Chapter 1

Baire spaces

Chapter Structure

1.1 Introduction

In this chapter, we shall introduce definition and various examples of Baire spaces. We shall also introduce Baire category theorem which has application in Open Mapping Theorem, Uniform Boundedness Principal and in later chapter of Banach spaces. Various applications of Baire spaces are there is analysis and branch of topology called *Dimension theory*. The term "Baire spaces " were coined by Nicolas Bourbaki. G_δ sets are also introduced in this chapter.

1.2 Objectives

After going through this chapter you will be able to:
- Define Baire spaces.
- Identify which spaces are Baire spaces.
- Learn that open subspace of Baire space is Baire.
- Learn about Baire category theorem.

• Learn to prove Hausdroff spaces which are compact or locally compact are Baire spaces.
• Learn about G_δ sets.
• Application to a sequence of continuous real valued functions converging point-wise to a limit function on complete metric space.

1.3 Few definitions with examples

Definition 1.1. Let X be a topological space and $A \subseteq X$ be any subset. The interior A° of A in the space X is defined as the union of all open subsets of X which are contained in A.

Examples

1) Interior of $[0, 1] = ([0, 1])^\circ = (0, 1)$.
2) $\mathbb{R}^\circ = \mathbb{R}$.
3) $\mathbb{Q}^\circ$ in $\mathbb{R} = \phi$.
4) $\mathbb{Q}^\circ$ in $\mathbb{Q} = \mathbb{Q}$.

Definition 1.2. A subset K of a topological space X is dense if $\bar{\mathrm{K}} = X$.

Example

- Set of rational numbers $\mathbb{Q}$ is dense in $\mathbb{R}$.

Remark: Let A be a subset of X. Then $A^\circ = \phi$ if and only if $X \setminus A$ is dense in X.

Definition 1.3. A topological space X is called a Baire space if given any countable collection $\{A_n\}$ of closed sets of X, each having empty interior in X then the union $\underset{n\in\mathbb{N}}{\cup} A_n$ also has empty interior in X.

Examples

1. The space of rational numbers $\mathbb{Q}$ is not a Baire space.
2. The space of integers $\mathbb{Z}$ is a Baire space.
3. The space of all irrational numbers $\mathbb{R}\setminus\mathbb{Q}$ is a Baire space.
4. Any open subset of Baire space is Baire space.
5. Any complete metric space is a Baire space.

6. Any compact Hausdroff space X is a Baire space.

Note: All the above examples are proved later.

Example 1. The space of rational numbers $\mathbb{Q}$ is not a Baire space.

Solution: Consider the field $\mathbb{Q}$ of rational numbers, a metric space with the metric $d(x, y) = |x - y|$ for all $x, y \in \mathbb{Q}$. Now for all $q \in \mathbb{Q}$, $\{q\}$ is closed . Also $\{q\}^\circ = \phi$ (since for $a < b \in \mathbb{R}$, $(a, b) \cap \mathbb{Q}$ is infinite, therefore $(a, b) \cap \mathbb{Q} \subsetneq \{q\}$). So $\{q\}_{q \in \mathbb{Q}}$ is a countable collection of closed subsets of $\mathbb{Q}$, each having empty interior in $\mathbb{Q}$. Now $\bigcup_{q \in \mathbb{Q}} \{q\} = \mathbb{Q}$. Thus $(\bigcup_{q \in \mathbb{Q}} \{q\})^\circ = (\mathbb{Q})^\circ$. But $(\mathbb{Q})^\circ = \mathbb{Q} \neq \phi$. Therefore $(\bigcup_{q \in \mathbb{Q}} \{q\})^\circ \neq \phi$. Hence by definition of Baire spaces, $\mathbb{Q}$ is not a Baire space.

Example 2. The space of integers $\mathbb{Z}$ is a Baire space.

Solution: Let $A_n = \{n\} \subseteq \mathbb{Z}$.
So A_n is open in $\mathbb{Z}$ (since $\mathbb{Z}$ is discrete metric space).
So $\{n\}^\circ = \{n\}$.
Also A_n is closed in $\mathbb{Z}$ (singletons are closed set).
So there is no closed set in $\mathbb{Z}$ with empty interior except for empty set.
Therefore $\mathbb{Z}$ is Baire space vaciously.

1.4 Baire Category Theorem

Theorem 1.4.1. *Any non-empty complete metric space is a Baire space.*

Proof. Suppose $\{A_n \mid n \in N\}$ is countable collection of closed subsets of the space X such that each A_n has empty interior in X.
Then we show that $\cup_{n=1}^{\infty} A_n$ has empty interior in X.
Assume the contrary, i.e. there exists a non- empty open subset U of X contained in $\cup_{n=1}^{\infty} A_n$. Put $U_n = X \setminus A_n$ for each $n \in \mathbb{N}$. Since each A_n is closed thus each U_n is open. Also since each ${A_n}^\circ = \phi$, so each U_n is dense, so the intersection of any nonempty open subset of X with each U_n is nonempty. Thus we get $U \cap U_n \neq \phi$, for all $n \in \mathbb{N}$.
So there exists an open ball $V_1 = B_d(x_1, r_1)$ in X with $r_1 < 1$ such that $\bar{\mathrm{V}}_1 \subset U \cap U_1$. Now since U_2 is dense so $V_1 \cap U_2 \neq \phi$. So there exist $V_2 = B_d(x_2, r_2)$ in X with $r_2 < 1/2$ such that $\bar{\mathrm{V}}_2 \subset V_1 \cap U_2$.

Continuing this way inductively we get open set $V_n = B_d(x_n, r_n)$, for n $\in \mathbb{N}$, such that $r_n < 1/n$ and $\bar{\mathrm{V}}_n \subset V_{n-1} \cap U_n$, for $n \in \mathbb{N}$.
Thus we have nested sequence of closed sets $\bar{\mathrm{V}}_1 \supseteq \bar{\mathrm{V}}_2 \cdots$ in X with diameter $(\bar{\mathrm{V}}_n) < 1/n$. Since X is complete so by Cantor's intersection theorem $\cap_{n=1}^{\infty} \bar{\mathrm{V}}_n \neq \phi$. Fix any $p \in \cap_{n=1}^{\infty} \bar{\mathrm{V}}_n$. Then $p \in U$ (since $p \in \bar{\mathrm{V}}_1$

$\subset U$). Now $p \in \bar{V}_n \subset U_n$ for all $n \in \mathbb{N}$, so $p \notin A_n$ for each $n \in \mathbb{N}$. This implies that $p \notin \cup_{n=1}^{\infty} A_n$. But $p \in U \subset \cup_{n=1}^{\infty} A_n$. This is not possible. So our assumption was wrong. Thus $\cup_{n=1}^{\infty} A_n$ has empty interior in X. Hence proved. □

Theorem 1.4.2. *Let U be any non-empty open subset of a compact Hausdroff space X and $x \in U$. Then there exist an open neighbourhood V of $x \in X$ such that $x \in \bar{V} \subset U$.*

Theorem 1.4.3. *Any compact Hausdroff space is a Baire space.*

Proof. Let X be a compact Hausdroff space. Suppose $\{A_n \mid n \in N\}$ is countable collection of closed subsets of the space X such that each A_n has empty interior in X.
Then in order to show that X is Baire space we will prove that $\cup_{n=1}^{\infty} A_n$ has empty interior in X.
Assume the contrary, i.e. $(\cup_{n=1}^{\infty} A_n)^{\circ} \neq \phi$.
Let V be any proper non-empty open set in X such that $V \subset \cup_{n=1}^{\infty} A_n$.
Put $U_n = X \setminus A_n$, for $n \in \mathbb{N}$. As each A_n is closed, hence each U_n is an open subset of X. Also since each $A_n^{\circ} = \phi$, thus each U_n is dense, hence intersection of any nonempty open subset of X with each U_n is nonempty. Thus $V \cap U_1 \neq \phi$. So by Theorem 1.4.2, there exists open set V_1 such that $\bar{V}_1 \subset V \cap U_1$.
Since U_2 is dense, thus $V_1 \cap U_2 \neq \phi$, so by Theorem 1.4.2, there exists open set V_2 in X such that $\bar{V}_2 \subset V_1 \cap U_2$.
Continuing this way inductively, we get open set V_n of X, for $n \in \mathbb{N}$ such that $\bar{V}_n \subset V_{n-1} \cap U_n$ and $\bar{V}_1 \supset \bar{V}_2 \supset \cdots \supset \bar{V}_n \supset \cdots$
Thus $\{\bar{V}_n\}_{n \in \mathbb{N}}$ has finite intersection property. Since X is compact, hence we get $\cap_{n=1}^{\infty} \bar{V}_n \neq \phi$.
Let $p \in \cap_{n=1}^{\infty} \bar{V}_n$. Then $p \in V$ (since $p \in \bar{V}_1 \subset V$).
Also $p \in \bar{V}_n \subseteq U_n$, for all $n \in \mathbb{N}$. Now since $p \in U_n$, for all $n \in \mathbb{N}$. Thus $p \notin A_n$, for all $n \in \mathbb{N}$. This implies $p \notin \cup_{n=1}^{\infty} A_n$. But $p \in V \subseteq \cup_{n=1}^{\infty} A_n$. This is not possible. Hence our assumption was wrong. Thus $(\cup_{n=1}^{\infty} A_n)^{\circ} = \phi$. Hence proved. □

1.5 Theorems on Baire spaces

Theorem 1.5.1. *Any open subset of a Baire Space is a Baire space.*

Proof. Let U be any non-empty proper open subset of a Baire space X. Suppose $\{A_n \mid n \in \mathbb{N}\}$ is countable collection of closed subsets of U such that each A_n has empty interior. Then we will prove that $(\cup_{n=1}^{\infty} A_n)^{\circ} = \phi$.
Let $\bar{A}_n$ denote the closure of A_n in the space X. Since A_n is closed in

U, so $A_n = U \cap \bar{\mathrm{A}}_n$ for all $n \in \mathbb{N}$.
Claim :$(\bar{\mathrm{A}}_n)^\circ = \phi$, for $n \in \mathbb{N}$.
If we prove this claim then since X is Baire space so we will get $(\cup_{n=1}^{\infty} \bar{\mathrm{A}}_n)^\circ = \phi$. Hence $(\cup_{n=1}^{\infty} A_n\ )^\circ = \phi$. Hence the result.
Let us proof the claim. Suppose for some $m \in \mathbb{N}$, $(\ \bar{\mathrm{A}}_m\ )^\circ \neq \phi$. Let W be any non-empty open subset of X contained in $\bar{\mathrm{A}}_m$, i.e. $W \subset \bar{\mathrm{A}}_m$, so $W \cap U \subset \bar{\mathrm{A}}_m \cap U = A_m$. Since $W \subset \bar{\mathrm{A}}_m$, so $W \cap \mathrm{A}_m \neq \phi$. Hence $W \cap U \neq \phi$ (since $A_m \subset$ U). Thus we get $W \cap U$ a non-empty, open subset of X contained in A_m. Hence $\mathrm{A}_m^\circ \neq \phi$, which is a contradiction, as A_m was choosen to be closed subset with empty interior. Hence our assumption that for some $m \in \mathbb{N}$, $(\bar{\mathrm{A}}_m\ )^\circ \neq \phi$ was wrong. Thus $(\bar{\mathrm{A}}_n)^\circ = \phi$, for all $n \in \mathbb{N}$. This completes the proof. □

Theorem 1.5.2. *A topological space X is a Baire space if and only if any countable intersection of open, dense subsets of X is a dense subset of X.*

Proof. Given X is a Baire space.
Let $\{U_n\}_{n\in\mathbb{N}}$ be any collection of open dense subsets of X.
To show that $\cap_{n\in\mathbb{N}}\ U_n$ is dense.
Let $A_n = X \setminus U_n$, for each $n \in \mathbb{N}$.
Since each U_n is open, therefore each A_n is closed.
Also since each U_n is dense implies that $A_n^\circ = \phi$ for each $n \in \mathbb{N}$.
So $\{A_n\}_{n\in\mathbb{N}}$ is a countable collection of closed sets with each having empty interior in X.
Since X is Baire space so we get $(\cup_{n\in\mathbb{N}} A_n)^\circ = \phi$.
Thus $\cap_{n\in\mathbb{N}} U_n$ is dense in X.
Conversely: Given that $\{U_n\}_{n\in\mathbb{N}}$ is a countable collection of open dense subsets of X such that $\cap_{n\in\mathbb{N}} U_n$ is dense in X.
To show that X is Baire space .
Choose $A_n = X \setminus U_n$.
Then each A_n is closed (since each U_n is open) and $A_n^\circ = \phi$ (since each U_n is dense). Hence we get $\{A_n\}_{n\in\mathbb{N}}$ countable collection of closed sets with empty interior. Also since $\cap_{n\in\mathbb{N}} U_n$ is dense implies $(\cup_{n=1}^{\infty} An)^\circ = \phi$. So by definition of Baire space, X is a Baire space. □

1.6 *G− delta set* (G_δ set)

Definition 1.4. G_δ set: Let X be a topological space. A subset S of X is called a G_δ subset of X if it can be written as a countable intersection of open subsets of X.

Theorem 1.6.1. *If Y is a dense G_δ set in X and if X is a Baire space, then Y is a Baire space in the subspaace topology.*

Proof. Given Y is a G_δ set of X. So $Y = \cap_{n\in\mathbb{N}} G_n$, where each set G_n is open in X. Now let $\{V_m\}_{m\in\mathbb{N}}$ be a countable collection of open dense subsets of Y. Inorder to show Y is Baire space we will prove $\cap_{m\in\mathbb{N}} V_m$ is dense in Y.
Since each V_m is open in Y hence there exist open set W_m of X such that $V_m = Y \cap W_m$.
Claim 1: Each W_m is dense in X.
Let U be any nonempty open subset of X. Then $U \cap Y \neq \phi$ (since Y is dense in X).
Now $U \cap Y$ is open in Y, so $V_m \cap (U \cap Y) \neq \phi \ldots$ (i).
Now $V_m \cap (U \cap Y) = (Y \cap W_m) \cap (U \cap Y) = (W_m \cap U) \cap Y \subseteq W_n \cap U \ldots$ (ii)
Hence $W_n \cap U \neq \phi$ (from (i)and (ii)). Since U was arbitrary open set in X. Hence W_m is dense in X, for each $m \in \mathbb{N}$.
So we get $\{W_m\}_{m\in\mathbb{N}}$ a collection in which each W_m is nonempty open dense set in X. Therefore $(\cap_{n\in\mathbb{N}} G_n) \cap W_m$ is dense in X, for each $m \in \mathbb{N}$. Hence $\cap_{m\in\mathbb{N}} ((\cap_{n\in\mathbb{N}} G_n) \cap W_m)$ is dense in X. $\ldots$ (iii)
Now $\cap_{m\in\mathbb{N}} V_m = \cap_{m\in\mathbb{N}} (Y \cap W_m) = \cap_{m\in\mathbb{N}} ((\cap_{n\in\mathbb{N}} G_n) \cap W_m) \ldots$ (iv).
So $\cap_{m\in\mathbb{N}} V_m$ is dense in X (from (iii),(iv)).
Claim 2: $\cap_{m\in\mathbb{N}} V_m$ is dense in Y.
Let U_1 be a nonempty open set of Y. Then there exist an open set U' of X such that $U_1 = U' \cap Y$.
Now $\cap_{m\in\mathbb{N}} V_m \cap U' \neq \phi$ (since $\cap_{m\in\mathbb{N}} V_m$ is dense in X.)
Thus we get $U_1 \cap (\cap_{m\in\mathbb{N}} V_m) \neq \phi$.
Since U_1 was arbitrary open set in Y. Hence $\cap_{m\in\mathbb{N}} V_m$ is dense in Y. Hence proved. □

Theorem 1.6.2. *The space of irrational numbers is a Baire space.*

Proof. We know that space of rational $\mathbb{Q} = \cup_{q\in\mathbb{Q}} \{q\}$.
So the space of irrationals i.e. $\mathbb{Q}^c = \cap_{q\in\mathbb{Q}} (\mathbb{R} \setminus \{q\})$.
Since each $\{q\}$ is closed hence each $\mathbb{R} \setminus \{q\}$ is open.
Also since $\{q\}^\circ = \phi$, so $\mathbb{R} \setminus \{q\}$ is dense. Therefore $\mathbb{Q}^c$ is countable intersection of open sets in $\mathbb{R}$, hence a G_δ set $\ldots$ (i)
We know that $\mathbb{Q}^c$ is dense in $\mathbb{R} \ldots$ (ii)
Now $\mathbb{R}$ is complete metric space hence a Baire space (by Theorem 1.4.1)$\ldots$ (iii).
Thus from (i),(ii),(iii) and by Theorem 1.6.1, we get that space of of irrational numbers is a Baire space. □

Theorem 1.6.3. *Set of rational numbers is not a G_δ subset of $\mathbb{R}$.*

Proof. Suppose there exists a countable collection $\{U_n\}_{n\in\mathbb{N}}$ of open set in $\mathbb{R}$ such that $\mathbb{Q} = \cap_{n=1}^{\infty} U_n$.
Put $A_n = \mathbb{R} \setminus U_n$, for all $n \in \mathbb{N}$. This implies that A_n is closed.
Since $\mathbb{Q} \subseteq U_n$, for each $n \in \mathbb{N}$. Thus $A_n^\circ = \phi$.
So $\mathbb{R} = (\cup_{n=1}^{\infty} A_n) \cup (\cup_{q\in\mathbb{R}} \{q\})$.

i.e. $\mathbb{R}$ is expressed as union of closed subset each having empty interior. We know that $\mathbb{R}$ is complete metric space so $\mathbb{R}$ is a Baire space (by theorem 1.4.1), so by definition of Baire space, $\mathbb{R}^\circ = ((\cup_{n=1}^{\infty} A_n) \cup (\cup_{q\in\mathbb{R}} \{q\}))^\circ = \phi$. But $\mathbb{R}^\circ = \mathbb{R} \neq \phi$. Hence a contradiction.
So our assumption was wrong. Hence $\mathbb{Q}$ is not a G_δ subset of $\mathbb{R}$. □

Theorem 1.6.4. *f: $\mathbb{R} \longrightarrow \mathbb{R}$ be a function then the set of points at which f is continuous is a G_δ set in $\mathbb{R}$.*

Proof. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function.
Let S_f be the set of all points of continuity of f.
To show that S_f is G_δ set in $\mathbb{R}$.
Let $B_n = \{V \subseteq \mathbb{R}$, where V is open in $\mathbb{R}$ and $\text{diam}(f(V)) < 1/n\}$.
Also let $V_n = \cup_{V\in B_n} V$.
Claim 1: V_n is open for each $n \in \mathbb{N}$.
Let $x \in V_n$, this implies $x \in V$ for some $V \in B_n$. Since V is open in $\mathbb{R}$ so for some $r > 0$, $B_d(x, r) \subset V \subseteq V_n$. Thus for $x \in V_n$, we get $B_d(x, r) \subset V_n$. Hence V_n is open for each $n \in \mathbb{N}$.
Claim 2: $S_f = \cap_{n\in\mathbb{N}} V_n$
First we show that $S_f \subseteq \cap_{n\in\mathbb{N}} V_n$
Let $x \in S_f$, i.e. f is continuous at x. Then for each $n \in \mathbb{N}$, we have $\delta_n > 0$, such that $|f(x) - f(y)| < 1/2n$, whenever $|x - y| < \delta_n$.
Thus $\text{diam}(f(B_d(x, \delta_n))) < 1/2n < 1/n$.
Therefore $x \in B_d(x, \delta_n) \subseteq B_n$, for each $n \in \mathbb{N}$.
Hence $x \in V_n$, for each $n \in \mathbb{N}$.
Therefore $x \in \cap_{n\in\mathbb{N}} V_n$.
Since x was arbitrary element of S_f, therefore $S_f \subseteq \cap_{n\in\mathbb{N}} V_n$.
Conversely, let $x \in \cap_{n\in\mathbb{N}} V_n$.
Choose $\epsilon > 0$ such that for $m \in \mathbb{N}$, $1/m < \epsilon$. Now since $x \in V_m$ so there is an open set $V \in B_m$ such that $x \in V$.
Since V is open, hence for some $\delta > 0$, $B_d(x, \delta) \subseteq V$.
Now for any $y \in B_d(x, \delta)$,we get $y \in V$ and since diam $(f(V)) < 1/m < \epsilon$, therefore $|f(x) - f(y)| < 1/m < \epsilon$. Hence f is continuous at x. Therefore $x \in S_f$. Since $x \in \cap_{n\in\mathbb{N}} V_n$ was arbitrary element, hence $\cap_{n\in\mathbb{N}} V_n \subseteq S_f$. Therefore $S_f = \cap_{n\in\mathbb{N}} V_n$. □

Theorem 1.6.5. *If A is countable dense subset of $\mathbb{R}$, then there is no continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$ which is continuous precisely at the points of A.*

Proof. First we will claim that any countable dense subset of $\mathbb{R}$ is not a G_δ set.
If possible, let A is a G_δ set with $A = \cap_{n\in\mathbb{N}} U_n$, where each U_n is open subset in $\mathbb{R}$.
Since A is dense in $\mathbb{R}$. So we have $\mathbb{R} = \bar{A}$.
Since $A \subseteq U_n$, for each $n \in \mathbb{N}$. Therefore $\bar{A} \subseteq \bar{U}_n \subseteq \mathbb{R}$.
Hence $\mathbb{R} = \bar{A} \subseteq \bar{U}_n \subseteq \mathbb{R}$. Therefore $\mathbb{R} = \bar{U}_n$.

Hence for each $n \in \mathbb{N}$, U_n is dense in $\mathbb{R}$.
Thus for each $n \in \mathbb{N}$, $(\mathbb{R} \setminus U_n)^\circ = \phi$. Also $\mathbb{R} \setminus U_n$ is closed subset for each $n \in \mathbb{N}$ (Since U_n is open).
So $\mathbb{R} = (\cup_{d \in A} \{d\}) \cup (\cup_{n \in \mathbb{N}} (\mathbb{R} \setminus U_n))$.
i.e. $\mathbb{R}$ is countable union of closed set, each having empty interior.
Since $\mathbb{R}$ is complete metric space hence Baire space (by Theorem 1.4.1).
So $(\mathbb{R})^\circ = ((\cup_{d \in A} \{d\}) \cup (\cup(\mathbb{R} \setminus U_n)))^\circ = \phi$.
But $\mathbb{R}^\circ = \mathbb{R} \neq \phi$.
Hence a contradiction. So our assumption was wrong .
Thus A cannot be a G_δ set . . . (i)
Suppose there exists a function which is precisely continuous at the points of A.
Then A is G_δ set (by Theorem 1.6.4). . . (ii)
From (i) and (ii), we get a contradiction. So our assumption was wrong.
Hence no such function exists which is continuous at precisely at the points of A.

□

Example 3. Is it possible to find a function $f : \mathbb{R} \longrightarrow (Y, d)$ which is continuous precisely at rational numbers.

Solution: Let us suppose that there exists a function which is continuous at rational numbers. Then $S_f = \{ x \in \mathbb{R} \mid f \text{ is continuous on } x\} = \mathbb{Q}$. We have proved that S_f is G_δ set (by Theorem 1.6.4). Hence we will get $\mathbb{Q}$ as G_δ set, which is a contradiction as we have proved that $\mathbb{Q}$ is not a G_δ set (by Theorem 1.6.3). Hence no such function is possible which is continuous precisely at rational numbers.

1.7 Applications

Theorem 1.7.1. *Let $X = (X, d)$ be a complete metric space. Let $\{f_\alpha\}_{\alpha \in \mathbb{N}}$ be a family of continuous function from X to $\mathbb{R}$ such that for each $x \in X$ $\exists$ a constant $M_x \in \mathbb{R}$ such that $|f_\alpha(x)| \leq M_x$, $\forall\ \alpha \in \wedge$. Then $\exists$ a constant $M \in \mathbb{R}$ and a non-empty open set $B_d(x_0, r)$ in X such that $|f_\alpha(x)| \leq M$, $\forall\ \alpha \in \wedge$, $\forall\ x \in B_d(x_0, r)$.*

Proof. Let $A_n = \{ x \in X \mid |f_\alpha(x)| \leq n, \text{ for all } \alpha \in \Lambda\}$, for each $n \in \mathbb{N}$.
Then $A_n = \cap_{\alpha \in \Lambda} f_\alpha^{-1}[0, n]$.
Since $[0, n]$ is closed and each f_α is continuous, hence $f_\alpha^{-1}[0, n]$ is closed.
Since arbitrary intersection of closed set is closed. Thus A_n is closed in X.
As X is complete, so X is a Baire space (by Theorem 1.4.1). Also since $\cup_{n=1}^{\infty} A_n = X$, so there exist $m \in \mathbb{N}$ such that $A_m^\circ \neq \phi$.
So there exists $B_d(x_0, r) \subseteq A_m$.

Let $x \in B_d(x_0, r) \subseteq A_m$. Therefore $|f_\alpha(x)| \leq m$.
Let us take $m = M$. Hence we get $|f_\alpha(x)| \leq M$, for all $x \in B_d(x_0, r)$, for all $\alpha \in \Lambda$. Hence proved. □

Theorem 1.7.2. *$f_n : X \longrightarrow \mathbb{R}$ for ($n \in \mathbb{N}$) be continuous function defined on a Baire space X converging pointwise to a limit function $f : X \longrightarrow \mathbb{R}$ then the points of continuity of f contains a dense subset of X.*

Proof. Let us fix $\epsilon > 0$ and $n \in \mathbb{N}$ and take $A_n = \{x \in X| \mid f_k(x) - f_l(x) \mid \leq \epsilon \text{ for all } k, l \geq n \ \} = \cap_{n=1}^{\infty} \ (\cap_{k,l \geq n} |f_k - f_l|^{-1}[0, \epsilon] \ )$
As $[0, \epsilon]$ is closed and f_k's are continuous functions so $|f_k - f_l|^{-1}[0, \epsilon]$ is closed for all $(k, l) \in \mathbb{N} \times \mathbb{N}$.
Since arbitrary intersection of closed subsets in X is again closed, hence $A_n(\epsilon)$ is closed in X.
Claim: $\cup_{n=1}^{\infty} \ A_n(\epsilon) = X$.
Since $A_n \subseteq X$, for all $n \in \mathbb{N}$. Thus $\cup_{n=1}^{\infty} A_n \subseteq X$.
Enough to show $X \subseteq \cup_{n=1}^{\infty} A_n$.
Let $x_0 \in X$ be arbitrary, since $f_n(x_0) \longrightarrow f(x_0)$ as $n \longrightarrow \infty$, so for given $\epsilon > 0$, there exists $n_0 = n_0(x, \epsilon) \in \mathbb{N}$ such that $|f_k(x_0) - f_l(x_0)| \leq \epsilon$ for all $k, l \geq n_0$. Therefore $x_0 \in A_{n_0}(\epsilon)$. Thus $x_0 \in \cup_{n=1}^{\infty} A_n(\epsilon)$.
Therefore $X \subseteq \cup_{n=1}^{\infty} A_n(\epsilon)$ (as x_0 was an arbitrary element of X).
Hence $\cup_{n=1}^{\infty} \ A_n(\epsilon) = X \ \ldots$ (i)
Now from (i) and since X is Baire space, so not every $A_n(\epsilon)$ can have empty interior. So there exists $m \in \mathbb{N}$ such that $A_m^\circ \neq \phi$.
Let $U(\epsilon) = \cup_{n=1}^{\infty} \ A_n^\circ(\epsilon) \ldots$ (ii)
Then $U(\epsilon)$ is nonempty open subset of X.
Claim: $U(\epsilon)$ is dense in X.
Let V be any nonempty open subset of X.
So we will prove that $U(\epsilon) \cap V \neq \phi$.
Since V is open in X. Hence V is Baire (by Theorem 1.5.1)... (iii)
Since $X = \cup_{n=1}^{\infty} \ A_n(\epsilon)$. Thus $V = \cup_{n=1}^{\infty} \ (V \cap A_n(\epsilon))$.
So V is countable union of closed subsets. (... (iv)
From (iii),(iv) we get that there exists $m \in \mathbb{N}$ such that $(V \cap A_m(\epsilon))^\circ \neq \phi$.
But $(V \cap A_m(\epsilon))^\circ = V^\circ \cap A_m^\circ(\epsilon) = V \cap A_m^\circ(\epsilon)$ (as $V^\circ = V$).
So $V \cap A_m^\circ(\epsilon) \neq \phi$.
Also $V \cap A_m^\circ(\epsilon) \subseteq V \cap U(\epsilon)$
So $V \cap U(\epsilon) \neq \phi$. Hence the claim.
So for $\epsilon > 0$, $U(\epsilon)$ is nonempty open dense subset of X.
Thus $\{U(1/n)\}_{n \in \mathbb{N}}$ is a countable collection of open dense subset of the Baire space X. Hence $\cap_{n \in \mathbb{N}} \ U(1/n)$ is dense in X.
Inorder to prove the Theorem it is enough to show that f is continuous at every $x \in \cap_{n=1}^{\infty} U(1/n)$.
Let $C = U(1) \cap U(1/2) \ldots U(1/n) \ldots$
Also let x_0 be arbitrary element in C.
Then we will show that f is continuous at x_0.

i.e. To show that for given $\epsilon > 0$, we will find a neighbourhood B of x_0 such that $d(f(x), f(x_0)) < \epsilon$, for all $x \in B$.
First lets choose $k \in \mathbb{N}$ such that $1/k < \epsilon/3$.
Since $x_0 \in C$. So $x_0 \in U(1/k)$. So there exists some $N \in \mathbb{N}$ such that $x_0 \in (A_N(1/k))^\circ$ (from ii).
Given that f_N is continuous, so continuity of f_N enables us to choose a neighbourhood B of x_0 contained in $A_N(1/k)$ such that
$d(f_N(x), f_N(x_0)) < \epsilon/3$, for all $x \in B \ldots$ (v)
Now $B \subset A_N(1/k)$, so we get $d(f_n(x), f_N(x)) \leq 1/k < \epsilon/3$, for all $x \in B$. Letting $n \longrightarrow \infty$, we obtain that $d(f(x), f_N(x)) \leq 1/k < \epsilon/3$, for all $x \in B \ldots$ (vi)
In particular, since $x_0 \in B$, so we have $d(f(x_0), f_N(x_0)) < \epsilon/3 \ldots$ (vii)
Applying triangle inequality and using (v),(vi) and (vii) we obtain $d(f(x), f(x_0)) < \epsilon$ for all $x \in B$.
Hence f is continuous at $x_0 \in C$
Since x_0 was arbitrary chosen, therefore f is continuous at all points of C. Hence proved. □

1.8 LET US SUM UP

- Given a countable collection of closed sets with each having empty interior in a topological space X, if Union of those sets also has empty interior, then such a topological space is called Baire space.
- The space of integers $\mathbb{Z}$, irrational numbers $\mathbb{R} \setminus \mathbb{Q}$ are Baire spaces.
- The space of rational numbers is not a Baire space.
- Any complete metric space is a Baire space.
- Any compact Haudroff space is a Baire space.
- Any open subset of a Baire space is Baire space.
- The set of rationals $\mathbb{Q}$ is not a G_δ set.
- If f: (X, d) to $\mathbb{R}$ is any function, then set of points of continuity of f is a G_δ set.
- There is no function f : $\mathbb{R}$ to $\mathbb{R}$ which is precisely continuous on rationals.
- If Y is a dense G_δ set in X and if X is a Baire space, then Y is a Baire space in the subspaace topology.

- If X is a nonempty Baire space and (f_n) be a sequence of continuous maps converging point-wise to a limit function f. Then points of contiunity of f is dense in X.

1.9 Chapter End Exercise

1. Let $A_1 \supset A_2 \supset \ldots$ be a nested sequence of nonempty closed sets in the complete metric space X. If diam$A_n \to 0$, then $\bigcap A_n \neq \phi$.
2. See Theorem 1.4.2.
3. Show that every locally compact Hausdroff space is a Baire space.
4. Let X equal the countable union $\cup_{n\in\mathbb{N}} U_n$. Show that if X is a nonempty Baire space then at least one of the sets $\bar{\mathrm{U}}_n$ has a nonempty interior.
5. Show that if every point x of X has a neighbourhood that is a Baire space, then X is a Baire space.

Chapter 2

Hilbert spaces

Unit Structure:

2.1 Introduction

This chapter introduces Hilbert spaces which are special type of Banach spaces. Hilbert spaces have additional structure which enable us to know when two vectors are orthogonal. The whole theory was initiated by the work of D.Hilbert(1912) on integral equation. The currently used geometrical notations and terminology is analogous to that of the Euclidean geometry. These are most useful spaces in practical applications of functional analysis.

2.2 Objectives

After going through this chapter you will be able to:
• Define Hilbert spaces.
• Identify which spaces are Hilbert spaces.
• Understand and apply Parallelogram equality.
• Understand and apply Schwarz inequality.
• Understand various properties of Hilbert spaces.
• Understand Complete othonormal sets.
• Understand Bessel's inequality and Parseval's Indentity.
• Define Separable Hilbert space.

2.3 Definition of Hilbert Space

Definition 2.1. Inner product Space:
An Inner product space is a vector space X with an inner product defined on X.

Definition 2.2. Complete inner product Space:
An inner product space X is said to be complete if every Cauchy sequence in X has a limit that is also in X,(i.e. every Cauchy sequence in X is convergent in X.)

Definition 2.3. Hilbert space:
A Hilbert space is a complete inner product space (complete in the metric defined by the inner product), where inner product on X is a mapping of $X \times X$ into the scaler field K of X.
i.e. with every pair of vectors x, y there is a associated scalar which is written as $< x, y >$ and is called the inner product of x and y such that for all the vectors x, y, z and for scalar $\alpha \in K$, we have
1) $< x, x > \geq 0$, $< x, x > = 0$ if and only if $x = 0$.
2) $< x, y > = \overline{< y, x >}$.
3) $< \alpha x, y > = \alpha < x, y >$.
4) $< x + y, z > = < x, z > + < y, z >$.

2.4 Examples of Hilbert spaces

1) The Euclidean space $\mathbb{R}^n$ is a Hilbert space with inner product defined by
$< x, y > = < (x_1, \ldots, x_n), (y_1, \ldots, y_n) >$
$= x_1y_1 + x_2y_2 + \ldots + x_ny_n$.
Where $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$.

2) The space $\mathbb{C}^n$ is Hilbert space with inner product defined by
$< x, y > = < (x_1, \ldots, x_n), (y_1, \ldots, y_n) >$
$= x_1\bar{y_1} + x_2\bar{y_2} + \ldots + x_n\bar{y_n} = \sum_{i=1}^{n} x_i\bar{y_i}$.
Where $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, each $x_i, y_i \in \mathbb{C}^n$, for all $1 \leq i \leq n$.

3) The space l^2 is a Hilbert space with inner product space defined by $< x, y > = \sum_{i=1}^{\infty} x_i\bar{y_i}$.
Where $x = (x_1, \ldots, x_n, \ldots)$, $y = (y_1, \ldots, y_n, \ldots) \in l^2$.

4) The space $L^2(\mathbb{R}^n)$ space is a Hilbert space with inner product space defined by
a) If functions are assumed to be real valued then
$< x, y > = \int x(t)y(t)dt$.
b) If functions are assumed to be complex valued then
$< x, y > = \int x(t)\overline{y(t)}dt$.

5) The space $L^2[-\pi, \pi]$ space is a Hilbert space with inner product space defined by
a) If functions are assumed to be real valued then
$< x, y > = \int_{-\pi}^{\pi} x(t)y(t)dt$.
b) If function are assumed to be complex valued then
$< x, y > = \int_{-\pi}^{\pi} x(t)\overline{y(t)}dt$.

2.5 Parallelogram equality

An inner product on X defines a norm on X given by

$$||x|| = \sqrt{< x, x >} \tag{2.1}$$

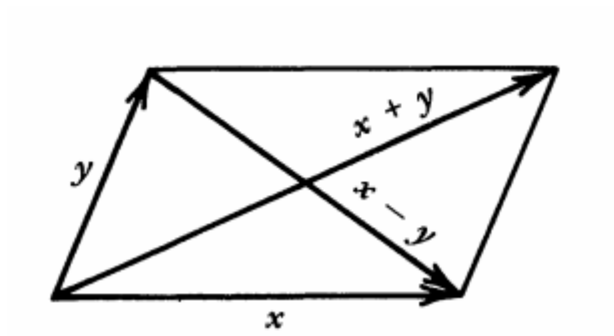
and a metric on X given by

$$d(x, y) = ||x - y|| = \sqrt{< x - y, x - y >}. \tag{2.2}$$

Hence inner product spaces are normed spaces.
The norm generalizes the elementary concept of the length of the vector. From the elementary geometry we know that the sum of the squares of the sides of a parallelogram equals the sum of the squares of its diagonals. In any Hilbert space we have an analogue to it. There is an important equality given by

$$||x + y||^2 + ||x - y||^2 = 2(||x||^2 + ||y||^2). \tag{2.3}$$

This is called parallelogram equality.



Parallelogram with sides x and y in the plane.

If a norm does not satisfy parallelogram law then that norm cannot be obtained from inner product (from equation (2.1)) .
Hence not all normed spaces are inner product spaces.

Example 4. The normed space l^p with $p \neq 2$ is not an inner product space, hence not a hilbert space .

Solution: The norm on l^p with $p \neq 2$ cannot be obtained from inner product we prove this by showing that norm does not satisfy parallelogram equality.
Let $x = (1, 1, 0, 0, \dots) \in l^p$.
$||x|| = (|1|^p + |1|^p + |0|^p + \dots)^{1/p} = (1 + 1 + 0 + 0 \dots)^{1/p} = 2^{1/p} \dots$ (i)
$y = (1, -1, 0, 0, \dots) \in l^p$.
$||y|| = (|1|^p + |-1|^p + |0|^p + \dots)^{1/p} = (1 + 1 + 0 + 0 \dots)^{1/p} = 2^{1/p} \dots$ (ii)
$x + y = (2, 0, 0, \dots)$
$||x + y|| = (|2|^p + 0 + 0, \dots)^{1/p} = (2^p)^{1/p} = 2 \dots$ (iii)
similarly $||x - y|| = 2 \dots$ (iv)
For $p = 2$, let us check the parallelogram equality.
$||x + y||^2 + ||x - y||^2 = 2^2 + 2^2 = 8$ (from (iii) and (iv))... (v)
Now 2 $(||x||^2 + ||y||^2) = 2(2+2) = 8$ (from (i),(ii))... (vi)
Thus from (v) and (vi) $||x + y||^2 + ||x - y||^2 = 2\ (||x||^2 + ||y||^2)$.
i.e. parallelogram equality holds.
For $p \neq 2$, let us check whether parallelogram equality holds.
$||x + y||^2 + ||x - y||^2 = 2^2 + 2^2 = 8$ (from (iii) and (iv))... (vii)
Now 2 $(||x||^2+||y||^2) = 2(2^{2/p} + 2^{2/p}) \neq 8$ (from (i)(ii) for $p \neq 2$)... (viii)
Thus from (vii) and (viii) $||x + y||^2 + ||x - y||^2 \neq 2\ (||x||^2 + ||y||^2)$.
i.e. for $p \neq 2$ parallelogram equality does not hold.
Hence for $p \neq 2$, l^p is a normed linear space but is not a Hilbert space.

We know that to each inner product there corresponds a norm which is given by equation (2.1).
One can rediscover inner product from the corresponding norm by some straight forward calculations.
For a real Inner product space:

$$< x, y > = \frac{1}{4}(||x+y||^2 - ||x-y||^2)$$

Calculation: Consider $\frac{1}{4}(||x+y||^2 - ||x-y||^2)$
$= \frac{1}{4}\ (< x+y, x+y >$ - $< x-y, x-y >\ )$
$= \frac{1}{4}\ (< x, x > + < x, y > + < y, x > + < y, y >$ - $< x, x > +< x, y >$
$+ < y, x >$ - $< y, y >\ )$
$= \frac{4}{4} < x, y > = < x, y >$.

