^n(x) = x$ for $x \in [0,1]^n$.

Definitions and Properties:
1. The vector field $\vec{F}$ is known as solenoidal if Div$\vec{F} = 0$.
2. The vector field $\vec{F}$ is known as irrotational if Curl$\vec{F} = 0$.
3. If the vector field $\vec{F}$ is solenoidal then by Divergence theorem

$$\int\limits_M \text{div } F dv = \int\limits_{\partial M} \langle F, n \rangle dA = 0.$$

4. If the vector field $\vec{F}$ is irrotational then by Stokes theorem

$$\int\limits_M \langle (\nabla \times F), n \rangle dA = \int\limits_{\partial M} \langle F, T \rangle ds = 0.$$

5. If the line integral of a vector field is independent of path then such a vector fields are called conservative.
6. A conservative vector fields are irrotational and an irrotational vector fields are also conservative if domain is simply connected.

5.3 The Integral of $k-$form

The Integral of $k-$form on the cube $[0,1]^k$: If ω is a $k-$form on $[0,1]^k$, then $\omega = f dx^1 \wedge dx^2 \wedge \cdots \wedge dx^k$ for a unique function f. We define

$$\int\limits_{[0,1]^k} \omega = \int\limits_{[0,1]^k} f.$$

We could also write this as

$$\int\limits_{[0,1]^k} f dx^1 \wedge dx^2 \wedge \cdots dx^k = \int\limits_{[0,1]^k} f(x^1, x^2, \cdots, x^k) dx^1 dx^2 \cdots dx^k.$$

The Integral of $k-$form on the singular $k-$cube c: If ω is a $k-$form on A and c is a singular $k-$cube in A, we define

$$\int\limits_c \omega = \int\limits_{[0,1]^k} c^* \omega.$$

Note, in particular, that

$$\int_{I^k} f dx^1 \wedge dx^2 \wedge \cdots \wedge dx^k = \int_{[0,1]^k} (I^k)^* f(dx^1 \wedge dx^2 \wedge \cdots \wedge dx^k)$$
$$= \int_{[0,1]^k} f(x^1, x^2, \cdots, x^k) dx^1 dx^2 \cdots dx^k. \quad (1)$$

Note: (1) A $0-$form ω is a function; if $c : \{0\} \to A$ is a singular $0-$cube in A. We define

$$\int_c \omega = \omega(c(0))$$

(2) The integral of ω over a $k-$chain $c = \sum a_i c_i$ is defined by

$$\int_c \omega = \sum a_i \int_{c_i} \omega$$

(3) The integral of a $1-$form over a $1-$ chain is often called a line integral.
If $Pdx + Qdy$ is a $1-$form on $\mathbb{R}^2$ and $c : [0,1] \to \mathbb{R}^2$ is a singular $1-$cube (a curve), then one can prove that

$$\int_c Pdx + Qdy = \lim \sum_{i=1}^{n} [c^1(t_i) - c^1(t_{i-1})] \cdot P(c(t^i)) + [c^2(t_i) - c^2(t_{i-1})] \cdot Q(c(t^i))$$

where $t_0, t_1, \cdots, t_n$ is a partition of $[0,1]$, the choice of t^i in $[t_{i-1}, t_i]$ is arbitrary, and the limit is taken over all partition as the maximum of $[t_{i-1}, t_i]$ goes to 0.

5.4 Stokes's Theorem for Integral of $k-$forms

Theorem-15: Stokes Theorem If ω is a $(k-1)-$form on an open set $A \subset \mathbb{R}^n$ and c is a $k-$chain in A, then

$$\int_c d\omega = \int_{\partial c} \omega.$$

Proof: Suppose first that $c = I^k$ and ω is a $(k-1)-$form on $[0,1]^k$. Then ω is the sum of $(k-1)-$forms of the type

$$\omega = f dx^1 \wedge dx^2 \wedge \cdots \widehat{dx^i} \wedge \cdots dx^k$$

Note that

$$\int\limits_{[0,1]^{k-1}} {I^k_{(j,\alpha)}}^* (f dx^1 \wedge dx^2 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^k)$$

$$= \begin{cases} 0 & \text{if } i \neq j, \\ \int\limits_{[0,1]^k} f(x^1, x^2, \cdots, \alpha, \cdots, x^k) dx^1 dx^2 \cdots dx^k & \text{if } j = i. \end{cases}$$

Therefore

$$\int\limits_{\partial I^k} f dx^1 \wedge dx^2 \wedge \cdots \widehat{dx^i} \wedge \cdots \wedge dx^k$$

$$= \sum_{j=1}^{k} \sum_{\alpha=0,1} (-1)^{j+\alpha} \int\limits_{[0,1]^{k-1}} I^k_{(j,\alpha)} * (f dx^1 \wedge dx^2 \wedge \cdots \widehat{dx^i} \wedge \cdots \wedge dx^k)$$

on expanding summation and using equation (1)

$$= (-1)^{i+1} \int\limits_{[0,1]^k} f(x^1, x^2, \cdots, 1, \cdots, x^k) dx^1 dx^2 \cdots dx^k$$
$$+ (-1)^i \int\limits_{[0,1]^k} f(x^1, x^2, \cdots, 0, \cdots, x^k) dx^1 dx^2 \cdots dx^k. \qquad (2)$$

On the other hand,

$$\int\limits_{I^k} d(f dx^1 \wedge dx^2 \wedge \cdots \widehat{dx^i} \wedge \cdots \wedge dx^k) = \int\limits_{[0,1]^k} D_i f dx^i \wedge dx^1 \wedge dx^2 \wedge \cdots \widehat{dx^i} \wedge \cdots \wedge dx^k$$

$$= (-1)^{i-1} \int\limits_{[0,1]^k} D_i f.$$

By Fubini theorem and the fundamental theorem of calculus in one dimension

$$\int\limits_{I^k} d(f dx^1 \wedge dx^2 \wedge \cdots \widehat{dx^i} \wedge \cdots dx^k)$$

$$= (-1)^{i-1} \int\limits_{[0,1]} \int\limits_{[0,1]} \cdots (\int\limits_{[0,1]} D_i f(x^1, x^2, \cdots, \alpha, \cdots, x^k) dx^i) dx^1 dx^2 \cdots \widehat{dx^i} \cdots dx^k$$

$$= (-1)^{i-1} \int\limits_0^1 \int\limits_0^1 \cdots \int\limits_0^1 [f(x^1, x^2, \cdots, 1, \cdots, x^k) - f(x^1, x^2, \cdots, 0, \cdots, x^k)] dx^1 dx^2 \cdots dx^k.$$

$$= (-1)^{i-1} \int\limits_{[0,1]^k} f(x^1, x^2, \cdots, 1, \cdots, x^k) dx^1 dx^2 \cdots dx^k$$

$$+ (-1)^i \int\limits_{[0,1]^k} f(x^1, x^2, \cdots, 0, \cdots, x^k) dx^1 dx^2 \cdots dx^k.$$

Thus by equation (2) we have

$$\int\limits_{I^k} d\omega = \int\limits_{\partial I^k} \omega.$$

Note: If c is an arbitrary singular $k-$cube, working through the definitions will show that

$$\int\limits_{\partial c} \omega = \int\limits_{\partial I^k} c^* \omega.$$

Therefore

$$\int\limits_c d\omega = \int\limits_{I^k} c^*(d\omega) = \int\limits_{I^k} d(c^*\omega) = \int\limits_{\partial I^k} c^*\omega = \int\limits_{\partial c} \omega.$$

Finally, if c is a $k-$chain $\sum a_i c_i$, we have

$$\int\limits_c d\omega = \sum a_i \int_{c_i} d\omega = \sum a_i \int\limits_{\partial c_i} \omega = \int\limits_{\partial c} \omega.$$

5.5 Stokes's Theorem on Manifolds

If ω is a $p-$form on a $k-$dimensional manifold with boundary M and c is a singular $p-$cube in M, we define

$$\int\limits_c \omega = \int\limits_{[0,1]^p} c^* \omega \qquad (3)$$

Note: (1) In the case $p = k$ it may happen that there is an open set $W \supset [0,1]^k$ and a coordinate system $f : W \to \mathbb{R}^n$ such that $c(x) = f(x)$ for $x \in [0,1]^k$.
(2) If M is oriented, the singular $k-$cube c is called orientation-preserving if f is orientation-preserving.

Theorem (16): If $c_1, c_2 : [0,1]^k \to M$ are two orientation preserving singular $k-$cubes in the oriented $k-$dimensional manifold M and ω is a $k-$form on M such that $\omega = 0$ outside of $c_1([0,1]^k) \cap c_2([0,1]^k))$, then

$$\int_{c_1} \omega = \int_{c_2} \omega$$

Proof: We have

$$\int_{c_1} \omega = \int_{[0,1]^k} c_1^*(\omega) \quad \text{by equation (3)}$$
$$\int_{c_1} \omega = \int_{[0,1]^k} (c_2^{-1} \circ c_1)^* c_2^*(\omega)$$

Note that $c_2^{-1} \circ c_1$ is defined only on a subset of $[0,1]^k$ and the second equality depends on the fact that $\omega = 0$ outside of $c_1([0,1]^k) \cap c_2([0,1]^k)).)$

It therefore suffices to show that

$$\int_{[0,1]^k} (c_2^{-1} \circ c_1)^* c_2^*(\omega) = \int_{[0,1]^k} c_2^*(\omega) = \int_{c_2} \omega.$$

If $c_2^*(\omega) = f dx^1 \wedge f dx^2 \wedge \cdots \wedge f dx^k$ and $c_2^{-1} \circ c_1$, is denoted by g, then by Theorem (9) we have

$$\begin{aligned}
&(c_2^{-1} \circ c_1)^* c_2^*(\omega) = g^*(f dx^1 \wedge f dx^2 \wedge \cdots \wedge f dx^k) \\
&= (f \circ g) \cdot \det g' . dx^1 \wedge dx^2 \wedge \cdots \wedge dx^k \\
&= (f \circ g) \cdot |\det g'| . dx^1 \wedge dx^2 \wedge \cdots \wedge dx^k,
\end{aligned}$$

where $\det g' = \det(c_2^{-1} \circ c_1)' > 0$.
On integrating both sides over $[0,1]^k$, we obtain

$$\int_{[0,1]^k} (c_2^{-1} \circ c_1)^* c_2^*(\omega) = \int_{[0,1]^k} (f \circ g) \cdot |\det g'| . dx^1 \wedge dx^2 \wedge \cdots \wedge dx^k \quad (4)$$

Now lets apply following theorem to equation (4)
Let $A \subset \mathbb{R}^n$ be an open set and $g : A \longrightarrow \mathbb{R}^n$ is $1-1$ continuously

differentiable function such that $\det g'(x) \neq 0$ for all $x \in A$. If $f : g(A) \longrightarrow \mathbb{R}$ is integrable then

$$\int_{g(A)} f = \int_A (fog) \mid detg' \mid$$

Above theorem and equation (4) shows that

$$\int\limits_{[0,1]^k} (c_2^{-1} \circ c_1)^* c_2^*(\omega) = \int\limits_{[0,1]^k} f dx^1 \wedge dx^2 \wedge \cdots \wedge dx^k$$
$$\int\limits_{[0,1]^k} (c_2^{-1} \circ c_1)^* c_2^*(\omega) = \int\limits_{[0,1]^k} c_2^*(\omega) = \int\limits_{c_2} \omega$$

Note: (1) Let ω be a $k-$form on an oriented $k-$dimensional manifold M. If there is an orientation-preserving singular $k-$cube c in M such that $\omega = 0$ outside of $c([0,1]^k)$, we define

$$\int\limits_M \omega = \int\limits_c \omega.$$

Theorem (15) shows $\int\limits_M \omega$ does not depend on the choice of c.

(2) Suppose that ω is an arbitrary $k-$form on M. There is an open cover O of M such that for each $U \in O$ there is an orientation-preserving singular $k-$cube c with $U \subset c([0,1]^k)$. Let Φ be a partition of unity for M subordinate to this cover. We define

$$\int\limits_M \omega = \sum_{\varphi \in \Phi} \int \varphi \cdot \omega$$

Theorem-16: Stokes Theorem on Manifolds: If M is a compact oriented $k-$dimensional manifold with boundary and ω is a $(k-1)-$form on M, then

$$\int\limits_M d\omega = \int\limits_{\partial M} \omega.$$

(Here M is given the induced orientation.)

Proof: Case I: Suppose that there is an orientation-preserving singular $k-$cube in $M - \partial M$ such that $\omega = 0$ outside of $c((0,1)^k)$.

By Theorem (15) and the definition of $d\omega$ we have

$$\begin{aligned}\int\limits_c d\omega &= \int\limits_{[0,1]^k} c^*(d\omega) \text{ by equation (3)}\\ &= \int\limits_{[0,1]^k} d(c^*\omega) \text{ by theorem (14)}\\ &= \int\limits_{\partial I^k} (c^*\omega) \text{ by theorem (15)}\\ &= \int\limits_{\partial c} \omega \text{ by equation (3)}\end{aligned}$$

Then

$$\int\limits_M d\omega = \int\limits_c d\omega = \int\limits_{\partial c} \omega = 0.$$

since $\omega = 0$ on ∂c.
On the other hand, $\int\limits_{\partial M} \omega = O$ since $\omega = 0$ on ∂M.

Suppose that there is an orientation-preserving singular $k-$cube in M such that $c_(k, 0)$ is the only face in ∂M, and $\omega = 0$ outside of $c([0,1]^k)$
Then

$$\int\limits_M d\omega = \int\limits_c (d\omega) = \int\limits_{\partial c} \omega = \int\limits_{\partial M} \omega.$$

Case II: The general case: There is an open cover O of M and a partition of unity Φ for M subordinate to O such that for each $\varphi \in \Phi$ the form $\varphi \cdot \omega$ is of one of the two sorts already considered. We have

$$0 = d(1) = d\left(\sum_{\varphi\in\Phi} \varphi\right) = \sum_{\varphi\in\Phi} d(\varphi)$$

so that

$$\sum_{\varphi\in\Phi} d(\varphi) \wedge \Phi = 0.$$

Since M is compact, this is a finite sum and we have

$$\int_M \sum_{\varphi\in\Phi} d(\varphi) \wedge \Phi = 0.$$

Therefore

$$\begin{aligned}
\int\limits_M d\omega &= \sum_{\varphi\in\Phi}\int\limits_M \varphi\cdot d\omega \\
&= \sum_{\varphi\in\Phi}\int\limits_M d\varphi\wedge\omega+\varphi\cdot d\omega \quad \text{since } d\varphi=0 \\
&= \sum_{\varphi\in\Phi}\int\limits_M d(\varphi\cdot\omega) \\
&= \sum_{\varphi\in\Phi}\int\limits_{\partial M} \varphi\cdot\omega \\
&= \int\limits_{\partial M}\omega.
\end{aligned}$$

5.6 The Volume Element

The Volume Element Let M be a $k-$dimensional manifold (or manifold with boundary) in R^n, with an orientation μ. If $x \in M$, then μ_x and the inner product T_x we defined previously determine a volume element $\omega(x) \in \Lambda^k(M_x)$. We therefore obtain a nowhere-zero $k-$form ω on M, which is called the volume element on M (determined by μ) and denoted dV, even though it is not generally the differential of a $(k-1)-$form.