For Complex valued Inner product space:

$$\text{Re}< x, y > = \frac{1}{4}(||x+y||^2 - ||x-y||^2).$$

$$\text{Im}< x, y > = \frac{1}{4}(||x+iy||^2 - ||x-iy||^2).$$

The above is called **polarization identity.**

2.6 Few more inequalities

Theorem 2.6.1. *If x, y are any two vectors in Hilbert space, then*
$| < x, y > | \leq ||x||\ ||y||$ *(Schwarz inequality).*

It can be easily proved that the inner product in a Hilbert space is continuous by using Schwarz inequality.

Theorem 2.6.2. *(Continuity of inner product) If in an inner product space sequence $x_n \to x$ and $y_n \to y$ as $n \to \infty$. Then $< x_n, y_n > \to < x, y >$ as $n \to \infty$.*

Proof. Consider $| < x_n, y_n > - < x, y > |$
$= | < x_n, y_n > - < x_n, y > + < x_n, y > - < x, y > |$
$\leq | < x_n, y_n > - < x_n, y > | + | < x_n, y > - < x, y > |$
$= | < x_n, y_n - y > | + | < x_n - x, y > |$
$\leq ||x_n||||y_n - y|| + ||x_n - x||||y||$ (by Schwarz inequality)... (i)
As $n \to \infty$, $x_n \to x$ and $y_n \to y$ (given)
Thus $\lim_{n\to\infty} | < x_n, y_n > - < x, y > | \leq ||x_n|| \lim_{n\to\infty}||y_n - y|| + \lim_{n\to\infty} ||x_n - x||||y|| = 0$.
Therefore $< x_n, y_n > \to < x, y >$ as $n \to \infty$. Hence proved.

□

The norm also satisfies one more inequality
$||x+y|| \leq ||x|| + ||y||$ (Triangle inequality).

2.7 Theorems on Hilbert spaces

Theorem 2.7.1. *Let Y be a finite dimensional subspace of a Hilbert space H then Y is complete.*

Proof. Given that Y is a finite dimensional subspace of a Hilbert space H. Let dim $Y = n$ and $\{e_1, \ldots, e_n\}$ is a basis for Y.
Let $(y_m)_{m\in\mathbb{N}}$ be a Cauchy sequence in Y.
Now $y_m = \alpha_1^m e_1 + \alpha_2^m e_2 + \ldots + \alpha_n^m e_n$, for each $m \in \mathbb{N}$, where α_i^m are scalers. Since $(y_m)_{m\in\mathbb{N}}$ is Cauchy sequence, so by definition, for given $\epsilon > 0$, there exist $N \in \mathbb{N}$ such that $||y_m - y_r|| < \epsilon$, for all $m, r \geq N$.
So we have $\epsilon > ||y_m - y_r|| = ||(\alpha_1^m - \alpha_1^r)e_1 + (\alpha_2^m - \alpha_2^r)e_2 + \ldots + (\alpha_n^m - \alpha_n^r)e_n||$. Hence $\epsilon > || \sum_{i=1}^n (\alpha_i^m - \alpha_i^r)e_i|| \ldots (1)$
Define $||y||_1 = \sum_{i=1}^n |\alpha_i|$, for all $y \in Y$, where $y = \sum_{i=1}^n \alpha_i e_i$.
We know that any two norms on finite dimensional space are equivalent.
Thus there exist $c > 0$ such that $||y|| > c||y||_1 \ldots (2)$
So from (1)and (2) we get, $\epsilon > || \sum_{i=1}^n (\alpha_i^m - \alpha_i^r)e_i|| \geq c \; || \sum_{i=1}^n (\alpha_i^m - \alpha_i^r)e_i||_1 \geq c \sum_{i=1}^n |\alpha_i^m - \alpha_i^r|$.
For fix i, $|\alpha_i^m - \alpha_i^r| < \epsilon/c$, for all $m, r > N$ and for each $\alpha_i \in H$.
Thus for each i, the sequence of scalers (α_i^m) is a Cauchy sequence in H. Since H is Hilbert space so H is complete.
Thus $\alpha_i^m \to \alpha_i$ as $m \to \infty$, for $1 \leq i \leq n$.
Consider $y = \alpha_1 e_1 + \alpha_2 e_2 + \ldots + \alpha_n e_n \in Y$.
Claim: $y_m \to y$ as $m \to \infty$.
Consider $||y_m - y|| = || \sum_{i=1}^n (\alpha_i^m - \alpha_i)e_i||$
Take $K = \max ||e_i||$. So we get $||y_m - y|| \leq K \sum_{i=1}^n |\alpha_i^m - \alpha_i|$.
As $m \to \infty$, $\alpha_i^m \to \alpha_i$ hence $y_m \to y$. Thus $(y_m)_{m\in\mathbb{N}}$ is convergent in Y, therefore Y is complete. Hence Proved. □

Theorem 2.7.2. *Let Y be a subspace of a Hilbert space H, then Y is complete if and only if Y is closed in H.*

Theorem 2.7.3. *A closed convex subset D of a Hilbert Space H contains a unique vector of smallest norm.*

Proof. Given that D is convex, so whenever $u, v \in D$, $(u + v)/2 \in D$.
Let $d = \inf \{||u|| : u \in D \}$.
So there exist a sequence $\{u_n\}$ of vectors in D such that $||u_n|| \to d$ as $n \to \infty$. i.e. there exist $n_0 \in \mathbb{N}$ such that
$\lim_{n\to\infty} ||u_n|| = d$, for all $n \geq n_0 \ldots (1)$
By convexity of D, $(u_m + u_n) /2 \in D$ and $||(u_m + u_n)/2|| \geq d$.
So $||(u_m + u_n)|| \geq 2d$.
Using the parallelogram law we get, $||u_m - u_n||^2 = 2||u_m||^2 + 2||u_n||^2$ - $||u_m + u_n||^2 \leq 2||u_m||^2 + 2||u_n||^2$ -$4d^2$.

Now from (1) we get, $2||u_m||^2 + 2||u_n||^2 - 4d^2 \to 2d^2 + 2d^2 - 4d^2 = 0$, for all $n, m \geq n_0$.
Hence we get $||u_m - u_n|| \to 0$, all $n, m \geq n_0$. Therefore $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in D. Since H is complete and D is closed hence D is complete by Theorem 2.7.2.
So there exist $u \in D$ such that $\lim_{n\to\infty} u_n = u$.
Now $||u|| = || \lim_{n\to\infty} u_n|| = \lim_{n\to\infty}||u_n|| = d$. Hence u is the vector in D with smallest norm.
Claim: u is unique.
Suppose that u' is another vector in D other than u which also has norm d. Then $(u + u')/2$ is also in D.
Again by using parallelogram law, we get $||(u + u')/2||^2 = ||u||^2/2 + ||u'||^2/2 - ||(u - u')/2||^2 < ||u||^2/2 + ||u'||^2/2 = d^2$.
Hence we get $||(u+u')/2|| < d$, which is a contradiction to the definition of d. Thus our assumption was wrong. Therefore u is unique. Hence proved. $\square$

2.8 Orthogonal complements

Definition 2.4. An element x of an inner product space X is said to be orthogonal to an element $y \in X$ if $< x, y > = 0$.
We write x is orthogonal to y as follows $x \perp y$.
Note: If $x \perp y$ then $y \perp x$.

Pythagorean Theorem If $x \perp y$ in an inner product space X. Then
$$||x + y||^2 = ||x||^2 + ||y||^2.$$

Consider $||x + y||^2 = < x + y, x + y >$
$= < x, x > + < x, y > + < y, x > + < y, y >$
$= < x, x > + 0 + 0 + < y, y >$ (since $x \perp y$ so $< x, y > = 0$, also $< y, x >= 0$)
$= ||x||^2 + ||y||^2$. Hence proved.

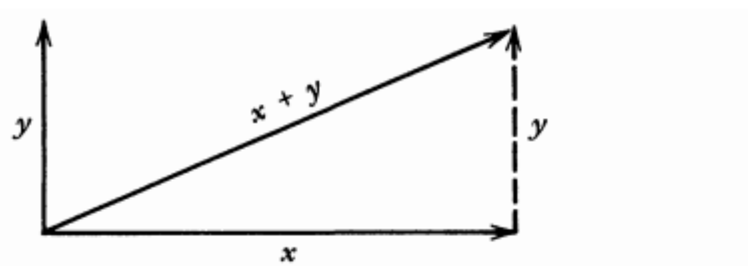


Illustration of the Pythagorean theorem in the plane

Definition 2.5. A vector x is said to be orthogonal to a non-empty set A (written $x \perp A$) if $x \perp y$ for every $y \in A$.

Definition 2.6. The Orthogonal complement of A is denoted by $A^{\perp}$ is the set of all vectors orthogonal to A.

Theorem 2.8.1. *If H is a Hilbert space then*

1) $\{0\}^{\perp} = H$, $H^{\perp} = \{0\}$.

2) $A \cap A^{\perp} \subseteq \{0\}$.

Theorem 2.8.2. *If $A_1 \subseteq A_2$ then $A_2^{\perp} \subseteq A_1^{\perp}$.*

Proof. Let $y \in A_2^{\perp}$. This implies that $< x, y > = 0$ for all $x \in A_2$.
Thus $< x, y > = 0$ for all $x \in A_1$(since $A_1 \subseteq A_2$).
So $y \in A_1^{\perp}$. Hence $A_2^{\perp} \subseteq A_1^{\perp}$. □

Theorem 2.8.3. *$A^{\perp}$ is closed linear subspace.*

Theorem 2.8.4. *Let A be a closed linear subspace of a Hilbert space H. Let x be a vector not in A and let d be the distance from x to A. Then there exists a unique vector y_0 in A such that $||x - y_0|| = d$.*

Proof. Let $C = x + A$. Then C is closed (since A is closed).
We first show that C is a convex set.
Let $u, v \in C$. Then $u = x + u'$ and $v = x + v'$, where $u', v' \in A$.
To show that $(1 - t)u + tv \in C$, where $0 \leq$ t ≤ 1.
Consider $(1-t)u + tv = (1-t)(x+u') + t(x+v') = (1-t)x + (1-t)u'$ $+ tx + tv' = x + (1 - t)u' + tv'$.
Since $u', v' \in A$, so $(1 - t)u'$ and $tv' \in A$ (since A is a subspace).
Hence $(1 - t)u' + tv' \in A$.
So $x + (1 - t)u' + tv' \in x + A = C$.
Therefore $(1 - t)u + tv \in C$. Hence C is convex.
Thus C is closed and convex set.
Let d be the distance from the origin to C.
So by Theorem 2.7.3, there exists a unique vector z_0 in C such that $||z_0|| = d$. Since $z_0 \in C$ so $z_0 = x + (-y_0)$, where $y_0 \in A$.
The vector $y_0 = x$ - z_0 is in A and $||x - y_0|| = ||z_0|| = d$.
Claim: y_0 is unique.
Suppose y_1 is a vector in A such that $y_1 \neq y_0$ and $||x - y_1|| = d$ then $z_1 = x - y_1$ is a vector in C such that $z_1 \neq z_0$ and $||z_1|| =$ d which is contradiction to the uniqueness of z_0. Hence our assumption was wrong. Thus we get a unique vector y_0 in A such that $||x - y_0|| = d$.
Hence proved. □

Theorem 2.8.5. *If A is a proper closed linear subspace of a Hilbert space H, then there exist a non-zero vector $z_0 \in H$ such that $z_0 \perp A$.*

Proof. Let x be a vector not in A. Let d be the distance of x from A. Then by Theorem 2.8.4 there exists a unique vector y_0 in A such that $||x - y_0|| = d$.
We define z_0 as follows $z_0 = x - y_0$. So $||z_0|| = ||x - y_0|| = d$.
Since $d > 0$, z_0 is a non-zero vector.
Claim: $z_0 \perp A$. It is enough to show that $z_0 \perp y$, where y is arbitrary element of A.
For any scaler α, consider $||z_0 - \alpha y|| = ||(x - y_0) - \alpha y||$
$= ||x - (y_0 + \alpha y)|| \geq d = ||z_0||$.
Therefore $||z_0 - \alpha y|| \geq ||z_0||$.
Thus $||z_0 - \alpha y||^2$ - $||z_0||^2 \geq 0$.
So $||z_0||^2$ - $\bar{\alpha} < z_0, y >$ - $\alpha \overline{< z_0, y >}$ + $|\alpha|^2 ||y||^2$ - $||z_0||^2 \geq 0$.
Thus - $\bar{\alpha} < z_0, y >$ - $\alpha \overline{< z_0, y >}$ + $|\alpha|^2||y||^2 \geq 0 \ldots (1)$
Put $\alpha = \beta < z_0, y >$ for any arbitrary real number $\beta \ldots (2)$
From (1) and (2) we get - $\beta \overline{< z_0, y >} < z_0, y >$ -$\beta < z_0, y > \overline{< z_0, y >}$ + $\beta^2 | < z_0, y > |^2 ||y||^2 \geq 0$.
Thus $-2\beta \ | < z_0, y > |^2 + \beta^2 | < z_0, y > |^2 ||y||^2 \geq 0$.
We will now put $a = | < z_0, y > |^2$ and b $= ||y||^2$ to obtain
$-2\beta a + \beta^2 ab \geq 0$.
Thus $\beta a \ (\beta b - 2) \geq 0$, for real $\beta \ldots (3)$.
However if $a > 0$ then (3) is false for sufficiently small positive β.
Thus $a = 0$, so $| < z_0, y > |^2 = 0$, hence $< z_0, y > = 0$.
Therefore $z_0 \perp y$. Since y is arbitrary element of A.
Therefore $z_0 \perp A$. □

Theorem 2.8.6. *If A and B are closed linear subspace of a Hilbert space H such that A is perpendicular to B i.e. $A \perp B$. Then the linear subspace $A + B$ is also closed.*

Proof. Let z be the limit point of $A + B$.
It suffices to show that z is in $A + B$.
Let $\{z_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $A+B$ such that $z_n \to z$ as $n \to \infty$.
It is given that $A \perp B$, so $< x, y > = 0$ for all $x \in A$ and $y \in B$.
Also $A \cap B = \{0\}$ and so we get A and B are disjoint.
Now each z_n can be written uniquely in the form $z_n = x_n + y_n$, where $x_n \in A$ and $y_n \in B$.
Now $||z_m - z_n||^2 = ||(x_m + y_m) - (x_n + y_n)||^2$
$= ||(x_m - x_n) + (y_m - y_n)||^2$.
Thus by pythagorean theorem we get $||z_m - z_n||^2 = ||x_m - x_n||^2 + ||y_m - y_n||^2$.
As $\{z_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence so by definition, there exists $n_0 \in \mathbb{N}$ such that $||z_m - z_n|| \to 0$, for all $n, m \geq n_0$. Therefore $||x_m - x_n|| \to 0$, for all $n, m \geq n_0$, and $||y_m - y_n|| \to 0$, for all $n, m \geq n_0$.
Hence $\{x_n\}_n \in \mathbb{N}$, $\{y_n\}_n \in \mathbb{N}$ are Cauchy sequence in A and B respectively. Now A and B are closed and therefore complete by Theorem 2.7.2. So there exists vectors x and y in A and B such that $x_n \to x$

and $y_n \to y$ as $n \to \infty$.
Now $x + y$ is in $A + B$.
Also $z = \lim_{n\to\infty} (x_n + y_n) = \lim_{n\to\infty} (x_n) + \lim_{n\to\infty} (y_n) = x + y$.
Thus $z \in A + B$. Hence $A + B$ is closed. □

Definition 2.7. Direct sum:
A vector space X is said to be the direct sum of two subspaces A and B of X, if each $x \in X$ can be represented uniquely as $x = a + b$, where $a \in A$ and $b \in B$, and $A \cap B = \{0\}$.
It is written as

$$X = A \oplus B$$

where B is called complement of A in X and vice-versa.

Theorem 2.8.7. *If A is closed linear subspace of a Hilbert space H, then $H = A \oplus A^{\perp}$.*

Proof. Given that A is closed. Also $A^{\perp}$ is closed (by Theorem 2.8.3.)
Thus $A + A^{\perp}$ is closed linear subspace of H (by Theorem 2.8.6.)
Claim: $A + A^{\perp} = H$.
Suppose $A + A^{\perp} \neq H$. Then $A + A^{\perp} \subseteq H$. So there exists a nonzero vector z_0 such that $z_0 \perp (A + A^{\perp})$ (by Theorem 2.8.5).
Thus $z_0 \in ((A + A^{\perp}))^{\perp} = A^{\perp} \cap A^{\perp\perp}$ which is a contradiction as $A^{\perp} \cap A^{\perp\perp} = \{0\}$. Hence $H = A + A^{\perp} \ldots (1)$
Also since $A \cap A^{\perp} = \{0\} \ldots (2)$
So, from (1),(2) and by definition of direct sum $H = A \oplus A^{\perp}$. □

2.9 Orthonormal sets

Definition 2.8. An Orthonormal set in an Hilbert space X is a subset of X in which each element has unit norm and elements are orthogonal to each other.

Theorem 2.9.1. *Let $\{e_1, e_2, \ldots, e_n\}$ be a finite orthonormal set in a Hilbert space H. If x is any vector in H then, $\sum_{i=1}^{n} |< x, e_i >|^2 \leq ||x||^2$. Further $x - \sum_{i=1}^{n} < x, e_i > e_i \perp e_j$, for each j .*

Proof. $0 \leq ||x - \sum_{i=1}^{n} < x, e_i > e_i||^2$

$= < x - \sum_{i=1}^{n} < x, e_i > e_i \ , \ x - \sum_{i=1}^{n} < x, e_i > e_i >$

$= < x, x >$ - $\sum_{i=1}^{n} < x, e_i > \overline{< x, e_i >}$ - $\sum_{i=1}^{n} < x, e_i > \overline{< x, e_i >} + \sum_{i=1}^{n} \sum_{j=1}^{n}$

$< x, e_i > \overline{< x, e_j >} < e_i, e_j >$.
$= < x, x >$ - $\sum\limits_{i=1}^{n} < x, e_i > \overline{< x, e_i >}$ - $\sum\limits_{i=1}^{n} < x, e_i > \overline{< x, e_i >}$
(since $< e_i, e_j > = 0$ if $i \neq j$ and $< e_i, e_j > = 1$ if $i = j$).
Thus $0 \leq ||x||^2$ - $\sum\limits_{i=1}^{n} | < x, e_i > |^2$
Hence $\sum\limits_{i=1}^{n} | < x, e_i > |^2 \leq ||x||^2$.
Claim: $x - \sum\limits_{i=1}^{n} < x, e_i > e_i \perp e_j$, for each j.
Consider $< x - \sum\limits_{i=1}^{n} < x, e_i > e_i, e_j > =$
$< x, e_j >$ - $\sum\limits_{i=1}^{n} < x, e_i >< e_i, e_j >$
$= < x, e_j > - < x, e_j >$ (since $< e_i, e_j > = 0$ if $i \neq j$ and $< e_i, e_j > = 1$ if $i = j$)
$= 0$. Hence we get $x - \sum\limits_{i=1}^{n} < x, e_i > e_i \perp e_j$ for each j. □

Theorem 2.9.2. *If $\{e_i\}$ is an orthonormal set in a Hilbert space H and if x is any vector in H then the set S=$\{e_i, < x, e_i > \neq 0\}$ is either empty or countable.*

Proof. If $x = 0$
then $< x, e_i >$ =0 for each e_i.
Therefore S is empty.
If $x \neq 0$. For $n \in \mathbb{N}$, we define $S_n = \{e_j \in S : | < x, e_j > |^2 > ||x||^2/n\}$.
Claim 1: $|S_n| \leq n - 1$.
Suppose there exists $e_{i1}, e_{i2}, \ldots, e_{in} \in S_n$.
Now we know, $\sum_{i=1}^{n} | < x, e_i > |^2 \leq ||x||^2$ (by Theorem 2.9.1) $\ldots (1)$
Also for each $e_{ij} \in S_n$ we have $| < x, e_j > |^2 > ||x||^2/n$.
Thus we get $n(||x||^2/n) < \sum_{i=1}^{n} | < x, e_i > |^2 \leq ||x||^2$.
Hence $||x||^2 < ||x||^2$, a contradiction. Hence $|S_n| \leq n - 1$.
Claim 2: $S = \cup_{n \in \mathbb{N}} S_n$.
Since $S_n \subseteq S$ for all $n \in \mathbb{N}$. So $\cup_{n \in \mathbb{N}} S_n \subseteq S$.
Now to show $S \subseteq \cup_{n \in \mathbb{N}} S_n$
Let $e \in S$. We will show that $e \in S_n$ for some n.
Enough to prove that there exists $n \in \mathbb{N}$ such that
$| < x, e > |^2 > ||x||^2/n$.
Assume that such n does not exist then we get,
$| < x, e > |^2 \leq ||x||^2/n$. As $n \to \infty$, $||x||^2/n = 0$.
Thus $| < x, e > |^2 = 0$. Hence $< x, e > = 0$.
Therefore $e \notin S$, a contradiction.
Hence there exists $n \in \mathbb{N}$ such that $| < x, e > |^2 > ||x||^2/n$.
Therefore $e \in S_n$.
Since e was arbitrary element of S. Thus $S \subseteq S_n$.

Hence $S \subseteq \cup_{n\in\mathbb{N}} S_n$.
Therefore $S = \cup_{n\in\mathbb{N}} S_n$.
Now since each S_n is countable so S is countable. Hence proved. $\square$

Theorem 2.9.3. *(Bessel's inequality) If $\{e_i\}$ is an orthonormal set in a Hilbert space H, then $\sum |< x, e_i >|^2 \leq ||x||^2$ for every vector $x \in H$.*

Proof. Consider $S = \{\ e_i : < x, e_i > \neq 0\}$.
Then S is either empty or countable(by Theorem 2.9.2).
If S is empty. Then $< x, e_i > = 0$, for all e_i.
Therefore $\sum | < x, e_i > |^2 = 0$.
Since $0 \leq ||x||^2$. Hence $\sum | < x, e_i > |^2 \leq ||x||^2$.
Now assume S is non-empty then S can be finite or countably infinite.
If S is finite, i.e. $S = \{e_1, e_2, \dots, e_n\}$.
Then $\sum | < x, e_i > |^2 = \sum_{i=1}^{n} | < x, e_i > |^2 \leq ||x||^2$ (by Theorem 2.9.1)
If S is countably infinite .
Let us arrange the elements of S as follows $S = \{e_1, e_2, \dots, e_n, \dots\}$.
Now $\sum_{i=1}^{\infty} | < x, e_i > |^2 = \lim_{n\to\infty} \sum_{i=1}^{n} | < x, e_i > |^2$ and
$\sum_{i=1}^{n} | < x, e_i > |^2 \leq ||x||^2$ (by Theorem 2.9.1).
So by Theory of absolute convergence $\sum_{i=1}^{\infty} | < x, e_i > |^2$ is convergent.
We therefore define $\sum | < x, e_i > |^2 = \sum_{i=1}^{\infty} | < x, e_i > |^2$.
Now $\sum | < x, e_i > |^2 = \sum_{i=1}^{\infty} | < x, e_i > |^2 = \lim_{n\to\infty} \sum_{i=1}^{n} | < x, e_i > |^2$
$\leq \lim_{n\to\infty} ||x||^2 = ||x||^2$. Thus $\sum | < x, e_i > |^2 \leq ||x||^2$.
Hence proved. $\square$

Theorem 2.9.4. *If $\{e_i\}$ is an orthonormal set in a Hilbert space H and if x is an arbitrary vector in H then, $x - \sum < x, e_i > e_i \perp e_j$ for each j.*

Proof. Let $S = \{e_i :< x, e_i > \neq 0\}$.
Then S is either empty or countable (by Theorem 2.9.2)
When S is empty, we get $< x, e_i > = 0$, for each e_i.
Consider $< x - \sum < x, e_i > e_i, e_j >$
$= < x, e_j >$ - $\sum < x, e_i >< e_i, e_j > = 0$.
Thus $x - \sum < x, e_i > e_i \perp e_j$ for each j.
When S is finite say $S = \{e_1, e_2, \dots, e_n\}$.
We define $\sum < x, e_i > e_i$ to be $\sum_{i=1}^{n} < x, e_i > e_i$. Then the result holds by Theorem 2.9.1.
When S is countably infinite .
Let us arrange the vectors in S in definite order say $S = \{e_1, e_2, \dots, e_n, \dots\}$.
Put $S_n = \sum_{i=1}^{n} < x, e_i > e_i$.
For $m > n$, $||S_m - S_n||^2 = || \sum_{i=n+1}^{m} < x, e_i > e_i||^2 =$
$\sum_{i=n+1}^{m} | < x, e_i > |^2$.
From Bessel's inequality $\sum_{n=1}^{\infty} | < x, e_n > |^2$ converges.
So $\{S_n\}_{n\in\mathbb{N}}$ is cauchy sequence in H.
Since H is complete, So $S_n \to S'$ as $n \to \infty$.
Let $S' = \sum_{n=1}^{\infty} < x, e_n > e_n$.

We define $\sum < x, e_i > e_i$ to be $\sum_{i=1}^{\infty} < x, e_i > e_i$.
Consider $< x - \sum < x, e_i > e_i, e_j > = < x - \sum_{i=1}^{\infty} < x, e_i > e_i, e_j > =$
$< x - S', e_j > = < x, e_j >$ - $< S', e_j > = < x, e_j >$ - $< lim_{n\to\infty} S_n, e_j >$
$= < x, e_j >$- $lim_{n\to\infty} < S_n, e_j > = < x, e_j >$ - $< x, e_j > = 0$.
Thus $< x - \sum < x, e_i > e_i, e_j > = 0$.
Hence we get x - $\sum < x, e_i > e_i \perp e_j$ for each j.

□

2.10 Complete orthonormal set

Definition 2.9. Complete orthonormal set:
An orthonormal set $\{e_i\}$ in a Hilbert space H is said to be complete if it is maximal in its partially ordered set. i.e. if it is impossible to adjoint a vector e to $\{e_i\}$ in such a way that $\{e, e_i\}$ is an orthonormal set which properly contains e_i.

Definition 2.10. Maximal orthonormal set:
An orthonormal set A in a Hilbert space H is maximal if the only point in H which is orthogonal to every $x \in A$ is 0, i.e. A cannot be extended to a larger orthonormal set.

Note: Maximal orthonormal set and Complete orthonormal set are equivalent. This is proved in Theorem 2.10.2

Theorem 2.10.1. *Every non-zero Hilbert space H contains a complete orthonormal set.*

Proof. It is given that Hilbert space H is non- zero, so there exist a non-zero vector $x_1 \in H$. Let $e_1 = x_1 \ / \ ||x_1||$
Now by using Gram Schmith process we can get orthonormal elements
Let A = collection of orthonormal set .
i.e. $A = \{E_i$— where $i \in \wedge$ and E_i is orthonormal set in H $\}$
Then A is non-empty as $\{e_1\} \in A$. Now the elements of A can be ordered in inclusion. Thus each chain in A has an upper bound given by union of all elements in that chain.
Therefore by Zorn's lemma, A has a maximal element say E.
To show that E is complete orthonormal set
Let us assume that there exists say $\{e, E\}$ an orthonormnal set in H. Then $\{e, E\}$ belongs to A and $E \subseteq \{e, E\}$ but this is a contradiction as E is maximal element of A. Thus our assumption was wrong. Hence E has to be complete orthonormal set. Hence proved. □

Theorem 2.10.2. *let H be a Hilbert space and $\{e_i\}$ be an orthonormal set in H. Then the following conditions are equivalent:*

1) $\{e_i\}$ *is complete.*
2) $x\perp \{ e_i \} \Longrightarrow x = 0$.
3)*If* x *is an arbitrary vector in* H *then* $x = \sum < x, e_i > e_i$.
4) *If* x *is an arbitrary vector in* H *then* $||x||^2 = \sum|< x, e_i >|^2$.

Proof. a) To prove 1) implies 2)
Let us assume that 2) is not true. Then there exists a vector $x \neq 0$ such that $x \perp \{e_i\}$. Define $e = \frac{x}{||x||}$ then, $\{e_i, e\}$ is an orthonormal set which properly contains $\{e_i\}$. This is a contradiction as it is given that $\{e_i\}$ is complete. Hence our assumption was wrong.
Therefore $x\perp \{ e_i \} \Longrightarrow x = 0$.
b) To prove 2) implies 3)
We know that $x - \sum < x, e_i > e_i$ is orthogonal to $\{e_i\}$ (by Theorem 2.9.4) So by assumption 2) we get $x - \sum < x, e_i > e_i = 0$.
Thus $x = \sum < x, e_i > e_i$. Hence proved.
c) To prove 3) implies 4)
$||x||^2 = < x, x > = < \sum < x, e_i > e_i, \sum < x, e_i > e_i >$ (by assumption 3)) $= \sum\sum < x, e_i > \overline{< x, e_j >} < e_i, e_j >$
$= \sum < x, e_i > \overline{< x, e_i >}$ (as $\{e_i\}$ is orthonormal)
$= \sum | < x, e_i > |^2$.
d) To prove 4) implies 1)
Suppose that $\{e_i\}$ is not complete.
i.e. it is a proper subset of an orthnormal set say $\{e_i, e\}$
Since e is orthogonal to all $e_i's$ so we get $< e, e_i > = 0 \ldots$(I).
Now using the assumption 4) we get $||e||^2 = \sum | < e, e_i > |^2 \ldots$(II)
From (I) and (II) we get $||e||^2 = 0$. Thus $e = 0$, which is contradiction, as e is a unit vector. Hence our assumption is wrong.
Thus $\{e_i\}$ is complete.
e) To prove 2) implies 1)
Let us suppose that $\{e_i\}$ is not complete so $\{e_i\}$ is contained in an orthonormal set say $\{e, e_i\}$. Now e is orthogonal to every element in $\{e_i\}$. So by assumption in 2) we get $e = 0$, which is a contradiction. Hence our assumption was wrong. Therefore $\{e_i\}$ is complete. □

The equation, $||x||^2 = \sum|< x, e_i >|^2$ is called Parseval's Identity.

2.11 Separable Hilbert space

Definition 2.11. Dense set:
A subset A of a metric space X is said to be dense in X if $\bar{A} = X$

Definition 2.12. Separable space:
A metric space X is said to be seprable if it has a countable subset which is dense in X.

Example 5. The space of real numbers is seprable

Solution: The set $\mathbb{Q}$ of all rational numbers is countable and is dense in $\mathbb{R}$.

Theorem 2.11.1. *If H is separable, every orthonormal set in H is countable.*

Proof. Given that H is seperable.
Let A be any dense set in H and N be any orthonormal set in H.
To show that N is countable.
Let x, y be two distinct element in N so $< x, y > = 0 \ldots (1)$
Now $||x-y||^2 = < x-y, x-y > = < x, x >$ - $< x, y >$ - $< y, x > + < y, y > = < x, x > + < y, y >$ (from (1)).
$= ||x||^2 + ||y||^2 = 2$. Hence $||x-y||^2 = 2$. Thus $||x-y|| = \sqrt{2}$.
Let B_x and B_y be the neighbourhood of x, y with radius $\sqrt{2/3}$.
Since A is dense in H, so $B_x \cap A \neq \phi$ and $B_y \cap A \neq \phi \ldots (2)$
Also $B_x \cap B_y = \phi \ldots (3)$
If N is uncountable we would have uncountably many such pairwise disjoint neighbourhood hence A would become uncountable. Thus we will not be able to get dense set which is countable in H, which is contradiction to the seperability of H. Therefore our assumption that N is uncountable is wrong. Thus N is countable. Hence proved. $\square$

Theorem 2.11.2. *Let Y be a subspace of a separble Hilbert space H then Y is also seperable.*

2.12 LET US SUM UP

- A space which is complete with respect to metric defined by inner product is called Hilbert space.
- The equation $||x+y||^2 + ||x-y||^2 = 2(||x||^2 + ||y||^2)$ is called parallelogram equality.
- If a norm does not satisfy Parallelogram law then that norm cannot be obtained from inner product. So all normed spaces are not Hilbert spaces.
- For x, y be any two vectors in a Hilbert space
  $| < x, y > | \leq ||x|| \, ||y||$ is called Schwarz iequality.
- For x, y be any two vectors in a Hilbert space
  $||x+y|| \leq ||x|| + ||y||$ is called triangle inequality.

- If Y be a finite dimensional subspace of a Hilbert space H then Y is complete.
- If Y be a subspace of a Hilbert space H, then Y is complete if and only if Y is closed in H.
- There exists a unique vector of smallest norm in a closed convex subset C of a Hilbert Space H.
- $A^{\perp}$ is closed linear subspace.
- If M is a closed linear subspace of a Hilbert space H, x is a vector not in M such that d be the distance from x to M. Then there exists a unique vector y_0 in M such that $||x - y_0|| = d$.
- If M is a proper closed linear subspace of a Hilbert space H, then there exists a non-zero vector $z_0 \in H$ such that $z_0 \perp M$.
- If A is closed linear subspace of a Hilbert space H, then $H = A \oplus A^{\perp}$.
- If $\{e_i\}$ is an orthonormal set in a Hilbert space H and if x is any vector in H then the set S=$\{e_i, < x, e_i > \neq 0\}$ is either empty or countable.
- If $\{e_i\}$ is an orthonormal set in a Hilbert space H then $\sum |< x, e_i >|^2 \leq ||x||^2$ $\forall$ every vector $x \in H$.. This is called Bessel's inequality.
- An orthonormal set $\{e_i\}$ in H is said to be complete if it is maximal in its partially ordered set.
- Every non-zero Hilbert space H contains a complete orthonormal set.
- H be a Hilbert space, $\{e_i\}$ be an orthonormal set in H and x is an arbitrary vector in H then $x = \sum < x, e_i > e_i$. This is called Parseval's Identity.
- A metric space X is said to be seprable if it has a countable subset which is dense in X.

2.13 Chapter End Exercise

1. Any two norms on finite dimensional space are equivalent.
2. Prove Theorem 2.7.2

3. Prove Theorem 2.8.1

4. Prove Theorem 2.8.3

5. State and Prove Schwarz inequality.

6. Prove Theorem 2.11.2

Chapter 3

Normed Spaces

Unit Structure :

3.1 Introduction

In this chapter, you will be introduced to the notion of a norm on a vector space. The concept of norm of a vector is a generalization of the notion of length. The definition of a normed space (a vector space equipped with a norm on it) was given (independently) by S. Banach, H. Hahn and N. Wiener in 1922. In one section of this chapter, you will study the concept of normed spaces which is fundamental to the development of the theory of Banach spaces. You will come to know the relation between a normed space and a metric space. In another section of this chapter, you will learn about convergent sequences and Cauchy sequences in a normed space.

3.2 Objectives

The main objective of this chapter is to learn the normed spaces and Cauchy sequences in it.

After going through this chapter you will be able to:

- Define a norm on a vector space.

- Define a normed space.
- Learn how to check a vector space is a normed space under the given norm.
- Learn Hölder's inequality and Minkowski's inequality (for finite sums and for integrals).
- Prove that every normed space is a metric space.
- Show that a metric space need not be a normed space.
- Define a convergent sequence and Cauchy sequence in a normed space.
- Prove that the quotient space is a normed space under the given norm.
- Prove that every convergent sequence in a normed space is a Cauchy sequence.

3.3 Few definitions and examples

You have learnt the definition of a norm (and properties of a norm) on an inner product space V in your B.Sc. So you can guess the definition of norm on a vector space and get convinced with the following definition.

Definition 3.1. Let V be a vector space over the field $\mathbb{F}(=\mathbb{R}$ or $\mathbb{C})$. A *norm* $\| \ \|$ on V is a real valued function (i.e. $\| \ \| : V \longrightarrow \mathbb{R}$), satisfying the following 4 properties/axioms:

(N 1) $\|x\| \geqslant 0 \qquad \forall x \in V$

(N 2) $\|x\| = 0$ if and only if $x = 0_v \qquad \forall x \in V$

(N 3) $\|x+y\| \leqslant \|x\| + \|y\| \qquad \forall x,\ y \in V$

(N 4) $\|\alpha x\| = |\alpha|\ \|x\| \qquad \forall x \in V$ and $\forall \alpha \in \mathbb{F}$

Now, you will come to know when a vector space is called, a normed space.

Definition 3.2. A *normed space* V is a vector space over the field $\mathbb{F}(=\mathbb{R}$ or $\mathbb{C})$ with a norm $\| \ \|$ defined on it. In such a case, we say, $(V, \| \ \|)$ is a normed space over $\mathbb{F}$. Here, if $\mathbb{F} = \mathbb{R}$ then V is called a *real normed space* and if $\mathbb{F} = \mathbb{C}$ then V is called a *complex normed space.*

Now, you will notice that examples of normed spaces are in abundance.

Example 6. Show that the vector space $\mathbb{R} = \{x | x \in \mathbb{R}\}$ over $\mathbb{R}$ is a normed space under the norm $\|x\| = |x| =$ absolute value of $x \in \mathbb{R}$.

Solution: Using the properties of absolute value of a real number, we have

(1) $\forall x \in \mathbb{R}$, $\|x\| = |x| \geqslant 0$ and hence property (N 1) of norm is satisfied.

(2) $\forall x \in \mathbb{R}$, $\|x\| = 0 \iff |x| = 0 \iff x = 0$ and hence property (N 2) of norm is satisfied.

(3) $\forall x,\ y \in \mathbb{R}$, $\|x+y\| = |x+y| \leqslant |x| + |y| = \|x\| + \|y\|$ and hence property (N 3) of norm is satisfied.

(4) $\forall \alpha \in \mathbb{R}$ and $x \in \mathbb{R}$, $\|\alpha x\| = |\alpha\ x| = |\alpha|\ |x| = |\alpha|\ \|x\|$ and hence property (N 4) of norm is satisfied.

Thus, $\mathbb{R}$ is a normed space under defined norm.

Can you recall the name of the following inequality?

$$\sum_{i=1}^{n} |\ x_i\ y_i\ | \quad \leqslant \quad \left(\sum_{i=1}^{n} |\ x_i\ |^2\right)^{1/2} \quad \left(\sum_{i=1}^{n} |\ y_i\ |^2\right)^{1/2}$$

for any complex (or real) numbers $x_1, \cdots, x_n;\ \ y_1, \cdots, y_n$

It's the Cauchy-Schwarz inequality in an inner product space which you have learnt in your B.Sc. You will revisit this inequality in further example(s).

Now, recall the vector space $\mathbb{C}^n = \{(x_1, \cdots, x_n) \mid x_i \in \mathbb{C}\}$ over $\mathbb{C}$, where the vector addition and scalar multiplication is defined as follows, respectively: for every scalar $\alpha \in \mathbb{C}$ and $(x_1, \cdots, x_n),\ (y_1, \cdots, y_n) \in \mathbb{C}^n$

$$(x_1, \cdots, x_n) + (y_1, \cdots, y_n) = (x_1 + y_1, \cdots, x_n + y_n)$$
$$\alpha\ (x_1, \cdots, x_n) = (\alpha x_1, \cdots, \alpha x_n)$$

Also, $(x_1, \cdots, x_n) = (y_1, \cdots, y_n) \iff x_i = y_i \quad \forall\ i = 1, \cdots, n$

Example 7. Show that the vector space $\mathbb{C}^n = \{(x_1, \cdots, x_n) \mid x_i \in \mathbb{C}\}$ over $\mathbb{C}$ is a normed space under the norm $\|x\| = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$.

Solution: Using the properties of modulus of a complex number, we have

(1) $\forall x = (x_1, \cdots, x_n) \in \mathbb{C}^n$, each $|x_i| \geqslant 0$, and hence $\left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2} = \|x\| \geqslant 0$. So property (N 1) of norm is satisfied.

(2) $\forall x = (x_1, \cdots, x_n) \in \mathbb{C}^n$,

$\|x\| = 0 \iff \|x\|^2 = 0 \iff \sum_{i=1}^{n} |x_i|^2 = 0 \iff |x_i| = 0 (1 \leqslant i \leqslant n)$

$\iff x_i = 0\ (1 \leqslant i \leqslant n) \iff x = (x_1, \cdots, x_n) = (0, \cdots, 0) = 0$ and hence property (N 2) of norm is satisfied.