The volume of M is defined as $\int\limits_M dV$, provided this integral exists, which is certainly the case if M is compact.

Note: (1) Volume is usually called length or surface area for one and two-dimensional manifolds, and dV is denoted ds (the "element of length") or dA [or ds] (the "element of (surface) area"). (2) Consider the volume element of an oriented surface (two-dimensional manifold) M in $\mathbb{R}^3$. Let $n(x)$ be the unit outward normal at $x \in M$. If $\omega \in \Lambda^2(M_x)$ is defined by

$$\omega(v,w) = \det\begin{bmatrix} v \\ w \\ n(x)\end{bmatrix},$$

then $\omega(v,w) = 1$ if v and w are an orthonormal basis of M_x with $[v,w] = \mu_x$. Thus $dA = \omega$.

On the other hand, $\omega(v,w) = \langle v\times w, n(x)\rangle$ by definition of $v\times w$. Thus we have $dA(v,w) = \langle v\times w, n(x)\rangle$. Since $v\times w$ is a multiple of $n(x)$

for $v, w \in M$, we conclude that $dA(v,w) = |v \times w|$ if $[v,w] = \mu_x$. (3) If we wish to compute the area of M, we must evaluate $\int_{[0,1]^2} c^*(dA)$ for orientation-preserving singular 2−cubes c. Define

$$E(a) = [D_1c^1(a)]^2 + [D_1c^2(a)]^2 + [D_1c^3(a)]^2.$$
$$F(a) = [D_1c^1(a) \cdot D_2c^1(a)] + [D_1c^2(a) \cdot D_2c^2(a)] + [D_1c^3(a) \cdot D_2c^3(a)]$$
$$.G(a) = [D_2c^1(a)]^2 + [D_2c^2(a)]^2 + [D_2c^3(a)]^2.$$

Then

$$\begin{aligned} c^*(dA)((e_1)_a, (e_2)_a,) &= dA(c_*(e_1)_a, c_*(e_2)_a,) \\ &= |(D_1c^1(a), D_1c^2(a), D_1c^3(a)) \cdot (D_2c^1(a), D_2c^2(a), D_2c^3(a))| \\ &= \sqrt{E(a)G(a) - F(a)^2} \end{aligned}$$

Thus

$$\int_{[0,1]^2} c*(dA) = \int_{[0,1]^2} \sqrt{E(a)G(a) - F(a)^2}.$$

Theorem-18: Let M be an oriented two-dimensional manifold (or manifold with boundary) in R^3 and let n be the unit outward normal. Then

(1) $\quad dA = n^1dy \wedge dz + n^2dz \wedge dx + n^3dx \wedge dy.$

Moreover, on M we have

(2) $\quad n^1dA = dy \wedge dz.$

(3) $\quad n^2dA = dz \wedge dx.$

(4) $\quad n^3dA = dx \wedge dy.$

Proof: Equation (1) is equivalent to the equation

$$dA(v,w) = \det \begin{bmatrix} v \\ w \\ n(x) \end{bmatrix},$$

This is seen by expanding the determinant by minors along the bottom row.

To prove the other equations, let $z \in \mathbb{R}^3_x$. Since $v \times w = \alpha n(x)$ for some $\alpha \in R$, we have

$$\langle z, n(x)\rangle \cdot \langle v \times w, n(x)\rangle = \langle z, n(x)\rangle\alpha = \langle z, \alpha n(x)\rangle = \langle z, v \times w\rangle.$$

Choosing $z = e_1, e_2$, and e_3 we obtain $(2), (3)$ and (4).

A word of caution; if $\omega \in \Lambda^2(\mathbb{R}^3_a)$ is defined by

$$\omega = n^1(a) \cdot dy(a) \wedge dz(a) + n^2(a) \cdot dz(a) \wedge dx(a) + n^3(a) \cdot dx(a) \wedge dy(a),$$

it is not true, for example, that $n^1(a).w = dy(a) \wedge dz(a)$. The two sides give the same result only when applied to $v, w \in M_a$.

5.7 Chapter End Exercise

1. State and prove the Stokes theorem for any $3-$forms ω.
2. Consider vector field $\vec{F} = (y+z)i + (z+x)j + (x+y)k$. Is vector field $\vec{F}$ solenoidal and irrotational? Justify your answer.
3. Let M be a two-dimensional manifold in $\mathbb{R}^3$. Compute the area of M over orientation preserving singular $2-$cubes c.
4. Consider an orientation-preserving singular $k-$cube in $M - \partial M$ such that $\omega = 0$ outside of $c((0,1)^k)$ where M is a compact oriented $k-$dimensional manifold with boundary and ω is a $(k-1)-$form on M then show that $\int\limits_M d\omega = \int\limits_{\partial M} \omega$.

Chapter 6

Classical Theorems

Unit Structure :
6.1 Objective
6.2 Classical Theorems
6.3 Applications of classical theorem
6.4 Chapter End Exercise

6.1 Objectives

After going through this chapter you will be able to:

1. Evaluation of a line integral using Green's Theorem.
2. Evaluation of a volume integral using Divergence Theorem.
3. Evaluation of a surface integral using Stoke's Theorem.
4. Learn a concept of conservative fields.

6.2 Classical Theorems

Theorem-19: Green's Theorem: Let $M \subset \mathbb{R}^2$ be a compact two-dimensional manifold with boundary. Suppose that $\alpha, \beta : M \to \mathbb{R}$ are differentiable. Then

$$\int_{\partial M} \alpha dx + \beta dy = \int_{M} (D_1\beta - D_2\alpha) dx \wedge dy = \iint_{M} \left(\frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y} \right) dxdy$$

(Here M is given the usual orientation, and ∂M the induced orientation, also known as the counter clockwise orientation.)

Proof: We have the Stoke's theorem on Manifolds as
If M is a compact oriented $k-$dimensional manifold with boundary and ω is a $(k-1)-$form on M, then

$$\int_{M} d\omega = \int_{\partial M} \omega.$$

Let $\omega = \alpha dx + \beta dy$
$\Rightarrow d\omega = D_1\alpha dx \wedge dx + D_2\alpha dy \wedge dx + D_1\beta dx \wedge dy + D_2\beta dy \wedge dy$
$\Rightarrow d\omega = -D_2\alpha dx \wedge dy + D_1\beta dx \wedge dy$
$\Rightarrow d\omega = (D_1\beta - D_2\alpha) dx \wedge dy$
Substitute in above toke's theorem on Manifolds we obtain

$$\int_{\partial M} \alpha dx + \beta dy = \int_{M} (D_1\beta - D_2\alpha) dx \wedge dy = \iint_{M} \left(\frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y} \right) dxdy$$

Theorem-20: Divergence Theorem: Let $M \subset \mathbb{R}^3$ be a compact three-dimensional manifold with boundary and n the unit outward normal on ∂M. Let F be a differentiable vector field on M. Then

$$\int_{M} \text{div } F dv = \int_{\partial M} \langle F, n \rangle dA.$$

This equation is also written in terms of three differentiable functions $\alpha, \beta, \gamma : M \to \mathbb{R}$:

$$\iiint_{M} \left(\frac{\partial \alpha}{\partial x} + \frac{\partial \beta}{\partial y} + \frac{\partial \gamma}{\partial z} \right) dV = \iint_{\partial M} (n^1\alpha + n^2\beta + n^3\gamma) dS.$$

Proof: Define ω on M by $\omega = F^l dy \wedge dz + F^2 dz \wedge dx + F^3 dx \wedge dy$
Then $d\omega = \text{div } F dV$. See example $III(3)$ of Unit 2
According to Theorem-18, on ∂M we have

$$\begin{aligned} n^1 dA &= dy \wedge dz, \\ n^2 dA &= dz \wedge dx, \\ n^3 dA &= dx \wedge dy. \end{aligned}$$

Therefore on ∂M we have

$\langle F, n \rangle dA = F^1 n^1 dA + F^2 n^2 dA + F^3 n^3 dA,$
Since $F = (F^1, F^2, F^3)$ and $n = (n^1, n^2, n^3)$
$\langle F, n \rangle dA = F^1 dy \wedge dz + F^2 dz \wedge dx + F^3 dx \wedge dy,$
$\langle F, n \rangle dA = \omega.$

We have the Stoke's theorem on Manifolds as
If M is a compact oriented $k-$dimensional manifold with boundary and ω is a $(k-1)-$form on M, then

$$\int\limits_{M} d\omega = \int\limits_{\partial M} \omega.$$

Thus using values of ω and $d\omega$ in the above theorem, we obtain

$$\int\limits_{M} \operatorname{div} F \, dV = \int\limits_{\partial M} \langle F, n \rangle dA.$$

Theorem-21: Stokes' Theorem: Let $M \subset \mathbb{R}^3$ be a compact oriented two-dimensional manifold with boundary and n the unit outward normal on M determined by the orientation of M. Let ∂M have the induced orientation. Let T be the vector field on ∂M with $ds(T) = 1$ and let f be a differentiable vector field in an open set containing M. Then

$$\int\limits_{M} \langle (\nabla \times F), n \rangle dA = \int\limits_{\partial M} \langle F, T \rangle ds.$$

This equation also written as

$$\int\limits_{\partial M} \alpha dx + \beta dy + \gamma dz = \iint\limits_{M} \left[n^1 \left(\frac{\partial \gamma}{\partial y} - \frac{\partial \beta}{\partial z} \right) + n^2 \left(\frac{\partial \alpha}{\partial z} - \frac{\partial \gamma}{\partial x} \right) + n^3 \left(\frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y} \right) \right] dS$$

Proof: Define ω on M by $\omega = F^l dx + F^2 dy + F^3 dz$.
Since $\nabla \times F = (D_2F^3 - D_3F^2, D_3F^1 - D_1F^3, D_1F^2 - D_2F^1)$
it follows that on M we have

$$\langle (\nabla \times F), n \rangle dA = (D_2F^3 - D_3F^2) n^1 dA + (D_3F^1 - D_1F^3) n^2 dA + (D_1F^2 - D_2F^1) n^3 dA$$

According to Theorem-18, on ∂M we have

$$n^1 dA = dy \wedge dz,$$
$$n^2 dA = dz \wedge dx,$$
$$n^3 dA = dx \wedge dy.$$

Therefore on M we have

$\langle (\nabla \times F), n \rangle dA$
$= (D_2F^3 - D_3F^2) dy \wedge dz + (D_3F^1 - D_1F^3) dz \wedge dx + (D_1F^2 - D_2F^1) dx \wedge dy$
$= d\omega.$ See example III(2) of Unit 2

On the other hand, since $ds(T) = 1$, on ∂M we have

$$T_1 ds = dx,$$
$$T_2 ds = dy,$$
$$T_3 ds = dz.$$

Therefore on ∂M we have

$$\langle F, T\rangle ds = F^l T^1 ds + F^2 T^2 ds + F^3 T^3 ds = F^l dx + F^2 dy + F^3 dz = \omega$$

We have the Stoke's theorem on Manifolds as
If M is a compact oriented $k-$dimensional manifold with boundary and ω is a $(k-1)-$form on M, then

$$\int\limits_M d\omega = \int\limits_{\partial M} \omega.$$

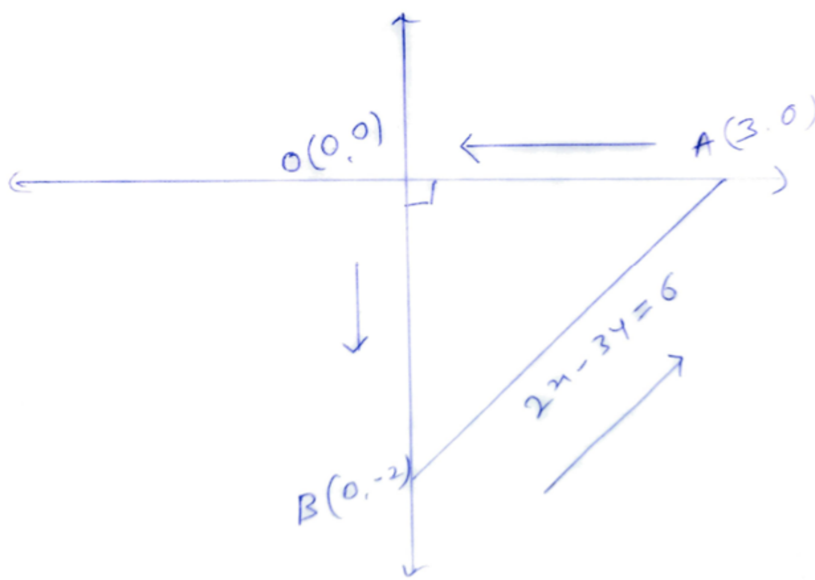
Thus using values of ω and $d\omega$ in the above theorem, we obtain

$$\int\limits_M \langle(\nabla \times F), n\rangle dA = \int\limits_{\partial M} \langle F, T\rangle ds.$$

6.3 Applications of classical theorem

Example 1: State and verify Green's Theorem in the plane for $\oint(3x^2 - 8y^2)dx + (4y - 6xy)dy$ where C is boundary of the region bounded by $x \geq 0$, $y \leq 0$ and $2x - 3y = 6$.