(3) $\forall x = (x_1, \cdots, x_n),\ y = (y_1, \cdots, y_n) \in \mathbb{C}^n$,
consider

$$\begin{aligned}
\|x+y\|^2 &= \sum_{i=1}^{n} |x_i + y_i|^2 \\
&= \sum_{i=1}^{n} |x_i + y_i| \ \ |x_i + y_i| \\
&\leqslant \sum_{i=1}^{n} |x_i + y_i| \ \ (|x_i| + |y_i|) \qquad \textit{by triangle inequality} \\
&= \sum_{i=1}^{n} |x_i + y_i| \ \ |x_i| \quad + \quad \sum_{i=1}^{n} |x_i + y_i| \ \ |y_i| \\
&= \sum_{i=1}^{n} |(x_i + y_i) \ x_i| \quad + \quad \sum_{i=1}^{n} |(x_i + y_i) \ y_i| \\
&\leqslant \|x+y\| \ \|x\| \quad + \quad \|x+y\| \ \|y\| \\
&\qquad\qquad\qquad\qquad \textit{by Cauchy} - \textit{Schwarz inequality} \\
&= \|x+y\| \ \Big(\|x\| \ + \ \|y\| \Big)
\end{aligned}$$

If $\|x+y\| \neq 0$ then dividing both sides by $\|x+y\|$, we get, $\|x+y\| \leqslant \|x\|+\|y\|$. If $\|x+y\| = 0$ then the inequality $\|x+y\| \leqslant \|x\| + \|y\|$ is trivial, since both sides reduce to zero and hence in any case, property (N 3) of norm is satisfied.

(4) $\forall \alpha \in \mathbb{C}$ and $x = (x_1, \cdots, x_n) \in \mathbb{C}^n$,
$$\|\alpha x\|^2 = \sum_{i=1}^{n} |\alpha \ x_i|^2 = \sum_{i=1}^{n} |\alpha|^2 \ |x_i|^2 = |\alpha|^2 \left(\sum_{i=1}^{n} |x_i|^2 \right) = |\alpha|^2 \ \|x\|^2$$
Taking positive square root on both sides, we get, $\|\alpha \ x\| = |\alpha| \ \|x\|$ and hence property (N 4) of norm is satisfied.

Thus, $\mathbb{C}^n$ is a normed space under defined norm.

Note: This norm is referred as Euclidean norm in $\mathbb{C}^n$ and is denoted as $\|x\|_2$. If $n = 1$ then $\|x\| = \sqrt{x_1^2 + x_2^2}$ where $x = x_1 + i \ x_2$. Clearly, the notion of norm is actually a generalisation of the concept of (Euclidean) length.

Recall the inequality between arithmetic mean and geometric mean. It states that $\quad \forall \ \alpha, \ \beta \in \mathbb{R}^+, \quad \sqrt{\alpha\beta} \leqslant \frac{1}{2} \ (\alpha + \beta)$

The generalisation of inequality between arithmetic mean and geometric mean is given in the following lemma.

Lemma 3.3.1. *Let $0 < \lambda < 1$. Then $\alpha^{\lambda}\ \beta^{1-\lambda} \leqslant \lambda\alpha + (1-\lambda)\beta$ holds good for every pair of non-negative real numbers α and β.*

Proof. If either $\alpha = 0$ or $\beta = 0$ then we are done. So, assume that $\alpha > 0$ and $\beta > 0$. For every non-negative real number t, define a function ϕ as $\phi(t) = (1-\lambda) + \lambda t - t^{\lambda}$. For extreme value of ϕ, we must have, $\phi'(t) = 0$ which implies $\lambda\ (1 - t^{\lambda-1}) = 0$ and hence $t^{\lambda-1} = 1$ as $\lambda \neq 0$. Thus $t = 1$. So, at $t = 1$, we have extreme value of ϕ. It is easy to see that

$$\phi'(t) \begin{cases} < 0 & \text{if} \quad t < 1 \\ > 0 & \text{if} \quad t > 1 \end{cases}$$

This implies that ϕ attains minimum at $t = 1$. Thus, $\phi(t) \geqslant \phi(1)$ which gives $(1-\lambda) + \lambda t \geqslant t^{\lambda}$. In this last inequality, put $t = \dfrac{\alpha}{\beta}$ where α and β are non-negative real numbers. Then $(1-\lambda) + \lambda\left(\dfrac{\alpha}{\beta}\right) \geqslant \left(\dfrac{\alpha}{\beta}\right)^{\lambda}$ and on multiplying by β throughout, we are done. □

Definition 3.3. Let p and q be non-negative extended real numbers. For $p \geqslant 1$, q is said to be *conjugate* of p if

$$\begin{aligned} \frac{1}{p} + \frac{1}{q} = 1, &\quad for\ \ 1 < p < \infty \\ q = \infty, &\quad for\ \ p = 1 \\ q = 1, &\quad for\ \ p = \infty \end{aligned}$$

Let $1 \leqslant p < \infty$. To show that $\mathbb{C}^n$ is a normed space under the norm $\|x\| = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$, we need a special inequality called as Minkowiski's inequality for finite sums. To prove this Minkowiski's inequality, we need another special inequality called as Hölder's inequality for finite sums. You will see these inqualities as following two Lemma's.

Lemma 3.3.2. ***Hölder's inequality*** *for finite sums:*
Let $1 < p,\ q < \infty$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$. For any complex (or real) numbers $x_1, \cdots, x_n;\ \ y_1, \cdots, y_n$

$$\sum_{i=1}^{n} |\ x_i\ y_i\ | \quad \leqslant \quad \left(\sum_{i=1}^{n} |\ x_i\ |^p\right)^{1/p} \quad \left(\sum_{i=1}^{n} |\ y_i\ |^q\right)^{1/q}$$

Proof. If x_i and y_i are all zero then the result is obvious. So let atleast one $x_i \neq 0$ and atleast one $y_i \neq 0$.
By Lemma 3.3.1, for $\alpha \geqslant 0$ and $\beta \geqslant 0$, we have $\alpha^{\lambda}\ \beta^{1-\lambda} \leqslant \lambda\alpha + (1-\lambda)\beta$.

In this inequality, take $\lambda = \dfrac{1}{p}$, $\alpha = \left(\dfrac{|\, x_i \,|}{(\sum_{i=1}^n |\, x_i \,|^p)^{1/p}}\right)^p$ and $\beta = \left(\dfrac{|\, y_i \,|}{(\sum_{i=1}^n |\, y_i \,|^q)^{1/q}}\right)^q$. Then we get, $1-\lambda = \dfrac{1}{q}$ and

$$\frac{|\, x_i \,|}{(\sum_{i=1}^n |\, x_i \,|^p)^{1/p}} \frac{|\, y_i \,|}{(\sum_{i=1}^n |\, y_i \,|^q)^{1/q}} \leqslant \frac{1}{p}\frac{|\, x_i \,|^p}{(\sum_{i=1}^n |\, x_i \,|^p)} + \frac{1}{q}\frac{|\, y_i \,|^q}{(\sum_{i=1}^n |\, y_i \,|^q)}$$

$$\forall \;\; i = 1, \cdots, n.$$

Adding all these inequalities, we get

$$\frac{\sum_{i=1}^n |\, x_i \,| \;\; |\, y_i \,|}{(\sum_{i=1}^n |\, x_i \,|^p)^{1/p} \; (\sum_{i=1}^n |\, y_i \,|^q)^{1/q}} \leqslant \frac{1}{p}\frac{\sum_{i=1}^n |\, x_i \,|^p}{(\sum_{i=1}^n |\, x_i \,|^p)} + \frac{1}{q}\frac{\sum_{i=1}^n |\, y_i \,|^q}{(\sum_{i=1}^n |\, y_i \,|^q)}$$

Thus, $$\sum_{i=1}^n |\, x_i \; y_i \,| \;\; \leqslant \;\; \left(\sum_{i=1}^n |\, x_i \,|^p\right)^{1/p} \;\; \left(\sum_{i=1}^n |\, y_i \,|^q\right)^{1/q}. \qquad \square$$

Note: If we take $p = 2$ in Hölder's inequality then $q = 2$ and we get the Cauchy-Schwarz inequality. So, Cauchy-Schwarz inequality is a special case of Hölder's inequality.

Lemma 3.3.3. ***Minkowski's inequality*** *for finite sums: Let* $1 \leqslant p < \infty$*. For any complex (or real) numbers* $x_1, \cdots, x_n; \quad y_1, \cdots, y_n$

$$\left(\sum_{i=1}^n |\, x_i + y_i \,|^p\right)^{1/p} \;\; \leqslant \;\; \left(\sum_{i=1}^n |\, x_i \,|^p\right)^{1/p} \;\; + \;\; \left(\sum_{i=1}^n |\, y_i \,|^p\right)^{1/p}$$

Proof. If $p = 1$ then

$$\left(\sum_{i=1}^n |\, x_i + y_i \,|^p\right)^{1/p} = \sum_{i=1}^n |\, x_i + y_i \,| \;\; \leqslant \;\; \sum_{i=1}^n (|\, x_i \,| + |\, y_i \,|)$$

$$= \sum_{i=1}^n |\, x_i \,| \; + \; \sum_{i=1}^n |\, y_i \,| = \left(\sum_{i=1}^n |\, x_i \,|^p\right)^{1/p} \; + \; \left(\sum_{i=1}^n |\, y_i \,|^p\right)^{1/p}$$

Therefore the inequality holds for $p = 1$. So let $p > 1$ and $\dfrac{1}{q} = 1 - \dfrac{1}{p}$ so that $q > 1$. Then $p = (p-1)q$ and $p - \dfrac{p}{q} = 1$

Now by Hölder's inequality in Lemma 3.3.2 (for finite sums), we have

$$\begin{aligned}
\sum_{i=1}^{n} | x_i | \quad | x_i + y_i |^{p-1} &\leqslant \left(\sum_{i=1}^{n} | x_i |^p \right)^{1/p} \left[\sum_{i=1}^{n} \left(| x_i + y_i |^{p-1} \right)^q \right]^{1/q} \\
&= \left(\sum_{i=1}^{n} | x_i |^p \right)^{1/p} \left[\sum_{i=1}^{n} | x_i + y_i |^{q(p-1)} \right]^{1/q} \\
&= \left(\sum_{i=1}^{n} | x_i |^p \right)^{1/p} \left[\sum_{i=1}^{n} | x_i + y_i |^{p} \right]^{1/q} \\
&= \left(\sum_{i=1}^{n} | x_i |^p \right)^{1/p} \left[\left(\sum_{i=1}^{n} | x_i + y_i |^{p} \right)^{1/p} \right]^{p/q} \\
&\qquad \longrightarrow (*)
\end{aligned}$$

$$\begin{aligned}
Now \quad \sum_{i=1}^{n} | x_i + y_i |^p &= \sum_{i=1}^{n} | x_i + y_i | \cdot | x_i + y_i |^{p-1} \\
&\leqslant \sum_{i=1}^{n} (| x_i | + | y_i |) \, | x_i + y_i |^{p-1} \\
&= \sum_{i=1}^{n} | x_i | \quad | x_i + y_i |^{p-1} + \sum_{i=1}^{n} | y_i | \quad | x_i + y_i |^{p-1}
\end{aligned}$$

On using $(*)$ to the two summation terms on RHS, we get,

$$\begin{aligned}
\sum_{i=1}^{n} | x_i + y_i |^p &\leqslant \left(\sum_{i=1}^{n} | x_i |^p \right)^{1/p} \left[\left(\sum_{i=1}^{n} | x_i + y_i |^{p} \right)^{1/p} \right]^{p/q} \\
&\quad + \left(\sum_{i=1}^{n} | y_i |^p \right)^{1/p} \left[\left(\sum_{i=1}^{n} | x_i + y_i |^{p} \right)^{1/p} \right]^{p/q} \\
\therefore \sum_{i=1}^{n} | x_i + y_i |^p &\leqslant \left(\sum_{i=1}^{n} | x_i + y_i |^p \right)^{1/q} \left[\left(\sum_{i=1}^{n} | x_i |^{p} \right)^{1/p} + \right. \\
&\qquad \left. \left(\sum_{i=1}^{n} | y_i |^{p} \right)^{1/p} \right]
\end{aligned}$$

Hence,

$$\left(\sum_{i=1}^{n} | x_i + y_i |^p \right)^{1/p} \leqslant \left(\sum_{i=1}^{n} | x_i |^p \right)^{1/p} + \left(\sum_{i=1}^{n} | y_i |^p \right)^{1/p}. \qquad \square$$

Example 8. Show that the vector space $\mathbb{C}^n = \{(x_1, \cdots, x_n) \mid x_i \in \mathbb{C}\}$ over $\mathbb{C}$ is a normed space under the norm $\|x\| = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$ where

$1 \leqslant p < \infty$.

Solution: Using the properties of modulus of a complex number, we have

(1) $\forall x = (x_1, \cdots, x_n) \in \mathbb{C}^n$, each $|x_i| \geqslant 0$, and hence $\left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$ $= \|x\| \geqslant 0$. So property (N 1) of norm is satisfied.

(2) $\forall x = (x_1, \cdots, x_n) \in \mathbb{C}^n$,

$$\begin{aligned} \|x\| = 0 &\Longleftrightarrow \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} = 0 \\ &\Longleftrightarrow \sum_{i=1}^{n} |x_i|^p = 0 \\ &\Longleftrightarrow |x_i| = 0 \qquad \forall \ i = 1, \cdots, n \\ &\Longleftrightarrow x_i = 0 \qquad \forall \ i = 1, \cdots, n \\ &\Longleftrightarrow x = (x_1, \cdots, x_n) = (0, \cdots, 0) = 0 \end{aligned}$$

and hence property (N 2) of norm is satisfied.

(3) $\forall x = (x_1, \cdots, x_n), \ y = (y_1, \cdots, y_n) \in \mathbb{C}^n$, by Minkowski's inequality in Lemma 3.3.3 (for finite sums), we have,

$$\left(\sum_{i=1}^{n} | \, x_i + y_i \, |^p \right)^{1/p} \leqslant \left(\sum_{i=1}^{n} | \, x_i \, |^p \right)^{1/p} + \left(\sum_{i=1}^{n} | \, y_i \, |^p \right)^{1/p}$$

$\Longrightarrow \ \|x + y\| \leqslant \|x\| + \|y\|$ and hence property (N 3) of norm is satisfied.

(4) $\forall \alpha \in \mathbb{C}$ and $x = (x_1, \cdots, x_n) \in \mathbb{C}^n$,

$$\begin{aligned} \|\alpha x\| &= \left(\sum_{i=1}^{n} |\alpha \ x_i|^p \right)^{1/p} \\ &= \left(\sum_{i=1}^{n} (|\alpha| \ |x_i|)^p \right)^{1/p} \\ &= \left(|\alpha|^p \ \sum_{i=1}^{n} |x_i|^p \right)^{1/p} \\ &= |\alpha| \ \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \\ &= |\alpha| \ \|x\| \end{aligned}$$

and hence property (N 4) of norm is satisfied.

Thus, $\mathbb{C}^n$ is a normed space under defined norm.

Example 9. Show that the vector space $\mathbb{C}^n = \{(x_1, \cdots, x_n) \mid x_i \in \mathbb{C}\}$ over $\mathbb{C}$ is a normed space under the norm $\|x\| = max\{| x_1 |, \cdots, | x_n |\}$. (This norm is referred as $\|x\|_\infty$ on $\mathbb{C}^n$).

Solution: Using the properties of of modulus of complex number, we have

(1) $\forall x = (x_1, \cdots, x_n) \in \mathbb{C}^n$, each $|x_i| \geqslant 0$, and hence $max\{| x_1 |, \cdots, | x_n |\} = \|x\| \geqslant 0$. So property (N 1) of norm is satisfied.

(2) $\forall x = (x_1, \cdots, x_n) \in \mathbb{C}^n$,

$$\begin{aligned} \|x\| = 0 &\Longleftrightarrow max\{| x_1 |, | x_2 |, \cdots, | x_n |\} = 0 \\ &\Longleftrightarrow |x_i| = 0 \qquad\qquad \forall \;\; i = 1, \cdots, n \\ &\Longleftrightarrow x_i = 0 \qquad\qquad \forall \;\; i = 1, \cdots, n \\ &\Longleftrightarrow x = (x_1, \cdots, x_n) = (0, \cdots, 0) = 0 \end{aligned}$$

and hence property (N 2) of norm is satisfied.

(3) $\forall x = (x_1, \cdots, x_n),\; y = (y_1, \cdots, y_n) \in \mathbb{C}^n$,

$$\begin{aligned} \|x + y\| &= max\{| x_1 + y_1 |, \cdots, | x_n + y_n |\} \\ &\leqslant max\{| x_1 | + | y_1 |, \cdots, | x_n | + | y_n |\} \\ &\leqslant max\{| x_1 |, \cdots, | x_n |\} + max\{| y_1 |, \cdots, | y_n |\} \\ &= \|x\| + \|y\| \end{aligned}$$

and hence property (N 3) of norm is satisfied.

(4) $\forall \alpha \in \mathbb{C}$ and $x = (x_1, \cdots, x_n) \in \mathbb{C}^n$,

$$\begin{aligned} \|\alpha x\| &= max\{| \alpha x_1 |, \cdots, | \alpha x_n |\} \\ &= max\{| \alpha || x_1 |, \cdots, | \alpha || x_n |\} \\ &= | \alpha | \, max\{| x_1 |, \cdots, | x_n |\} \\ &= |\alpha| \; \|x\| \end{aligned}$$

and hence property (N 4) of norm is satisfied.

Thus, $\mathbb{C}^n$ is a normed space under defined norm.

Remark 3.3.1. In view of norms defined on $\mathbb{C}^n$ in examples 8 and 9, we have

$$\|x\|_p = \begin{cases} \left(\displaystyle\sum_{i=1}^{n} |x_i|^p \right)^{1/p} & \text{if} \quad 1 \leqslant p < \infty \\ max\{| x_1 |, \cdots, | x_n |\} & \text{if} \quad p = \infty \end{cases}$$

These norms are referred as p-norms on $\mathbb{C}^n$.

Remark 3.3.2. From examples 7, 8 and 9, it is clear that on a vector space V, we can define more than one norm and accordingly different normed spaces are obtained from same vector space V.

Recall a sequence space which is a vector space whose elements are infinite sequences of real or complex numbers where the vector addition and scalar multiplication, respectively are defined as follows:

$$\{x_1, \cdots, x_n, \cdots\} + \{y_1, \cdots, y_n, \cdots\} = \{x_1 + y_1, \cdots, x_n + y_n, \cdots\}$$
$$\alpha\{x_1, \cdots, x_n, \cdots\} = \{\alpha x_1, \cdots, \alpha x_n, \cdots\}$$

for every scalar $\alpha \in \mathbb{C}$ *or* $\mathbb{R}$
Also, $\{x_1, \cdots, x_n, \cdots\}$=$\{y_1, \cdots, y_n, \cdots\}$ if and only if $x_i = y_i \quad \forall\ i = 1, \cdots, n, \cdots$.

Example 10. Let $1 \leqslant p < \infty$. Show that the sequence space $l^p = \left\{\{x_1, \cdots, x_n, \cdots\} \,\middle|\, \sum_{i=1}^{\infty} \mid x_i \mid^p < \infty \text{ and } x_i \in \mathbb{C}\right\}$ over $\mathbb{C}$ is a normed space under the norm $\|x\| = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p}$.

Solution: Using the properties of modulus of a complex number, we have

(1) $\forall x = \{x_1, \cdots, x_n, \cdots\} \in l^p$, each $|x_i| \geqslant 0$, and hence $\left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p}$ $= \|x\| \geqslant 0$. So property (N 1) of norm is satisfied.

(2) $\forall x = \{x_1, \cdots, x_n, \cdots\} \in l^p$,

$$\begin{aligned}
\|x\| = 0 &\iff \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p} = 0 \\
&\iff \sum_{i=1}^{\infty} |x_i|^p = 0 \\
&\iff |x_i| = 0 \qquad\qquad \forall\ \ i = 1, \cdots, n, \cdots \\
&\iff x_i = 0 \qquad\qquad\ \ \forall\ \ i = 1, \cdots, n, \cdots \\
&\iff x = \{x_1, \cdots, x_n, \cdots\} = \{0, \cdots, 0, \cdots\} = 0
\end{aligned}$$

and hence property (N 2) of norm is satisfied.

(3) Let $x = \{x_1, \cdots, x_n, \cdots\},\ y = \{y_1, \cdots, y_n, \cdots\} \in l^p$.
By Minkowski's inequality in Lemma 3.3.3 (for finite sums), we have,

$$\left(\sum_{i=1}^{n} \mid x_i + y_i \mid^p\right)^{1/p} \leqslant \left(\sum_{i=1}^{n} \mid x_i \mid^p\right)^{1/p} + \left(\sum_{i=1}^{n} \mid y_i \mid^p\right)^{1/p}$$
$$\leqslant \left(\sum_{i=1}^{\infty} \mid x_i \mid^p\right)^{1/p} + \left(\sum_{i=1}^{\infty} \mid y_i \mid^p\right)^{1/p} \longrightarrow (*)$$

This is true for all $n \in \mathbb{N}$. As the 2 series on RHS of $(*)$ converge, it is clear that the series on LHS of $(*)$ must converge. Therefore,

$$\left(\sum_{i=1}^{\infty} \mid x_i + y_i \mid^p\right)^{1/p} \leqslant \left(\sum_{i=1}^{\infty} \mid x_i \mid^p\right)^{1/p} + \left(\sum_{i=1}^{\infty} \mid y_i \mid^p\right)^{1/p}$$

$\implies \|x + y\| \leqslant \|x\| + \|y\|$ and hence property (N 3) of norm is satisfied.

(4) $\forall \alpha \in \mathbb{C}$ and $x = \{x_1, \cdots, x_n, \cdots\} \in l^p$,

$$\begin{aligned}\|\alpha x\| &= \left(\sum_{i=1}^{\infty} |\alpha \; x_i|^p\right)^{1/p} \\ &= \left(\sum_{i=1}^{\infty} (|\alpha| \; |x_i|)^p\right)^{1/p} \\ &= \left(|\alpha|^p \sum_{i=1}^{\infty} |x_i|^p\right)^{1/p} \\ &= |\alpha| \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p} \\ &= |\alpha| \; \|x\|\end{aligned}$$

and hence property (N 4) of norm is satisfied.

Thus, l^p is a normed space under defined norm.

Example 11. Show that the sequence space $l^p = \left\{ \{x_1, \cdots, x_n, \cdots\} \middle| sup \right.$ $\left. \{\mid x_1 \mid, \cdots, \mid x_n \mid, \cdots\} < \infty \; and \; x_i \in \mathbb{C} \right\}$ over $\mathbb{C}$ is a normed space under the norm $\|x\| = sup\{\mid x_1 \mid, \cdots, \mid x_n \mid, \cdots\}$.
(This norm is denoted as $\|x\|_\infty$ on l^p).

Solution: Using the properties of modulues of a complex number, we have

(1) $\forall x = \{x_1, \cdots, x_n, \cdots\} \in l^p$, each $|x_i| \geqslant 0$, and hence $sup\{| x_1 |, \cdots, | x_n | \cdots\} = \|x\| \geqslant 0$. So property (N 1) of norm is satisfied.

(2) $\forall x = \{x_1, \cdots, x_n, \cdots\} \in l^p$,

$$\begin{aligned} \|x\| = 0 &\Longleftrightarrow sup\{| x_1 |, | x_2 |, \cdots, | x_n |, \cdots\} = 0 \\ &\Longleftrightarrow |x_i| = 0 \qquad\qquad \forall \;\; i = 1, \cdots, n, \cdots \\ &\Longleftrightarrow x_i = 0 \qquad\qquad \forall \;\; i = 1, \cdots, n, \cdots \\ &\Longleftrightarrow x = \{x_1, \cdots, x_n \cdots\} = \{0, \cdots, 0, \cdots\} = 0 \end{aligned}$$

and hence property (N 2) of norm is satisfied.

(3) $\forall x = \{x_1, \cdots, x_n, \cdots\},\; y = \{y_1, \cdots, y_n, \cdots\} \in l^p$,

$$\begin{aligned} \|x + y\| &= sup\{| x_1 + y_1 |, \cdots, | x_n + y_n |, \cdots\} \\ &\leqslant sup\{| x_1 | + | y_1 |, \cdots, | x_n | + | y_n |, \cdots\} \\ &\leqslant sup\{| x_1 |, \cdots, | x_n |, \cdots\} + sup\{| y_1 |, \cdots, | y_n |, \cdots\} \\ &= \|x\| + \|y\| \end{aligned}$$

and hence property (N 3) of norm is satisfied.

(4) $\forall \alpha \in \mathbb{C}$ and $x = \{x_1, \cdots, x_n, \cdots\} \in l^p$,

$$\begin{aligned} \|\alpha x\| &= sup\{| \alpha x_1 |, \cdots, | \alpha x_n |, \cdots\} \\ &= sup\{| \alpha || x_1 |, \cdots, | \alpha || x_n |, \cdots\} \\ &= |\alpha| \; sup\{| x_1 |, \cdots, | x_n |, \cdots\} \\ &= |\alpha| \; \|x\| \end{aligned}$$

and hence property (N 4) of norm is satisfied.

Thus, l^p is a normed space under defined norm.

Remark 3.3.3. In view of norms defined on l^p in examples 10 and 11, we have

$$\|x\|_p = \begin{cases} \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p} & \text{if} \quad 1 \leqslant p < \infty \\ sup\{| x_1 |, \cdots, | x_n |, \cdots\} & \text{if} \quad p = \infty \end{cases}$$

These norms are referred as p-norms on l^p.

Recall the measure theory and Lebesgue integration that you learnt. Consider a measure space (E, S, μ), where E is a measurable set, S is a σ-algebra and μ is a measure on S. For $1 \leqslant p < \infty$, let $L^p(E, \mu) = \left\{ f : E \longrightarrow \mathbb{R} \,\middle|\, f \text{ is a measurable function on } E \;\&\; \int_E | f(x) |^p \, dx < \infty \right\}$

For convenience, (1) we denote $\int_E \mid f(x) \mid^p dx < \infty$ as $\int_E \mid f \mid^p < \infty$. (2) we take $E = [a, b]$ and μ as a Lebesgue measure. We write this space as $L^p(E)$.

It is easy to verify that $L^p(E)$ is a vector space over $\mathbb{R}$. i.e., $f + g \in L^p(E)$ and $\alpha f \in L^p(E) \quad \forall\ f, g \in L^p(E)$, $\forall \alpha \in \mathbb{R}$. Note that the elements of $L^p(E)$ are equivalence classes of those functions, where f is equivalent to g if $\int_E \mid f - g \mid^p = 0$. i.e. the elements of $L^p(E)$ are equivalence classes of measurable functions which are equal almost everywhere (a.e.).

To show that $L^p(E)$ is a normed space under the norm $\|f\| = \left(\int_E |f|^p\right)^{1/p}$ where $1 \leqslant p < \infty$, we need a special inequality called as Minkowiski's inequality for integrals. To prove this Minkowiski's inequality, we need another special inequality called as Hölder's inequality for integrals. You will see these inqualities as following two Lemma's.

Lemma 3.3.4. ***(Hölder's inequality for integrals)***:
Let $1 < p,\ q < \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$. *For* $f \in L^p(E)$ *and* $g \in L^q(E)$ *(where* E *is bounded closed interval in* $\mathbb{R}$*),*

$$\int_E \mid f\ g \mid \quad \leqslant \quad \left(\int_E \mid f \mid^p\right)^{1/p} \quad \left(\int_E \mid g \mid^q\right)^{1/q}$$

Proof. The inequality is trivial if either $f = 0$ a.e. or $g = 0$ a.e. So let $f \neq 0$ a.e. and $g \neq 0$ a.e. Then $\int_E \mid f \mid^p \ > 0$ and $\int_E \mid g \mid^q \ > 0$. By Lemma 3.3.1, for $\alpha \geqslant 0$ and $\beta \geqslant 0$, we have $\alpha^{\lambda}\ \beta^{1-\lambda} \leqslant \lambda\alpha + (1-\lambda)\beta$. In this inequality, take $\lambda = \frac{1}{p}$, $\alpha = \left(\frac{\mid f \mid}{(\int_E \mid f \mid^p)^{1/p}}\right)^p$ and $\beta = \left(\frac{\mid g \mid}{(\int_E \mid g \mid^q)^{1/q}}\right)^q$. Then we get, $1 - \lambda = \frac{1}{q}$ and

$$\frac{\mid f \mid}{(\int_E \mid f \mid^p)^{1/p}} \frac{\mid g \mid}{(\int_E \mid g \mid^q)^{1/q}} \leqslant \frac{1}{p}\frac{\mid f \mid^p}{(\int_E \mid f \mid^p)} + \frac{1}{q}\frac{\mid g \mid^q}{(\int_E \mid g \mid^q)}$$

On integrating, we get

$$\frac{\int_E \mid f \mid \ \mid g \mid}{(\int_E \mid f \mid^p)^{1/p}\ (\int_E \mid g \mid^q)^{1/q}} \leqslant \frac{1}{p}\frac{\int_E \mid f \mid^p}{(\int_E \mid f \mid^p)} + \frac{1}{q}\frac{\int_E \mid g \mid^q}{(\int_E \mid g \mid^q)}$$

Thus, $$\int_E \mid f\ g \mid \quad \leqslant \quad \left(\int_E \mid f \mid^p\right)^{1/p} \quad \left(\int_E \mid g \mid^q\right)^{1/q}$$ □

Note: Recall that if f and g are Lebesgue measurable over E then the product $f\ g$ is also Lebesgue measurable over E. If $\int_E \mid f \mid^p < \infty$

and $\int_E \mid g \mid^q < \infty$ then by Hölder's inequality, $\int_E \mid f\ g \mid < \infty$.
i.e. if $f \in L^p$ and $g \in L^q$ then $f\ g \in L$

Lemma 3.3.5. ***(Minkowski's inequality for integrals)****:*
Let $1 \leqslant p < \infty$. If $f \in L^p(E)$ and $g \in L^p(E)$ (where E is bounded closed interval in $\mathbb{R}$), then

$$\left(\int_E \mid f+g \mid^p \right)^{1/p} \leqslant \left(\int_E \mid f \mid^p \right)^{1/p} + \left(\int_E \mid g \mid^p \right)^{1/p}$$

Proof. It is easy to see that if f and g are Lebesgue measurable over E then $f+g$ is also Lebesgue measurable over E. Also, if $\displaystyle\int_E \mid f \mid^p < \infty$ and $\displaystyle\int_E \mid g \mid^p < \infty$ then $\displaystyle\int_E \mid f+g \mid^p < \infty$.
If $p=1$ then as $\mid f+g \mid \leqslant \mid f \mid + \mid g \mid$, we are done. So let $p>1$ and $\dfrac{1}{q} = 1 - \dfrac{1}{p}$ so that $q>1$. Then $p=(p-1)q$ and $p - \dfrac{p}{q} = 1$.
Clearly, $\displaystyle\int_E \left(\mid f+g \mid^{p-1} \right)^q = \int_E \mid f+g \mid^p < \infty$.
Also, $\displaystyle\left(\int_E (\mid f+g \mid^{p-1})^q \right)^{1/q} = \left(\int_E \mid f+g \mid^p \right)^{1/q}$. $\longrightarrow (1)$

Now by Hölder's inequality in Lemma 3.3.4 (for integrals),

$$\int_E \mid f \mid \ \mid f+g \mid^{p-1} \leqslant \left(\int_E \mid f \mid^p \right)^{1/p} \left(\int_E (\mid f+g \mid^{p-1})^q \right)^{1/q} \longrightarrow (2)$$

and

$$\int_E \mid g \mid \ \mid f+g \mid^{p-1} \leqslant \left(\int_E \mid g \mid^p \right)^{1/p} \left(\int_E (\mid f+g \mid^{p-1})^q \right)^{1/q} \longrightarrow (3)$$

Now, consider

$$\int_E \mid f+g \mid^p = \int_E \mid f+g \mid^{p-1} \ \mid f+g \mid \leqslant \int_E \mid f+g \mid^{p-1} \ (\mid f \mid + \mid g \mid)$$

$$\implies \int_E \mid f+g \mid^p \leqslant \int_E \mid f \mid \ \mid f+g \mid^{p-1} + \int_E \mid g \mid \ \mid f+g \mid^{p-1}$$

Using $(1),(2),(3)$, we get,

$$\int_E \mid f+g \mid^p \leqslant \left\{ \left(\int_E \mid f \mid^p \right)^{1/p} + \left(\int_E \mid g \mid^p \right)^{1/p} \right\} \left(\int_E \mid f+g \mid^p \right)^{1/q}$$

If $\displaystyle\int_E \mid f+g \mid^p \neq 0$ then on dividing throughout by $\displaystyle\int_E \mid f+g \mid^p$ and using $1 - \dfrac{1}{q} = \dfrac{1}{p}$, we get,

$$\left(\int_E \mid f+g \mid^p \right)^{1/p} \leqslant \left(\int_E \mid f \mid^p \right)^{1/p} + \left(\int_E \mid g \mid^p \right)^{1/p}.$$

If $\displaystyle\int_E \mid f+g \mid^p = 0$ then there is nothing to prove and we are done. $\square$

Example 12. Let $1 \leqslant p < \infty$ & E be a (bounded closed interval in $\mathbb{R}$) measurable set. Show that the vector space $L^p(E) = \{f : E \longrightarrow \mathbb{R} | f$ is Lebesgue measurable function on E and $| f |^p$ is Lebesgue integrable over $E\}$ over $\mathbb{R}$ is a normed space under norm $\|f\|_p = \left(\int_E | f |^p \right)^{1/p}$. (This norm is referred as p-norms on $L^p(E)$).

Solution: Using the properties of Lebesgue measurable and Lebesgue integrable functions, we have

(1) $\forall f \in L^p(E)$, $\|f\|_p = \left(\int_E | f |^p \right)^{1/p} \geqslant 0$. So property (N 1) of norm is satisfied.

(2) Let $f \in L^p(E)$. If $f = 0$ a.e. then $\|f\|_p = \left(\int_E | 0 |^p \right)^{1/p} = 0$.
Conversely,

$$\begin{aligned} \|f\|_p = 0 &\Longrightarrow \left(\int_E |f|^p \right)^{1/p} = 0 \\ &\Longrightarrow |f|^p = 0 \quad a.e. \\ &\Longrightarrow |f| = 0 \quad a.e. \\ &\Longrightarrow f = 0 \quad a.e. \end{aligned}$$

(note that the condition $\|f\| = 0 \iff f = 0$ is not satisfied)
If we do not distinguish between equivalent functions then the property (N 2) of norm is satisfied.

(3) Let $f,\ g \in L^p(E)$.

By Minkowski's inequality 3.3.5 (for integrals), we have,

$$\left(\int_E | f + g |^p \right)^{1/p} \leqslant \left(\int_E | f |^p \right)^{1/p} + \left(\int_E | g |^p \right)^{1/p}$$

Therefore, $\|f + g\|_p \leqslant \|f\|_p + \|g\|_p$ and hence property (N 3) of norm is satisfied.

(4) $\forall \alpha \in \mathbb{R}$ and $f \in L^p(E)$,

$$\begin{aligned}\|\alpha f\|_p &= \left(\int_E |\alpha \ f|^p\right)^{1/p} \\ &= \left(\int_E (|\alpha| \ |f|)^p\right)^{1/p} \\ &= \left(|\alpha|^p \ \int_E |f|^p\right)^{1/p} \\ &= |\alpha| \ \left(\int_E |f|^p\right)^{1/p} \\ &= |\alpha| \ \|f\|_p\end{aligned}$$

and hence property (N 4) of norm is satisfied.

Thus, $L^p(E)$ is a normed space under defined norm.

Definition 3.4. Let E be a (bounded closed interval in $\mathbb{R}$) measurable set. A measurable function $f : E \longrightarrow \mathbb{R}$ is said to be *essentially bounded* on E if there exists a finite real number $m > 0$ such that $|f(x)| \leqslant m$ a.e. on E. Here m is called essential (upper) bound for f. (i.e.f is bounded except possibly on a set of measure zero)

If f has an essential upper bound then least upper bound exists. The least such bound is denoted by *ess sup* $|f|$. If f does not have any essential bound, then its essential supremum is defined to be ∞.

We define $L^\infty(E)$ = the class of all those measurable functions defined on E which are essentially bounded on E. The elements of $L^\infty(E)$ are equivalence classes of f. It is easy to verify that $L^\infty(E)$ is a vector space over $\mathbb{R}$.

Example 13. Let E be a (bounded closed interval in $\mathbb{R}$) measurable set. Show that the vector space $L^\infty(E) = \{f : E \longrightarrow \mathbb{R} | \text{f is a measurbale function on } E \text{ and } ess\ sup|f| < \infty\}$ over $\mathbb{R}$ is a normed space under norm $\|f\|_\infty = ess\ \sup_E |f(x)| = inf\{m > 0 \mid |f(x)| \leqslant m \text{ a.e. on } E\}$.

Hint.(Check)

(1) $\forall f \in L^\infty(E)$, $\|f\|_p \geqslant 0$.

(2) $\forall f \in L^\infty(E)$, $\|f\|_\infty = 0$ if and only if $f = 0$ a.e.

(3) Let $f, \ g \in L^\infty(E)$. Clearly, $|f| \leqslant \|f\|_\infty$ a.e. and $|g| \leqslant \|g\|_\infty$ a.e. As $|f + g| \leqslant |f| + |g| \leqslant \|f\|_\infty + \|g\|_\infty$ a.e., it follows that, $\|f + g\|_\infty \leqslant \|f\|_\infty + \|g\|_\infty$

(4) $\forall \alpha \in \mathbb{R}$ and $f \in L^\infty(E)$, $\|\alpha f\|_\infty = |\alpha| \ \|f\|_\infty$

Remark 3.3.4. In view of norms defined on $L^p(E)$ in examples 12 and 13, we have

$$\|f\|_p = \begin{cases} \left(\displaystyle\int_E |f|^p\right)^{1/p} & \text{if} \quad 1 \leqslant p < \infty \\ ess\ sup|f| & \text{if} \quad p = \infty \end{cases}$$

Recall that the set $C[0,1] = \{f : [0,1] \longrightarrow \mathbb{R}(or\ \mathbb{C})|\ f$ is a continuous function$\}$ is a vector space over $\mathbb{R}$ *or* $\mathbb{C}$ under the operations $(f+g)(x) = f(x) + g(x) \quad \forall\, x \in [0,1]$

$$(\alpha\ f)(x) = \alpha\ f(x) \qquad \forall\, x \in [0,1]$$

Example 14. Show that the vector space $C[0,1] = \{f : [0,1] \longrightarrow \mathbb{R}|$ f is a continuous function$\}$ over $\mathbb{R}$ is a normed space under norm $\|f\| = \displaystyle\int_0^1 |f(t)|dt$.

Solution: Using the properties of Riemann integration and absolute value of a real number , we have

(1) Let $f \in C[0,1]$. As $|f(t)| \geqslant 0\ \ \forall\, t \in [0,1]$, we have $\displaystyle\int_0^1 |f(t)|dt \geqslant 0$ and hence $\|f\| \geqslant 0$. So property (N 1) of norm is satisfied.

(2) Let $f \in C[0,1]$.

$$\begin{aligned} \|f\| = 0 &\iff \int_0^1 |f(t)|dt = 0 \\ &\iff |f(t)| = 0 \quad \forall\, t \in [0,1] \\ &\iff f(t) = 0 \quad \forall\, t \in [0,1] \\ &\iff f = 0\ (zero\ function) \end{aligned}$$

and hence property (N 2) of norm is satisfied.

(3) Let $f,\ g \in C[0,1]$.

$$\begin{aligned} \|f+g\| &= \int_0^1 |(f+g)(t)|dt \\ &= \int_0^1 |f(t)+g(t)|dt \\ &\leqslant \int_0^1 |f(t)|dt + \int_0^1 |g(t)|dt \\ &= \|f\| + \|g\| \end{aligned}$$

Therefore, $\|f+g\| \leqslant \|f\| + \|g\|$ and hence property (N 3) of norm is satisfied.

(4) $\forall \alpha \in \mathbb{R}$ and $f \in C[0,1]$,

$$\begin{aligned} \|\alpha f\| &= \int_0^1 |(\alpha f)(t)|dt \\ &= \int_0^1 |\alpha\ f(t)|dt \\ &= \int_0^1 |\alpha|\ |f(t)|dt \\ &= |\alpha| \int_0^1 |f(t)|dt \\ &= |\alpha|\ \|f\| \end{aligned}$$

and hence property (N 4) of norm is satisfied.