Solution: Here closed curve C consists of straight lines OB, BA and AO, where coordinates of A and B are (3, 0) and (0, -2) respectively.
Let R be the region bounded by C.
Then by Green's Theorem in plane, we have,
$\oint(3x^2 - 8y^2)dx + (4y - 6xy)dy = \iint_R [\frac{\partial}{\partial x}(4y - 6xy) - \frac{\partial}{\partial y}(3x^2 - 8y^2)]dxdy$........(1)
$= \iint_R (-6y + 16y)dxdy$
$= \iint_R (10y)dxdy$
$= 10 \int_0^3 dx \int_{\frac{1}{3}(2x-6)}^0 ydy$
$= 10 \int_0^3 dx =$ -20
Now we evaluate L.H.S. of (1) along OB, BA and AO.
Along OB, $x = 0$, $dx = 0$and y varies from 0 to -2.
Along BA, $x = \frac{1}{2}(6 + 3y)$, $dx = \frac{3}{2}$dy and y varies -2 to 0.
and along AO, $y = 0$, $dy = 0$ and x varies from 3 to 0



L.H.S of (1) = $\oint(3x^2$ - $8y^2)dx + (4y - 6xy)dy$
= $\int_{OB}(3x^2$ - $8y^2)dx$+$(4y-6xy)dy$+$\int_{BA}(3x^2$ - $8y^2)dx$+$(4y-6xy)dy$+$\int_{AO}(3x^2$ - $8y^2)dx + (4y - 6xy)dy$
= $\int_0^{-2}4ydy$ + $\int_{-2}^{0}$ [$\frac{9}{8}(6+3y)^2$ - $12y^2$+$4y$-$18y$-$9y^2$]dy+$\int_3^0 3x^2dx$
= $[2y^2]_0^{-2}$ + $\int_{-2}^{0}$ [$\frac{9}{8}(6+3y)^2$ - $12y^2$+$4y$-$18y$-$9y^2$]dy + $[x^3]_3^0$
= [2(4)] + $\int_{-2}^{0}$ [$\frac{9}{8}(6+3y)^2$ - $21y^2$-$14y$]dy + [0-27]
= -19+27-56+28
= -20
with help of (2) and (3), we find that (1) is true and so Green's Theorem is verified.

Example 2: Verify Stoke's theorem for the vector field $\vec{F} = (2x-y)\hat{i}$ - y$z^2\hat{j}$ - y^2z$\hat{k}$ over the upper half of the surface x^2+y^2+z^2=1 bounded by its projection on xy-plane.

Solution: Let S be the upper half of the surface x^2+y^2+z^2=1. The boundary Cor$S$ is a circle in the xy plane of radius unity and centre O. The equation of C are x^2+$y^2 = 1$, $z = 0$
whose parametric form is $x = cos(t)$, $y = sin(t)$, $z = 0$, 0 < t < 2π.
$\int_C \vec{F} \cdot \vec{dr} = \int_C$ [$(2x-y)\hat{i}$ - $yz^2\hat{j}$ - y^2z$\hat{k}$] $\cdot$ [$dx\hat{i}$+$dy\hat{j}$+$dz\hat{k}$]
= $\int_C$ [$(2x-y)dx$ - yz^2dy - y^2zdz]
= $\int_C$ [$(2x-y)dx$since on C, $z = 0$ and $2z = 0$
= $\int_0^{2\pi}$ [$2cos(t) - sin(t)$]$\frac{dx}{dt}dt$
= $\int_0^{2\pi}$[$2cos(t) - sin(t)$]$(-sin(t))dt$
= $\int_0^{2\pi}$ [$-sin(2t)$-$sin^2(t)$]dt
= $\int_0^{2\pi}$ [$-sin(2t)$+$\frac{1-cos(2t)}{2}$]dt
= $[\frac{cos(2t)}{2}$+$\frac{t}{2}$-$\frac{sin(2t)}{4}]_0^{2\pi}$

$= \frac{1}{2} + \pi - \frac{1}{2} = \pi$...........(1)

Consider,

$$\text{Curl}\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x - y & -yz^2 & -zy^2 \end{vmatrix} = (-2yz+2yz)\hat{i}+(0-0)\hat{j}+(0+1)\hat{k}=\hat{k}$$

$\text{Curl}\vec{F} \cdot \hat{n} = \hat{k} \cdot \hat{n} = \hat{n} \cdot \hat{k}$

$\iint_S \text{Curl}\vec{F} \cdot \hat{n}\text{ds} = \iint_S \hat{n} \cdot \hat{k} ds = \iint_R \hat{n} \cdot \hat{k} \, \frac{dx}{\hat{n}} \, \frac{dy}{\hat{k}}$

where R is the projection of S on xy-plane.

$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dxdy$

$= \int_{-1}^{1} 2\sqrt{1-x^2}dx$

$= 4\int_0^1 \sqrt{1-x^2}dx$

$= 4[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}sin^{-1}(x)]_0^1$

$= 4[\frac{1}{2}][\frac{\pi}{2}]$

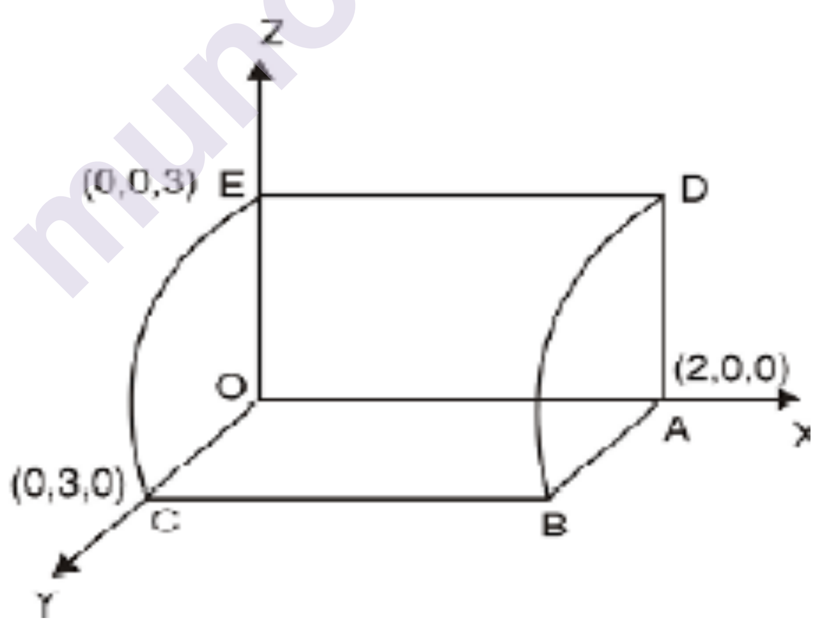
$= \pi$

From (1) and (2), we have,

$\int_C \vec{F} \cdot \vec{dr} = \text{Curl}\vec{F} \cdot \hat{n}ds$ which is the stoke's theorem.

Example 3: Verify the divergence theorem for the function $\vec{F} = 2x^2y\hat{i}$-$y^2\hat{j}$+$4xz^2\hat{k}$ taken over the region in the first octant bounded by $y^2+z^2 = 9$ and $x = 2$.