Thus, $C[0,1]$ is a normed space under defined norm.

Example 15. Show that the vector space $C[0,1] = \{f : [0,1] \longrightarrow \mathbb{R}(or\ \mathbb{C}) |\ f$ is a continuous function$\}$ over $\mathbb{R}(or\ \mathbb{C})$ is a normed space under norm $\|f\| = \sup\limits_{x\in[0,1]} \{|f(x)|\}$.

(This norm is referred as $\|f\|_\infty$ or sup norm on $C[0,1]$).

Solution: Using the properties of supremum and absolute value (or modulus in $\mathbb{C}$) of a real number , we have

(1) Let $f \in C[0,1]$. As $|f(x)| \geqslant 0\ \forall\, x \in [0,1]$, we have $\sup\limits_{x\in[0,1]} \{|f(x)|\} \geqslant 0$ and hence $\|f\| \geqslant 0$. So property (N 1) of norm is satisfied.

(2) Let $f \in C[0,1]$.

$$\begin{aligned} \|f\| = 0 &\iff \sup_{x\in[0,1]} \{|f(x)|\} = 0 \\ &\iff |f(x)| = 0 \quad \forall\ x \in [0,1] \\ &\iff f(x) = 0 \quad \forall\ x \in [0,1] \\ &\iff f = 0\ (zero\ function) \end{aligned}$$

and hence property (N 2) of norm is satisfied.

(3) Let $f,\ g \in C[0,1]$.

$$\begin{aligned} \|f+g\| &= \sup_{x\in[0,1]} \{|(f+g)(x)|\} \\ &= \sup_{x\in[0,1]} \{|(f(x)+g(x)|\} \\ &\leqslant \sup_{x\in[0,1]} \{|(f(x)| + |g(x)|\} \\ &= \sup_{x\in[0,1]} \{|f(x)| + \sup_{x\in[0,1]} \{|g(x)|\} \\ &= \|f\| + \|g\| \end{aligned}$$

Therefore, $\|f+g\| \leqslant \|f\| + \|g\|$ and hence property (N 3) of norm is satisfied.

(4) $\forall \alpha \in \mathbb{R}(or\ \mathbb{C})$ and $f \in C[0,1]$,

$$\begin{aligned}\|\alpha f\| &= \sup_{x\in[0,1]} \{|(\alpha f)(x)|\} \\ &= \sup_{x\in[0,1]} \{|\alpha\ f(x)|\} \\ &= \sup_{x\in[0,1]} \{|\alpha|\ |f(x)|\} \\ &= |\alpha| \sup_{x\in[0,1]} \{|f(x)|\} \\ &= |\alpha|\ \|f\|\end{aligned}$$

and hence property (N 4) of norm is satisfied.

Thus, $C[0,1]$ is a normed space under defined norm.

Remark 3.3.5. 1. You can mimic Examples 14 and 15 for the vector space $C[a,b]$ over $\mathbb{R}$ or $\mathbb{C}$.

2. In view of norms on $C[a,b]$ defined in examples 14 and 15, we have

$$\|f\|_p = \begin{cases} \left(\displaystyle\int_a^b |f(t)|^p\ dt\right)^{1/p} & \text{if} \quad 1 \leqslant p < \infty \\ \displaystyle\sup_{t\in[a,b]} \{|f(t)|\} & \text{if} \quad p = \infty \end{cases}$$

3. Note that the vector space $C[a,b]$ is a particular case of the vector space $C(X)$ where X is a compact space. So you can have similar versions of Examples 14 and 15 for the vector space $C(X)$ over $\mathbb{R}$ or $\mathbb{C}$.

Definition 3.5. A normed space is called *finite dimensional* if the underlying vector space is finite dimensional, otherwise it is called *infinite dimensional*.

Remark 3.3.6. The normed spaces in examples 6, 7, 8, 9 are *finite dimensional* and the normed spaces in examples 10, 11, 12, 14 are *infinite dimensional*.

In a given normed space $(V, \|\ \|)$, for $x,\ y \in V$, $x - y \in V$. (Why?). This suggests us to give following definition.

Definition 3.6. In a given normed space $(V, \|\ \|)$, a function $d : V \times V \longrightarrow \mathbb{R}^+$ defined as $d(x,y) = \|x - y\|$ is called the *distance* from x to y, where $x,\ y \in V$. Here d is referred as distance function on V.

Clearly, $\|x\|$ is the distance from the zero vector to vector $x \in V$. Recall the definition of a metric space. In the next results, you will see the relation between normed space and metric space.

Theorem 3.3.1. *Every normed space is a metric space with respect to the distance function*

Proof. Let $(V, \| \ \|)$ be a normed space and d be distance function on V. So for $x, \ y \in V$, $d(x, y) = \|x - y\|$. Let $x, \ y, \ z \in V$

(1) By property (N 1) of norm, $\|x - y\| \geqslant 0$ and hence $d(x, y) \geqslant 0$.

(2) By property (N 2) of norm,

$$\begin{aligned} \|x - y\| = 0 &\Longleftrightarrow x - y = 0 \\ &\Longleftrightarrow x = y \\ Thus, \ d(x, y) = 0 \ &\Longleftrightarrow x = y \end{aligned}$$

(3) By property (N 4) of norm, $\|x-y\| = \|-(y-x)\| = |-1| \ \|y-x\| = \|y - x\|$. Thus, $d(x, y) = d(y, x)$.

(4) By property (N 3) of norm,
$\|x - y\| = \|(x - z) + (z - y)\| \leqslant \|x - z\| + \|z - y\|$ and thus
$d(x, y) \leqslant d(x, z) + d(z, y)$

Therefore, all the conditions of the metric are satisfied by d. So d is a metric on V and hence (V, d) is a metric space. ☐

The metric defined in this way is called as the natural metric induced by the norm.

Remark 3.3.7. You will come to know from the following example that the converse of Theorem 3.3.1 need not be true.

Example 16. Let $0 < p < 1$. Consider the sequence space $l^p = \left\{ x = \{x_1, \cdots, x_n, \cdots\} \middle| \ x_i \in \mathbb{R} \text{ and } \sum_{i=1}^{\infty} |x_i|^p < \infty \right\}$ over $\mathbb{R}$. For $x = \{x_1, \cdots, x_n, \cdots\}$ and $y = \{y_1, \cdots, y_n, \cdots\}$ in l^p, define a function $d : l^p \times l^p \longrightarrow \mathbb{R}^+$ as $d(x, y) = \sum_{k=1}^{\infty} |x_k - y_k|^p$. Then (l^p, d) is a metric space (Check!) but not a normed space as property (N 4) of norm is not satisfied as shown below:
for $z = \{0, 1, 0, \cdots\} \in l^p$ and $\alpha = 2 \in \mathbb{R}$, we have,
$\|\alpha z\| = \|\alpha z - 0\| = d(\alpha z, 0) = |0-0|^p + |\alpha - 0|^p + |0-0|^p + \cdots = |\alpha|^p = \alpha^p$
& $|\alpha| \|z\| = \alpha \|z - 0\| = \alpha \, d(z, 0) = \alpha \left(|0-0|^p + |1-0|^p + |0-0|^p + \cdots \right) = \alpha$

3.4 Convergent Sequence and Cauchy Sequence in a Normed Space

You are already familiar with the definition of convergence of sequences in a set of points and its related results. Now, you will learn the concepts of convergent sequences and Cauchy sequences in a normed space to obtain analogous results, which you learnt in your B.Sc.

Definition 3.7. Let $(V, \| \ \|)$ be a normed space. A sequence $\{x_n\}$ of vectors in V is *convergent* to $x \in V$ if $\forall\ \epsilon > 0\ \exists\ n_0 \in \mathbb{N}$ such that $\forall\ n \geqslant n_o$ we have $\|x_n - x\| < \epsilon$. (or equivalently, if $\|x_n - x\|$ converges to 0 as $n \longrightarrow \infty$). Here, we say, x is the *limit* of the sequence $\{x_n\}$ or $\{x_n\}$ is a convergent sequence, converging to x. In this case, we write, $\lim\limits_{n\longrightarrow\infty} x_n = x$ or ($x_n \longrightarrow x$ as $n \longrightarrow \infty$) or $\lim\limits_{n\longrightarrow\infty} \|x_n - x\| = 0$ or ($\|x_n - x\| \longrightarrow 0$ as $n \longrightarrow \infty$)

In the following result, you will notice that, in a normed space, a sequence can have at most one limit.

Theorem 3.4.1. *The limit of a convergent sequence in a normed space is unique.*

Proof. Consider a convergent sequence $\{x_n\}$ in normed space $(V, \| \ \|)$. Assume that $\{x_n\}$ converges to $x \in V$ and $\{x_n\}$ converges to $y \in V$. Then as $n \longrightarrow \infty$, we have, $\|x_n - x\| \longrightarrow 0$ and $\|x_n - y\| \longrightarrow 0$. Clearly, $\|x_n - x\| = \|x - x_n\|$.
Consider

$$\begin{aligned}
\|x - y\| &= \|(x - x_n) + (x_n - y)\| \\
&\leqslant \|x - x_n\| + \|x_n - y\| \quad \textit{by property } (N\ 3) \textit{ of norm} \\
&\leqslant 0 \quad \textit{as } n \longrightarrow \infty
\end{aligned}$$

$$\begin{aligned}
\textit{Thus,} \quad & \|x - y\| = 0 \qquad \textit{by property } (N\ 1) \textit{ of norm} \\
& \therefore x - y = 0 \qquad \textit{by property } (N\ 2) \textit{ of norm}
\end{aligned}$$

$$\therefore x = y \ \text{ and we are done.} \qquad \square$$

Example 17. In a normed space $(V, \| \ \|)$, show that $\Big|\|x\| - \|y\|\Big| \leqslant \|x - y\| \quad \forall\ x,\ y \in V$.

Solution: Using property (N 3) of norm, we have, $\|x\| = \|(x-y)+y\| \leqslant \|x - y\| + \|y\|$. So, $\|x\| - \|y\| \leqslant \|x - y\| \qquad \longrightarrow\ (1)$
Similarly, $\|y\| - \|x\| \leqslant \|x - y\|$ as $\|y - x\| = \|x - y\|$

So, $-(\|x\| - \|y\|) \leqslant \|x - y\|$ $\longrightarrow$ (2)

From (1) and (2), we get, $\Big| \|x\| - \|y\| \Big| \leqslant \|x - y\|$

Now, you will learn the definition of a continuous function on normed spaces and then prove that norm is a continuous function on $\mathbb{R}$.

Definition 3.8. Consider the normed spaces $(V, \| \|_V)$ and $(W, \| \|_W)$. The function $f : V \longrightarrow W$ is continuous at $x_0 \in V$ if $\forall\ \epsilon > 0\ \exists\ \delta > 0$ such that $\|x - x_0\|_V < \delta \implies \|f(x) - f(x_0)\|_W < \epsilon$
Equivalently, we write, as $n \longrightarrow \infty$, $x_n \longrightarrow x_o \implies f(x_n) \longrightarrow f(x_o)$ i.e. for every sequence $\{x_n\}$ in V converging to $x_0 \in V$, the sequence $\{f(x_n)\}$ in W converges to $f(x_0) \in W$.
Here, the notations $\| \|_V$ and $\| \|_W$ mean the norms in normed spaces V and W, respectively.

Theorem 3.4.2. *Let $(V, \| \|)$ be a normed space. Define a function $f : V \longrightarrow \mathbb{R}$ as $f(x) = \|x\|$. Then the norm $\| \|$ on V is a real valued continuous function.*

Proof. Let $x_0 \in V$. Consider a sequence $\{x_n\}$ in V such that as $n \longrightarrow \infty$, $x_n \longrightarrow x_o$. i.e. as $n \longrightarrow \infty$, $\|x_n - x_0\| \longrightarrow 0$. Then using Example 17, we get, $\Big| f(x_n) - f(x_0) \Big| = \Big| \|x_n\| - \|x_0\| \Big| \leqslant \|x_n - x_0\| \longrightarrow 0$ as $n \longrightarrow \infty$ and thus as $n \longrightarrow \infty$, we have $f(x_n) \longrightarrow f(x_0)$. Therefore, for every sequence $\{x_n\}$ in V converging to $x_0 \in V$, the sequence $\{f(x_n)\}$ in $\mathbb{R}$ converges to $f(x_0) \in \mathbb{R}$. Hence, the norm $\| \|$ on V is a real valued continuous function. $\square$

Definition 3.9. A *subspace* M of a normed space V is a subspace of V considered as a vector space, with the norm obtained by restricting the norm on V to the subset M. This norm on M is said to be induced by the norm on V.

Note that, if M is closed in a normed space V, then M is called a closed subspace of V.

Recall the quotient space (or factor space). Let M be a subspace of a vector space V. The coset of an element $x \in V$ with respect to M is $x + M = \{x + m \mid m \in M\}$. Under the following algebraic operations $(x + M) + (x' + M) = (x + x') + M$ and $\alpha\ (x + M) = \alpha x + M$, the quotient space of V by M, denoted by V/M is a vector space. Note that, $x + M = M$ if and only if $x \in M$.

Given a normed space, you will learn how to form a new normed space.

Theorem 3.4.3. *Let M be a closed subspace of a normed space $(V, \| \|)$. For each coset $x+M$ in quotient space V/M, define $\|x+M\| = inf\{\|x+m\| \mid m \in M\}$. Then V/M is a normed space under the norm $\|x+M\|$.*

Proof. Using the properties of infimum and properties in normed space $(V, \| \|)$, we have

(1) As $\|x+m\|$ is a non-negative real number and every set of non-negative real numbers is bounded below, it follows that $inf\{\|x+m\| \mid m \in M\}$ exists and is non-negative. i.e.$\|x+M\| \geqslant 0$ $\forall\ x+M \in V/M$. So property (N 1) of norm is satisfied.

(2) Let $x+M = M$(zero element of V/M). Then $x \in M$. So

$$\begin{aligned}\|x+M\| =& inf\{\|x+m\| \mid m \in M\} \\ =& inf\{\|x+m\| \mid m \in M, x \in M\} \\ =& inf\{\|y\| \mid y \in M\} \quad \textit{where } y = x+m \\ =& 0\end{aligned}$$

Thus, $x+M = M$(zero element of V/M) $\implies$ $\|x+M\| = 0$.

Conversely, let $\|x+M\| = 0$
$\implies inf\{\|x+m\| \mid m \in M\} = 0$
$\implies \exists$ a sequence $\{m_k\}$ in M such that $\|x+m_k\| \longrightarrow 0$ as $k \longrightarrow \infty$
$\implies \lim\limits_{k \longrightarrow \infty} m_k = -x$
$\implies -x \in M$ as M is closed
$\implies x \in M$ as M is subspace
$\implies x+M = M$
Thus, $\|x+M\| = 0 \implies x+M = M$(zero element of V/M)
$\therefore \|x+M\| = 0 \iff x+M = M$(zero element of V/M) and hence property (N 2) of norm is satisfied.

(3) Let $x+M,\ y+M \in V/M$.

$$\begin{aligned}\|(x+M)+(y+M)\| &= \|(x+y)+M\| \\ &= inf\{\|x+y+m\| \mid m \in M\} \\ &= inf\{\|x+y+m_1+m_2\| \mid m_1, m_2 \in M\} \\ &= inf\{\|(x+m_1)+(y+m_2)\| | m_1, m_2 \in M\} \\ &\leqslant inf\{\|x+m_1\| + \|y+m_2\| \mid m_1, m_2 \in M\} \\ &= inf\{\|x+m_1\| \mid m_1 \in M\} \\ &\qquad\qquad + inf\{\|x+m_2\| \mid m_2 \in M\} \\ &= \|x+M\| + \|y+M\|\end{aligned}$$

Therefore, $\|(x+M)+(y+M)\| \leqslant \|x+M\| + \|y+M\|$ and hence property (N 3) of norm is satisfied.

(4) If $\alpha = 0$ then obviously $\|\alpha(x+M)\| = |\alpha|\ \|x+M\|$.
So, let $\alpha \neq 0$. Then

$$\begin{aligned}
\|\alpha(x+M)\| &= \|\alpha x + M\| \\
&= inf\{\|\alpha x + m\| \mid m \in M\} \\
&= inf\{\|\alpha(x+m')\| \mid m/\alpha = m' \in M\} \\
&= inf\{|\alpha|\ \|x+m'\| \mid m' \in M\} \\
&= |\alpha|\ inf\{\|x+m'\| \mid m' \in M\} \\
&= |\alpha|\ \|x+M\|
\end{aligned}$$

and hence property (N 4) of norm is satisfied.

Thus, V/M is a normed space under defined norm. □

As with metric spaces, you can understand the concept of norms from a geometrical point of view. In a normed space $(V, \|\ \|)$, the open ball, centered at a with radius ϵ is defined by the set $B(a;\epsilon) = \{x \in V \mid \|x-a\| < \epsilon\}$ and the open unit ball is given by $B(0;1) = \{x \in V \mid \|x\| < 1\}$. Also, the closed ball, centered at a with radius ϵ is defined by the set $B[a;\epsilon] = \{x \in V \mid \|x-a\| \leqslant \epsilon\}$ and the closed unit ball is given by $B[0;1] = \{x \in V \mid \|x\| \leqslant 1\}$.

Recall that $\overline{M}$=set of all limit points of M. For each $x \in M$ and for each $\epsilon > 0$, the open ball $\{y \mid \|y-x\| < \epsilon\}$ must contain a point of M. Hence, for each element to be in $\overline{M}$, it suffices to show that for any $\epsilon > 0$, $\exists$ some element of M which is within ϵ distance from it.

Theorem 3.4.4. *If M is a subspace of a normed space $(V, \|\ \|)$ then $\overline{M}$ is a closed subspace of V.*

Proof. Initially, we show that, $\overline{M}$ is a subspace of V. Let $\epsilon > 0$. Consider $x,\ y \in \overline{M}$ and non-zero scalars $\alpha,\ \beta$. Then $\exists\ x_1,\ y_1 \in M$ such that $\|x - x_1\| < \dfrac{\epsilon}{2\ |\alpha|}$ and $\|y - y_1\| < \dfrac{\epsilon}{2\ |\beta|}$.
Using the properties (N 3) and (N 4) of norm, we get,

$$\begin{aligned}
\|(\alpha x + \beta y) - (\alpha x_1 + \beta y_1)\| &= \|\alpha(x-x_1) + \beta(y-y_1)\| \\
&\leqslant \|\alpha(x-x_1)\| + \|\beta(y-y_1)\| \\
&= |\alpha|\ \|x-x_1\| + |\beta|\ \|y-y_1\| \\
&< |\alpha|\ \frac{\epsilon}{2\ |\alpha|} + |\beta|\ \frac{\epsilon}{2\ |\beta|} \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

So $\exists\ (\alpha\ x_1 + \beta\ y_1) \in M$ (being subspace) such that $\|(\alpha x + \beta y) - (\alpha x_1 + \beta y_1)\| < \epsilon$. Thus $(\alpha x + \beta y) \in \overline{M}$ if $x,\ y \in \overline{M}$ and $\alpha,\ \beta$ are non-zero scalars.(if $\alpha = 0 = \beta$ then $\alpha x + \beta y = 0 \in \overline{M}$). Hence, $\overline{M}$ is a subspace of V. Further, as $\overline{M}$ is closed, $\overline{M}$ is a closed subspace of V. $\square$

You have learnt the definition of a Cauchy sequence in $\mathbb{R}^n$ at your B.Sc. So you can guess the definition of a Cauchy sequence in a normed space and get convinced with the following definition.

Definition 3.10. Let $(V, \| \ \|)$ be a normed space. A sequence $\{x_n\}$ of vectors in V is said to be a *Cauchy sequence* if $\forall\ \epsilon > 0\ \exists\ n_0 \in \mathbb{N}$ such that $\forall\ m,\ n \geqslant n_o$ we have $\|x_n - x_m\| < \epsilon$. (or equivalently, if $\|x_n - x_m\| \longrightarrow\ 0$ as $m,\ n \longrightarrow \infty$).

You will come to know that the relation between convergent sequences and Cauchy sequences in normed spaces.

Theorem 3.4.5. *Every convergent sequence in a normed space is a Cauchy sequence.*

Proof. In normed space $(V, \| \ \|)$, consider a convergent sequence $\{x_n\}$, converging to $x \in V$. Then as $n \longrightarrow \infty$, we have, $\|x_n - x\| \longrightarrow 0$. Clearly, $\|x_m - x\| = \|x - x_m\|$.
Consider

$$\begin{aligned}
\|x_n - x_m\| &= \|(x_n - x) + (x - x_m)\| \\
&\leqslant \|x_n - x\| + \|x_m - x\| \qquad by\ property\ (N\ 3)\ of\ norm \\
&\longrightarrow 0 \quad as\ \ m, n \longrightarrow \infty
\end{aligned}$$

Hence, $\{x_n\}$ is a Cauchy sequence. $\square$

Remark 3.4.1. The converse of Theorem 3.4.5 need not be true. i.e. A Cauchy sequence in a normed space need not be a convergent sequence. You will see its counter example in next chapter. In the next chapter, you will come to know that Cauchy sequences play a vital role in the theory of normed spaces.

Recall the concept of subsequences in $\mathbb{R}$ that you learnt at your B.Sc. and think of it in a normed space $(V, \| \ \|)$. Can you guess, what happens to a Cauchy sequence, having a convergent subsequence?. The next result is related to it.

Theorem 3.4.6. *In a normed space, every Cauchy sequence having a convergent subsequence is convergent.*

Proof. In normed space $(V, \| \ \|)$, consider a Cauchy sequence $\{x_n\}$, having a convergent subsequence $\{x_{n_k}\}$. Let $\{x_{n_k}\}$ converge to $x \in V$. Then as $n_k \longrightarrow \infty$, we have, $\|x_{n_k} - x\| \longrightarrow 0$. Also, since $\{x_n\}$ is a Cauchy sequence, as $n,\ n_k\ \longrightarrow \infty$, we have, $\|x_{n_k} - x_n\| \longrightarrow 0$.
As $n \longrightarrow \infty$, it follows that,

$$\|x_n - x\| = \|x_n - x_{n_k} + x_{n_k} - x\| \leqslant \|x_n - x_{n_k}\| + \|x_{n_k} - x\| \longrightarrow 0.$$

Hence, the Cauchy sequence $\{x_n\}$ is convergent. $\square$

3.5 LET US SUM UP

1. Let V be a vector space over the field $\mathbb{F}(=\mathbb{R}$ or $\mathbb{C})$. A norm $\| \; \|$ on V is a real valued function (i.e. $\| \; \| : V \longrightarrow \mathbb{R}$), satisfying the following 4 properties/axioms:

   (N 1) $\|x\| \geqslant 0 \qquad \forall x \in V$

   (N 2) $\|x\| = 0$ if and only if $x = 0_v \qquad \forall x \in V$

   (N 3) $\|x+y\| \leqslant \|x\| + \|y\| \qquad \forall x, \; y \in V$

   (N 4) $\|\alpha x\| = |\alpha| \; \|x\| \qquad \forall x \in V$ and $\forall \alpha \in \mathbb{F}$

2. A normed space V is a vector space over the field $\mathbb{F}(=\mathbb{R}$ or $\mathbb{C})$ with a norm $\| \; \|$ defined on it. In such a case, we say, $(V, \; \| \; \|)$ is a normed space over $\mathbb{F}$. Here, if $\mathbb{F} = \mathbb{R}$ then V is called a *real normed space* and if $\mathbb{F} = \mathbb{C}$ then V is called a *complex normed space.*

3. A normed space is called finite dimensional if the underlying vector space is finite dimensional, otherwise a normed space is called infinite dimensional.

4. The vector space $\mathbb{C}^n = \{(x_1, \cdots, x_n) \mid x_i \in \mathbb{C}\}$ over $\mathbb{C}$ is a normed space under the Euclidean norm $\|x\| = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2}$.

5. Let p and q be non-negative extended real numbers. For $p \geqslant 1$, q is said to be conjugate of p if

$$\frac{1}{p} + \frac{1}{q} = 1, \quad for \;\; 1 < p < \infty$$
$$q = \infty, \quad for \;\; p = 1$$
$$q = 1, \quad for \;\; p = \infty$$

6. Hölder's inequality for finite sums:
   Let $1 < p, \; q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. For any complex (or real) numbers $x_1, \cdots, x_n; \;\; y_1, \cdots, y_n$

$$\sum_{i=1}^{n} | \; x_i \; y_i \; | \quad \leqslant \quad \left(\sum_{i=1}^{n} | \; x_i \; |^p \right)^{1/p} \quad \left(\sum_{i=1}^{n} | \; y_i \; |^q \right)^{1/q}$$

   (Cauchy-Schwarz inequality is a special case of Hölder's inequality for $p = 2 = q$)

7. Minkowski's inequality for finite sums: Let $1 \leqslant p < \infty$. For any complex (or real) numbers $x_1, \cdots, x_n;\quad y_1, \cdots, y_n$

$$\left(\sum_{i=1}^{n} \mid x_i + y_i \mid^p \right)^{1/p} \leqslant \left(\sum_{i=1}^{n} \mid x_i \mid^p \right)^{1/p} + \left(\sum_{i=1}^{n} \mid y_i \mid^p \right)^{1/p}$$

8. The vector space $\mathbb{C}^n = \{(x_1, \cdots, x_n) \mid x_i \in \mathbb{C}\}$ over $\mathbb{C}$ is a normed space under the norms

$$\|x\|_p = \begin{cases} \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} & \text{if} \quad 1 \leqslant p < \infty \\ max\{\mid x_1 \mid, \cdots, \mid x_n \mid\} & \text{if} \quad p = \infty \end{cases}$$

These norms are referred as p-norms on $\mathbb{C}^n$.

9. On a vector space V, one can define more than one norm and accordingly different normed spaces are obtained from same vector space V.

10. The sequence space $l^p = \left\{ \{x_1, \cdots, x_n, \cdots\} \,\middle|\, \sum_{i=1}^{\infty} \mid x_i \mid^p < \infty \text{ and } x_i \in \mathbb{C} \right\}$ over $\mathbb{C}$ is a normed space under the norms

$$\|x\|_p = \begin{cases} \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p} & \text{if} \quad 1 \leqslant p < \infty \\ sup\{\mid x_1 \mid, \cdots, \mid x_n \mid, \cdots\} & \text{if} \quad p = \infty \end{cases}$$

These norms are referred as p-norms on l^p.

11. Hölder's inequality for integrals:
Let $1 < p,\ q < \infty$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$. For $f \in L^p(E)$ and $g \in L^q(E)$ (where E is bounded closed interval in $\mathbb{R}$),

$$\int_E \mid f\ g \mid \quad \leqslant \quad \left(\int_E \mid f \mid^p \right)^{1/p} \quad \left(\int_E \mid g \mid^q \right)^{1/q}$$

12. Minkowski's inequality for integrals:
Let $1 \leqslant p < \infty$. If $f \in L^p(E)$ and $g \in L^p(E)$ (where E is bounded closed interval in $\mathbb{R}$), then

$$\left(\int_E \mid f + g \mid^p \right)^{1/p} \leqslant \left(\int_E \mid f \mid^p \right)^{1/p} + \left(\int_E \mid g \mid^p \right)^{1/p}$$

13. The vector space $L^p(E) = \{f : E \longrightarrow \mathbb{R} | f$ is Lebesgue measurable function on E and $| f |^p$ is Lebesgue integrable over $E\}$ over $\mathbb{R}$ is a normed space under norms

$$\|f\|_p = \begin{cases} \left(\int_E |f|^p \right)^{1/p} & \text{if} \quad 1 \leqslant p < \infty \\ ess\ sup|f| & \text{if} \quad p = \infty \end{cases}$$

where $ess\ \sup\limits_E |f(x)| = inf\{m > 0 \mid |f(x)| \leqslant m$ a.e. on $E\}$ and $L^\infty(E) =$ the class of all those measurable functions f defined on E which are essentially bounded on E with $ess\ sup|f| < \infty$.

14. The vector space $C(X) = \{f : X \longrightarrow \mathbb{R}(or\ \mathbb{C})|\ f$ is bounded continuous function on $X\}$ over $\mathbb{R}(or\ \mathbb{C})$ is a normed space under norm $\|f\| = \sup\limits_{x \in X}\{|f(x)|\}$.

15. The vector space $C[0,1] = \{f : [0,1] \longrightarrow \mathbb{R}|\ f$ is a continuous function$\}$ over $\mathbb{R}$ is a normed space under norm $\|f\| = \displaystyle\int_0^1 |f(t)|dt$. Note that C[0,1] is a particular case of C(X).

16. In a given normed space $(V, \| \ \|)$, a function $d : V \times V \longrightarrow \mathbb{R}^+$ defined as $d(x,y) = \|x - y\|$ is called the <u>distance</u> from x to y, where $x,\ y \in V$. Here d is referred as distance function on V.

17. Every normed space is a metric space (with respect to the distance function).

18. Every metric space need not be a normed space as shown in following example:-
Let $0 < p < 1$. Consider the sequence space $l^p = \left\{ x = \{x_1, \cdots, x_n, \cdots\} \middle| \ x_i \in \mathbb{R} \text{ and } \sum\limits_{k=1}^{\infty} |x_i|^p < \infty \right\}$ over $\mathbb{R}$. For $x = \{x_1, \cdots, x_n, \cdots\}$ and $y = \{y_1, \cdots, y_n, \cdots\}$ in l^p, define a function $d : l^p \times l^p \longrightarrow \mathbb{R}^+$ as $d(x,y) = \sum\limits_{k=1}^{\infty} |x_k - y_k|^p$. Then (l^p, d) is a metric space but not a normed space.

19. Let $(V, \| \ \|)$ be a normed space. A sequence $\{x_n\}$ of vectors in V is <u>convergent</u> to $x \in V$ if $\forall\ \epsilon > 0\ \exists\ n_0 \in \mathbb{N}$ such that $\forall\ n \geqslant n_o$ we have $\|x_n - x\| < \epsilon$. (or equivalently, if $\|x_n - x\|$ converges to 0 as $n \longrightarrow \infty$). Here, we say, x is the *limit* of the sequence $\{x_n\}$ or $\{x_n\}$ is a convergent sequence, converging to x. In this case, we write, $\lim\limits_{n \longrightarrow \infty} x_n = x$ or ($x_n \longrightarrow x$ as $n \longrightarrow \infty$) or $\lim\limits_{n \longrightarrow \infty} \|x_n - x\| = 0$ or ($\|x_n - x\| \longrightarrow 0$ as $n \longrightarrow \infty$).

20. The limit of a convergent sequence in a normed space is unique.

21. In a normed space $(V, \| \ \|)$, $\Big| \|x\| - \|y\| \Big| \leqslant \|x - y\| \quad \forall\ x,\ y \in V$.

22. Consider the normed spaces $(V, \| \ \|_V)$ and $(W, \| \ \|_W)$. The function $f : V \longrightarrow W$ is continuous at $x_0 \in V$ if $\forall\ \epsilon > 0\ \exists\ \delta > 0$ such that $\|x - x_0\|_V < \delta \implies \|f(x) - f(x_0)\|_W < \epsilon$
Equivalently, we write, as $n \longrightarrow \infty$, $x_n \longrightarrow x_o \implies f(x_n) \longrightarrow f(x_o)$. i.e. for every sequence $\{x_n\}$ in V converging to $x_0 \in V$, the sequence $\{f(x_n)\}$ in W converges to $f(x_0) \in W$.

23. Let $(V, \| \ \|)$ be a normed space. Define a function $f : V \longrightarrow \mathbb{R}$ as $f(x) = \|x\|$. Then the norm $\| \ \|$ on V is a real valued continuous function.

24. Let M be a closed subspace of a normed space $(N, \| \ \|)$. For each coset $x + M$ in quotient space N/M, define $\|x + M\| = inf\{\|x + m\| \mid m \in M\}$. Then N/M is a normed space under the norm $\|x + M\|$.

25. If M is a subspace of a normed space $(V, \| \ \|)$ then $\overline{M}$ is a closed subspace of V.

26. Let $(V, \| \ \|)$ be a normed space. A sequence $\{x_n\}$ of vectors in V is said to be a Cauchy sequence if $\forall\ \epsilon > 0\ \exists\ n_0 \in \mathbb{N}$ such that $\forall\ m,\ n \geqslant n_o$ we have $\|x_n - x_m\| < \epsilon$. (or equivalently, if $\|x_n - x_m\| \longrightarrow 0$ as $m,\ n \longrightarrow \infty$).

27. Every convergent sequence in a normed space is a Cauchy sequence.

3.6 Chapter End Exercise

1. Prove that every inner product space V is a normed space with respect to the norm $\|x\| = \sqrt{< x, x >}\ \ \forall\ x \in V$ where $< x, x >$ denotes the inner product of vector x with itself.

2. Show that the vector space $\mathbb{C} = \{z | z \in \mathbb{C}\}$ over $\mathbb{C}$ is a normed space under the norm $\|z\| = |z|$=absolute value of $z \in \mathbb{C}$.

3. Show that the vector space $\mathbb{R}^n = \{(x_1, \cdots, x_n) \mid x_i \in \mathbb{R}\}$ over $\mathbb{R}$ is a normed space under the (Euclidean) norm $\|x\| = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2}$.

4. Show that the vector space $\mathbb{R}^n = \{(x_1, \cdots, x_n) \mid x_i \in \mathbb{R}\}$ over $\mathbb{R}$ is a normed space under the norm $\|x\| = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$ where $1 \leqslant p < \infty$.

5. Show that the vector space $\mathbb{R}^n = \{(x_1, \cdots, x_n) \mid x_i \in \mathbb{R}\}$ over $\mathbb{R}$ is a normed space under the norm $\|x\| = max\{| x_1 |, \cdots, | x_n |\}$.

6. Show that the vector space $L^p([a, b]) = \{f : [a, b] \longrightarrow \mathbb{R} | f$ is Lebesgue measurable function on $[a, b]$ and $| f |^p$ is Lebesgue integrable over $[a, b]\}$ over $\mathbb{R}$ is a normed space under the norm $\|f\| = \left(\int_a^b |f|^2\right)^{1/2}$

7. Show that the vector space $C([a, b]) = \{f : [a, b] \longrightarrow \mathbb{R} | f$ is continuous function$\}$ over $\mathbb{R}$ is a normed space under the norm $\|f\| = \left(\int_a^b |f|^2\right)^{1/2}$

8. Show that the vector space $\mathbb{R}^2 = \{(x_1, x_2) \mid x_i \in \mathbb{R}\}$ over $\mathbb{R}$ is a normed space under the norm $\|x\| = |x_1| + |x_2|$.

9. If $\| \|_1$ and $\| \|_2$ are 2 norms on a vector space V then check whether the function $\| \| : V \longrightarrow \mathbb{R}$ defined as $\|x\| = \|x\|_1 + \|x\|_2$ is a norm on V.

10. Let $(X, \| \|_X)$ and $(Y, \| \|_Y)$ be normed spaces. Then prove that $X \times Y$ is a normed space under the norm $\|(x, y)\| = \|x\|_X + \|y\|_Y$.

11. Show that a metric d induced by a norm on a normed space $(V, \| \|)$ is translation invariant. (Hint: to show (a) $d(x + u, y + u) = d(x, y)$ and (b) $d(ax, ay) = |a|\ d(x, y)\ \forall x, y \in V$, for every scalar a where u is a fixed vector in V).

12. Let $\{x_n\}$ and $\{y_n\}$ be sequences in a normed space $(V, \| \|)$. If $\lim\limits_{n \longrightarrow \infty} x_n = x \in V$ and $\lim\limits_{n \longrightarrow \infty} y_n = y \in V$ then prove that $\lim\limits_{n \longrightarrow \infty} x_n + y_n = x + y$

13. Let $\{x_n\}$ be a sequence in a normed space $(V, \| \|)$ and $\{\lambda_n\}$ be a sequence of real numbers. If $\lim\limits_{n \longrightarrow \infty} x_n = x \in V$ and $\lim\limits_{n \longrightarrow \infty} \lambda_n = \lambda \in \mathbb{R}$ then prove that $\lim\limits_{n \longrightarrow \infty} \lambda_n\ x_n = \lambda\ x$

14. Prove that in a normed space, if $\{x_n\}$ is a Cauchy sequence then $\{\|x_n\|\}$ is a Cauchy sequence of real numbers.

15. Prove that in a normed space, every Cauchy sequence is bounded.

16. Prove that in a normed space, a Cauchy sequence is convergent if and only if it has a convergent subsequence.

Chapter 4

Banach Space

Unit Structure :

4.1 Introduction

In this chapter, you will be introduced to the notion of a Banach Space. The concept of Banach space was introduced by the Polish mathematician Stefan Banach in 1922. Banach spaces are fundamental parts of functional analysis. Banach thought of, when a norm is defined on a vector space, how to deal with Cauchy sequences and hence about completeness. This chapter has 3 sections, of which in the first section, you will find several examples on Banach spaces, along with a characterization of Banach Spaces. In an attempt to obtain a criterion for determining when a Cauchy sequence with respect to one norm will also be a Cauchy sequence with respect to other norm, you will be introduced to the notion of *equivalent norms* in the second section and interesting results on finite dimensional normed spaces are obtained. Further, through *Riesz lemma*, the concept of compactness is linked to subspaces of finite dimensional normed spaces. In the last section of this chapter, you will be introduced to *Ascoli-Arzela theorem* and the purpose of this theorem is to show a sequence of continuous functions on campact space has a uniformly convergent subsequence, under certain conditions.

4.2 Objectives

The main objective of this chapter is to learn the Banach spaces and interesting results of finite dimensional normed spaces. After going through this chapter you will be able to:

- Define a Banach space.
- Learn how to check a normed space is a Banach space under the given norm.
- Prove that L^p spaces are Banach spaces (Riesz-Fisher theorem).
- Define equivalent norms on a normed space.
- Prove that on a finite dimensional normaed space, any two norms are equivalent.
- Learn Riesz Lemma and results related to it.
- Prove Ascoli-Arzela theorem.

4.3 Few definitions and examples

In previous chapter, you have learnt the concept of a normed space and at the end, it was mentioned that "A Cauchy sequence in a normed space need not be a convergent sequence". Here is a counter example for it.

Example 18. In the normed space $C[0,1]$ under the norm $\|f\| = \int_0^1 |f(t)|dt$, consider the sequence $\{f_n(t)\}$ where each function $f_n : [0,1] \longrightarrow \mathbb{R}$ is defined as follows:

$$f_n(t) = \begin{cases} 0 & \text{if} \quad 0 \leqslant t \leqslant \dfrac{1}{2} \\ 2nt - n & \text{if} \quad \dfrac{1}{2} \leqslant t \leqslant \left(\dfrac{1}{2} + \dfrac{1}{2n}\right) \\ 1 & \text{if} \quad \left(\dfrac{1}{2} + \dfrac{1}{2n}\right) \leqslant t \leqslant 1 \end{cases}$$

Show that the sequence $\{f_n(t)\}$ is a Cauchy sequence in $C[0,1]$. Is this sequence convergent? Justify.

Solution: Clearly, $f_n(t) \in C[0,1] \quad \forall n \in \mathbb{N}$.
Also, $\forall t \in \left[\dfrac{1}{2}, \left(\dfrac{1}{2} + \dfrac{1}{2n}\right)\right]$, it is easy to see that $|f_n(t)| \leqslant 1$. So, $|f_m(t) - f_n(t)| \leqslant |f_m(t)| + |f_n(t)| \leqslant 1 + 1 = 2$. $\longrightarrow (*)$
Now, with $m > n$, we have,

$$\|f_m - f_n\| = \int_0^1 |f_m(t) - f_n(t)|dt$$
$$= \int_0^{\frac{1}{2}} |f_m(t)-f_n(t)|dt + \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{2n}} |f_m(t)-f_n(t)|dt + \int_{\frac{1}{2}+\frac{1}{2n}}^{1} |f_m(t)-f_n(t)|dt$$
$$= 0 + \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{2n}} |f_m(t) - f_n(t)|dt + 0 \qquad \text{by definition of } f_n(t)$$
$$\leqslant 2\int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{2n}} dt = 2\left(\frac{1}{2n}\right) \qquad \text{using inequality } (*)$$

Thus, $\|f_m - f_n\| \leqslant \dfrac{1}{n} \longrightarrow 0$ as $m, n \longrightarrow \infty$

Hence, the sequence $\{f_n(t)\}$ is a Cauchy sequence in $C[0,1]$.

Assume that this Cauchy sequence $\{f_n\}$ is convergent in $C[0,1]$. Then $\exists$ a function $g \in C[0,1]$ such that $\lim\limits_{n\longrightarrow\infty} f_n = g$. It is easy to see that $\lim\limits_{n\longrightarrow\infty} \int_0^{\frac{1}{2}+\frac{1}{2n}} |f_n(t) - g(t)|dt = 0$ and $\lim\limits_{n\longrightarrow\infty} \int_{\frac{1}{2}+\frac{1}{2n}}^{1} |1 - g(t)|dt = 0$. Thus,

$$g(t) = \begin{cases} 0 & \text{if } \quad 0 \leqslant t < \dfrac{1}{2} \\ 1 & \text{if } \quad \dfrac{1}{2} < t \leqslant 1 \end{cases}$$

It is clear that the function g is discontinuous at $t = \dfrac{1}{2}$ and hence $g \notin C[0,1]$, which contadicts $g \in C[0,1]$. So, our assumption that the Cauchy sequence $\{f_n\}$ is convergent in $C[0,1]$ must be wrong. Hence, the Cauchy sequence $\{f_n\}$ is not convergent in normed space $C[0,1]$ under norm $\|f\| = \int_0^1 |f(t)|dt$.

Now, you will come to know when a normed space is called, a Banach space.

Definition 4.1. A *Banach space* is a normed space in which every Cauchy sequence is convergent.

You know that, a metric space is called a *complete space* if every Cauchy sequence of points in it converges to a point in the space. In view of this definition, the normed space $(V, \|.\|)$ is said to be *complete* if V is complete as a metric space with the metric $d(u,v) = \|u - v\| \; \forall \; u, v \in V$. Hence, you can redefine the definition 4.1 as a *Banach space* is a complete normed space or a *Banach space* is a normed space which is a complete metric space.

Now, you will notice that examples of Banach spaces are in abundance. Initially, you will find an example of a normed space which is not a Banach space.

Example 19. Show that the vector space $C[0,1] = \{f : [0,1] \longrightarrow \mathbb{R} |$ f is a continuous function$\}$ over $\mathbb{R}$ is not a Banach space under norm

$\|f\| = \int_0^1 |f(t)|dt.$

Solution: Refer Example 14 to show that $C[0,1]$ is a normed space under defined norm. Refer Example 18 to show that a Cauchy sequence in $C[0,1]$ is not convergent on it.

Thus, $C[0,1]$ is not a Banach space under defined norm.

You can generalize above Example 19 to get the following result.

Theorem 4.3.1. *Prove that the vector space* $C[a,b] = \{f : [a,b] \longrightarrow \mathbb{R}(or\ \mathbb{C}) |\ f$ *is a continuous function*$\}$ *over* $\mathbb{R}(or\ \mathbb{C})$ *is not a Banach space under norm* $\|f\| = \left(\int_a^b |f(t)|^p\ dt\right)^{1/p}$ *where* $1 \leqslant p < \infty$

Proof. Left to the reader. □

But you will see that $C[0,1]$ is a Banach space with respect to sup norm.

Example 20. Show that the vector space $C[0,1] = \{f : [0,1] \longrightarrow \mathbb{R}(or\ \mathbb{C}) |\ f$ is a continuous function$\}$ over $\mathbb{R}(or\ \mathbb{C})$ is a Banach space under norm $\|f\| = \sup_{x\in[0,1]} \{|f(x)|\}$.

Solution: Refer Example 15 to show that $C[0,1]$ is a normed space under defined norm. Now, consider a Cauchy sequence $\{f_n\}$ in $C[0,1]$. Let $\epsilon > 0$. Then $\{f_n\}$ being a Cauchy sequence, $\exists\ m_0 \in \mathbb{N}$ such that $\forall\ l, n \geq m_0$ and $\forall\ x \in [0,1]$, we have, $\|f_l - f_n\| < \frac{\epsilon}{3}$ i.e. $\sup_{x\in[0,1]} \{|f_l(x) - f_n(x)|\} < \frac{\epsilon}{3}$ and hence $\forall\ l, n \geq m_0$ and $\forall\ x \in [0,1]$; we have, $|f_l(x) - f_n(x)| < \frac{\epsilon}{3}$ $\longrightarrow (*)$

This shows that for fixed but arbitrary $x \in [0,1]$, $\{f_n(x)\}$ is a Cauchy sequence in $\mathbb{R}(or\ \mathbb{C})$. As every Cauchy sequence in $\mathbb{R}(or\ \mathbb{C})$ is convergent, $\exists$ a function $f : [0,1] \longrightarrow \mathbb{R}(or\ \mathbb{C})$ such that $\lim_{n\longrightarrow\infty} f_n(x) = f(x)$, for each fixed $x \in [0,1]$. So, making $l \longrightarrow \infty$ in inequality $(*)$, we get, $\forall\ n \geq m_0$ and for each fixed $x \in [0,1]$; $|f(x) - f_n(x)| < \frac{\epsilon}{3}$. Taking supremum over $x \in [0,1]$, (as m_0 is independent of x and x is arbitrary), we get, $\|f - f_n\| < \frac{\epsilon}{3}$ $\quad \forall\ n \geq m_0$ $\longrightarrow (**)$

Thus, $f_n \longrightarrow f$ uniformly, i.e. $\|f_n - f\| \longrightarrow 0$ as $n \longrightarrow \infty$.

Claim: $f \in C[0,1]$.

Consider a sequence $\{x_n\}$ in $[0,1]$ such that $x_n \longrightarrow x$ where $x \in [0,1]$. From inequality $(**)$, in particular $n = m_0$ gives, $\|f - f_{m_0}\| < \frac{\epsilon}{3}$. And therefore, for n sufficiently large, we have,

$|f(x_n) - f(x)| \leq |f(x_n) - f_{m_0}(x_n)| + |f_{m_0}(x_n) - f_{m_0}(x)| + |f_{m_0}(x) - f(x)|$

which implies $|f(x_n) - f(x)| < \dfrac{\epsilon}{3} + |f_{m_0}(x_n) - f_{m_0}(x)| + \dfrac{\epsilon}{3}$
and hence $|f(x_n) - f(x)| < \dfrac{\epsilon}{3} + \dfrac{\epsilon}{3} + \dfrac{\epsilon}{3} = \epsilon$ as $f_{m_0} \in C[0,1]$ and $x_n \longrightarrow x$

So, $f(x_n) \longrightarrow f(x)$ if $x_n \longrightarrow x$ which implies f is continuous and we are done. Thus, the Cauchy sequence $\{f_n\}$ in $C[0,1]$ is convergent to $f \in C[0,1]$. Hence, $C[0,1]$ is a Banach space under defined norm.

Remark 4.3.1. From above Example 20, it is clear that in the space $C[0,1]$ under norm $\|f\| = \sup\limits_{x\in[0,1]} \{|f(x)|\}$,
$f_n \longrightarrow f$
$\iff f_n(x) \longrightarrow f(x)$ uniformly on [0,1]
$\iff \forall \epsilon > 0,\ \exists\ m_0 \in \mathbb{N}$ (independent of x) such that $\forall x \in [0,1]$ and $\forall n \geqslant m_0$, we have, $|f_n(x) - f(x)| < \epsilon$

You can generalize above Example 20 to get the following results.

Theorem 4.3.2. *Prove that the vector space $C[a,b] = \{f : [a,b] \longrightarrow \mathbb{R}(or\ \mathbb{C})|\ f$ is a continuous function$\}$ over $\mathbb{R}(or\ \mathbb{C})$ is a Banach space under norm $\|f\| = \sup\limits_{x\in[a,b]} \{|f(x)|\}$.*

Proof. Left to the reader. □

Theorem 4.3.3. *Prove that the vector space $C(X)$ over $\mathbb{R}(or\ \mathbb{C})$ is a Banach space under norm $\|f\| = \sup\limits_{x\in X}\{|f(x)|\}$ where X is a compact space.*

Proof. Left to the reader. □

Example 21. Show that the vector space $\mathbb{R} = \{x | x \in \mathbb{R}\}$ over $\mathbb{R}$ is a Banach space under the norm $\|x\| = |x|$=absolute value of $x \in \mathbb{R}$.

Solution: Refer Example 6 of previous chapter to show that $\mathbb{R}$ is a normed space under defined norm.
Claim: Every Cauchy sequence in $\mathbb{R}$ is convergent in $\mathbb{R}$.
Let $\{x_n\}$ be a Cauchy sequence. Then for any $\epsilon > 0\ \exists\ n_0 \in \mathbb{N}$ such that $\forall n, m \geq n_0$ we have $|x_n - x_m| < \dfrac{\epsilon}{2}$ ⟵(1)
$\{x_n\}$ being Cauchy, is bounded and hence $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ ⟵ (using Bolzano Weirstrass theorem)
So suppose $\{x_{n_k}\}$ converges to $l \in \mathbb{R}$. Then $\exists\ k_0 \in \mathbb{N}$ such that $\forall k \geq k_0$, we have, $|x_{n_k} - l| < \frac{\epsilon}{2}$ ⟵(2)
Note that $n_k \geq k \geq k_0$
Choose p=max$\{k_0, n_0\}$. Then $p \geq k_0$ and $p \geq n_0$ $\forall n, m \geq p$ using (1), we have, $|x_n - x_m| < \frac{\epsilon}{2}$ ⟵(3)
$\forall k \geq p$ using (2), we have, $|x_{n_k} - l| < \frac{\epsilon}{2}$
So in particular, for k=p, we have, $|x_{n_p} - l| < \frac{\epsilon}{2}$

Now, in particular, for m=$n_p \geq p$, $\forall n \geq p$ by (3), we get, $|x_n - x_{n_p}| < \frac{\epsilon}{2}$

Consider $|x_n - l| = |(x_n - x_{n_p}) + (x_{n_p} - l)| \leq |x_n - x_{n_p}| + |x_{n_p} - l|$
$< \frac{\epsilon}{2} + \frac{\epsilon}{2} \qquad \forall n \geq p$

$\therefore$ $\forall \epsilon > 0$ there exists $p \in \mathbb{N}$ such that $\forall n \geq p$, we have, $|x_n - l| < \epsilon$
Thus, the Cauchy sequence $\{x_n\}$ in $\mathbb{R}$ is convergent in $\mathbb{R}$ and hence $\mathbb{R}$ is a Banach space under defined norm.

Example 22. Show that the vector space $\mathbb{C}^n = \{(x_1, \cdots, x_n) \mid x_i \in \mathbb{C}\}$ over $\mathbb{C}$ is a Banach space under the norm $\|x\| = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$ where $1 \leqslant p < \infty$.

Solution: Refer Example 8 to show that $\mathbb{C}^n$ is a normed space under defined norm. Now, consider a Cauchy sequence $\{x_m\}$ in $\mathbb{C}^n$. As $x_m \in \mathbb{C}^n$ is an n-tuple, denote $x_m = (x_1^{(m)}, \cdots, x_n^{(m)})$. Let $\epsilon > 0$. Then $\{x_m\}$ being a Cauchy sequence, $\exists$ $m_0 \in \mathbb{N}$ such that $\forall$ $l, m \geq m_0$, we have, $\|x_m - x_l\| < \epsilon$ which implies $\|(x_1^{(m)} - x_1^{(l)}, \cdots, x_n^{(m)} - x_n^{(l)})\| =$

$$\left(\sum_{i=1}^{n} |x_i^{(m)} - x_i^{(l)}|^p\right)^{1/p} < \epsilon \qquad \longrightarrow (*)$$

Eq.$(*)$ implies that $|x_i^{(m)} - x_i^{(l)}| < \epsilon \quad \forall\, l, m \geq m_0$ and $\forall\, i = 1, \cdots, n$
This shows that for fixed but arbitrary i, $\{x_i^{(r)}\}$ is a Cauchy sequence in $\mathbb{C}$. As every Cauchy sequence in $\mathbb{C}$ is convergent, $\{x_i^{(r)}\}$ must converge, say to, $z_i \in \mathbb{C}$. Thus $\lim\limits_{r \longrightarrow \infty} x_i^{(r)} = z_i \in \mathbb{C} \quad \forall\, i = 1, \cdots, n \qquad \longrightarrow (**)$
Making $l \longrightarrow \infty$ in Eq.$(*)$ and then using Eq.$(**)$, we get, $\forall$ $m \geq m_0$, $\left(\sum_{i=1}^{n} |x_i^{(m)} - z_i|^p\right)^{1/p} < \epsilon$ which implies $\|x_m - z\| < \epsilon$ where $z = (z_1, \cdots, z_n) \in \mathbb{C}^n$. It follows that the Cauchy sequence $\{x_m\}$ in $\mathbb{C}^n$ is convergent to $z \in \mathbb{C}^n$. Hence, $\mathbb{C}^n$ is a Banach space under defined norm.

Example 23. Show that the vector space $\mathbb{C}^n = \{(x_1, \cdots, x_n) \mid x_i \in \mathbb{C}\}$ over $\mathbb{C}$ is a Banach space under the norm $\|x\| = max\{\mid x_1 \mid, \cdots, \mid x_n \mid\}$. (This norm is referred as $\|x\|_\infty$ on $\mathbb{C}^n$).

Solution: Refer Example 9 to show that $\mathbb{C}^n$ is a normed space under defined norm. Now, consider a Cauchy sequence $\{x_m\}$ in $\mathbb{C}^n$. As $x_m \in \mathbb{C}^n$ is an n-tuple, denote $x_m = (x_1^{(m)}, \cdots, x_n^{(m)})$. Let $\epsilon > 0$. Then $\{x_m\}$ being a Cauchy sequence, $\exists$ $m_0 \in \mathbb{N}$ such that $\forall$ $l, m \geq m_0$, we have, $\|x_m - x_l\| < \epsilon$ which implies $\|(x_1^{(m)} - x_1^{(l)}, \cdots, x_n^{(m)} - x_n^{(l)})\| = max\{|x_1^{(m)} - x_1^{(l)}|, \cdots, |x_n^{(m)} - x_n^{(l)}|\} < \epsilon \qquad \longrightarrow (*)$
Eq.$(*)$ implies that $|x_i^{(m)} - x_i^{(l)}| < \epsilon \quad \forall\, l, m \geq m_0$ and $\forall\, i = 1, \cdots, n$
This shows that for fixed but arbitrary i, $\{x_i^{(r)}\}$ is a Cauchy sequence in $\mathbb{C}$. As every Cauchy sequence in $\mathbb{C}$ is convergent, $\{x_i^{(r)}\}$ must converge, say to, $z_i \in \mathbb{C}$. Thus $\lim\limits_{r \longrightarrow \infty} x_i^{(r)} = z_i \in \mathbb{C} \quad \forall\, i = 1, \cdots, n \qquad \longrightarrow (**)$
Making $l \longrightarrow \infty$ in Eq.$(*)$ and then using Eq.$(**)$, we get, $\forall$ $m \geq m_0$,

$max\{|x_1^{(m)} - z_1|, \cdots, |x_n^{(m)} - z_n|\} < \epsilon$ which implies $\|x_m - z\| < \epsilon$ where $z = (z_1, \cdots, z_n) \in \mathbb{C}^n$. It follows that the Cauchy sequence $\{x_m\}$ in $\mathbb{C}^n$ is convergent to $z \in \mathbb{C}^n$. Hence, $\mathbb{C}^n$ is a Banach space under defined norm.

Example 24. Show that the vector space $\mathbb{C}^n = \{(x_1, \cdots, x_n) \mid x_i \in \mathbb{C}\}$ over $\mathbb{C}$ is a Banach space under the norm $\|x\| = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$.

Solution: Left to the reader.

Example 25. Let $1 \leqslant p < \infty$. Show that the sequence space $l^p = \left\{\{x_1, \cdots, x_n, \cdots\} \,\middle|\, \sum_{i=1}^{\infty} \mid x_i \mid^p < \infty \text{ and } x_i \in \mathbb{C}\right\}$ over $\mathbb{C}$ is a Banach space under the norm $\|x\| = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p}$.

Solution: Refer Example 10 to show that l^p is a normed space under defined norm. Now, consider a Cauchy sequence $\{x_m\}$ in l^p. As $x_m \in l^p$, denote $x_m = \{x_1^{(m)}, \cdots, x_n^{(m)}, \cdots\}$ where $\sum_{i=1}^{\infty} \mid x_i^{(m)} \mid^p < \infty$. Let $\epsilon > 0$. Then as $\{x_m\}$ is a Cauchy sequence, $\exists\, m_0 \in \mathbb{N}$ such that $\forall\, l, m \geq m_0$, we have, $\|x_m - x_l\| < \epsilon$ which implies $\|\{x_1^{(m)} - x_1^{(l)}, \cdots, x_n^{(m)} - x_n^{(l)}, \cdots\}\| =$

$$\left(\sum_{i=1}^{\infty} |x_i^{(m)} - x_i^{(l)}|^p\right)^{1/p} < \epsilon. \qquad \longrightarrow (*)$$

Eq.$(*)$ implies that $|x_i^{(m)} - x_i^{(l)}| < \epsilon \quad \forall\, l, m \geq m_0$ and $\forall\, i$

This shows that for fixed but arbitrary i, $\{x_i^{(r)}\}$ is a Cauchy sequence in $\mathbb{C}$. As every Cauchy sequence in $\mathbb{C}$ is convergent, $\{x_i^{(r)}\}$ must converge, say to, $z_i \in \mathbb{C}$. Thus $\lim_{r \longrightarrow \infty} x_i^{(r)} = z_i \in \mathbb{C} \quad \forall\, i \qquad \longrightarrow (**)$

From Eq.$(*)$, it is clear that, $\forall\, k \in \mathbb{N}$, $\sum_{i=1}^{k} |x_i^{(m)} - x_i^{(l)}|^p < \epsilon^p \;\; \forall\, l, m \geq m_0$.

Making $l \longrightarrow \infty$ and then using Eq.$(**)$, we get, $\left(\sum_{i=1}^{k} |x_i^{(m)} - z_i|^p\right) < \epsilon^p$ $\forall\, m \geq m_0$. Further, making $k \longrightarrow \infty$, we get, $\left(\sum_{i=1}^{\infty} |x_i^{(m)} - z_i|^p\right) < \epsilon^p$ $\forall\, m \geq m_0$. This implies that $(z - x_m) \in l^p$ where $z = \{z_1, \cdots, z_n, \cdots\}$. So, $z = ((z - x_m) + x_m) \in l^p$.

Now, $\|x_m - z\| = \left(\sum_{i=1}^{\infty} |x_i^{(m)} - z_i|^p\right)^{1/p} < \epsilon \;\; \forall\, m \geq m_0$. It follows that the Cauchy sequence $\{x_m\}$ in l^p is convergent to $z \in l^p$. Hence, l^p is a Banach space under defined norm.

Example 26. Show that the sequence space $l^p = \left\{ \{x_1, \cdots, x_n, \cdots\} \middle| sup \{| x_1 |, \cdots, | x_n |, \cdots\} < \infty \ and \ x_i \in \mathbb{C} \right\}$ over $\mathbb{C}$ is a Banach space under the norm $\|x\| = sup\{| x_1 |, \cdots, | x_n |, \cdots\}$.
(This norm is denoted as $\|x\|_\infty$ on l^p).
Solution: Left to the reader.

To show that the normed space $L^p(E)$ is a Banach space, we first prove characterization of a Banach space in the forem of lemma, for which you are introduced with following terms.

Definition 4.2. A sequence $\{x_k\}$ in a normed space $(V, \| \ \|)$ is said to be *summable* to the sum s if the sequence $\{s_n\}$ of the partial sums of the series $\sum_{k=1}^{\infty} x_k$ converges to $s \in V$. i.e. $\|s_n - s\| \longrightarrow 0$ as $n \longrightarrow \infty$.
In this case, we write, $s = \sum_{k=1}^{\infty} x_k$.

Definition 4.3. The sequence $\{x_k\}$ in a normed space $(V, \| \ \|)$ is said to be *absolutely summable* if $\sum_{k=1}^{\infty} \|x_k\| < \infty$.

Lemma 4.3.1. *A normed space $(V, \| \ \|)$ is a Banach space if and only if every absolutely summable sequence in V is summable.*

Proof. Let the normed space $(V, \| \ \|)$ be a Banach space. Consider an absolutely summable sequence $\{x_k\}$ in V. Then $\sum_{k=1}^{\infty} \|x_k\| = M < \infty$ where $M > 0$. Thus, $\forall \ \epsilon > 0$, $\exists \ r \in \mathbb{N}$ such that $\sum_{k=r}^{\infty} \|x_k\| < \epsilon$. So, if $s_n = \sum_{k=1}^{n} x_k$ is n^{th} partial sum of the series $\sum_{k=1}^{\infty} x_k$ then $\forall \ n > m \geqslant r$, we have, $\|s_n - s_m\| = \left\| \sum_{k=m+1}^{n} x_k \right\| \leqslant \sum_{k=m+1}^{n} \|x_k\| \leqslant \sum_{k=r}^{\infty} \|x_k\| < \epsilon$. It follows that, the sequence $\{s_n\}$ of partial sums of the series $\sum_{k=1}^{\infty} x_k$ is a Cauchy sequence in Banach space V and hence must converge to some element; say $s \in V$. Thus, $\{x_k\}$ is summable in V and we are done.

Conversely, in a normed space $(V, \| \ \|)$, assume that every absolutely summable sequence in V is summable. Consider a Cauchy sequence $\{x_k\}$ in V. Then, for each $k \in \mathbb{N}$, $\exists \ n_k \in \mathbb{N}$ such that $\forall \ n, m \geqslant n_k$, we have, $\|x_n - x_m\| < \frac{1}{2^k}$ $\longrightarrow (*)$

Choose n_k such that $n_{k+1} > n_k$. Then $\{x_{n_k}\}$ is a subsequence of Cauchy sequence $\{x_n\}$.
Define $y_1 = x_{n_1}$

$$
\begin{aligned}
y_2 &= x_{n_2} - x_{n_1}\\
&\vdots\\
y_k &= x_{n_k} - x_{n_{k-1}}\\
&\vdots
\end{aligned}
$$

Then as $n_k > n_{k-1}$, by $(*)$, we have, $\|y_k\| < \dfrac{1}{2^{k-1}}$ where $k > 1$.

Consider the series $\displaystyle\sum_{k=1}^{\infty} y_k$. Its k^{th} partial sum is $s_k = \displaystyle\sum_{i=1}^{k} y_i = x_{n_k}$. As $\displaystyle\sum_{k=2}^{\infty} \frac{1}{2^{k-1}}$ is a geometric series with first term 0.5 and common ratio 0.5, we have, $\displaystyle\sum_{k=2}^{\infty} \frac{1}{2^{k-1}} = \frac{0.5}{1-0.5} = 1$.

Now $\displaystyle\sum_{k=1}^{\infty} \|y_k\| = \|y_1\| + \sum_{k=2}^{\infty} \|y_k\| \leqslant \|y_1\| + \sum_{k=2}^{\infty} \frac{1}{2^{k-1}} = \|y_1\| + 1 = M < \infty$ which implies that the sequence $\{y_k\}$ is absolutely summable and hence summable (by hypothesis). So, the sequence $\{s_k\}$ converges to some $s \in V$ and hence the subsequence $\{x_{n_k}\}$ is convergent. By Theorem 3.4.6, it follows that, the Cauchy sequence $\{x_n\}$ in V is convergent. Therefore, the normed space $(V, \| \ \|)$ is a Banach space. $\square$

Recall the following results which you studied in measure theory.
Fatou's Lemma : Let $\{f_n\}$ be a sequence of non-negative measurable functions and $\displaystyle\lim_{n\longrightarrow\infty} f_n = f$ a.e. on E. Then $\displaystyle\int_E f \leqslant \varliminf_{n\longrightarrow\infty} \int_E f_n$

Lesbesgue Dominated Convergence Theorem : Let g be an (Lesbesgue) integrable function on E. Let $\{f_n\}$ be a sequence of measurable functions such that $|f_n| \leqslant g$ on E and $\displaystyle\lim_{n\longrightarrow\infty} f_n = f$ a.e. on E. Then $\displaystyle\int_E f = \lim_{n\longrightarrow\infty} \int_E f_n$

Example 27. Let $1 \leqslant p < \infty$ & E be a (bounded closed interval in $\mathbb{R}$) measurable set. Show that the vector space $L^p(E) = \{f : E \longrightarrow \mathbb{R} | f$ is Lebesgue measurable function on E and $| f |^p$ is Lebesgue integrable over $E\}$ over $\mathbb{R}$ is a Banach space under norm $\|f\|_p = \left(\displaystyle\int_E | f |^p \right)^{1/p}$.
(This norm is referred as p-norms on $L^p(E)$).
Solution: Refer Example 12 to show that $L^p(E)$ is a normed space under defined norm.

Consider an absolutely summable sequence $\{f_k\}$ in $L^p(E)$. Then

$\sum_{k=1}^{\infty} \|f_k\|_p = M < \infty$ where $M > 0$.

1. Define a sequence $\{g_n\}$ of functions where $g_n(x) = \sum_{k=1}^{n} |f_k|$.
   Clearly, for each x, $\{g_n(x)\}$ is an increasing sequence of (extended) real numbers and $\exists$ some (extended) real number $g(x)$ such that $g_n(x) \longrightarrow g(x) \quad \forall x \in E$

2. The function g is measurable, since the functions $g_n(x)$ are measurable.

3. By Minkowski's inequality (for finite sums) in Lemma 3.3.3, we have,
   $\|g_n\|_p = \left\| \sum_{k=1}^{n} |f_k| \right\|_p \leqslant \sum_{k=1}^{n} \|f_k\|_p < M$ and hence $\int_E |g_n|^p \leqslant M^p$.

4. As $g_n \geqslant 0$ and $\lim_{n \longrightarrow \infty} g_n^p = g^p$, by Fatou's lemma, we have,
   $\int_E g^p \leqslant \underline{\lim}_{n \longrightarrow \infty} \int_E g_n^p \leqslant M^p$
   It follows that g^p is integrable and hence $g(x)$ is finite a.e. on E.

5. Thus, we find that, for each x, for which $g(x)$ is finite, the sequence $\{f_n(x)\}$ is an absolutely summable sequence of real numbers and therefore, must be summable to a real number, say $s(x)$.

6. Define $s(x) = 0$ for those x where $g(x) = \infty$. Then, the function s so defined is the limit a.e. of partial sums $s_n(x) = \sum_{k=1}^{n} f_k(x)$.
   i.e. $s_n(x) \longrightarrow s(x)$ a.e. Hence, s is a measurable function.
   (note that $|s_n(x) - s(x)|^p \longrightarrow 0$ a.e. $\forall\ x$)

7. Clearly, $|s_n(x)| \leqslant \sum_{k=1}^{n} |f_k(x)| = g_n(x) \leqslant g(x)$. Then as $g \in L^p(E)$, we have, $s \in L^p(E)$. $\longleftarrow$ since, if $h \in L^p$ & $|f| \leqslant |h|$ then $f \in L^p$.

8. It is easy to see that $|s_n(x) - s(x)|^p \leqslant 2^p\ (g(x))^p$.

9. As $2^p g^p$ is an integrable function and $|s_n(x) - s(x)|^p \longrightarrow 0$ a.e. $\forall\ x$, by Lesbesgue Dominated Convergence Theorem, we have, $\int_E |s_n - s|^p \longrightarrow 0$ and hence $\|s_n - s\|_p \longrightarrow 0$

10. So, the sequence $\{s_n\}$ of partial sums of series $\sum_{k=1}^{\infty} f_k$ converges to $s \in L^p(E)$. i.e. the absolutely summable sequence $\{f_k\}$ in $L^p(E)$ is summable in $L^p(E)$.

Thus, by Lemma 4.3.1, $L^p(E)$ is a Banach space under defined norm.

Example 28. Let E be a (bounded closed interval in $\mathbb{R}$) measurable set. Show that the vector space $L^\infty(E) = \{f : E \longrightarrow \mathbb{R}|\text{f is a measurbale function on } E \text{ and } ess\ sup|f| < \infty\}$ over $\mathbb{R}$ is a Banach space under norm $\|f\|_\infty = ess\ sup_E|f(x)| = inf\{m > 0 \mid |f(x)| \leqslant m$ a.e. on $E\}$.

Solution: Refer Example 13 to show that $L^\infty(E)$ is a normed space under defined norm.

Consider a Cauchy sequence $\{f_n\}$ in $L^\infty(E)$. Then $|f_n(x)-f_m(x)| \leqslant \|f_n - f_m\|_\infty$ except on a set $A_{n,m} \subset [a,b] = E$ with $m(A_{n,m}) = 0$.
If $A = \cup_{n,m} A_{n,m}$ then $m(A) = 0$ and $|f_n(x) - f_m(x)| \leqslant \|f_n - f_m\|_\infty$ $\forall\ n,m$ and $\forall x \in (E - A)$.

Therefore, it follows that, $\{f_n\}$ converges uniformly to a bounded limit f outside A and the result is proved by observing the fact that the convergence in $L^\infty(E)$ is equivalent to uniform convergence outside a set of measure zero.

Thus, as $\{f_n\} \longrightarrow f$ outside A, we have, $L^\infty(E)$ is a Banach space under defined norm.

The Examples 27 and 28 are well known by following theorem.

Theorem 4.3.4. ***Riesz-Fischer theorem:***
For $1 \leqslant p \leqslant \infty$, L^p spaces are Banach spaces.

Proof. Combine the answers to the Examples 27 and 28. □

Now, you will see next two results related to quotient space.

Theorem 4.3.5. *Let M be a closed subspace of a Banach space $(V, \|\ \|)$. For each coset $x+M$ in quotient space V/M, define $\|x+M\| = inf\{\|x+m\| \mid m \in M\}$. Then V/M is a Banach space under the norm $\|x+M\|$.*

Proof. Refer Theorem 3.4.3 to show that V/M is a normed space under defined norm. Now, let $\{s_n+M\}$ be a Cauchy sequence in V/M. Then $s_n \in V$. We know that, a Cauchy sequence is convergent if and only if it has a convergent subsequence. So in order to show that $\{s_n + M\}$ is convergent, it is sufficient to show that it has a convergent subsequence. We construct a subsequence in the following manner:

As $\{s_n + M\}$ is Cauchy, given $\epsilon = \frac{1}{2} > 0$, $\exists\ n_1 \in \mathbb{N}$ such that $\forall\ n,m \geqslant n_1$, we have, $\|(s_n + M) - (s_m + M)\| < \epsilon = \frac{1}{2}$

Set $s_{n_1} = x_1 \in V$.

Similarly, $\epsilon = \dfrac{1}{2^2} > 0$, $\exists\ n_2 \in \mathbb{N}$ with $n_2 > n_1$ such that $\forall\ n, m \geqslant n_1$, we have, $\|(s_n + M) - (s_m + M)\| < \epsilon = \dfrac{1}{2^2}$

Set $s_{n_2} = x_2 \in V$.

In general, having chosen $x_1, \cdots, x_k$ and $n_1, \cdots, n_k$, let $n_{k+1} > n_k$ be such that $\forall\ n, m \geqslant n_k$, we have, $\|(s_n + M) - (s_m + M)\| < \epsilon = \dfrac{1}{2^{k+1}}$

Set $s_{n_{k+1}} = x_{k+1} \in V$.

Thus, we have obtained a subsequence $\{x_k + M\}$ of $\{s_n + M\}$ such that $\|(x_k + M) - (x_{k+1} + M)\| < \dfrac{1}{2^k}$ for $k = 1, 2, \cdots$

Claim: This subsequence converges to an element of V/M.

Let $y_1 \in x_1 + M$. Choose $y_2 \in x_2 + M$ such that $\|y_1 - y_2\| < \dfrac{1}{2}$. Next, choose $y_3 \in x_3 + M$ such that $\|y_2 - y_3\| < \dfrac{1}{2^2}$. Continuing this process, we obtain a sequence $\{y_n\}$ in V such that $\|y_n - y_{n+1}\| < \dfrac{1}{2^n}$. Note that given $\epsilon > 0$, we can choose $m_0 \in \mathbb{N}$ so large that $\dfrac{1}{2^{m_0 - 1}} < \epsilon$.

Then for $n > m \geqslant m_0$, we have,

$$\begin{aligned}
\|y_m - y_n\| &= \|(y_m - y_{m+1}) + (y_{m+1} - y_{m+2}) + \cdots + (y_{n-1} - y_n)\| \\
&\leqslant \|(y_m - y_{m+1})\| + \|(y_{m+1} - y_{m+2})\| + \cdots + \|(y_{n-1} - y_n)\| \\
&< \sum_{i=m}^{n-1} \frac{1}{2^i} \\
&< \sum_{i=m}^{\infty} \frac{1}{2^i} = \frac{1}{2^{m-1}} \quad \text{,being a geometric series} \\
&< \frac{1}{2^{m_0 - 1}} \\
&< \epsilon
\end{aligned}$$

$\therefore\ \{y_n\}$ is a Cauchy sequence in Banach space V and hence $\exists\ y \in V$ such that $\|y_n - y\| \longrightarrow 0$ as $n \longrightarrow \infty$.

Consider
$$\begin{aligned}
\|(x_n + M) - (y + M)\| &= \|(x_n - y) + M\| \\
&= inf\{\|(x_n - y) + m\| \mid m \in M\} \\
&\leqslant \|(x_n - y) + m\| \qquad \forall\ m \in M
\end{aligned}$$

As $y_n = x_n + m_n$ for some $m_n \in M$, we conclude that,
$$\|(x_n + M) - (y + M)\| \leqslant \|y_n - y\| \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

$\Longrightarrow$ $x_n + M \longrightarrow y + M \in V/M$

$\Longrightarrow$ The Cauchy sequence $\{s_n + M\}$ has a subsequence $\{x_n + M\}$ which is convergent in V/M.

$\Longrightarrow$ The Cauchy sequence $\{s_n + M\}$ is convergent in V/M.

$\Longrightarrow$ The normed space V/M is complete and we are done. □

Theorem 4.3.6. *Let M be a closed subspace of a normed space $(V, \|.\|)$. If M and V/M are Banach spaces then V is a Banach space*

Proof. Consider a Cauchy sequence in V. Let $\epsilon > 0$ be given. Then $\exists$

$n_0 \in \mathbb{N}$ such that $\forall m, n \geqslant n_0$, we have, $\|x_n - x_m\| < \epsilon$. Also, $\{x_n + M\}$ is a sequence in V/M.

Consider $\|(x_n + M) - (x_m + M)\| = \|(x_n - x_m) + M\|$

$$= inf\{\|(x_n - x_m) + y\| \mid y \in M\}$$
$$\leqslant \|(x_n - x_m) + y\| \qquad \forall\, y \in M$$

But $y = 0 \in M$. So, $\|(x_n + M) - (x_m + M)\| \leqslant \|(x_n - x_m)\| < \epsilon$ $\forall m, n \geqslant n_0$. This implies that $\{x_n + M\}$ is a Cauchy sequence in Banach space V/M and so it must converge to some $z + M \in V/M$ for some $z \in V$. Hence $\|(x_n + M) - (z + M)\| = \|(x_n - z) + M\| \longrightarrow 0$ as $n \longrightarrow \infty$

Now, for each $n \in \mathbb{N}$, $\exists\ y_n \in M$ such that $\|x_n - z + M\| = inf\{\|(x_n - z) + y_n\| \mid y_n \in M\} \leqslant \|(x_n - z) + y_n\|$ and thus $\|x_n - z + M\| < \|(x_n - z) + y_n\| + \dfrac{1}{n}$.

Consider $\|y_n - y_m\| = \|(x_n - z + y_n) - (x_m - z + y_m) - (x_n - x_m)\|$ $\therefore \|y_n - y_m\| \leqslant \|(x_n - z + y_n)\| + \|(x_m - z + y_m)\| + \|(x_n - x_m)\|$ and thus $\|y_n - y_m\| \leqslant \|x_n - x_m\| < \epsilon\ \ \forall m, n \geqslant n_0$. This implies that $\{y_n\}$ is a Cauchy sequence in Banach space M and so it must converge to some $y \in M$. Hence $y_n \longrightarrow y$ in M as $n \longrightarrow \infty$.

Now, since as $n \longrightarrow \infty$, we have $\|x_n - z + y_n\| \longrightarrow 0$ and hence $x_n \longrightarrow (z - y_n)$ in V. So, $\lim\limits_{n \longrightarrow \infty} x_n = \lim\limits_{n \longrightarrow \infty} (z - y_n) = z - y = x \in V$. $\therefore$ the Cauchy sequence $\{x_n\}$ in V is convergent to $x \in V$. Hence V is complete and we are done. $\square$

This section is concluded with following result.

Theorem 4.3.7. *Every complete subspace of a normed space is closed.*

Proof. Consider a normed space $(V, \|.\|)$. Let M be a complete subspace of V. Let z be any limit point of M. Then $\forall\ n \in \mathbb{N}$, the open ball $B\left(z, \dfrac{1}{n}\right) = \left\{x \,\middle|\, \|x - z\| < \dfrac{1}{n}\right\}$ must contain atleast one point y_n other than z. So, $\exists\ y_1, y_2, \cdots, y_n, \cdots$ in M such that

$$\|y_1 - z\| < 1$$
$$\|y_2 - z\| < \frac{1}{2}$$
$$\vdots$$
$$\|y_n - z\| < \frac{1}{n}$$
$$\vdots$$

<u>Claim</u>: the sequence $\{y_n\}$ of points in M converges to z.

Let $\epsilon > 0$. Choose $m \in \mathbb{N}$ such that $\dfrac{1}{m} < \epsilon$. Then

$\forall\ n \geqslant m$, we have, $\|y_n - z\| < \dfrac{1}{n} \leqslant \dfrac{1}{m} < \epsilon$

Thus, $\forall \epsilon > 0$, $\exists\ m \in \mathbb{N}$ such that $\|y_n - z\| < \epsilon\ \ \forall\ n \geqslant m$
$\Longrightarrow$ the sequence $\{y_n\}$ converges to z. Hence the claim.
As every convergent sequence in a metric space is a Cauchy sequence, the sequence $\{y_n\}$ is a Cauchy sequence in M. But, since M is complete, we have, every Cauchy sequence in M converges to a point in M. So, $y_n \longrightarrow z$ implies $z \in M$.

Thus, we have shown that, every limit point of M belongs to M and consequently, M is closed. $\square$

4.4 Equivalent Norms and Finite-Dimensional Spaces

You have seen that there are many norms on the same finite dimensional vector space X. It is interesting to see that all these norms on X lead to same topology for X, that is, the open subsets of X are the same, regardless of the particular choice of a norm on X. In this section, you will see the notion of equivalent norms and basic results related to it.

Definition 4.4. Two norms $\|\ \|_1$ and $\|\ \|_2$ on a normed space X are said to be *equivalent* and written as $\|\ \|_1 \sim \|\ \|_2$, if $\exists$ positive real numbers a and b (independent of $x \in X$) such that

$$a\ \|x\|_1 \leqslant \|x\|_2 \leqslant b\ \|x\|_1 \quad \forall\ x \in X$$

With little effort you can show that equivalent of norms is an equivalence relation on the set of all norms over a given space.

Two norms $\|\ \|_1$ and $\|\ \|_2$ on a normed space X are equivalent if and only if any (Cauchy) sequence in X converges with respect to $\|\ \|_1$ converges with respect to $\|\ \|_2$ and conversely.

From following result, you will come to know that although one can define many different norms on finite dimensional linear spaces, there is only one topology derived from these norms.

Theorem 4.4.1. *On a finite dimensional normed space X, any two norms are equivalent.*

Proof. Let X be a finite dimensional vector space over $\mathbb{F}$ with dim(X)=n. Then $X \cong \mathbb{F}^n$, since if $B = \{v_1, \cdots, v_n\}$ is a basis for X then each $x \in X$ can be uniquely represented as $x = \sum_{j=1}^{n} a_j\ v_j$ for some unique scalars $a_j \in \mathbb{F}$ which gives an element $\bar{x} = (a_1, \cdots, a_n) \in \mathbb{F}^n$.