Solution: $\iiint_V \nabla \cdot \vec{F}dV = \iiint (\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}) \cdot (2x^2y\hat{i}$-$y^2\hat{j}$+$4xz^2\hat{k})dV$



$= \iiint (4xy - 2y + 8xz)dxdydz$

$= \int_0^2 dx\int_0^3 dy\int_0^{\sqrt{9-y^2}}(4xy - 2y + 8xz)dz$

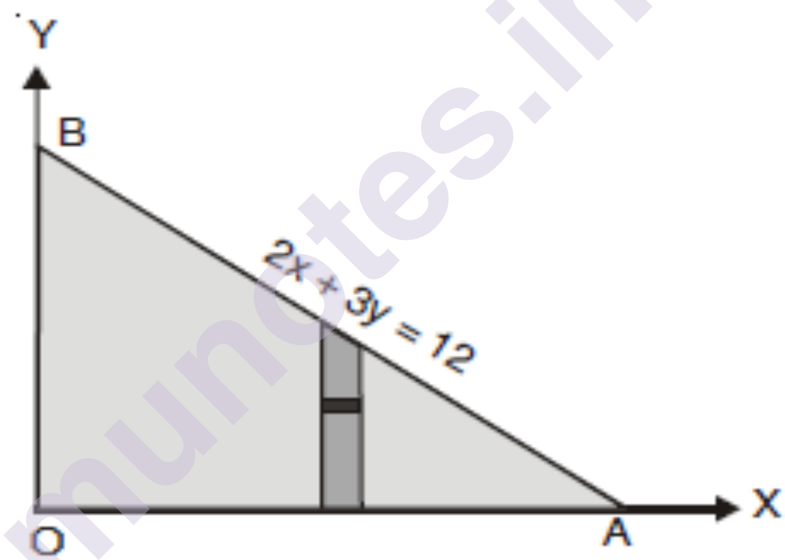
$= \int_0^2 dx\int_0^3 dy[(4xyz - 2yz + 4xz^2)]_0^{\sqrt{9-y^2}}$

$= \int_0^2 dx \int_0^3 [(4xy\sqrt{9-y^2} - 2y\sqrt{9-y^2} + 4x(9-y^2)]dy$
$= \int_0^2 dx[-\frac{4x}{2}\frac{2}{3}(9-y^2)^{\frac{3}{2}} + \frac{2}{3}(9-y^2)^{\frac{3}{2}} + 36xy - \frac{4xy^3}{3}]$
$= \int_0^2 \ (0+0+108x-36x+36x-18)dx$
$= \int_0^2 \ (108x-18)dx$
= 216-36
= 180
Here, $\iint_S \vec{F}\cdot\hat{n}\ ds = \iint_{OABC} \vec{F}\cdot\hat{n}\ ds + \iint_{OCE} \vec{F}\cdot\hat{n}\ ds + \iint_{OADE} \vec{F}\cdot\hat{n}\ ds + \iint_{ABD} \vec{F}\cdot\hat{n}\ ds + \iint_{BDEC} \vec{F}\cdot\hat{n}\ ds$
Consider,
$\iint_{BDEC} \vec{F}\cdot\hat{n}\ ds = \iint_{BDEC} (2x^2\text{y}\hat{i}\text{-}y^2\hat{j}\text{+}4\text{x}z^2\hat{k})\cdot\hat{n}\ ds$.........................(1)
Normal vector
$= \nabla\phi = (\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k})(y^2+z^2-9) = 2\text{y}\hat{j} + 2\text{z}\hat{k}$
Unit normal vector $= \hat{n} = \dfrac{2y\hat{j}+2z\hat{k}}{\sqrt{4y^2+4z^2}} = \dfrac{y\hat{j}+z\hat{k}}{\sqrt{y^2+z^2}} = \dfrac{y\hat{j}+z\hat{k}}{\sqrt{9}} = \dfrac{y\hat{j}+z\hat{k}}{3}$
From (1),
$\iint_{BDEC} (2x^2\text{y}\hat{i}\text{-}y^2\hat{j}\text{+}4\text{x}z^2\hat{k})\cdot\dfrac{y\hat{j}+z\hat{k}}{3}\ ds = \frac{1}{3}\iint_{BDEC} (\text{-}y^3\text{+}4xz^3)ds$
$= \frac{1}{3}\iint_{BDEC} (\text{-}y^3\text{+}4xz^3)\dfrac{dxdy}{\frac{z}{3}}$
$= \int_0^2 \ dx \ \int_0^3(\text{-}\dfrac{y^3}{z}\text{+}4xz^2)\ dy$
by substitution, y = 3sin(θ), z = 3cos(θ))
$= \int_0^2 \ dx \ \int_0^{\frac{\pi}{2}}[\frac{-27sin^3(\theta)}{3cos(\theta)}\text{+}4x(9cos^2\theta)]$
$= \int_0^2 \ dx((\text{-}27)(\frac{2}{3})\text{+}108x(\frac{2}{3}))$
$= \int_0^2 \ (\text{-}18\text{+}72x)dx$
= 108....................................(2)
Consider,
$\iint_{OABC} (2x^2\text{y}\hat{i}\text{-}y^2\hat{j} + 4xz^2\hat{k})\cdot(\text{-}\hat{k})ds$
$= \iint_{OABC} 4xz^2 = 0$.................(3) because in OABC xy-plane, $z = 0$
Consider,
$\iint_{OADE} (2x^2y\hat{i}\text{-}y^2\hat{j} + 4xz^2\hat{k})\cdot(\text{-}\hat{j})ds$
$= \iint_{OADE} y^2\ ds = 0$.................(4) because in OADE xz-plane, $y = 0$
Consider,
$\iint_{OCE} (2x^2\text{y}\hat{i}\text{-}y^2\hat{j} + 4\text{x}z^2\hat{k})\cdot(\text{-}\hat{i})ds$
$= \iint_{OCE} \text{-}\ 2x^2yds = 0$.................(5) because in OCE yz-plane, $x = 0$
Consider,
$\iint_{ABD} (2x^2y\hat{i}\text{-}y^2\hat{j} + 4xz^2\hat{k})\cdot(\hat{i})ds$
$= \iint_{ABD} 2x^2yds$
$= \iint_{ABD} 2x^2ydydz$
$= \int_0^3 \ dz \ \int_0^{\sqrt{9-z^2}} 2(2)^2\ ydy$ because in ABD plane, $x = 2$

$= 8 \int_0^3 dz[\frac{y^2}{2}]_0^{\sqrt{9-z^2}}$
$= 4 \int_0^3 dz(9 - z^2)$
$= 4[9z - \frac{z^3}{3}]_0^3$
$= 4[27\text{-}9]$
$= 72$....................................(6)
on adding (2), (3), (4), (5) and (6), we get
$\iint_S \vec{F} \cdot \hat{n}$ ds = 108 + 0 + 0 + 0 + 72 = 180......(7)
from (1) to (7), we have, $\iiint_V \nabla \cdot \vec{F}$dV $= \iint_S \vec{F}\cdot\hat{n}\ ds$
Hence the theorem is verified.

Example 4: Evaluate $\iint_S \vec{A} \cdot \hat{n}\ ds$ where $\vec{A} = 18z\hat{i} - 12\hat{j} + 3y\hat{k}$ and S is the part of the plane $2x + 3y + 6z = 12$ included in the first octant.