Now, by Euclidean norm in $\mathbb{F}^n$, we have, $\|\bar{x}\|_2 = \left(\sum_{j=1}^{n} |a_j|^2\right)^{1/2}$ and for each $x = \sum_{j=1}^{n} a_j\ v_j \in X$, define $\|x\|_2 = \left(\sum_{j=1}^{n} |a_j|^2\right)^{1/2}$. Then $\|\bar{x}\|_2 = \|x\|_2$.

Suppose $\|.\|$ is a norm on X. Then, for each $x \in X$,

$$\|x\| = \left\|\sum_{j=1}^{n} a_j\ v_j\right\| \leqslant \sum_{j=1}^{n} |a_j|\ \|v_j\| \leqslant \left(\sum_{j=1}^{n} \|v_j\|^2\right)^{1/2} \left(\sum_{j=1}^{n} |a_j|^2\right)^{1/2}$$

If $M = \left(\sum_{j=1}^{n} \|v_j\|^2\right)^{1/2}$ then $M > 0$ such that $\|x\| \leqslant M\ \|x\|_2\ \ \forall\ x \in X$ which gives the one half of the desired inequality.

Now, to establish the other inequlaity, define $S = \{\bar{x} = (a_1, \cdots, a_n) \in \mathbb{F}^n |\ \|\bar{x}\|_2 = 1\}$. Then S is closed and bounded, and hence is compact (by Heine-Borel theorem) with respect to the Euclidean norm.

Define $f : S \longrightarrow \mathbb{R}$ as $f(\bar{x}) = \|x\|$. As B is a linearly independent set and since $\bar{x} \in S$, i.e. $\bar{x} \neq 0$, all a_j cannot vanish simultaneously on S so that $f(\bar{x}) > 0$ on S.

Clearly, $|f(\bar{x}) - f(\bar{y})| = \Big|\|x\| - \|y\|\Big| \leqslant \|x - y\| \leqslant M\ \|x - y\|_2$.

It follows that f is continuous on S. Thus, f is a positive valued continuous function on the compact set S and therefore, f attains its minimum $m > 0$ at some point on the compact set S. Consequently, whenever $\bar{x} \in S$, we have, $f(\bar{x}) = \|x\| \geqslant m$.

Thus, for each $0 \neq \bar{u} = (c_1, \cdots, c_n) \in \mathbb{F}^n$,

$$\|u\| = \left\|\sum_{j=1}^{n} c_j\ v_j\right\| = \|\bar{u}\|_2\ f\left(\frac{\bar{u}}{\|\bar{u}\|_2}\right) \geqslant m\ \|\bar{u}\|_2 = m\ \|u\|_2$$

Therefore, $\exists$ positive real numbers m and M such that

$$m\ \|u\|_2 \leqslant \|u\| \leqslant M\ \|u\|_2\ \ \forall\ u \in X$$

This implies that, any given norm $\|.\|$ is equivalent to the 2-norm $\|.\|_2$. Since, equivalence of norms is an equivalence relation, it follows that any two norms on X are equivalent. □

Now, you will see some immediate consequences of this theorem in the form of following corollaries.

Corollary 4.4.1. *If V is a finite dimensional normed space, then V is complete.*

Proof. Let V be a finite dimensional vector space over $\mathbb{F}$ where $\mathbb{F} = \mathbb{R}$ *or* $\mathbb{C}$ with dim(V)=$n > 0$ and $\{e_1, \cdots, e_n\}$ be basis for V. Then each $x \in V$ can be uniquely represented as $x = \sum_{j=1}^{n} a_j\ e_j$ for some unique scalars $a_j \in \mathbb{F}$. It can be easily proved that $\|x\|_0 = \max_i |a_i|$ is a norm on V. This norm is called the zeroth norm.

Since, by Theorem 4.4.1, on a finite dimensional normed space V, any two norms are equivalent, to prove V is complete, it suffices to prove the completeness of V with respect to this zeroth norm.

Let $\{y_n\}$ be any Cauchy sequence in V. Consider the i^{th} term of this sequence $\{y_n\}$ which is $y_i = \sum_{k=1}^{n} a_k^{(i)} e_k$ for some uniquely determined scalars $a_1^{(i)}, \cdots, a_n^{(i)}$ in $\mathbb{F}$.

Since, $\{y_n\}$ is Cauchy, we have, $\|y_n - y_m\|_0 \longrightarrow 0 \quad \text{as} \quad m, n \longrightarrow \infty.$

$$\Longrightarrow \left\| \sum_{k=1}^{n} \left(a_k^{(n)} - a_k^{(m)} \right) e_k \right\|_0 \longrightarrow 0 \qquad \text{as} \quad m, n \longrightarrow \infty.$$

$$\Longrightarrow \max_k |a_k^{(n)} - a_k^{(m)}| \longrightarrow 0 \qquad \text{as} \quad m, n \longrightarrow \infty.$$

$$\Longrightarrow |a_k^{(n)} - a_k^{(m)}| \longrightarrow 0 \qquad \text{as} \quad m, n \longrightarrow \infty.$$

$\Longrightarrow \{a_k^{(m)}\}$ is Cauchy sequence in $\mathbb{F}$ for $k = 1, \cdots, n$,

As $\mathbb{F}$ is complete, $\exists$ scalars $a_1, \cdots, a_n$ in $\mathbb{F}$ such that

$$a_k^{(m)} \longrightarrow a_k \quad \text{as} \quad m \longrightarrow \infty \qquad \longrightarrow \quad (*)$$

$\therefore$ the Cauchy sequence $\{a_k^{(m)}\}_{m=1}^{\infty}$ converges for some a_k $(k = 1, \cdots, n)$.

Let $y = \sum_{k=1}^{n} a_k \, e_k$ then $y \in V$. To show that $y_m \longrightarrow y$ as $m \longrightarrow \infty$, let $\epsilon > 0$ be given. Then using $(*)$, we get,

$$\|y_m - y\|_0 = \left\| \sum_{k=1}^{n} \left(a_k^{(m)} - a_k \right) e_k \right\|_0 = \max_k |a_k^{(m)} - a_k| \longrightarrow 0 \text{ as } m \longrightarrow \infty$$

which implies $y_m \longrightarrow y$ as $m \longrightarrow \infty$

$\therefore$ the Cauchy sequence $\{y_m\}$ in V converges to $y \in V$ and hence V is complete. $\square$

Corollary 4.4.2. *If M is any finite dimensional subspace of a normed space V, then M is closed.*

Proof. Let M be a finite dimensional subspace of a normed space V. As by Corollary 4.4.1, every finite dimensional normed space is complete, we have, M is complete subspace of V. Further, as by Theorem 4.3.7, every complete subspace of a normed space is closed, we have, M is closed. $\square$

Now, you will see another interesting result (about closed subspaces) in the form of lemma, which is due to the famous Hungarian mathematician Riesz. This result/lemma is to prove very important theorem that relates finite dimensional normed spaces with compactness of its bounded subset.

Lemma 4.4.1. ***(Riesz Lemma)**: **(F.Riesz, 1918)***
Let M be a closed proper subspace of a normed space V and let $a \in \mathbb{R}$

be such that $0 < a < 1$. *Then* $\exists$ *a vector* $x_a \in V$ *such that* $\|x_a\| = 1$ *and* $\|x - x_a\| \geqslant a \quad \forall x \in M$

Proof. Let M be a closed proper subspace of a normed space V. Then we have $M = \overline{M}$ and $\exists\ x_1 \in (V - M)$. i.e. $x_1 \notin \overline{M} = M$.

Define $h = \inf\limits_{x \in M} \|x - x_1\| = d(x_1, M)$. Then $h \leqslant \|x - x_1\|\ \forall x \in M$.

Clearly, $h > 0$ because $h = 0$ implies $x_1 \in \overline{M} = M$, a contradiction.

Let $a \in \mathbb{R}$ be such that $0 < a < 1$. Then $\dfrac{h}{a} > h$.

By definition of infimum, $\exists\ x_0 \in M$ such that $\|x_0 - x_1\| \leqslant \dfrac{h}{a}$. Clearly, $x_0 \neq x_1$. Further, as $\|x_0 - x_1\| > h > 0$, we have, $\dfrac{1}{\|x_0 - x_1\|} > 0$.

Define $x_a = \dfrac{x_1 - x_0}{\|x_0 - x_1\|}$. Then $x_a \in V$ such that $\|x_a\| = 1$.

Let $x \in M$ be arbitrary. Then $(\|x_1 - x_0\|\ x + x_0) \in M$ as $x, x_0 \in M$ and hence $\|(\|x_1 - x_0\|\ x + x_0) - x_1\| \geqslant h$.

Now,

$$
\begin{aligned}
\|x - x_a\| &= \left\| x - \frac{x_1}{\|x_1 - x_0\|} + \frac{x_0}{\|x_1 - x_0\|} \right\| \\
&= \frac{1}{\|x_1 - x_0\|} \|(\|x_1 - x_0\|\ x + x_0) - x_1\| \\
&\geqslant \frac{h}{\|x_1 - x_0\|} \\
&\geqslant a
\end{aligned}
$$

Thus, $\exists\ x_a \in V$ such that $\|x_a\| = 1$ and $\|x - x_a\| \geqslant a \quad \forall x \in M$. $\square$

The Riesz Lemma 4.4.1 states that for any closed proper subspace M of a normed space V, $\exists$ points in the unit sphere $S(0,1) = \{x \in V |\ \|x\| = 1\}$ whose distance from M is as near as we please to 1(but not 1). There may not be a point, though, whose distance is exactly 1.

We conclude this section with the following required result, proved using Reisz lemma.

Theorem 4.4.2. *In a normed space* $(V, \|.\|)$, *if the set* $S = \{x \in V |\ \|x\| = 1\}$ *is compact then* V *is finite dimensional.*

Proof. We know that, in a metric space, a subset is compact if and only if every sequence has a convergent subsequence,

Let the set $S = \{x \in V |\ \|x\| = 1\}$ in normed space $(V, \|.\|)$ be compact. Then every sequence in S must have a convergent subsequence.

Suppose, if possible, V is not finite dimensional. Choose $x_1 \in S$. Then $\|x_1\| = 1$.

Let V_1 be the subspace spanned by x_1. Then V_1 is a proper subspace of V and V_1 is finite dimensional. It follows that V_1 is closed, since by Corollary 4.4.2 if M is any finite dimensional subspace of a normed space V then M is closed.

Applying Riesz Lemma 4.4.1 to this closed proper subspace V_1 of V, we get, $\exists\, x_2 \in V$ such that $\|x_2\| = 1$ and $\|x_2 - x\| \geqslant \dfrac{1}{2}\ \ \forall\, x \in V_1$.

$\implies \exists\, x_2 \in S$ and $\|x_2 - x_1\| \geqslant \dfrac{1}{2}\quad$ as $\ x_1 \in V_1$.

Let V_2 be the subspace spanned by x_1, x_2. Then V_2 is a proper subspace of V and V_2 is finite dimensional. So, by Corollary 4.4.2, V_2 is closed. Applying Riesz Lemma 4.4.1 to this closed proper subspace V_2 of V, we get, $\exists\, x_3 \in V$ such that $\|x_3\| = 1$ and $\|x_3 - x\| \geqslant \dfrac{1}{2}\ \ \forall\, x \in V_2$.

$\implies \exists\, x_3 \in S$ and $\|x_3 - x_2\| \geqslant \dfrac{1}{2}\quad$ as $\ x_2 \in V_2$.

Continuing this argument, we obtain an infinite sequence $\{x_n\}$ of vectors in S such that $\|x_n - x_m\| \geqslant \dfrac{1}{2}$.

$\implies$ the sequence $\{x_n\}$ can have no convergent subsequence, which contradicts the hypothesis that, S is compact.

Hence, the assumption that V is not finite dimensional must be wrong and we msut have V is finite dimensional. $\square$

4.5 Arzela-Ascoli theorem

You are already familiar with Bolzano-Weierstrass theorem which states that every bounded sequence of real/complex numbers contains a convergent subsequence. In this section, you will see something similar is true for sequence of functions, but in connection with the additional concept of equicontinuity.

Recall that the set $C(X)$ is the set of all K-valued continuous functions on a compact metric space X where $K = \mathbb{R}$ *or* $\mathbb{C}$. For $f, g \in C(X)$, let $d_\infty(f, g) = \sup_{x\in X}\{|f(x) - g(x)|\}$. It is easy to see that d_∞ is a metric on $C(X)$ and is called as sup metric on $C(X)$.

In the metric space (X, d), for $x \in X$ and $r > 0$, recall, the set $B(x, r) = \{y \in X | d(x, y) < r\}$, called as the open ball about x of radius r.

Also, recall that a subset E of a metric space X is said to be *totally bounded* if $\forall \epsilon > 0$, $\exists\, x_1, \cdots, x_n \in E$ such that $E \subset \cup_{j=1}^{n} B(x_j, \epsilon)$.

Further, recall that for functions, $f_n, f \in C(X)$, we say that the sequence $\{f_n\}$ converges uniformly to f on X if $\forall \epsilon > 0$, $\exists\, n_0 \in \mathbb{N}$(depends only on ϵ) such that $\forall\ x \in X$, $\forall\ n \geqslant n_0$, we have, $|f_n(x) - f(x)| < \epsilon$.

Now, you will see two kinds of boundedness.

Definition 4.5. Let S be subset of $C(X)$. We say that S is *pointwise bounded* on X if S is bounded at each $x \in X$, that is, if $\exists$ a finite-valued function ϕ defined on X such that $|f(x)| < \phi(x)\ \forall\ f \in S$.

Definition 4.6. Let S be subset of $C(X)$. We say that S is *uniformly bounded* on X if $\exists$ a number M such that $|f(x)| < M\ \forall\ f \in S$ and $\forall\ x \in X$.

It is easy to see that a uniformly bounded subset S of $C(X)$ is always pointwise bounded on X. The converse holds (as seen in *Ascoli's theorem*, the following result) under a certain condition which we now introduce.

Definition 4.7. A subset S of $C(X)$ is said to be *equicontinuous* at $x \in X$ if for every $\epsilon > 0$, $\exists\ \delta > 0$ such that for every $y \in X$ with $d(x, y) < \delta$ and $\forall f \in S$, we have, $|f(x) - f(y)| < \epsilon$, where δ may depend on x, but not on $f \in S$.
Here d denotes the metric of X.

A subset S of $C(X)$ is said to be equicontinuous on X if S is equicontinuous at every $x \in X$.

You can verify that the functions belonging to the equicontinuous collection are uniformly continuous.

Example 29. Define a sequence $\{f_n\}$ on $\mathbb{R}$ by $f_n(x) = sin(x + n)$. Show that the family $\{f_n | n \in \mathbb{N}\}$ is equicontinuous on $\mathbb{R}$.
Solution: Note that $|cos\theta| \leqslant 1$ and for small θ, $|sin\theta| \leqslant |\theta|$.
Also, $sin(n + x) - sin(n + x') = 2\ cos\left(\dfrac{2n + x + x'}{2}\right)\ sin\left(\dfrac{x - x'}{2}\right)$
Let $\epsilon > 0$ be given and $x \in \mathbb{R}$. With $x' \in \mathbb{R}$,
Conisder

$$\begin{aligned}
|f_n(x) - f_n(x')| &= 2\left|cos\left(\frac{2n + x + x'}{2}\right)\right|\left|sin\left(\frac{x - x'}{2}\right)\right| \\
&\leqslant 2\left|\left(\frac{x - x'}{2}\right)\right| \\
&= |x - x'|
\end{aligned}$$

If $|x - x'| < \delta = \epsilon$ then $|f_n(x) - f_n(x')| < \epsilon\ \forall\ n \in \mathbb{N}$.
Thus, the family $\{f_n | n \in \mathbb{N}\}$ is equicontinuous at $x \in \mathbb{R}$ and hence is equicontinuous on $\mathbb{R}$.

A condition sufficient to ensure that a sequence of continuous functions on compact space X has a uniformly convergent subsequence will come out of the following result (which is known as *Arzela's theorem*).

Theorem 4.5.1. *Let S be subset of $C(X)$ where X is a compact metric space. Suppose that S is pointwise bounded on X and is equicontinuous at every $x \in X$. Then*

(a) ***(Ascoli, 1883)*** *S is uniformly bounded on X. In fact, S is totally bounded in the sup metric on $C(X)$.*

(b) ***(Arzela, 1889)*** *Every sequence in S contains a uniformly convergent subsequence.*

Proof. (a) Let $\epsilon > 0$. Since (X, d) is a compact metric space, we have, $\exists\ x_1, \cdots, x_n \in X$ and positive numbers $\delta_1, \cdots, \delta_n$ such that $X = \cup\{B(x_i, \delta_i) \mid i = 1, \cdots, n\}$. So, to every $x \in X$ there corresponds at least one x_i with $x \in B(x_i, \delta_i)$. Also, since S is equicontinuous at every $x \in X$, $\forall f \in$S and $\forall\ x \in X$, we have, $d(x, x_i) < \delta_i\ (1 \leqslant i \leqslant n)$ which implies $|f(x) - f(x_i)| < \epsilon$. Further, since S is pointwise bounded on X, S is bounded at each $x \in X$. So, $\exists\ M_i < \infty\ (1 \leqslant i \leqslant n)$ such that $|f(x_i)| \leqslant M_i\ \ \forall f \in$S. Define $M = max\{M_1, \cdots, M_n\} + \epsilon$. Then it follows that $|f(x)| \leqslant M\ \ \forall f \in$S and $\forall\ x \in X$. Thus, S is uniformly bounded on X.

Now, let $E_M = \{k \in K |\ |k| \leqslant M\}$ where $K \in \mathbb{R}$ *or* $\mathbb{C}$ and for $f \in$S, define $e(f) = (f(x_1), \cdots, f(x_n)) \in E_M^n$. It is easy to see that E_M^n is totally bounded. Hence we can cover it by a finite union of open balls of radius ϵ, say $V_1, \cdots, V_m$. If $j = 1, \cdots, m$ and $V_j \cap \{e(f) | f \in S\} \neq \emptyset$, choose $f_j \in S$ such that $e(f_j) \in V_j$.
Claim: S is union of open balls of radius 5ϵ about these $f_1, \cdots, f_m$.
Let $f \in$S. Then $e(f) \in V_j$ for some $j = 1, \cdots, m$. Since, $e(f_j) \in V_j$ and the radius of $V_j's$ is ϵ, we see that $|f(x_i) - f_j(x_i)| < 2\epsilon\ \forall i = 1, \cdots, n$. Now, each $x \in X$ belongs to some $B(x_i, \delta_i)$, $i = 1, \cdots, n$, which implies $|f_j(x) - f_j(x_i)| < \epsilon$ and $|f(x) - f(x_i)| < \epsilon$.
Consider

$$\begin{aligned} |f(x) - f_j(x)| &= |f(x) - f(x_i) + f(x_i) - f_j(x_i) + f_j(x_i) - f_j(x)| \\ &\leqslant |f(x) - f(x_i)| + |f(x_i) - f_j(x_i)| + |f_j(x_i) - f_j(x)| \\ &< \epsilon + 2\epsilon + \epsilon \end{aligned}$$

so that $d_\infty(f, f_j) = \sup_{x \in X}\{|f(x) - f_j(x)|\} \leqslant 4\epsilon < 5\epsilon$.
This proves that S is totally bounded in the sup metric d_∞.

(b) We prove (b) using the following results of metric space.
(i) The subset A of metric space X is totally bounded if and only if every sequence of points of A contains a Cauchy subsequence.
(ii) The sequence $\{f_n\}$ in $C(X)$ converges to $f \in C(X)$ (with respect to sup metric on $C(X)$) if and only if $\{f_n\}$ converges uniformly to f on X.

Consider a sequence $\{f_n | f_n \in C(X),\ n = 1, 2, \cdots\}$ in S. Then by

Ascoli theorem, $\{f_n\} \subseteq C(X)$ is totally bounded. Hence, by result (i), $\{f_n\}$ has a Cauchy subsequence $\{f_{n_k}\}$ (with respect to sup metric on $C(X)$). Since, by Theorem 4.3.3, $C(X)$ is complete , the sequence $\{f_{n_k}\}$ is convergent to some $f \in C(X)$. By result (ii), this implies that $\{f_{n_k}\}$ converges uniformly to f on X and we are done.

(b) (Alternative proof of (b) without using Ascoli theorem)
Consider a sequence $\{f_n | f_n \in C(X),\ n = 1.2, \cdots\}$ in S. Let A be a countable dense subset of X. Then S is pointwise bounded on A.
<u>Claim</u>: $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ such that $\{f_{n_k}(x)\}$ converges for every $x \in A$.
Let $\{x_i\}$, $i = 1, 2, 3, \cdots$ be points of A, arranged in a sequence. Since, $\{f_n(x_1)\}$ is bounded, $\exists$ a subsequence, which we shall denote by $\{f_{1,k}\}$, such that $\{f_{1,k}(x_1)\}$ converges as $k \longrightarrow \infty$.

Now, consider sequences $S_1, S_2, S_3, \cdots$, which we represent by the array

$$\begin{array}{llllll} S_1: & f_{1,1} & f_{1,2} & f_{1,3} & f_{1,4} & \cdots \\ S_2: & f_{2,1} & f_{2,2} & f_{2,3} & f_{2,4} & \cdots \\ S_3: & f_{3,1} & f_{3,2} & f_{3,3} & f_{3,4} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

and which have the following properties:
(1) S_n is a subsequence of S_{n-1}, for $n = 2, 3, 4, \cdots$
(2) $\{f_{n,k}(x_n)\}$ converges, as $k \longrightarrow \infty$
(3) Order in which the functions appear is the same in each sequence; i.e., if one function precedes another in S_1, they are in same relation in every S_n, until one or the other is deleted. Hence, when going from one row in the above array to the next below, functions may move to the left but never to the right.

We now go down the diagonal of the array; i.e., we consider the sequence P: $f_{1,1}\ \ f_{2,2}\ \ f_{3,3}\ \ f_{4,4} \cdots$
By (3), the sequence P(except possibly its first n-1 terms) is a subsequence of S_n, for $n = 1, 2, 3, \cdots$. Hence, (2) implies that $\{f_{n,n}(x_i)\}$ converges, as $n \longrightarrow \infty$, for every $x_i \in A$.

For convenience, put $f_{n_i} = g_i$. We shall prove that $\{g_i\}$ converges uniformly on X.

Let $\epsilon > 0$ be given. As S is equicontinuous, choose $\delta > 0$ such that $d(x, y) < \delta$ implies $|f_n(x) - f_n(y)| < \epsilon \quad \forall n$. Since A is dense in X and X is compact, $\exists$ finitely many points $x_1, \cdots, x_m$ in A such that $X = \cup\{B(x_i, \delta) \mid i = 1, \cdots, m\}$.

Since $\{g_i(x)\}$ converges for every $x \in A$, $\exists\, n_0 \in \mathbb{N}$ such that $|g_i(x_s) - g_j(x_s)| < \epsilon$ whenever $i \geqslant n_0$, $j \geqslant n_0$, $1 \leqslant s \leqslant m$.

Also, if $x \in X$ then clearly, $x \in B(x_s, \delta)$ for some s, so that $|g_i(x) - g_i(x_s)| < \epsilon$ for every i. If $i \geqslant n_0$ and $j \geqslant n_0$ then it follows that

$$\begin{aligned}|g_i(x) - g_j(x)| &= |g_i(x) - g_i(x_s) + g_i(x_s) - g_j(x_s) + g_j(x_s) - g_j(x)| \\ &\leqslant |g_i(x) - g_i(x_s)| + |g_i(x_s) - g_j(x_s)| + |g_j(x_s) - g_j(x)| \\ &< \epsilon + \epsilon + \epsilon\end{aligned}$$

Thus, the subsequence $\{g_i\}$ of $\{f_n\}$ converges uniformly on X. □

4.6 LET US SUM UP

1. A *Banach space* is a normed space in which every Cauchy sequence is convergent. In other words, a Banach space is a complete normed space.

2. Every Banach space is a normed space but a normed space need not be a Banach space.

3. The normed space $C[0,1] = \{f : [0,1] \longrightarrow \mathbb{R} |\ f$ is a continuous function$\}$ over $\mathbb{R}$ is not a Banach space under norm $\|f\| = \displaystyle\int_0^1 |f(t)|dt$.

4. Prove that the vector space $C[a,b] = \{f : [a,b] \longrightarrow \mathbb{R}(or\ \mathbb{C}) |\ f$ is a continuous function$\}$ over $\mathbb{R}(or\ \mathbb{C})$ is not a Banach space under norm $\|f\| = \left(\displaystyle\int_a^b |f(t)|^p\ dt\right)^{1/p}$ where $1 \leqslant p < \infty$

5. The vector space $C(X) = \{f : X \longrightarrow \mathbb{R}(or\ \mathbb{C}) |\ f$ is bounded continuous function on $X\}$ over $\mathbb{R}(or\ \mathbb{C})$ is a Banach space under norm $\|f\| = \sup\limits_{x\in X}\{|f(x)|\}$.

6. The vector space $\mathbb{R} = \{x | x \in \mathbb{R}\}$ over $\mathbb{R}$ is a Banach space under the norm $\|x\| = |x|$=absolute value of $x \in \mathbb{R}$.

7. The vector space $\mathbb{C}^n = \{(x_1, \cdots, x_n) \mid x_i \in \mathbb{C}\}$ over $\mathbb{C}$ is a Banach space under the norms

$$\|x\|_p = \begin{cases} \left(\displaystyle\sum_{i=1}^n |x_i|^p\right)^{1/p} & \text{if} \quad 1 \leqslant p < \infty \\ max\{|\ x_1\ |, \cdots, |\ x_n\ |\} & \text{if} \quad p = \infty \end{cases}$$

8. The sequence space $l^p = \left\{ \{x_1, \cdots, x_n, \cdots\} \;\middle|\; \sum_{i=1}^{\infty} \mid x_i \mid^p < \infty \text{ and } x_i \in \mathbb{C} \right\}$ over $\mathbb{C}$ is a Banach space under the norms

$$\|x\|_p = \begin{cases} \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p} & \text{if} \quad 1 \leqslant p < \infty \\ sup\{\mid x_1 \mid, \cdots, \mid x_n \mid, \cdots\} & \text{if} \quad p = \infty \end{cases}$$

9. A sequence $\{x_k\}$ in a normed space $(V, \|\ \|)$ is said to be *summable* to the sum s if the sequence $\{s_n\}$ of the partial sums of the series $\sum_{k=1}^{\infty} x_k$ converges to $s \in V$. i.e. $\|s_n - s\| \longrightarrow 0$ as $n \longrightarrow \infty$. In this case, we write, $s = \sum_{k=1}^{\infty} x_k$.

10. The sequence $\{x_k\}$ in a normed space $(V, \|\ \|)$ is said to be *absolutely summable* if $\sum_{k=1}^{\infty} \|x_k\| < \infty$.

11. A normed space $(V, \|\ \|)$ is a Banach space if and only if every absolutely summable sequence in V is summable.

12. The vector space $L^p(E) = \{f : E \longrightarrow \mathbb{R} | f$ is Lebesgue measurable function on E and $\mid f \mid^p$ is Lebesgue integrable over $E\}$ over $\mathbb{R}$ is a Banach space under norms

$$\|f\|_p = \begin{cases} \left(\int_E |f|^p \right)^{1/p} & \text{if} \quad 1 \leqslant p < \infty \\ ess\ sup|f| & \text{if} \quad p = \infty \end{cases}$$

where $ess\ sup_E|f(x)| = inf\{m > 0 \mid |f(x)| \leqslant m$ a.e. on $E\}$ and $L^\infty(E) =$ the class of all those measurable functions f defined on E which are essentially bounded on E with $ess\ sup|f| < \infty$.

13. **(Riesz-Fischer theorem)**: L^p spaces are Banach spaces where $1 \leqslant p \leqslant \infty$.

14. Let M be a closed subspace of a Banach space $(V, \|\ \|)$. For each coset $x + M$ in quotient space V/M, define $\|x + M\| = inf\{\|x + m\| \mid m \in M\}$. Then V/M is a Banach space under the norm $\|x + M\|$.

15. Let M be a closed subspace of a normed space $(V, \|.\|)$. If M and V/M are Banach spaces then V is a Banach space.

16. Every complete subspace of a normed space is closed.

17. Two norms $\| \ \|_1$ and $\| \ \|_2$ on a normed space X are said to be *equivalent* and written as $\| \ \|_1 \sim \| \ \|_2$, if $\exists$ positive real numbers a and b (independent of $x \in X$) such that
$$a \|x\|_1 \leqslant \|x\|_2 \leqslant b \|x\|_1 \quad \forall\, x \in X.$$

18. On a finite dimensional normed space X, any two norms are equivalent.

19. If V is a finite dimensional normed space, then V is complete.

20. **(Riesz Lemma)**: Let M be a closed proper subspace of a normed space V and let $a \in \mathbb{R}$ be such that $0 < a < 1$. Then $\exists$ a vector $x_a \in V$ such that $\|x_a\| = 1$ and $\|x - x_a\| \geqslant a \quad \forall x \in M$.

21. In a normed space $(V, \|.\|)$, if the set $S = \{x \in V | \ \|x\| = 1\}$ is compact then V is finite dimensional.

22. Let S be subset of $C(X)$. We say that S is *pointwise bounded* on X if S is bounded at each $x \in X$, that is, if $\exists$ a finite- valued function ϕ defined on X such that $|f(x)| < \phi(x)\ \forall\ f \in S$.

23. Let S be subset of $C(X)$. We say that S is *uniformly bounded* on X if $\exists$ a number M such that $|f(x)| < M\ \forall\ f \in S$ and $\forall\ x \in X$.

24. A subset S of $C(X)$ is said to be *equicontinuous* at $x \in X$ if for every $\epsilon > 0$, $\exists\ \delta > 0$ such that for every $y \in X$ with $d(x, y) < \delta$ and $\forall f \in S$, we have, $|f(x) - f(y)| < \epsilon$, where δ may depend on x, but not on $f \in S$. Here d denotes the metric of X.

25. Let S be subset of $C(X)$ where X is a compact metric space. Suppose that S is pointwise bounded on X and is equicontinuous at every $x \in X$. Then

    (a) **(Ascoli, 1883)** S is uniformly bounded on X. In fact, S is totally bounded in the sup metric on $C(X)$.

    (b) **(Arzela, 1889)** Every sequence in S contains a uniformly convergent subsequence.

4.7 Chapter End Exercise

1. Show that the vector space $\mathbb{C} = \{z | z \in \mathbb{C}\}$ over $\mathbb{C}$ is a Banach space under the norm $\|z\| = |z|$=absolute value of $z \in \mathbb{C}$.

2. Show that the vector space $\mathbb{R}^n = \{(x_1, \cdots, x_n) \mid x_i \in \mathbb{R}\}$ over $\mathbb{R}$ is a Banach space under the (Euclidean) norm $\|x\| = \left(\sum\limits_{i=1}^{n} |x_i|^2 \right)^{1/2}$.

3. Show that the vector space $\mathbb{R}^n = \{(x_1, \cdots, x_n) \mid x_i \in \mathbb{R}\}$ over $\mathbb{R}$ is a Banach space under the norm $\|x\| = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$ where $1 \leqslant p < \infty$.

4. Show that the vector space $\mathbb{R}^n = \{(x_1, \cdots, x_n) \mid x_i \in \mathbb{R}\}$ over $\mathbb{R}$ is a Banach space under the norm $\|x\| = max\{| x_1 |, \cdots, | x_n |\}$.

5. Show that the normed space $C([a, b]) = \{f : [a, b] \longrightarrow \mathbb{R} | f$ is continuous function$\}$ over $\mathbb{R}$ under the norm $\|f\| = \left(\int_a^b |f|^2\right)^{1/2}$ is not a Banach space.

6. Let $(X, \| \|_X)$ and $(Y, \| \|_Y)$ be normed spaces. Then prove that $X \times Y$ is a Banach space under the norm $\|(x, y)\| = \|x\|_X + \|y\|_Y$.

7. Let V be a non-zero normed space and let $S = \{x \in V | \|x\| \leqslant 1\}$. Prove that V is complete if and only if S is complete.

8. Prove that every finite dimensional normed space is Banach and hence deduce each finite dimensional subspace of a normed space is closed.

9. Give a counter example to show that any two norms on an infinite dimensional normed space are not equivalent.

10. Show that the p-norms on $\mathbb{R}^n$ are equivalent where $1 \leqslant p \leqslant \infty$.

11. Let M be a closed proper subspace of the normed space V. Then prove that for every real number $a > 0$, $\exists$ an element $y \in V$ with $\|y\| = 1$ such that $\|x - y\| > 1 - a \ \forall \ x \in M$.

12. Let M be a closed proper subspace of a normed space V. Then prove that for each $a \in (0, 1)$, $\exists$ a point x_a in V but not in M (not necessarily unique) such that $\|x_a\| = 1$ and $\text{dist}(x_a, M) = \inf_{y \in M} \|x_a - y\| \geqslant a$.

13. Let V be a finite dimensional normed space and $r > 0$. Then prove that the closed ball $B[0; r] = \{x \in V | \|x\| \leqslant r\}$ is compact.

14. Let V be a normed space such that the closed ball $B[x_0; r] = \{x \in V | \|x - x_0\| \leqslant r\}$ is compact for some $x_0 \in V$ and $r > 0$. Then prove that V is finite dimensional.

15. Prove that in a finite dimensional normed space V, any proper subset M of V is compact if and only if M is closed and bounded.

16. Let V be a normed space. Prove that the closed unit ball in V is compact if and only if V is finite dimensional.

17. Prove that a normed space V is finite dimensional if and only if every bounded closed subset in V is compact.

18. If M is a compact subset of a Banach space V then show that $\overline{M}$ is also compact.

19. If M is any finite dimensional subspace of a normed space V, then M is closed.

20. Let $f_n \in C(X)$ for $n = 1, 2, 3, \cdots$ where X is a compact metric space. If $\{f_n\}$ converges uniformly on X then prove that $\{f_n\}$ is equicontinuous on X.

21. If $\{f_n\}$ is a pointwise bounded sequence of complex functions on a countable set E, then prove that $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ such that $\{f_{n_k}(x)\}$ converges for every $x \in E$.

22. Let X be a compact metric space. Prove that a closed subspace of $C(X)$ is compact if and only if it is uniformly bounded and equicontinuous.

23. Show that the family $\{sin\ nx\}_{n=1}^{\infty}$ is not an equicontinuous subset of $C[0, \pi]$.

Chapter 5

BOUNDED LINEAR TRANSFORMATIONS AND DUAL SPACES

Unit Structure :

5.1 Introduction

The bounded linear transformations on normed linear spaces are important operators , that satisfy many properties as a function between two metric spaces like continuity and their collections $B(X, Y)$ can be made into a normed linear space under pointwise addition and scalar multiplication.Completeness of the normed space $B(X, Y)$ is inherited via the completeness of the space Y.The Dual space of X is a complete metric space even if X is complete or not and hence it satisfies the properties of being a complete space.The significance of dual spaces of $l^p, L^p, \mathbb{R}^n$ is that it is useful to know the general form of bounded linear functionals on spaces of practical importance.For Hilbert spaces, Riesz's theorem elucidates the form of such bounded linear functionals in simple manner.The separable spaces are somewhat simpler than the non separable spaces.The separability of the dual space $X^{'}$ implies that the space X is separable.

5.2 Objectives

After going through this Chapter, you will be able to
• Define a bounded linear transformations between two normed linear spaces.
• Characterise the bounded linear transformations as continuous functions
• Identify the algebraic structure of the bounded linear transformations as normed linear space
• Define the dual space of X and describe its properties like completeness and seperability

5.3 Definitions, notations, theorems

Definition 1: A metric space is a pair (X, d), where X is a set and d is a metric on X, that is d is a distance function defined on $X \times X$ such that for all $x, y, z \epsilon X$ we have
(1) d is real valued , finite and nonnegative.
(2) $d(x, y) = 0$ if and only if $x = y$
(3) $d(x, y) = d(y, x)$
(4) $d(x, y) \leq d(x, z) + d(z, y)$
Example 1: Euclidean space $\mathbb{R}^n$ with metric d defined as
$d(x, y) = \sqrt{(x_1 - y_1)^2 + ... + (x_n - y_n)^2}$
Example 2: Sequence space l^∞ with metric d defined as
$d(x, y) = sup\{|x_i - y_i| : i \geq 1\}$
Example 3: l^p space with metric d defined as $d(x,y)$=$(\sum_{i=1}^{\infty} |x_i - y_i|^p)^{\frac{1}{p}}$

Definition 2: Let X, Y be normed linear spaces and and T be a linear transformation of X into Y then T is said to be bounded linear transformation, if there exists a real number $K \geq 0$ such that
$||T(x)|| \leq K||x||$ for every xϵ X. The set of all continuous or bounded linear transformations from X into Y is denoted by B(X,Y)

Example 1: The identity operator $I : X \longrightarrow X$ defined by $I(x) = x$ on a nonzero normed space X is bounded linear operator with $||I(x)|| = ||x|| \leq K||x||$ with $K = 1$
Example 2: The 0 operator $0 : X \longrightarrow Y$ defined by $0(x) = 0_Y$ is bounded linear operator with $||0(x)|| = ||0_Y|| = 0 \leq 0||x||$ with $K = 0$

Now we shall see equivalent characterisations of a bounded linear transformation T existing between two normed spaces.

Theorem 5.3.1. *Let X, Y be normed linear spaces and T is a linear transformation of X into Y . Then the following statements are equivalent*
(1)T is continuous
(2)T is continuous at the origin, in other words $T(x_n) \longrightarrow 0$ as $x_n \longrightarrow 0$
(3)There exists a real number $K \geq 0$ such that $||T(x)|| \leq K||x||$ for every
$x\epsilon$ X
(4) T carries the closed unit sphere in X to a bounded set in Y

Proof: (1) $\Rightarrow$ (2)
Suppose that T is continuous . Since we have T(0)=0 , therefore T is continuous at the point x=0. If T is continuous at the origin then $x_n \longrightarrow x$ if and only if $x_n - x \longrightarrow 0$. This implies that $T(x_n - x) \longrightarrow 0$.This is true if and only if $T(x_n) \longrightarrow$T(x). So T is continuous.
(2) $\Rightarrow$ (3)
Assume that for each positive integer n , we can find a point x_n in X such that $||T(x_n)|| \geq n||x_n||$. This implies that $||T(x_n/nx_n)|| > 1$. Put $y_n = x_n/n||x_n||$. Then observe that $y_n \rightarrow 0$ as $n \rightarrow \infty$ but $T(y_n) \nrightarrow 0$. Therefore, T is not continuous at the origin.This is a contradiction to our hypothesis. Therefore There exists a real number $K \geq 0$ such that $||T(x)|| \leq K||x||$ for every
x ϵ X
(3) $\Rightarrow$ (4)
Let x be any point belonging to the closed unit sphere in X. This implies that $||x|| \leq 1$.Hence,by hypothesis $||T(x)|| \leq$ K$||x|| \leq$ K. Therefore, by definition, T carries the closed unit sphere in X to a bounded set in Y.
(4) $\Rightarrow$ (1)
We first show that (4) $\Rightarrow$ (3). If $x = 0$ then $T(x)$=0, beacuse T is Linear transformation. Hence $||T(x)|| \leq$K$||x||$. If $x \neq 0$ then $x/||x|| = 1$. Put $y = x/||x||$, then y belongs to the closed unit sphere and therefore by hypothesis, $||T(y)|| \leq$K for some real number K$\geq$0. This implies that $||T(x/||x||) \leq$K. Thus, $||T(x)|| \leq$K$||x||$. Therefore, $||T(x)|| \leq$K$||x||$for every x ϵX. Now, if x_n is any convergent sequence in X such that $x_n \longrightarrow x$, then $||T(x_n - x)|| \leq$K$||x_n - x||$ implies that $T(x_n) - T(x) \longrightarrow 0$ as $x_n - x \longrightarrow 0$. Hence T is continuous.
Note:(1) T is continuous iff T is bounded
(2) If T is continuous then T carries the closed unit sphere to a bounded set in Y , in this case, we denote the norm of T by $||T||$ and it is defined as
$||T||$= sup { $||T(x)||$: $||x|| \leq$1 } .
We also have $||T||$= inf {K : K$\geq$0 and $||T(x)|| \leq$K$||x||$. From this, we conclude that $||T(x)|| \leq ||T||\ ||x||$ for all x

Now we shall see that $B(X, Y)$ forms a normed linear space and it is complete , when Y is complete space.