Solution: Here $\vec{A} = 18z\hat{i} - 12\hat{j} + 3y\hat{k}$

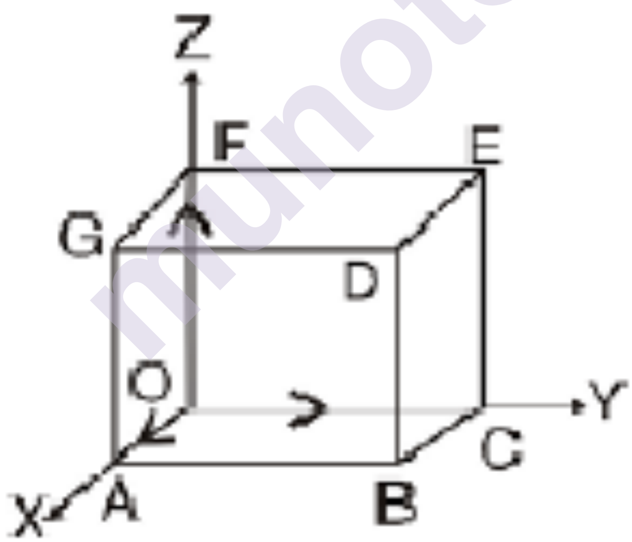


Given surface $f(x, y, z) = 2x + 3y + 6z - 12$
Normal vector = ∇ f = $(\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k})(2x + 3y + 6z - 12) = 2\hat{i}+3\hat{j}+6\hat{k}$
$\hat{n}$ = unit normal vector at any point (x, y, z) of $2x + 3y + 6z = 12$
$= \frac{2\hat{i} + 3\hat{j} + 6\hat{k}}{\sqrt{4 + 9 + 16}} = \frac{1}{7}(2\hat{i} + 3\hat{j} + 6\hat{k})$
and $dS = \frac{dxdy}{\hat{n}\cdot\hat{k}} = \frac{dxdy}{\frac{1}{7}(2\hat{i} + 3\hat{j} + 6\hat{k})\cdot\hat{k}} = \frac{dxdy}{\frac{6}{7}} = \frac{7}{6}dxdy$
Consider,
$\iint_S \vec{A} \cdot \hat{n}\ ds = \iint\ (18z\hat{i} - 12\hat{j} + 3y\hat{k})\frac{1}{7}(2\hat{i} + 3\hat{j} + 6\hat{k})\ \frac{7}{6}\ dxdy$
$= \iint\ (36z - 36 + 18y)\frac{dxdy}{6}$

$= \iint (6z - 6 + 3y)dxdy$
putting the value of $6z = 12 - 2x - 3y$, we get,
$= \int_0^6 \int_0^{\frac{1}{3}(12-2x)}$ (12-2x-3y-6+3y)$dxdy$
$= \int_0^6 \int_0^{\frac{1}{3}(12-2x)} (6 - 2x)dxdy$
$= \int_0^6 (6 - 2x)dx \int_0^{\frac{1}{3}(12-2x)} dy$
$= \int_0^6 (6 - 2x)dx \ (y)_0^{\frac{1}{3}(12-2x)}$
$= \int_0^6 (6 - 2x) \ \frac{1}{3}(12 - 2x)dx$
$= \dfrac{1}{3} \int_0^6 (4x^2 - 36x + 72)dx$
$= \dfrac{1}{3} [\dfrac{4x^3}{3} - 18x^2 + 72x]_0^6$
$= \dfrac{72}{3}$ [4-9+6]
$= 24$

Example 5: Show that $\iint_S \vec{F}\cdot\hat{n} \ ds = \frac{3}{2}$, where $\vec{F} = 4xz\hat{i}$ - $y^2\hat{j}$ + $yz\hat{k}$ and S is the surface of the cube bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$ and $z = 1$.

Solution: $\iint_S \vec{F}\cdot\hat{n} \ ds$



$= \iint_{OABC} \vec{F}\cdot\hat{n} \ ds + \iint_{DEFG} \vec{F}\cdot\hat{n} \ ds + \iint_{OAGF} \vec{F}\cdot\hat{n} \ ds + \iint_{BCED} \vec{F}\cdot\hat{n} \ ds + \iint_{ABDG} \vec{F}\cdot\hat{n} \ ds + \iint_{OCEF} \vec{F}\cdot\hat{n} \ ds$.........(1)

Consider,
$\iint_{OABC} \vec{F}\cdot\hat{n} \ ds$
$= \iint_{OABC} (4xz\hat{i}$ - $y^2\hat{j}$ + $yz\hat{k})$(-$\hat{k}$) $dxdy$

$= \int_0^1 \int_0^1 (-yz)dxdy$
$= 0$ (as $z = 0$)

Consider,
$\iint_{DEFG} \vec{F} \cdot \hat{n}\ ds$
$= \iint_{DEFG} (4xz\hat{i}$ - $y^2\hat{j}$ + yz$\hat{k}) \cdot (\hat{k})\ dxdy$
$= \iint_{DEFG} yzdxdy$
$= \int_0^1 \int_0^1 y(1)\ dxdy$
$= \int_0^1 dx\ [\frac{y^2}{2}]_0^1$
$= \frac{1}{2}$

Consider, $\iint_{OAGF} \vec{F} \cdot \hat{n}\ ds$
$= \iint_{OAGF} (4xz\hat{i}$ - $y^2\hat{j} + yz\hat{k}) \cdot$(-$\hat{j})\ dxdz$
$= 0$

Consider, $\iint_{BCED} \vec{F} \cdot \hat{n}\ ds = \iint_{BCED} (4xz\hat{i}$ - $y^2\hat{j}$ + yz$\hat{k}) \cdot (\hat{j})\ dxdz$
$= \iint_{BCED}$ (-y^2) $dxdz$
$= \int_0^1 \int_0^1$ (-1) $dxdz$......(as $y = 1$)
= -1

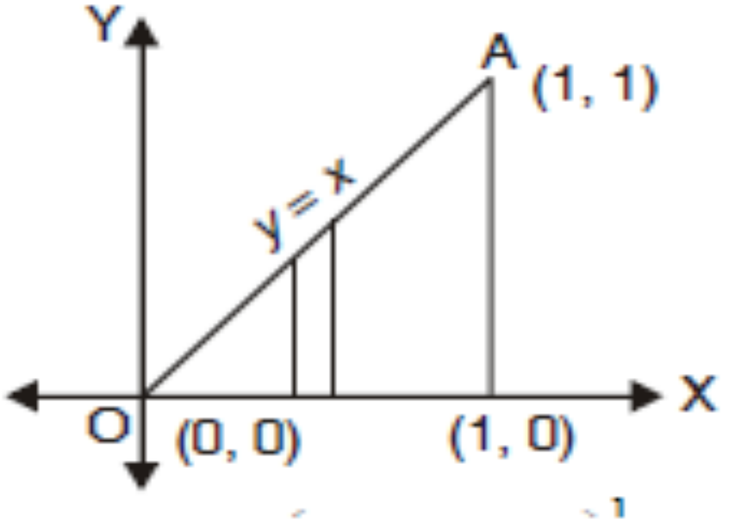
Consider, $\iint_{ABDG} \vec{F} \cdot \hat{n}\ ds$
$= \iint_{ABDG} (4xz\hat{i}$ - $y^2\hat{j}$ + yz$\hat{k}) \cdot (\hat{i})\ dydz$
$= \iint 4xzdydz = \int_0^1 \int_0^1 4(1)\ zdydz$......(as $x = 1$)
$= 2$

Consider, $\iint_{OCEF} \vec{F} \cdot \hat{n}$ ds $= \iint_{OCEF} (4xz\hat{i}$ - $y^2\hat{j}$ + yz$\hat{k}) \cdot$(-$\hat{i})\ dydz$
$= \int_0^1 \int_0^1$ - $4xzdydz$......(as $x = 0$)
$= 0$
putting all values in equation (1),
$\iint_S \vec{F} \cdot \hat{n}\ ds = \frac{3}{2}$.