Theorem 5.3.2. *If X, Y are normed linear spaces, then B(X,Y) is a normed linear space with respect to pointwise addition and scalar multiplication and the norm defined as* $||T||=$ *sup* $\{ \ ||T(x)|| : ||x|| \leq 1 \}$*.Further, if Y is a Banach space then B(X,Y) is also a Banach space.*

Proof: To show that B(X,Y) is a normed linear space, let T, U be any two linear transformations belonging to B(X,Y), then T+U is defined as $(T+U)(x)$:=$T(x)+U(x)$ and for any scalar α in F , we have $(\alpha\ T)(x)$:=$\alpha\ T(x)$.Therefore we have (U+T)(x_n)=U(x_n)+T(x_n) and $(\alpha\ T)(x_n)$:=$\alpha\ T(x_n)$.
Therefore $(T+U)(x_n) \longrightarrow 0$ and $(\alpha\ T)(x_n) \longrightarrow 0$ as $x_n \longrightarrow 0$
Hence, T+U and αT both are continuous at the origin. This implies that T+Uϵ B(X,Y) and αTϵ B(X,Y)
We have (T+U)(x)=T(x)+U(x)=U(x)+T(x)=(U+T)(x), therefore vector addition is commutative.
For any S,T,Uϵ B(X,Y), we have [(S+T)+U](x)=(S+T)(x)+U(x)= S(x)+T(x)+U(x)=[S+(T+U)](x) for every $x\ \epsilon$ X.Therefore, (S+T)+U=S+(T+U)
There exist 0 linear transformation in B(X,Y) defined as 0(x)=0_Y for every $x\ \epsilon$ X.
For every TϵB(X,Y), there exist an additive inverse -T ϵB(X,Y) such that T+(-T)=0.
we have (-T)(x)=-T(x) for every $x\ \epsilon$X.
Scalar multiplication is associative and distributive.For all α, $\beta\ \epsilon$ F and T, U ϵ B(X,Y), [α(βU)](x)=α(βU(x))=$\alpha\ \beta$U(x)= ($\alpha\ \beta$)U(x).This implies that α(βU)=($\alpha\ \beta$)U. [(α+β)U](x)=(α+β)U(x)=αU(x)+βU(x)=(αU+βU)(x).
Hence, (α+β)U=αU+βU. Now [α(T+U)](x)=α(T+U)(x)=αT(x)+αU(x)= (αT+αU)(x). Therefore, we have α(T+U)=αT+αU.
Further, we have (1.U)(x)=1.U(x)=U(x).Hence 1.U=U.
Now we shall show that for Tϵ B(X,Y), $||T||$= sup $\{ \ ||T(x)|| : ||x|| \leq 1 \}$ is norm on the linear space B(X,Y).
(a) For every Tϵ B(X,Y), we have sup $\{ \ ||T(x)|| : ||x|| \leq 1 \ \} \geq 0$, therefore $||T|| \geq 0$
(b)$||T||$=0 if and only if sup $\{ \ ||T(x)|| : ||x|| \leq 1\}$=0. This is possible iff $||T$(x)$||$=0. This is true iff T(x)=0 and therefore T=0.
(c)$||\ \alpha$T$||$=sup { $||(\alpha$T)(x)$||$: $||x|| \leq 1$}=$|\ \alpha\ |$sup $\{ \ ||T(x)|| : ||x|| \leq 1\}$=$|\ \alpha\ |\ ||T||$ for every $\alpha\ \epsilon$F and For every Tϵ B(X,Y)
(d)$||T+U||$= sup { $||$(T+U)(x)$||$: $||x|| \leq 1$}=sup { $||$T(x)+U(x)$||$: $||x|| \leq 1$}
$\leq$sup { $||$T(x)$||$ +$||$U(x)$||$: $||x|| \leq 1$}$\leq$ sup { $||$T(x)$||$:$||x|| \leq 1$}+sup { $||$U(x)$||$:$||x|| \leq 1$}=$||T||$+$||U||$

Therefore, $||T+U|| \leq ||T||+||U||$ for every T, U ϵ B(X,Y)
Next, we show that B(X,Y) is complete, when Y is complete.
Let T_n be any Cauchy sequence of linear transformations in B(X,Y). For any vector x ϵ X , we have $||T_n(x)$-$T_m(x)||$=|| $(T_n - T_m)(x)|| \leq ||$ T_n-T_m || —— x ||.This implies that $\{T_n(\ x)\}$ is a Cauchy sequence in Y.Since Y is Complete therefore, there exist a vector T(x) in Y such that $T_n(\ x)\longrightarrow$T(x) .This defines a function T: X$\longrightarrow$Y by x $\longrightarrow$T(x).This function is a linear transformation from X into Y , for if x_1,x_2 ϵ X, we have $T_n(x_1 + x_2)$=$T_n(x_1)$+$T_n(x_2)$ and $T_n(\alpha\ x)$=α $T_n(x)$.Hence T(x_1+x_2)=T(x_1)+T(x_2) and T($\alpha\, x$)=αT(x).Now we show that T is continuous and T_n $\longrightarrow$T as $n \longrightarrow \infty$
It is enough to show that T is bounded linear transformation.
Consider $||T$(x)||=|| $lim_{n\to\infty}$ $T_n(x)$||=$lim_{n\to\infty}$ $||T_n$(x)|| $\leq$ sup($||T_n||$ ||x||)= (sup$||T_n||$) ||x||. Since the norms of the terms of the Cauchy Sequence in a normed linear space is a bounded set, therefore there exists K=sup$||T_n|| \geq$0 such that
$||T$(x)|| $\leq$K ||x||.Hence, T is a bounded linear transformation.Now we show that $||T_n - T|| \longrightarrow$0.Let ϵ >0 be any number , let N be a positive integer such that $n, m \geq$N $\Rightarrow$ $||T_n - T_m|| < \epsilon$. Now, if $||x|| \leq$1 and m, n $\geq$N, then
|| $T_n(x)$-$T_m(x)$||=||(T_n-T_m)(x)|| $\leq$ $||T_n - T_m||$ $||x|| \leq ||T_n - T_m|| < \epsilon$
Now hold n fixed and letting $m \longrightarrow \infty$, we obtain
|| $T_n(x)$-$T_m(x)$|| $\longrightarrow$ || $T_n(x)$-T(x)||.This implies that || $T_n(x)$-T(x)|| < ϵ for all $n \geq$N and every x such that $||x|| \leq$1.Hence, $||T_n - T|| < \epsilon$ for every $n \geq$N.Therefore, we have $||T_n - T|| \longrightarrow$0 as $n \longrightarrow \infty$.
Definition 3: If X is any normed linear space then then the set of all continuous linear transformations from X into $\mathbb{R}$ or $\mathbb{C}$ is denoted by B(X,$\mathbb{R}$) or B(X,$\mathbb{C}$), according to X is real or complex vector space. Denote it by X', it is called as the dual space of X.The elements of X' are called as continuous linear functionals.
Note: (1) functional defined on normed linear space X is a scalar- valued continuous linear functional defined on X.
(2) X' is a normed linear space with norm defined by
$||f||$:=sup$\{\frac{|f(x)|}{||x||}$: x ϵX,$x \neq$0 $\}$=sup$\{$ $|f(x)|$:$||x|| = 1$ $\}$
(3) Since $\mathbb{R}$ and $\mathbb{C}$ are complete normed linear spaces therefore X' is a Banach space.
Definition 4: A bijective linear operator from a normed space X onto the normed space Y is called as an isomorphism if it preserves the norm, that is, for every x ϵX, $||T(x)||$=$||x||$. In this case, X is said to be isomorphic to Y and X,Y are called isomorphic normed spaces.

Theorem 5.3.3. *The dual space of* $\mathbb{R}^n$ *is* $\mathbb{R}^n$

Proof: $\mathbb{R}^n$ is a normed linear space with the norm defined as follows: For every x=(x_1,x_2,...,x_n)ϵ $\mathbb{R}^n$, $||x||$=$\sqrt{x_1^2 + x_2^2 + ... + x_n^2}$. We recall the theorem , which states that if the dimension of a normed linear space

is finite, then every linear operator on X is bounded. Therefore, we have $\mathbb{R}^{n'}$=$\mathbb{R}^{n*}$.Given any $f \epsilon \quad \mathbb{R}^n$, if $\{e_1, e_2, ..., e_n\}$ is a standard basis for $\mathbb{R}^n$, then $f(x) = f(x_1e_1 + x_2e_2 + ... + x_ne_n)$=$\sum_{i=1}^{n} x_if(e_i)$=$\sum_{i=1}^{n} x_i\gamma_i$, where γ_i=$f(e_i)$ for every i.Therefore, by the Cauchy-Schwarz inequality we have $|f(x)| \leq \sum_{i=1}^{n} |x_i\gamma_i| \leq (\sum_{i=1}^{n} x_i^2)^{1/2}\ (\sum_{i=1}^{n} \gamma_i^2)^{1/2}$=$||x||\ (\sum_{i=1}^{n} \gamma_i^2)^{1/2}$ Hence $\frac{|f|}{||x||} \leq (\sum_{i=1}^{n} \gamma_i^2)^{1/2}$.Now taking supremum over all x of norm 1, we get, $||f|| \leq (\sum_{i=1}^{n} \gamma_i^2)^{1/2}$, because of the equality obtained for x=$(\gamma_1, \gamma_2, ..., \gamma_n)$ in the above inequality, we must have $||f||$=$(\sum_{i=1}^{n} \gamma_i^2)^{1/2}$.This implies that the norm of f is the Euclidean norm on $\mathbb{R}^n$
Hence the mapping ϕ: $\mathbb{R}^{n'} \longrightarrow \mathbb{R}^n$, defined as $\phi(f)$=$(\gamma_1,\gamma_2,...,\gamma_n)$ is norm preserving bijective linear map, hence it is an isomorphism.

Here, we shall try to identify dual spaces of some of the normed linear spaces.

Theorem 5.3.4. *The dual space of* l^1 *is* l^∞.

Proof: Consider a Schauder basis for l^1, namely (e_k), where e_k=(δ_{kj}), where δ_{kj}=1 if $j = k$ and δ_{kj}=0, if $j \neq k$.Then every $x \ \epsilon \ l^1$ can be uniquely
represented as x=$\sum_{k=1}^{\infty} x_k \ e_k$. Let $f \ \epsilon \quad l^{1'}$ be any linear functional. Since f is linear and bounded , therefore, we have $f(x)$=$\sum_{k=1}^{\infty} x_k \ \gamma_k$, where $\gamma_k = f(e_k)$.Here the γ_k=$f(e_k)$ are uniquely determined by f. Also, $||e_k||$=1 and $|\gamma_k|$=$|f(e_k) \leq ||f|| \quad ||e_k||$=$||f||$.Taking supremum on both th sides, we get, $\sup\{|\gamma_k| : k \geq 1\} \leq ||f||$...(1)
Hence, $(\gamma_k) \ \epsilon \ l^\infty$. Further, if d=$(\delta_k) \ \epsilon \ l^\infty$ then define g on l^1 as follows: $g(x)$=$\sum_{k=1}^{\infty} x_k\delta_k$, where x=$(x_k) \ \epsilon \quad l^1$. Observe that g is linear as well as bounded map, because we have
$|g(x)| \leq \sum_{k=1}^{\infty} |x_k||\delta_k| \leq sup\{|\delta_k| : k \geq 1\} \ \sum_{k=1}^{\infty} |x_k|$=$||x|| \ sup\{|\delta_k| : k \geq 1\}$.Hence $g \ \epsilon \quad l^1$.Finally, we prove that the norm of f is the norm on the space l^∞.Since we have, $f(x)$=$\sum_{k=1}^{\infty} x_k \ \gamma_k$, where $\gamma_k = f(e_k)$, therefore,
$|f(x)|$=$|\sum_{k=1}^{\infty} x_k \ \gamma_k| \leq sup\{|\gamma_k| : k \geq 1\} \ \sum_{k=1}^{\infty} |x_k|$=$||x|| \ sup\{|\gamma_k| : k \geq 1\}$.Thus, we get
$\frac{|f(x)|}{||x||} \leq sup\{|\gamma_k| : k \geq 1\}$.Taking supremum over all x of norm 1, we get, $||f|| \leq sup\{|\gamma_k| : k \geq 1\}$...(2). From (1) and (2), we conclude that $||f||$=$sup\{|\gamma_k| : k \geq 1\}$, which is the norm on l^∞.This shows that the bijective linear mapping of $l^{1'}$ onto l^∞ defined by $f \longrightarrow (\gamma_k)$ is an isomorphism.

Theorem 5.3.5. *The dual space of* l^p *is* l^q *for* $1 < p < \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$

Proof: A Schauder basis for l^p is (e_k), where $e_k = (\delta_{kj})$. Then every $x \ \epsilon \quad l^p$ has a unique representation x=$\sum_{k=1}^{\infty} x_ke_k$. Let $f \ \epsilon \ l^{p'}$, since f is linear and bounded, therefore $f(x)$=$\sum_{k=1}^{\infty} x_k\gamma_k$, where γ_k=$f(e_k)$...(*)
Let q be the conjugate of p and define y_n=$(\beta_k^{(n)})$ with $\beta_k^{(n)}$=$\frac{|\gamma_k|^q}{\gamma_k}$, if k

$\leq n$ and $\gamma_k \neq 0$ and $\beta_k^{(n)}=0$, if $k > n$ or $\gamma_k = 0$. By substituting this in (*) we get,
$f(y_n)=\sum_{k=1}^{\infty} \beta_k^{(n)} \gamma_k=\sum_{k=1}^{n} |\gamma_k|^q$
Using the definition of β_k^n and $(q-1)p = q$, we obtain,
$f(y_n) \leq ||f||\ ||y_n||=||f||\ (\sum_{k=1}^{\infty} |\beta_k^{(n)}|^p)^{1/p}=||f||\ (\sum_{k=1}^{\infty} |\gamma_k|^{(q-1)p})^{1/p}=$ $||f||\ (\sum_{k=1}^{n} |\gamma_k|^q)^{1/p}$
Hence, $f(y_n)=\sum_{k=1}^{n} |\gamma_k|^q \leq ||f||\ (\sum_{k=1}^{n} |\gamma_k|^q)^{1/p}$.Therefore, $(\sum_{k=1}^{n} |\gamma_k|^q)^{1-1/p}=(\sum_{k=1}^{n} |\gamma_k|^q)^{1/q} \leq ||f||$.Since n was arbitrary, therefore letting $n \longrightarrow \infty$, we obtain $(\sum_{k=1}^{\infty} |\gamma_k|^q)^{1/q} \leq ||f||$...(**)
Hence $(\gamma_k)\ \epsilon\ \ l^q$.
Conversely, for any $(\eta_k)\ \epsilon\ \ l^q$, we can define corresponding bounded linear functional g on l^p as follows:
$g(x)=g(\psi_k)=\sum_{k=1}^{\infty} \psi_k \eta_k$, where $x=(\psi_k)\ \epsilon\ \ l^p$. Then by the Hölder inequality, we have
$|g(x)|=|\sum_{k=1}^{\infty} \psi_k \eta_k| \leq (\sum_{k=1}^{\infty} |\psi_k|^p)^{1/p}\ (\sum_{k=1}^{\infty} |\eta_k|^q)^{1/q}$.Thus g is linear and bounded. Hence, $g\ \epsilon\ \ l^{p'}$.Now, we prove that the norm of f is the norm on the space l^q. therefore, $|f(x)|=|\sum_{k=1}^{\infty} x_k \gamma_k| \leq (\sum_{k=1}^{\infty} |x_k|^p)^{1/p}$ $(\sum_{k=1}^{\infty} |\gamma_k|^q)^{1/q}=||x||\ (\sum_{k=1}^{\infty} |\gamma_k|^q)^{1/q}$.
Therefore, we have $\frac{|f(x)|}{||x||} \leq (\sum_{k=1}^{\infty} |\gamma_k|^q)^{1/q}$.Now taking supremum over all x of norm 1, we obtain , $||f|| \leq (\sum_{k=1}^{\infty} |\gamma_k|^q)^{1/q}$. From (**), we have $||f||=(\sum_{k=1}^{\infty} |\gamma_k|^q)^{1/q}$
Therefore, the mapping of $l^{p'}$ onto l^q defined by $f \longrightarrow (\gamma_k)$ is linear, bijective and norm preserving. Hence it is an isomorphism.

Its practically important to know the general form of bounded linear fucntionals on various spaces.For general Banach spaces, such formulas and their derivation can sometimes be complicated. But, for Hilbert space the situation is simple as described by the following result:
Representation of functionals on Hilbert spaces:

Riesz's representation of bounded linear functionals on Hilbert spaces:

Theorem 5.3.6. *Every bounded linear functional f on a Hilbert space H can be represented in terms of an inner product as follows: $f(x)=< x, z >$, where z is uniquely determined by f and $||z|| = ||f||$.*

Proof: We prove the following claims
(a) f has representation as $f(x)=< x, z >$
(b) z is uniquely determined by f
(c) $||z||=||f||$

(a) If $f=0$ then $f(x)=0=< x, 0 >$ and $||z||=||f||=0$ for $z=0$. Therefore, assume that $f \neq 0$. In this case, $z \neq 0$ since, otherwise $f=0$. Now $< x, z >= 0$ for every x in the nullspace of f, denoted by $\mathcal{N}(f)$.So, con-

sider $\mathcal{N}(f)$ and its orthogonal complement $\mathcal{N}\ (f)^{\perp}$. Since f is bounded functional, therefore $\mathcal{N}(f)$ is a closed vector subspace of H.Further f $\neq 0$ implies that $\mathcal{N}(f)\neq$H. So that $\mathcal{N}\ (f)^{\perp}\neq 0$ by the following theorem: Let Y be any closed subspace of a Hilbert space H .Then H=Y$\oplus$ $Y^{\perp}$
Hence $\mathcal{N}\ (f)^{\perp}$ contains some $z_0 \neq 0$.Put $v = f(x)z_0 - f(z_0)x$; where, x ϵ H is arbitrary.Applying f, we obtain $f(v)$=$f(x)f(z_0)-f(z_0)f(x)$=0.
This shows that $v\ \epsilon\ \ \mathcal{N}\ (f)$
Since $z_0 \perp \mathcal{N}\ (f)$, we have 0=$< v, z_0 >=< f(x)z_0 - f(z_0)x, z_0 >$
=$f(x) < z_0, z_0 > -f(z_0) < x, z_0 >$=$f(x)||z_0||^2 - f(z_0) < x, z_0 >$
Since $||z_0||^2 \neq 0$, we can solve for $f(x)$.The solution is given by
$f(x)$=$\frac{f(z_0)}{<z_0,z_0>} < x, z_0 >$. Therefore

z=$\frac{\overline{f(z_0)}}{<z_0,z_0>}z_0$. Since x was arbitrary , therefore $f(x)$=$< x, z >$.
(b) We prove that z is uniquely determined in (a). Suppose that for all $x\ \epsilon$ H, $f(x)$=$< x, z_1 >=< x, z_2 >$. Then $< x, z_1 - z_2 >= 0$ for all x.Choose $x = z_1 - z_2$, we have $< x, z_1 - z_2 >=< z_1 - z_2, z_1 - z_2 >=||z_1 - z_2||^2$=0.
Hence $z_1 - z_2 = 0$, this implies that z_1=z_2
(c) Now we show that $||z||$=$||f||$. Assume that $f\neq 0$ then $z\neq 0$
Put $x = z$ in $f(x)$=$< x, z >$. This implies that $||z||^2$=$< z, z >$=$f(z)$ $\leq ||f||\quad ||z||$
Dividing by $||z||\neq 0$, we get $||z|| \leq ||f||$. It remains to show that $||f|| \leq ||z||$, from Schwarz inequality, we have $|f(x)|$=$| < x, z > | \leq ||x||||z||$
This implies that $||f||$=$sup\{| < x, z > | : ||x|| = 1\} \leq ||z||$.Hence $||f||$=$||z||$.

Definition 5: Let $(X, \mathcal{A}, \mu)$ be measure space and $1 \leq p < \infty$.The space $L^p(X)$ consists of equivalence classes of measurable functions $f : X \longrightarrow \mathbb{R}$ such that $\int |f|^p\ d\mu < \infty$, where two measurable functions are equivalent, if they are equal μ a.e.The L^p norm of $f\epsilon L^p(X)$ is defined by $||f||_{L^p} = (\int |f|^p d\mu)^{1/p}$

We say that $f_n \longrightarrow f$ in L^p if $||f - f_n||_{L^p} \longrightarrow 0$.For example, the characteristic function $\chi_{\mathbb{Q}}$ of the rationals on $\mathbb{R}$ is equivalent to 0 in $L^p(\mathbb{R})$
Example 1: If $\mathbb{N}$ is equipped with counting measure, then $L^p(\mathbb{N})$ consists of all sequences $\{x_n\epsilon\mathbb{R} : n\epsilon\mathbb{N}\}$ such that $\sum_{n=1}^{\infty} |x_n|^p < \infty$ with norm
$||(x_n)||_{L^p} = (\sum_{n=1}^{\infty} |x_n|^p)^{1/p}$

Definition 6: Let $(X, \mathcal{A}, \mu)$ be a measure space.The space $L^{\infty}(X)$ consists of pointwise a.e.- equivalence classes of essentially bounded measurable functions $f : X \longrightarrow \mathbb{R}$ with norm $||f||_{L^{\infty}}$=ess sup $|f|$, where for any measurable function $f : X \longrightarrow \mathbb{R}$, the essential supre-

mum of f on X is
ess sup$f$=inf$\{a\epsilon\mathbb{R} : \mu\{x\epsilon X : f(x) > a\} = 0\}$, equivalently
ess sup f=inf{sup g: $g = f$ pointwise a.e}

Now we shall establish the isomorphism between $L^q(X)$ and the dual space of $L^p(X)$ for $1 < p < \infty$.

Proposition 5.3.1. *Suppose that $(X, \mathcal{A}, \mu)$ be a measure space and $1 < p \leq \infty$. If $f \epsilon L^q(X)$, then $F(g)$=$\int fgd\mu$ defines a bounded linear functional $F : L^p(X) \longrightarrow \mathbb{R}$ with $||F||_{L^{p'}}$=$||f||_{L^q}$.If X is σ-finite then the same results hold for $p = 1$.*

Proof: Using Hölder's inequality, we have for $1 \leq p \leq \infty$ that $|F(g)| \leq ||f||_{L^q}\ ||g||_{L^p}$.This implies that F is a bounded linear functional on L^p with $||F||_{L^{p'}} \leq ||f||_{L^q}$.In proving the reverse inequality, we may assume that $f \neq 0$, otherwise the result is trivial.
First assume that $1 < p < \infty$.Let $g = (sgnf)(\frac{|f|}{||f||_{L^q}})^{q/p}$, then, $g \epsilon L^p$, since $f \epsilon L^q$, and $||g||_{L^p}$=1.Also, since $\frac{q}{p}$=$q-1$, $F(g)$=$\int(sgnf)f(\frac{|f|}{||f||_{L^q}})^{q-1}$ $d\mu$=$||f||_{L^q}$
Since $||g||_{L^p} = 1$, we have $||F||_{L^{p'}} \geq |F(g)|$ so that $||F||_{L^{p'}} \geq ||f||_{L^q}$.
If $p = \infty$, we get the same conclusion by taking g=$sgn(f)\epsilon L^\infty$.Therefore, in these cases the supremum defining $||F||_{L^{p'}}$ is actually attained for suitable function g.
Now, suppose that $p = 1$ and X is σ-finite. For $\epsilon > 0$, let $A = \{x\epsilon X : |f(x)| > ||f||_{L^\infty} - \epsilon\}$. Then $0 < \mu(A) \leq \infty$. Moreover, since X is σ-finite, there is an increasing sequence of sets A_n of finite measure whose union is A such that $\mu(A_n) \longrightarrow \mu(A)$, so we can find a subset B$\subset$A such that $0 < \mu(B) < \infty$.
Let g=$sgn(f)\ \frac{\chi(B)}{\mu(B)}$. Then $g\epsilon\ L^1(X)$ and $||g||_{L^1}$=1 and $F(g) = \frac{1}{\mu(B)} \int_B |f|d\mu \geq ||f||_{L^\infty} - \epsilon$. This implies that , $||F||_{L^{1'}} \geq ||f||_{L^\infty} - \epsilon$ and therefore we have $||F||_{L^{1'}} \geq ||f||_{L^\infty}$, this is because ϵ was arbitrary.
This result shows that the map F:$L^q(X) \longrightarrow Ł^p(X)'$ defined by $F(g)$=$\int fgd\mu$ is an isometry from L^q into $L^{p'}$
F is onto, when $1 < p < \infty$, so that every bounded linear functional on L^p arises in this way from an L^q function.

Theorem 5.3.7. *Let $(X, \mathcal{A}, \mu)$ be a measure space.If $1 < p < \infty$ then F:$L^q(X) \longrightarrow Ł^p(X)'$ defined by $F(g)$=$\int fgd\mu$ is an isometric isomorphism of $L^q(X)$ onto the dual space of $L^p(X)$.*

Proof: Suppose that X has a finite measure and let F:$L^p \longrightarrow \mathbb{R}$ be a bounded linear functional on $L^p(X)$. If $A\ \epsilon\mathcal{A}$, then $\chi_A\ \epsilon L^p(X)$. Since, X has finite measure, define $\nu : \mathcal{A} \longrightarrow \mathbb{R}$ by $\nu(A) = F(\chi_A)$. If A=$\bigcup_{i=1}^{\infty} A_i$ is a disjoint union of measurable sets , then χ_A=$\sum_{i=1}^{\infty} \chi_{A_i}$ and the dominated convergence theorem implies that

$||\chi_A - \sum_{i=1}^{\infty} \chi_{A_i}||_{L^p} \longrightarrow 0$ as $n \longrightarrow \infty$.Hence, since F is continuous linear functional on L^p,
$\nu(A) = F(\chi_A) = F(\sum_{i=1}^{\infty} \chi_{A_i})$=$\sum_{i=1}^{\infty} F(\chi_{A_i})$=$\sum_{i=1}^{\infty} \nu(A_i)$. This implies that ν is a signed measure on $(X, \mathcal{A})$.
If $\mu(A) = 0$, then χ_A is equivalent to 0 in L^p and therefore $\nu(A) = 0$, by the linearity of F.Thus ν is absolutely continuous with respect to μ.
By the Radon-Nikodym theorem,that is stated as follows:
Let ν be a $\sigma-$finite signed measure and μ be a $\sigma-$ finite measure on a measurable space $(X, \mathcal{A})$.Then there exists unique $\sigma-$ finite signed measures ν_a, ν_s such that $\nu = \nu_a + \nu_s$, where $\nu_a << \mu$ and $\nu_s \perp \mu$. Further, there exists a measurable function $f : X \longrightarrow \overline{\mathbb{R}}$ uniquely defined upto μ a.e. equivalence, such that $\nu_a(A) = \int_A f d\mu$ for every $A \epsilon \mathcal{A}$, where the integral is well defined as an extended real number.
The decomposition $\nu = \nu_a + \nu_s$ is called the Lebesgue decomposition of ν and the representation of an absolutely continuous signed measure $\nu << \mu$ as $d\nu = f d\mu$ is the Radon-Nikodym theorem.We call the function f here as the Radon-Nikodym derivative of ν with respect to μ and denote it by $f = \frac{d\nu}{d\mu}$.

Thus there is a function $f : X \longrightarrow \mathbb{R}$ such that $d\nu = f d\mu$ and $F(\chi_A) = \int f \chi_A d\mu$ for every $A \; \epsilon \; \mathcal{A}$.Hence, by the linearity and boundedness of F, $F(\phi) = \int f \phi d\mu$ for all simple functions ϕ, and $|\int f \phi d\mu|$ $\leq M||\phi||_{L^p}$, where $M = ||F||_{L^{p'}}$.
Taking $\phi = sgn f$, which is a simple function, we observe that $f \; \epsilon$ $L^1(X)$.We may then extend the integral of f against bounded functions by continuity. If $g \; \epsilon \; L^{\infty}(X)$, then from the following theorem:[Suppose that $(X, \mathcal{A}, \mu)$ is a measure space and $1 \leq p \leq \infty$. Then the simple functions that belong to $L^p(X)$ are dense in $L^p(X)$.], there exist a sequence of simple fucntions ϕ_n with $|\phi_n| \leq |g|$ such that $\phi_n \longrightarrow g$ in L^{∞}, and therefore, also in L^p. Since $|f\phi_n| \leq ||g||_{L^{\infty}}|f| \; \epsilon \; L^1(X)$, the dominated convergence theorem and the continuity of F implies that
$F(g) = \lim_{n\to\infty} F(\phi_n)$=$\lim_{n\to\infty} \int f\phi_n d\mu$=$\int fg d\mu$ and that $|\int fg d\mu| \leq$ $M||g||_{L^p}$ for every $g \; \epsilon \; L^{\infty}(X)$
Now we prove that $f \; \epsilon \; L^q(X)$. Let $\{\phi_n\}$ be a sequence of simple functions such that $\phi_n \longrightarrow f$ pointwise a.e. as $n \longrightarrow \infty$ and $|\phi_n| \leq |f|$.
Define g_n=$(sgn f)(\frac{|\phi_n|}{||\phi_n||_{L^q}})^{q/p}$.Then $g_n \; \epsilon L^{\infty}(X)$ and $||g_n||_{L^p} = 1$.Further, fg_n=$|fg_n|$
and $\int |\phi_n g_n| d\mu = ||\phi_n||_{L^q}$.Therefore, by Fatou's lemma and inequality $|\phi_n| \leq |f|$, we have, $||f||_{L^q} \leq \lim_{n\to\infty} inf ||\phi_n||_{L^q} \leq \lim_{n\to\infty} inf \int |\phi_n g_n| d\mu$ $\leq \lim_{n\to\infty} \int |fg_n| d\mu \leq M$.
Thus $f \; \epsilon \; L^q$.Since the simple functions are dense in L^p and g is a continuous functional on L^p, if $f \; \epsilon \; L^q$, it follows that $F(g) = \int fg d\mu$ for every $g \; \epsilon \; L^p(X)$.By the previous proposition, this implies that $||F||_{L^{p'}}$=$||f||_{L^q}$.
This proves the result, when X has a finite measure.
If X is $\sigma - finite$ then there is an increasing sequence $\{A_n\}$ of sets

with finite measure such that $\bigcup_{n=1}^{\infty} A_n = X$.By the previous result, there is a unique function

$f_n \,\epsilon\, L^q(A_n)$ such that $F(g) = \int_{A_n} f_n g d\mu$ for all $g \,\epsilon\, L^p(A_n)$. If $m \geq n$ then the functions f_m, f_n are equal pointwise a.e. on A_n and the dominated convergence theorem implies that $f = \lim_{n\to\infty} f_n \,\epsilon\, L^q(X)$ is the required function.

Finally , if X is not $\sigma - finite$ then for each $\sigma - finite$ subset $A \subset X$, let $f_A \epsilon L^q(A)$ be the function such that $F(g) = \int_A f_A g d\mu$ for every $g \epsilon L^p(A)$. Define $M = sup\{||f_A||_{L^q(A)} : A \subset X \;\; is \;\; \sigma - finite\} \leq ||F||_{L^p(X)'}$, and choose an increasing sequence of sets A_n such that $||f_{A_n}||_{L^q(A_n)} \longrightarrow M$ as $n \longrightarrow \infty$

.Define $B = \bigcup_{n=1}^{\infty} A_n$, verify that f_B is the required function.

5.4 Separable spaces

Definition 7: Let (X, d) be a metric space then X is said to be separable if there exist a countable subset A of X such that $\overline{A} = X$

Examples:

1] the real line $\mathbb{R}$ is separable because $\mathbb{Q}$ is a countable subset of $\mathbb{R}$ such that $\overline{\mathbb{Q}} = \mathbb{R}$.

Now , we shall see some of the examples of separable and nonseparable spaces through the following results.

Theorem 5.4.1. *The space l^∞ is not separable.*

Proof: Let $y = (\eta_1, \eta_2, ...)$ be a sequence of 0,1.Then clearly, $y \,\epsilon\, l^\infty$.Corresponding to y, we associate the real number $\tilde{y}$, whose binary representation is $\frac{\eta_1}{2} + \frac{\eta_2}{2^2} + \frac{\eta_3}{2^3} + ...$

Observe that every $\tilde{y}\epsilon[0, 1]$ has a unique binary representation.Hence, there are uncountably many sequences made up of 0, 1.The metric on l^∞ shows that if $x \neq y$ then $d(x, y) = 1$.If we choose a small ball of radius $\frac{1}{3}$ centred at these sequences then these balls do not intersect and there are uncountably many such balls. If A is any dense set in l^∞ then every such non-intersecting balls contain an element of A. This shows that A cant be countable set.Since A is an arbitrary dense set, we conclude that l^∞ cant have countable dense set.Therefore, l^∞ is not separable.

Theorem 5.4.2. *The space l^p is separable for $1 \leq p < \infty$.*

Proof: Let M be the set of all sequences y of the form $y = (\eta_1, \eta_2, ..., \eta_n, 0, 0, ...)$ where n is any positive integer and η_i 's are rational.Observe that M is countable.We claim that M is dense in l^p.Let $x = (\zeta_i) \;\epsilon l^p$ be any element.Then for every $\epsilon > 0$ there is an n depending on ϵ such that

$\sum_{i=n+1}^{\infty} |\zeta_i|^p < \epsilon^p/2$.This is because LHS is the remainder of a convergent series. Since, the rationals are dense in $\mathbb{R}$, for each ζ_i there is rational number η_i close to it.Hence, we can find $y \epsilon M$ satisfying $\sum_{i=1}^{n} |\zeta_i - \eta_i|^p < \epsilon^p/2$.This implies that
$[d(x,y)]^p = \sum_{i=1}^{n} |\zeta_i - \eta_i|^p + \sum_{i=n+1}^{\infty} |\zeta_i|^p < \epsilon^p$.Thus, we have $d(x,y) < \epsilon$.Therefore, M is dense in l^p.

Theorem 5.4.3. *If the dual space X' of a normed space is separable , then X itself is separable.*

Proof: We assume that X' is separable.Then the unit sphere
$U = \{f \epsilon X' : ||f|| = 1\} \subset X'$ also contains a countable dense subset say (f_n) . Since $f_n \epsilon U$, we have $||f_n|| = sup\{|f_n(x)| : ||x|| = 1\} = 1$. Therefore, by definition of supremum, we can find points $x_n \epsilon X$ of norm 1 such that $|f_n(x_n)| \geq \frac{1}{2}$. Let $Y = \overline{span(x_n)}$.Then Y is separable, because Y has a countable dense subset $span(x_n)$, which consists of all linear combinations of $x_n's$ with coefficients, whose real and imaginary parts are rational numbers.We claim that $Y = X$.Suppose that $Y \neq X$.Since, Y is closed in X, therefore, by the following lemma:

Lemma 5.4.1. *(Existence of functional). Let Y be proper closed subspace of a normed space X.Let $x_0 \epsilon X - Y$ be arbitrary and $\delta = inf\{||\tilde{y} - x_0|| : \tilde{y} \epsilon Y\}$, the distance from x_0 to Y. Then there exists an $\tilde{f} \epsilon X'$ such that $||\tilde{f}|| = 1$, $\tilde{f}(y) = 0$ for all $y \epsilon Y$, $\tilde{f}(x_0) = \delta$*

We have $\tilde{f} \epsilon X'$ with $||\tilde{f}|| = 1$ and $\tilde{f}(y) = 0$ for all $y \epsilon Y$
Since $x_n \epsilon Y$, we have $\tilde{f}(x_n) = 0$ and for all n,
$\frac{1}{2} \leq |f_n(x_n)| = |f_n(x_n) - \tilde{f}(x_n)| = |(f_n - \tilde{f})(x_n)| \leq ||f_n - \tilde{f}||\ \ ||x_n||$, where $||x_n|| = 1$.Hence $||f_n - \tilde{f}|| \geq \frac{1}{2}$,but this contradicts the assumption that (f_n) is dense in U , because we have $\tilde{f} \epsilon U$

5.5 Let us Sum Up

(1) If T is a linear transformation existing between two normed spaces X and Y then T is bounded iff T is continuous iff T is continuous at the origin iff T maps closed unit sphere to a bounded set.
(2) $B(X,Y)$ is normed linear space and it is complete, if Y is complete.
(3) The dual space of $\mathbb{R}^n$ is $\mathbb{R}^n$.
(4) The dual space of l^1 is l^∞.
(5) For $1 < p < \infty$ the dual space of l^p is l^q.
(6) Every bounded linear functional on Hilbert space can be represented in terms of an inner product.
(7) For $1 < p < \infty$, the dual space of $L^p(X)$ is $L^q(X)$.If X is σ-finite then $L^1(X)' = L^\infty(X)$.
(8) The space l^∞ is not separable.

(9) For $1 \leq p < \infty$, the space l^p is separable.
(10)If the dual space X' of a normed space is separable then X is separable.

5.5 List of References:

(1) Introductory Functional Analysis with applications by Erwin Kreyszig, Wiley India

(2) Educational Resources , the University of California , Davis

5.6 CHAPTER END EXERCISES

(1) Show that $C([a, b])$ is separable.
(Hint: This follows from the Weierstrass approximation theorem that states that $\mathcal{P}([a, b])$=$\{f \epsilon C([a, b]) : f$ is a polynomial with real coefficients $\}$ is dense in $C([a, b])$.Further, $\mathbb{Q}([a, b]) = \{f \epsilon C([a, b]) : f$ is a polynomial with rational coefficients $\}$, we can show that $\mathbb{Q}([a, b]$ is countable and it is dense in $\mathcal{P}([a, b])$)

(2) $C([a, b], \mathbb{R})$ with supremum norm is Banach space.
Proof: Let $\{f_n\}$ be any Cauchy sequence in $C([a, b], \mathbb{R})$. This means that given $\epsilon > 0$ there exists an integer $N > 0$ such that $||f_n - f_m||_\infty < \frac{\epsilon}{2}$, whenever, $n, m \geq N$ .That is , given any $\epsilon > 0$ there exists an integer $N > 0$ such that $|f_n - f_m| < \frac{\epsilon}{2}$ for all $n, m \geq N$ and every $x \epsilon [a, b]$.Thus $\{f_k(x)\}$ is a Cauchy sequence of real numbers for every $x \epsilon [a, b]$.Since $\mathbb{R}$ is complete.Therefore $\{f_k(x)\}$ converges to some real number for each x; we will denote this value by $f(x)$.This defines a new function f such that $f_n \longrightarrow f$ pointwise. We prove that $f_n \longrightarrow f$ uniformly on $[a, b]$. Since f_n is a sequence of continuous function therefore its uniform limit f is also continuous.Let $\epsilon > 0$ be any number.Then there exists an N such that $|f_n(x) - f_m(x)| < \frac{\epsilon}{2}$ for every choice of $x \epsilon [a, b]$ and $n, m \geq N$. If $m \geq N$ and $x \epsilon [a, b]$ then $f_n(x) \epsilon (f_m(x) - \frac{\epsilon}{2}, f_m(x) + \frac{\epsilon}{2})$, for all $n \geq N$.Therefore $f(x) \epsilon [f_m(x) - \frac{\epsilon}{2}, f_m(x) + \frac{\epsilon}{2}]$, and hence $|f(x) - f_m(x)| \leq \frac{\epsilon}{2} < \epsilon$.Since x was arbitrary, therefore we are done.

(3) Let X be a normed space of all polynomials on $I = [0, 1]$ with norm given by $||x|| = max|x(t)|$, for $t \epsilon I$. A differentiation operator T is defined on X by $T(x(t)) = x'(t)$.Show that T is linear operator and it is not bounded.
(Hint: Consider $x_n(t) = t^n$, where $n \epsilon \mathbb{N}$. Then $||x_n|| = 1$ and $||T(x_n)|| = n$)

(4) Let T be a bounded linear operator from a normed space X onto a

normed space Y.If there is positive b such that $||T(x)|| \geq b||x||$ for all $x\epsilon X$ then show that T^{-1} exists and it is bounded.