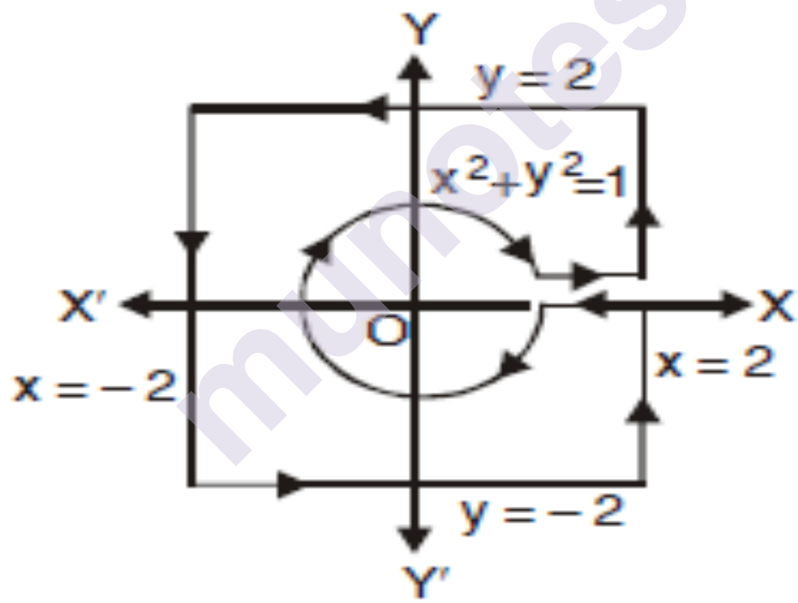
Example 6: Using Green's theorem, evaluate $\int_C (x^2y\ dx + x^2dy)$ where C is the boundary described counter clockwise of the triangle with vertices (0,0), (1,0) and (1,1).

Solution: By Green's theorem, we have,
$\int_C (x^2y\ dx + x^2dy) = \iint_R (2x$ - $x^2)$ dxdy
$= \int_0^1 (2x$ - $x^2)\ dx \int_0^x dy$
$= \int_0^1 (2x$ - $x^2)\ dx\ [y]_0^x$
$= \int_0^1 (2x$ - $x^2)(x)\ dx$
$= \frac{5}{12}$

Example 7: Evaluate $\oint_C$ -$\dfrac{y}{x^2+y^2}\ dx + \dfrac{x}{x^2+y^2}\ dy$ where $C = C_1 \cup C_2$ with C_1: $x^2 + y^2 = 1$ and C_2: x = 2, -2 and $y = 2$, -2.



Solution: Consider $\oint_C$ -$\dfrac{y}{x^2+y^2}\ dx + \dfrac{x}{x^2+y^2}\ dy$



$$= \iint \frac{\partial}{\partial x}\frac{x}{(x^2+y^2)} + \frac{\partial}{\partial y}\frac{y}{(x^2+y^2)}\ dxdy$$
$$= \iint \frac{(x^2+y^2)1-2x(x)}{(x^2+y^2)^2} + \frac{(x^2+y^2)1-2y(y)}{(x^2+y^2)^2}\ dxdy$$
$$= \iint \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} + \frac{x^2+y^2-2y^2}{(x^2+y^2)^2}\ dxdy$$
$$= \iint \frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2}\ dxdy$$

$= \iint \dfrac{0}{(x^2+y^2)^2} dxdy$
$= 0$

Example 8: Directly or by Stoke's theorem, evaluate $\iint_S$ curl $\vec{v} \cdot \hat{n}$ dS, $\vec{v}$ = y$\hat{i}$+z$\hat{j}$+x$\hat{k}$, S is the surface of the paraboloid z = 1 - x^2 - y^2, $z^3 \geq 0$ and $\hat{n}$ is the unit vector normal to S.

Solution:
$\nabla \times \vec{v}$ = - $\hat{i}$ - $\hat{j}$ - $\hat{k}$
Obviously, $\hat{n} = \hat{k}$
$(\nabla \times \vec{v}) \cdot \hat{n}$ = (- $\hat{i}$ - $\hat{j}$ - $\hat{k}$) $\cdot \hat{k}$ = -1
$\iint_S (\nabla \times \vec{v}) \cdot \hat{n}\ ds = \iint_S (-1)\ dxdy = - \iint_S dxdy = - \pi\ (1)^2 = - \pi.$

6.4 Chapter End Exercise

1. If $\vec{F}$ = $2y\hat{i}$ - $3\hat{j}$ + $x^2\hat{k}$ and S is the surface of parabolic cylinder $y^2 = 8x$ in the first octant bounded by the planes $y = 4$ and $z = 6$ then evaluate $\iint_S \vec{F} \cdot \hat{n}\ dS$. [**Ans. 132**]

2. If $\vec{F}$ = ($2x^2$-3z)$\hat{i}$ - 2xy$\hat{j}$ - 4x$\hat{k}$ then evaluate $\iiint_V \nabla \times \vec{F}\ dV$ where V is the closed region bounded by planes $x = 0$, $y = 0$, $z = 0$ and $2x + 2y + z = 4$.[Ans. $\frac{8}{3}(\hat{j}$ - $\hat{k})$]

3. Evaluate $\iiint_V (2x + y)dV$ where V is the closed region bounded by the cylinder $z = 4 - x^2$ and the planes $x = 0$, $y = 0$, $y = 2$ and $z = 0$.[**Ans.** $\frac{80}{3}$]

4. Either directly or by Green's theorem, evaluate the line integral $\int_C e^{-x}\,(cos(y)dx - sin(y)dy)$ where C is the rectangle with vertices (0, 0), (π, 0), (π, $\frac{\pi}{2}$) and (0, $\frac{\pi}{2}$).[**Ans.2(1-**$e^{-\pi}$)]

5. Use the Green's theorem in a plane to the evaluate the integral $\int_C[(2x^2$- $y^2)dx$+$(x^2$+$y^2)dy]$ where C is the boundary in the xy-plane of the area enclosed by the x-axis and the semi-circle $x^2 + y^2 = 1$ in the upper half xy-plane.[**Ans.** $\frac{4}{3}$]

6. If $\vec{F}$ = $3y\hat{i}$ - $xy\hat{j}$ + $yz^2\hat{k}$ and S is the surface of the parboloid $2z = x^2 + y^2$ bounded by $z = 2$, show by using Stoke's theorem that $\iint_S$ curl $\times\ \vec{F} \cdot dS = 20\ \pi$

7. If $\vec{F}$ = $(x - z)\ \hat{i}$ + $(x^3 + yz)\ \hat{j}$ + 3xy^2 $\hat{k}$ and S is the surface of the cone $z = a$ - $\sqrt{x^2+y^2}$ above the xy-plane, show that $\iint_S$ curl $\vec{F} \cdot dS = \frac{3\pi a^4}{4}$.

8. Let $M \subset \mathbb{R}^3$ be a compact three-dimensional manifold with boundary and n the unit outward normal on ∂M. Let F be a differentiable vector field on M. Then show that

$$\iiint\limits_{M} \left(\frac{\partial f^1}{\partial x} + \frac{\partial f^2}{\partial y} + \frac{\partial f^3}{\partial z} \right) dV = \iint\limits_{\partial M} (n^1 f^1 + n^2 f^2 + n^3 f^3) dS.$$

9. Let $M \subset \mathbb{R}^3$ be a compact three-dimensional manifold with boundary and n the unit outward normal on ∂M. Let F be a differentiable vector field on M. Then show that

$$\int\limits_{M} \operatorname{div} F dv = \int\limits_{\partial M} \langle F, n \rangle dA.$$