(5) Show that the dual space of c_0 is l^1

(6) Show that every bounded linear functional f on l^2 can be represented in the form $f(x) = \sum_{j=1}^{\infty} \xi_j \zeta_j$ for $z = (\zeta_j)\epsilon l^2$
(Hint: Use Riesz representation theorem to express any bounded linear functional f on Hibert space l^2 as $f(x) =< x, z >$)

(7) Show that any Hilbert Space is isomorphic with its second dual space $H'' = (H')'$(This property is called reflexivity of H)
Proof: We shall prove that the canonical mapping $C : H \longrightarrow H''$ defined by $C(x) = g_x$ is onto, where g_x is a functional on X' defined for fixed $x\epsilon X$ as $g_x(f) = f(x)$ for $f\epsilon X'$, by showing that for every $g\epsilon H''$ there exist an $x\epsilon H$ such that $g = C(x)$
Define $A : H' \longrightarrow H$ by $A(f) = z$, where z is determined by the Riesz representation theorem $f(x) =< x, z >$, we know that A is bijective and isometric.$A(\alpha f_1 + \beta f_2) = \overline{\alpha} Af_1 + \overline{\beta} Af_2$ implies that A is conjugate linear.Observe that H' is complete and its an Hilbert space with inner product defined by $< f_1, f_2 >=< Af_2, Af_1 >$.For all functionals f_1, f_2, f_3 and scalars α we have
$< f_1 + f_2, f_3 >=< Af_3, A(f_1 + f_2) >= \overline{< A(f_1 + f_2), Af_3} >=$
$\overline{< Af_1, Af_3 > + < Af_2, Af_3} >=< Af_3, Af_1 > + < Af_3, Af_2 >$
$=< f_1, f_3 > + < f_2, f_3 >$
$< \alpha f_1, f_2 >=< Af_2, A(\alpha f_1) >=< Af_2, \overline{\alpha} Af_1 >= \alpha < f_1, f_2 >$
$< f_1, f_2 >=< Af_2, Af_1 >= \overline{< Af_1, Af_2 >} = \overline{< f_2, f_1 >}$
$< f_1, f_1 >=< Af_1, Af_1 >=< z, z >\geq 0$ and $< f_1, f_1 >= 0$ implies that $z = 0$, hence $f_1 = 0$
Let $g\epsilon H$" be arbitrary.Let its Riesz representation be $g(f) =< f, f_0 >=< Af_0, Af >$.We know that $f(x) =< x, z >$, where $z = Af$. Writing $Af_0 = x$, we therefore have $< Af_0, Af >=< x, z >= f(x)$.Together with $g(f) = f(x)$ implies that $g = C(x)$, by the definition of C.Since $g\epsilon H$" was arbitrary, C is onto, so that H is reflexive.

Chapter 6

Four Pillars of Functional Analysis

Unit Structure :

6.1 Introduction

In this chapter we shall see four important theorems, which are also called sometimes called as four pillars of Functional Analysis. The Hahn-Banach theorem, the Open Mapping Theorem, Closed Graph Theorem and Uniform Boundedness Principle.

Hahn-Banach Theorems: It is so much important because it provides us with the linear functionals to work on various spaces as Functional Analysis is all about the study of functionals.

Open Mapping Theorem: It provides us with the open sets in the topology of the range of the mapping.

Uniform Boundedness Principle: An application of Baire Category theorem. It is further used many times as the uniformity is an important property.

Closed Graph Theorem: Closeness of the graph of a map is enough to prove its boundedness or continuity. This fact is further used many times.

6.2 Objectives

After going through this chapter you will be able to:
- State and prove Hahn-Banach theorems
- State and prove Open Mapping theorem
- State and prove Closed Graph theorem
- State and prove Uniform Boundedness theorem

6.3 Few Definitions and Notations

Let X be a normed space over the field $\mathbb{K}(\mathbb{R}$ or $\mathbb{C})$.

Definition 6.1. Let X be a normed space over $\mathbb{K}$. A mapping $f : X \to \mathbb{K}$ is said to be *linear functional* on X if:
$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y),\ \forall x, y \in X,\ \alpha, \beta \in \mathbb{K}$

Definition 6.2. A linear functional f, as defined in **Definition 6.1**, is said to be *bounded* if there exists $M > 0$ in $\mathbb{R}$ such that

$$|f(x)| \leq M \parallel x \parallel,\ \forall x \in X$$

.

Note: The branch of analysis of functionals as defined above was basically called as *functional analysis* initially! The bounded linear functional is a special case of bounded liner operator and hence all properties related to bounded linear operators holds true for bounded linear functionals also. Here is a small activity for you to recollect these properties.

Activity 1: Let f be a linear functional on a normed space X.

1. f is continuous iff ker (f) is closed in X.

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2. Consider $\mathbb{R}^n$ with usual norm and let $a = (a_1, a_2, \ldots, a_n)$ be a vector in $\mathbb{R}^n$. Define $f : \mathbb{R}^n \to \mathbb{R}$ such that $f(x) = x.a$, where $x = (x_1, x_2, \ldots, x_n)$ and $x.a$ denote the scalar product of x with a. Show that f is a bounded linear functional and $\| f \| = \| a \|$.

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3. f is continuous iff f is bounded.

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4. If f is bounded linear functional then $|f(x)| \leq \| f \| \| x \|$, $\forall x \in X$.

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Remark: If f is a bounded linear complex functional then $\overline{f}$ need not be linear.

Definition 6.3. Let X be a linear space over $\mathbb{R}$. A functional p is said to be *sublinear functional* on X if it satisfies the following properties:
(i) $p(x+y) \leq p(x) + p(y)$, $\forall x, y \in X$
(ii) $p(\alpha x) = \alpha p(x)$, $\forall \alpha \geq 0$ in $\mathbb{R}, x \in X$

Definition 6.4. Let X be a linear space over $\mathbb{K}$ and Z be it's subspace. Let $f : Z \to \mathbb{K}$ be a linear functional. Then $\tilde{f} : X \to \mathbb{K}$ is said to be an extension of f if $f(x) = \tilde{f}(x), \forall x \in Z$. Infact f is also called as restriction of $\tilde{f}$ on Z and is denoted as $\tilde{f}|_Z = f$.

6.4 Hahn-Banach Theorem

Theorem 6.4.1. *(***Hahn-Banach Lemma***) Let X be a real vector space and p be a sublinear functional on X. Let Z be a subspace of X and f be linear functional defined on Z such that*

$$f(x) \leq p(x), \forall x \in Z$$

Then, there exists a linear functional $\tilde{f}$ on X such that $\tilde{f}|_Z = f$ and

$$\tilde{f}(x) \leq p(x), \ \forall x \in X$$

Proof: Consider a set $\mathcal{L}$ of all linear extensions (Z_α, g_α) of (Z, f) such that,

$$g_\alpha(x) \leq p(x), \forall x \in Z_\alpha$$

Since $(M, f) \in \mathcal{L}$, clearly $\mathcal{L} \neq \phi$.
Now, we define a relation "$\leq$" on $\mathcal{L}$ such that:
$(Z_\alpha, g_\alpha) \leq (Z_\beta, g_\beta) \Leftrightarrow Z_\alpha \subset Z_\beta$ and
g_β is a extension of g_α i.e. $g_\beta|_{Z_\alpha} = g_\alpha$

Activity 2: Check that the relation " $\leq$ " defined above is reflexive, antisymmetric and transitive.

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From the above activity we can conclude that $(\mathcal{L}, \leq)$ is a partial ordered set. Let $\mathcal{Q}$ be any totally ordered subset of $\mathcal{L}$ and let

$$Z' = \beta \cup \{Z_\beta : (Z_\beta, g_\beta) \in \mathcal{Q}\}$$

We see that Z' is a subspace. Define $g' : Z' \to \mathbb{R}$ by

$$g'(x) = g_\beta(x), \ \forall x \in Z_\beta$$

Clearly, g' is a linear functional on Z' and $g'|_Z = f$.
Claim: (Z', g') is an upper bound of $\mathcal{Q}$ (Try it yourself)
By Zorn's lemma, $\mathcal{Q}$ has a maximal element, say (Z_0, g_0).

To prove that, $(Z_0, g_0) = (X, \tilde{f})$. It is enough to prove that $Z_0 = X$.
Suppose $Z_0 \neq X$. Then there exists $x_0 \in X - Z_0$. Consider the linear space spanned by Z_0 and x_0, $Z' = Z_0 + [x_0]$.
Each element $z \in Z'$ can be uniquely expressed as $z = x + \alpha x_0$, where $x \in Z_0$ and $\alpha \in \mathbb{R}$.
Define $g_1 : Z_1 \to \mathbb{R}$ as $g_1(x + \alpha x_0) = g_0(x) + \alpha K$, where K is a real constant.
We can see that g_1 is a linear functional on Z_1 (*Verify this*!) And $g_1|_Z = g_0$
Thus, Z_1 is a linear subspace of X containing Z and $g_1|_Z = f$
The constant K can be chosen appropriately so that $g_1(y) \leq p(y)$, $\forall y \in Z_1$.
This means, $9Z_1, g_1) \in \mathcal{Q}$ and $(Z_0, g_0) \leq (Z_1, g_1)$, $Z_0 \neq Z_1$. This contradicts the maximality of (Z_0, g_0). Hence our assumption that $Z_0 \neq X$ is not true. Hence the proof.

Theorem 6.4.2. *(***Hahn-Banach***) Let X be a normed space over a field $\mathbb{K}$ and Z be a subspace of X.Then, for every bounded linear functional f on Z, there exists a bounded linear functional $\tilde{f}$ on X such that, $\tilde{f}|_Z = f$ and $\|\tilde{f}\| = \|f\|$*

6.5 Uniform Boundedness Principle

The uniform boundedness principle (or uniform boundedness theorem) by S. Banach and H. Steinhaus (1927) is one of the fundamental results in functional analysis. Together with the Hahn-Banach theorem, the open mapping theorem and the closed graph theorem, it is considered as one of the cornerstones of the field.

The uniform boundedness principle answers the question of whether a "point-wise bounded" sequence of bounded linear operators must also be "uniformly bounded". As the proof of the Uniform Boundedness Principle is an application of Baire's Category Theorem. So, we shall prove the Baire's category theorem first. Following are the basic concepts needed for Baire's theorem:

Definition 6.5. (Nowhere Dense or Rare) A subset M of a metric space X is said to be Nowhere dense in X if its closure $\overline{M}$ has no interior points. That is, $int(\overline{M}) \neq \emptyset$, $\overline{M}$ contains no open ball.

Example 30. The set of all integers $\mathbb{Z}$ is nowhere dense set in $\mathbb{R}$

Example 31. Let $(\mathbb{R}, d)$ be the usual metric space, then every singleton is nowhere dense in $\mathbb{R}$ since $\{\overline{a}\} = \{a\}$ for every $a \in \mathbb{R}$. And $int(\overline{a}) = \emptyset$ since it contains no open interval.

Definition 6.6. (Meager or of First Category) A subset M in metric space X is said to be of First Category if M is the union of countably many sets which are all nowhere dense in X.

Example 32. Since $\mathbb{Q}$ is countable and for every $a \in \mathbb{Q}$, $\{a\}$ being nowhere dense,

$$Q = \bigcup_{a\in\mathbb{Q}} \{a\}$$

is of first category.

Definition 6.7. (Nonmeager or of Second Category) A subset M in metric space X is said to be of Second Category if M is not of first category in X.

Theorem 6.5.1. *(Baire's Category Theorem) If a metric space $X \neq \emptyset$ is complete, it is of second category. Hence, if $X \neq \emptyset$ is complete and*

$$X = \bigcup_{k=1}^{\infty} A_k \quad (A_k \ \ closed)$$

then atleast one A_k contains a nonempty open subset.

Proof. Suppose the metric space $X \neq \emptyset$ is of first category in itself. Then

$$X = \bigcup_{k=1}^{\infty} M_k,$$

where each M_k is rare in X. Since M_1 is rare in X, so by definition, $\overline{M_1}$ does not contain a nonempty open set. But (X, d) is complete, so it will contain a nonempty set. So, $\overline{M_1} \neq X$. Therefore, $\overline{M_1}^C = X - \overline{M_1}$ which is nonempty and open. At the point $p_1 \in \overline{M_1}^C$, we can get an open ball

$$B_1 = B(p_1, \epsilon_1) \subset \overline{M_1}^C,$$

where $\epsilon < \frac{1}{2}$.
Further, M_2 is rare in X, so that M_2 does not contain a nonempty open set. Hence, it does not contain open ball $B(p_1, \frac{\epsilon_1}{2})$. This implies that $\overline{M_2}^C \bigcap B(p_1, \frac{\epsilon_1}{2})$ is not empty and open. Now, we may choose an open ball in this set, say,

$$B_2 = B(p_2, \epsilon_2) \subset \overline{M_2}^C \bigcap B(p_1, \frac{\epsilon_1}{2}), \quad \epsilon_2 < \frac{\epsilon_1}{2}.$$

Continuing this process, we obtain a sequence of balls by induction,

$$B_k = B(p_k, \epsilon_k), \quad \epsilon_k < 2^{-k}$$

such that $B_k \bigcap M_k = \emptyset$ and $B_{k+1} \subset B(p_k, \frac{\epsilon_k}{2}) \subset B_k$, $k = 1, 2, ...$
As $\epsilon_k < 2^{-k}$ and the space X is complete, the sequence (p_k of the centers is Cauchy and converges, say, $p_k \to p \in X$. Also, for every m and $n > m$ we have $B_n \subset B(p_m, \frac{\epsilon_m}{2})$, so that

$$\begin{aligned} d(p_m, p) \leq & d(p_m, p_n) + d(p_n, p) \\ < & \frac{\epsilon_m}{2} + d(p_n, p) \to \frac{\epsilon_m}{2} \end{aligned}$$

as n approaches to ∞. Hence, p belongs to B_m for every m. Since $B_m \subset \overline{M_m}^C$, we get $p \notin M_m$ for every m, so that $p \notin \bigcup M_m = X$. This contradicts $p \in X$. Hence proved. □

Theorem 6.5.2. *Let $\{T_n\}$ be a sequence of bounded linear operators $T_n : X \to Y$ from a Banach space X into a normed space Y. If for every $x \in X$, $\{T_n(x)\}$ is bounded, say,*

$$\|T_n(x)\| \leq c_x, \quad n = 1, 2, 3, ... \ (pointwise\ boundedness),$$

where c_x is a real number, then the sequence $\|T_n\|$ is also bounded, that is, there is a positive real c such that

$$\|T_n\| \leq c, \quad n = 1, 2, 3, ... \ (uniform\ boundedness),$$

Proof. For every $x \in X$, $\{T_n(x)\}$ is bounded sequence in Y, that is,

$$\|T_n(x)\| \leq c_x \ \ where \ \ c_x \geq 0.$$

Suppose that $A_k = \{x \in X : \|T_n(x)\| \leq k\}$
Step 1: We claim that A_k is nonempty, closed and $X = \bigcup_{k=1}^{\infty} A_k$. Since $0 \in X$ and T_n is linear so $T_n(0) = 0 \leq k$ for each k. This implies that $0 \in A_k$ for each k which implies that A_k is nonempty.

Let $\{x_n\}$ be a convergent sequence in A_k with $x_m \to x$ as $n \to \infty$. This means that for every fixed n, we have $\|T_n(x_m)\| \leq k$ and obtain $\|T_n(x)\| \leq k$ by applying limits for $m \to \infty$ because T_n is continuous and so is the norm. Hence, $x \in A_k$ and therefore A_k is closed.

Also, $A_k \subseteq X$ (for all k) $\Rightarrow \bigcup_{k=1}^{\infty} A_k \subseteq X$. On the other hand, let $x \in X$, $\|T_n(x)\|$ is a real number. Using Archimedean property, there is a positive integer n_0 such that $T_n(x)\| \leq c_x \leq n_0$. So,

$$x \in A_{n_0} \subseteq \textstyle\bigcup_{k=1}^{\infty} A_k \Rightarrow X \subseteq \bigcup_{k=1}^{\infty} A_k$$

Hence,

$$X = \bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} \overline{A_k}, \quad (Since \quad A_k = \overline{A_k})$$

In view of Baire's Category Theorem, every complete space is of second category. That is, at least one of A_k is not nowhere dense. Hence, some

A_k contains an open ball, say, $B_0 = B(x_0, r) \subseteq A_{k_0}$.
Step 2: Let $x \in X$ be arbitrary, not zero. We set

$$z = x_0 + \gamma_x,$$

where $\gamma = \frac{r}{2\|x\|}$. Then

$$z = x_0 + \frac{r}{2\|x\|}x$$
$$z - x_0 = \frac{r}{2\|x\|}x$$
$$\|z - x_0\| = \frac{r}{2\|x\|}\|x\| = \frac{r}{2} < r$$

That is, $z \in B(x_0, r) \subseteq A_{k_0}$.
As, $z, x_0 \in A_{k_0} \Rightarrow \|T_n(z)\| \leq k_0, \|T_n(x_0)\| \leq k_0$. We have,

$$x = \frac{2\|x\|}{r}(z - x_0).$$

Therefore,

$$\begin{aligned}
\|T_n(x)\| &= \frac{2\|x\|}{r}\|T_n(z - x_0)\| \\
&= \frac{2\|x\|}{r}\|T_n(z) - T_n(x_0)\| \\
&\leq \frac{2\|x\|}{r}[\|T_n(z)\| + \|T_n(x_0)\|] \\
&\leq \frac{2\|x\|}{r}(2k_0) \\
&= \frac{4k_0}{r}\|x\|
\end{aligned}$$

Hence,

$$\|T_n(x)\| \leq \frac{4k_0}{r}\|x\| \Rightarrow \frac{\|T_n(x)\|}{\|x\|} \leq \frac{4k_0}{r} = c.$$

which implies

$$\sup_{x \in X, x \neq 0} \frac{\|T_n(x)\|}{\|x\|} \leq c \Rightarrow \|T_n\| \leq c.$$

□

6.6 Open Mapping Theorem

Open mapping theorem is one of the basic theorems for the development of the general theory of normed linear spaces. The theorem gives conditions under which a linear mapping is open. In this theorem, we begin to appreciate the importance of the completeness condition for normed linear spaces.

Definition 6.8. (Open Mapping) Let X and Y be metric spaces, a mapping $T : D(T) \to Y$ with domain $D(T) \subset X$ is called an open mapping if for every open set in $D(T)$ the image is an open set in Y.

Definition 6.9. (Continuous Mapping) A continuous mapping $T : X \to Y$ has the property that for every open set in Y the inverse image is an open set in X.

Remark 6.6.1. Continuous mappings are not necessarily open mappings.

Example 33. Consider a mapping $T : \mathbb{R} \to \mathbb{R}$ defined by $T(x) = Sinx$. As Sine function is continuous, T is a continuous mapping but $T[(0, 2\pi)] = [-1, 1]$ that is, T maps an open set $(0, 2\pi)$ onto $[-1, 1]$ which is not open.

Example 34. Define $T : \mathbb{R} \to \mathbb{R}$ by $T(x) = x^2$. Then T is continuous mapping but not an open mapping because it maps an open set $(-1, 1)$ onto $[0, 1)$ which is not open.

Lemma 6.6.1. *(Open Unit Ball) Let T be a bounded linear operator from a Banach space X onto a Banach space Y. Then the image of the open ball $B_0 = B_1(0) \subset X$, that is, $T(B_0)$, contains an open ball with center 0 in Y.*

Proof. The proof has three steps. We will prove
(a) The closure of the image of the open ball $B_1 = B(0; \frac{1}{2})$ contains an open ball B^*.
(b) $\overline{T(B_n)}$ contains an open ball V_n about $0 \in Y$, where $B_n = B(0; 2^{-n}) \subset X$.
(c) $T(B_0)$ contains an open ball about $0 \in Y$.
(a) Clearly, we have

$$\bigcup_{k=1}^{\infty} kB_1 \subset X$$

For any $x \in X$, there is $k(k > 2||x||)$ such that $x \in kB_1$. So,

$$X \subset \bigcup_{k=1}^{\infty} kB_1.$$

Thus, we have

$$X = \bigcup_{k=1}^{\infty} kB_1.$$

Since T is surjective,

$$Y = T(X) = T(\bigcup_{k=1}^{\infty} kB_1) = \bigcup_{k=1}^{\infty} T(kB_1).$$

Due to the linearity of T, we have

$$Y = \bigcup_{k=1}^{\infty} T(kB_1) = \bigcup_{k=1}^{\infty} kT(B_1) = \bigcup_{k=1}^{\infty} \overline{kT(B_1)}.$$

Since Y is complete, by the Baire's Category theorem, we conclude that a $\overline{kT(B_1)}$ contains an open ball. This implies that $\overline{T(B_1)}$ also contains an open ball, namely, there is $B^* = B(y_0; \epsilon)$ such that $B^* \subset \overline{T(B_1)}$. It follows that $B^* - y_0 = B(0; \epsilon) \subset \overline{T(B_1)} - y_0$.

(b) We will first prove that $B(0; \epsilon) = B^* - y_0 \subset \overline{T(B_0)}$. Since $B^* \subset \overline{T(B_1)}$ by (a), we have $B^* - y_0 \subset \overline{T(B_1)} - y_0$. It suffices to prove $\overline{T(B_1)} - y_0 \subset \overline{T(B_0)}$. Let $y \in \overline{T(B_1)} - y_0$. Then $y + y_0 \in \overline{T(B_1)}$. Notice that $y_0 \in \overline{T(B_1)}$ since $B^* \subset \overline{T(B_1)}$. Then there are sequences $u_n = Tw_n \in T(B_1)$ and $v_n = Tz_n \in T(B_1)$ such that $u_n \to y + y_0$, $v_n \to y_0$, where $w_n, z_n \in B_1$. Observing that

$$||w_n - z_n|| \leq ||w_n|| + ||z_n|| < \frac{1}{2} + \frac{1}{2} = 1.$$

So, $w_n - z_n \in B_0$. Also, $T(w_n - z_n) = Tw_n - Tz_n = u_n - v_n \to y$. Hence, $y \in \overline{T(B_0)}$. This proves that $B(0; \epsilon) = B^* - y_0 \subset \overline{T(B_0)}$. Let $B_n = B(0; \frac{1}{2^n})$. Since T is linear, we have $\overline{T(B_n)} = 2^{-n}\overline{T(B_0)}$. Let $V_n = B(0; \frac{\epsilon}{2^n})$. Then $V_n = \frac{1}{2^n}B(0; \epsilon) \subset \frac{1}{2^n}\overline{T(B_0)} = \overline{T(B_n)}$. This proves (b).

(c) We finally prove that $V_1 = B(0; \frac{1}{2}\epsilon) \subset T(B_0)$ by showing that every $y \in V_1$ is in $T(B_0)$. So, let $y \in V_1$. Since $V_1 \subset \overline{T(B_1)}$, there is $x_1 \in B_1$ such that $\|y - Tx_1\| < \frac{\epsilon}{4}$. Then we have $y - Tx_1 \in V_2$. Since $V_2 \subset \overline{T(B_2)}$, there is $x_2 \in B_2$ such that $\|y - Tx_1 - Tx_2\| < \frac{\epsilon}{8}$. Continuing in this manner, we have, for each n, there are $x_n \in B_n$ such that

$$\|y - \sum_{k=1}^{n} Tx_k\| < \frac{\epsilon}{2^{n+1}}.$$

Let $z_n = x_1 + x_2 + ... + x_n$. The above inequality becomes

$$\|y - Tz_n\| < \frac{\epsilon}{2^{n+1}}, \ \ \forall n.$$

Namely, $Tz_n \to y$. Since $x_k \in B_k$, we have $\|x_k\| < \frac{1}{2^k}$. So, for $n > m$,

$$\|z_n - z_m\| \leq \sum_{k=m+1}^{n} \|x_k\| < \sum_{k=m+1}^{\infty} \frac{1}{2^k} \to 0, \quad as \ m \to \infty.$$

Thus, the sequence $\{z_n\}$ is Cauchy. Since X is complete, there is $x \in X$ such that $z_n \to x$ and $x = x_1 + x_2 + ...$ Notice that

$$\|x\| \leq \sum_{k=1}^{\infty} \|x_k\| < \frac{1}{2} + \sum_{k=2}^{\infty} \|x_k\| \leq \frac{1}{2} + \frac{1}{2} = 1.$$

So, $x \in B_0$. Since T is continuous, we have $Tz_n \to Tx$. Hence, $y = Tx$. That is, $y \in T(B_0)$. $\square$

Theorem 6.6.1. *(Open Mapping Theorem, Bounded Inverse Theorem) Let X and Y be Banach spaces. Then any bounded linear operator T from X onto Y is an open mapping. Consequently, if T is bijective, then T^{-1} is continuous and hence bounded.*

Proof. Let $A \subset X$ be an arbitrary open subset of X. We will show that the image $T(A)$ is open in Y. That is, for any $y = Tx \in T(A)$, the set $T(A)$ contains an open ball centered at y. Let $y \in T(A)$. Then $y = Tx$ with $x \in A$. Since A is open, there is $r > 0$ such that $B_r(x) \subset A$. Thus

$$B_1(0) \subset \frac{1}{r}(A - x).$$

By Lemma 6.6.1, the image $T(\frac{1}{r}(A - x))$ contains an open ball with center 0. That is, there is $\epsilon > 0$, such that

$$B(0; \epsilon) \subset T(\frac{1}{r}(A - x))$$

Since T is linear, we have

$$B(0; \epsilon) \subset \frac{1}{r}(T(A) - Tx).$$

Since $y = Tx$, the above relation implies $B(y; r\epsilon) \subset T(A)$. Hence, $T(A)$ contains an open ball with center y. $\square$

6.7 Closed Graph Theorem

Definition 6.10. (Cartesian product of two normed spaces)
Let $(X, \|.\|_1)$ and $(X, \|.\|_2)$ be two normed spaces, then $X \times Y$ is also a normed space where the two algebraic operations of a vector space and the norm on $X \times Y$ are defined as usual, that is

- $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$
- $\alpha(x, y) = (\alpha x, \alpha y)$ $(\alpha \ a \ scalar)$
- $\|(x, y)\| = \|x\|_1 + \|y\|_2$

Theorem 6.7.1. *For any two Banach spaces X and Y, $X \times Y$ is also a Banach space.*

Proof. We show that for any two Banach spaces X and Y,$X \times Y = \{(x, y) : x \in X, y \in Y\}$ is also a Banach spaces. Let $\{z_n\}$ be a Cauchy sequence in $X \times Y$, where $z_n = (x_n, y_n)$. Then for every $\epsilon > 0$, there exists a positive integer n_0 such that for every $n \geq n_0$, we have

$$\begin{aligned}\|z_n - z_m\| &= \|(x_n, y_n) - (x_m, y_m)\| \\ &= \|(x_n - x_m, y_n - y_m)\| \\ &= \|x_n - x_m\| + \|y_n - y_m\| < \epsilon\end{aligned}$$

This implies that $\|x_n - x_m\| < \epsilon$, $\|y_n - y_m\| < \epsilon$ thereby proving that $\{x_n\}$ is a Cauchy sequence in X and $\{y_n\}$ is a Cauchy sequence in Y. Let $x_n \to x \in X$, $y_n \to y \in Y$, then

$$\|z_n - z\| = \|(x_n, y_n) - (x, y)\| = \|(x_n - x, y_n - y)\| = \|x_n - x\| + \|y_n - y\|$$

Applying $n \to \infty$,
$\|x_n - x\| \to 0$ and $\|y_n - y\| \to 0$. This implies that $\lim_{n\to\infty} \|z_n - z\| = 0$. Hence, $z_n \to z = (x, y) \in X \times Y$. Thus, $X \times Y$ is a Banach space. $\square$

Definition 6.11. (Closed Linear Operator) Let $(X, \|.\|_1\|)$ and $(Y, \|.\|_2\|)$ be normed spaces and let $T : D(T) \to Y$ a linear operator with domain $D(T) \subset X$. Then T is called a closed linear operator if its graph

$$G(T) = \{(x, y) : x \in D(T), y = Tx\} \subseteq X \times Y$$

Theorem 6.7.2. *Let X and Y be Banach spaces and $T : D(T) \to Y$ be a closed linear operator, where $D(T) \subset X$. If $D(T)$ is closed in X, then the operator T is bounded.*

Proof. First recall that for any two Banach spaces X and Y, $X \times Y$ is also a Banach space. By assumption, $G(T)$ is closed in $X \times Y$. Hence, $G(T)$ is complete and also $D(T)$ is closed in X. Therefore, $D(T)$ is also complete space. We now consider the mapping $P : G(T) \to D(T)$, defined by $P(x, Tx) = x$. We prove that P is bijective, linear and bounded. To show that P is linear, we consider

$$\begin{aligned}P((x_1, Tx_1) + (x_2, Tx_2)) &= P(x_1 + x_2, Tx_1 + Tx_2) \\ &= x_1 + x_2 \\ &= P(x_1, Tx_1) + P(x_2, Tx_2)\end{aligned}$$

And for $\alpha \in K$, we have

$$P(\alpha(x, Tx)) = P(\alpha x, \alpha Tx) = \alpha x = \alpha P(x, Tx)$$

Now, P is bounded, because

$$\|P(x, Tx)\| = \|x\| \leq \|x\| + \|Tx\| = \|(x, Tx)\|$$

This implies that $\|P(x, Tx)\| \leq 1.\|(x, Tx)\|$. P is onto, since for all $x \in D(T)$ $\exists (x, Tx) \in G(T)$ such that $P(x, Tx) = x$. P is one to one, since

$$P(x_1, Tx_1) = P(x_2, Tx_2) \Rightarrow x_1 = x_2 \Rightarrow (x_1, Tx_1) = (x_2, Tx_2).$$

Hence, P is a bounded linear operator from Banach space $G(T)$ to Banach space $D(T)$ and also it is bijective hence $P^{-1} : D(T) \to G(T)$ given by $P^{-1}(x) = (x, Tx)$ is a bounded linear operator by Bounded inverse theorem. That is, there exists $b > 0$ such that for every $x \in D(T)$, we have

$$\|P^{-1}(x)\| \leq b\|x\| \Rightarrow \|(x, Tx)\| \leq b\|x\| \Rightarrow \|x\| + \|Tx\| \leq b\|x\|$$

Therefore, $\|Tx\| \leq \|x\| + \|Tx\| \leq b\|x\| \Rightarrow \|Tx\| \leq b\|x\|$. This implies that T is bounded. $\square$

6.8 Applications of Hahn-Banach theorem

1. Given a normed linear space X over a field K and a non zero member $x_0 \in X$, there is a bounded linear functional F over X such that $F(x_0) = \|x_0\|$ and $\|F\| = 1$.

Proof. Let, $M = [x_0]$= the subspace spanned by $\{x_0\} = \{\alpha x_0 : \alpha \ is \ real\}$. Define $f : M \to \mathbb{R}$ by $f(\alpha x_0) = \alpha\|x_0\|$. Clearly, f is linear. Now, for $x \in M$, $x = \alpha x_0$ for some α. Now, $|fx| = |f(\alpha x_0)| = |\alpha\|x_0\|| = |\alpha|\|x_0\| = \|\alpha x_0\| = \|x\|$. Clearly, $\|f\| = 1$, that is, f is a bounded linear functional on M. So, there exists an extension F(a bounded linear functional) of f over X such that $\|f\| = \|F\|$. But $f(x) = F(x)$, $\forall x \in M$. Now, $x_0 \in M$. So, $f(x_0) = F(x_0)$. But $f(x_0) = \|x_0\|$. So, $F(x_0) = \|x_0\|$ with $\|F\| = 1$. $\square$

2. Let, X be a normed linear space over a field K and $x_0 \neq \theta$ be an arbitrary member of X and let M_0 be an arbitrary positive real. Then $\exists$ a bounded linear functional f on X, such that $\|f\| = M_0$ and $f(x_0) = \|f\|\|x_0\|$.

Proof. Let $G = [x_0] = \{tx_0 : t \ is \ real\}$. Clearly, G is a subspace of X. Define $\phi : G \to \mathbb{R}$ by $\phi(tx_0) = tM_0\|x_0\|$. Clearly, ϕ is linear. Now, for $x \in G$, $x = tx_0$ for some t. Now,

$$\begin{aligned}
|\phi(x)| &= |\phi(tx_0)| \\
&= |tM_0|\|x_0\| \\
&= M_0|t|\|x_0\| \\
&= M_0\|tx_0\| \\
&= M_0\|x\|.
\end{aligned}$$

So, $|\phi(x)| = M_0\|x\|$. Clearly, $\|\phi\| = M_0$. So, ϕ is a bounded linear functional on G. So, $\exists$ an extension f of ϕ over X such that $\|f\| = \|\phi\| = M_0$. Hence, $\|f\| = M_0$. But $\phi(x_0) = M_0\|x\|$. Now, $x_0 \in G$, $f(x_0) = \phi(x_0) = M_0\|x_0\| = \|f\|\|x_0\|$. So, $f(x_0) = \|f\|\|x_0\|$. This completes the proof. $\square$

3. For every $x \in X$, $\|x\| = \sup_{f \neq \Theta} \frac{f(x)}{\|f\|}$. [$\Theta$ is zero functional on X].

Proof. We find a non zero bounded linear functional $f_0 \in X'$ such that $f_0(x) = \|x\|$ and $\|f_0\| = 1$. Now,

$$\sup_{f \neq \Theta} \frac{|f(x)|}{\|f\|} \geq \frac{|f_0(x)}{\|f_0\|}$$

Again, $\forall f \neq \Theta \in X'$, $|f(x)| \leq \|f\|\|x\|$. So,

$$\frac{|f(x)|}{\|f\|} \leq \|x\|.$$

That is,

$$\sup_{f \neq \Theta \in X'} \frac{|f(x)|}{\|f\|} \leq \|x\|.$$

Hence,

$$\|x\| = \sup_{f \neq \Theta \in X'} \frac{|f(x)|}{\|f\|}.$$

□

4. Let M be a closed subspace of normed linear space X such that $M \neq X$. Let $u \in X \setminus M$ and let

$$d = dist(u, M) = \inf_{m \in M} \|u - M\|$$

Then there is a bounded linear functional $f \in X'$ such that
(i)$f(x) = 0 \ \forall x \in M$
(ii) $f(u) = 1$ and (iii) $\|f\| = \frac{1}{d}$

Proof. Clearly, $d > 0$. Let $N = [M \bigcup \{u\}]$. Clearly, N is a subspace of X. So, every member of N is of the form $m + tu$, where $m \in M, t \in \mathbb{R}$. Define $g : N \to \mathbb{R}$ by $g(m+tu) = t$. Clearly, g is linear. Now, $g(m) = 0$, for some $m \in M, g(u) = 1$. For $t \neq 0$,

$$\begin{aligned} |g(m + tu)| &= |t| \\ &= \frac{|t|\|m + tu\|}{\|m + tu\|} \\ &= \frac{\|m + tu\|}{\|u - (-m/t)\|} \leq \frac{1}{d}\|m + tu\|, \end{aligned}$$

as $(-m/t) \in M$. Let $x = m + tu \in N$. So, $|g(x)| \leq \frac{1}{d}\|x\|$ implying that g is bounded and $\|g\| \leq \frac{1}{d}$. So, $g \in N'$. Again, $d = \inf_{m \in M} \|u - m\|$. So, there exists a sequence $\{m_n\} \in M$ such that $\|u - m_n\| \to d$ as $n \to \infty$. Now,

$$|g(u - m_n)| \leq \|g\|\|u - m_n\| \Rightarrow 1 \leq \|g\|\|u - m_n\|,$$

since $g(u-m_n) = g(u) - g(m_n) = 1 - 0 = 1$. Letting $n \to \infty$, $\frac{1}{d} \leq \|g\|$, we get $\|g\| = \frac{1}{d}$. So, $\exists$ a bounded linear functional $f \in X'$ which is an extension of g on N such that $f(x) = g(x)$, $\forall x \in N$ and $\|f\| = \|g\|$. Thus, we have
(i) $f(x) = 0, \forall x \in N$
(ii) $f(u) = 1$
(iii) $\|f\| = \frac{1}{d}$. □

5. Let M be a subspace of a normed linear space X and $M \neq X$. If $u \in X\ M$ such that $d = dist(u, M) > 0$. Then there is a bounded linear functional $F \in X'$ such that
(i) $F(x) = 0\ \forall x \in M$
(ii) $F(u) = d$
(iii) $\|F\| = 1$.

Proof. Let, $N = [M \bigcup \{u\}]$ = a subspace of X spanned by M and $\{u\}$. Define $f : N \to \mathbb{R}$ by $f(x) = td$, where $x = m + tu$ for some $m \in M$ and for some t. That is, $f(m + tu) = td$. Clearly, f is linear. Now, $f(m) = 0$ for $m \in M$, $f(u) = d$. Now, for $t \neq 0$

$$\begin{aligned}\|m + tu\| &= \| - t\left(\frac{-m}{t} - u\right)\| \\ &= |t|\|\frac{-m}{t} - u\| \geq |t|d,\end{aligned}$$

as $\frac{-m}{t} \in M$ and $d = dist(u, M)$. So, $|f(m + tu)| = |t|d \leq \|m + tu\|$. Let $x = m + tu \in N$. So,

$$|f(x)| \leq \|x\| \Rightarrow \|f\| \leq 1. \tag{6.1}$$

Clearly, f is bounded with $\|f\| \leq 1$. So, $f \in N'$. We have $d = dist(u, M) = \inf_{m \in M} \|u - m\|$. Let $\epsilon > 0$. By infimum property there exists an element $m \in N$ such that $\|u - m\| < d + \epsilon$.
Put

$$v = \frac{m - u}{\|m - u\|} = \frac{m}{\|m - u\|} - \frac{1}{\|m - u\|}u \in N,$$

where $t = \frac{-1}{\|m-u\|}$ and $\frac{m}{\|m-u\|} \in M$. So, $\|v\| = 1$. But

$$|f(v)| = \frac{d}{\|m - u\|} > \frac{d}{d + \epsilon}\|v\|, \|v\| = 1.$$

As, $\epsilon > 0$ is arbitrary,

$$|f(v)| \geq \|v\| \Rightarrow \|f\| \geq 1. \tag{6.2}$$

So, by (6.1) and (6.2), $\|f\| = 1$. So, there exists a bounded linear functional F over X which is an extension of f over N such that $f(x) = F(X)$, $\forall x \in N$ and $\|f\| = \|F\|$. So, (i) $F(x) = 0, \forall x \in M$,
(ii) $F(u) = d$ and (iii) $\|F\| = 1$. □

6.9 Chapter End Exercises

1. Let f be an additive functional on a normed space X. Prove that if f is continuous then f is linear.

2. Prove that $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = f(x) + f(y), x, y \in \mathbb{R}$, is a bounded linear functional on $(\mathbb{R}, \| \, . \, \|_2)$.

3. If X be a non trivial real normed linear space, that is, $X \neq \{0\}$. Then its first conjugate space X' is also nontrivial.

4. Let X is a Banach space and $A \subset X$ a dense set. Can we find a function $f : X \to \mathbb{R}$ such that, for every $x \in A$, we have $\lim_{t \to x} |f(t)| = \infty$?

5. Let X and Y be Banach spaces and $T \in \mathbf{B}(X, Y)$. Suppose T is bijective. Show that there exist real numbers $a, b > 0$ such that $a\|x\| \leq \|Tx\| \leq b\|x\|, \forall x \in X$.

6. Let X, Y and Z be Banach spaces. Suppose that $T : X \to Y$ is linear, that $J : Y \to Z$ is linear, bounded and injective, and that $JT \equiv J \circ T : X \to Z$ is bounded. Show that T is also bounded.

7. Let $(X, \|.\|_1)$ and $(X, \|.\|_2)$ be Banach spaces. Suppose that

   $$\exists C \geq 0 : \|x\|_2 \leq C\|x\|_1, \forall x \in X.$$

   Show that the two norms $\|.\|_1$ and $\|.\|_2$ are equivalent.