1

GRAPHS

Unit Structure :

1.0 OBJECTIVES

- History and Introduction
- Areas of Application
- Various Definitions and examples pertaining to graphs
- Representation of Graph

1.1 INTRODUCTION

The first occurrence of graph in the Mathematical history is considered to be the classical "Konigsberg Bridge Problem". The problem is stated by the great mathematician L. Euler who lived in Konigsberg, as below:

"Konigsberg is divided into four parts by river Pregel and connected by seven bridges. Is it possible to tour Konigsberg along a path that crosses every bridge once and only once and return to the starting point?"

The diagram is given as below.

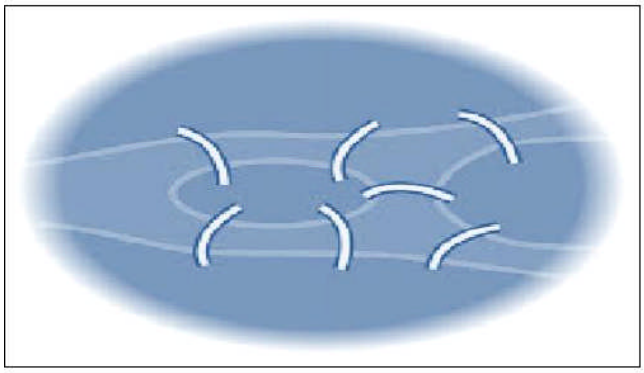


Fig. 1.1

In proving the fact that the problem is unsolvable, Euler represented above image in the form of a "graph", as follows.

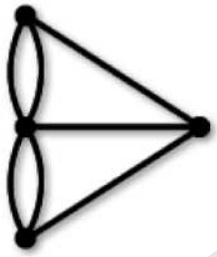


Fig. 1.2

The points denote the land and the lines denote bridges.

In fact, rather than only proving that above problem is unsolvable, Euler introduced type of graphs, which can be traceable by starting from one point, traversing every line once and returning to the starting point, which is termed as Euler Graph. In Chapter 4, we shall discuss Euler Graphs in detail.

In modern times, Graph theory is applicable in many areas, such as, Chemistry, Electronics, and Networks, to name a few.

In this chapter, we will get acquainted ourselves with terminology and basic concepts of Graph Theory.

1.2 APPLICATIONS OF GRAPH THEORY

Though graph "theory" appears to be a theoretical and hence pure mathematical term, we shall be amazed to know the areas in which it can be applied. In this section, we shall just quote a few in which graph theory is applied.

1. Graph Theory is helpful in making robots function autonomously.
2. Graph Theory is used to solve actual crimes.
3. Mathematics, often called the universal language, also forms a ridge between languages. Machine translators use Graph Theory to achieve good translations efficiently

4. Descriptions of cellular activity involve a combination of continuous models. The analysis of cells requires usage of Graph Theory as well.

5. Researchers use graph theory to find near- optimal solutions saving industry time and money. (Travelling salesman's problem, Chinese postman's problem)

6. The Graph Theory is applicable in Road and Rail Traffic network.

7. The Graph Theory is applicable in planning tournaments (such as football, chess)

8. Hierarchy in the office, such as Chairperson is the root and people work under him are at various level.

9. The Graph Theory is present in the virtual world of internet such www (world wide web, social networking, searching data, data mining, and so on.

1.3 BASIC GRAPH THEORY DEFINITIONS AND NOTATIONS

Definition 1.3.1: Simple Graph: Simple graph is a set G (V(G), E(G)), V(G) the set of vertices (points) and E(G) the set of edges (lines) disjoint from V(G), together with an incidence function □G, that associates with each edge of G a distinct unordered pair of vertices of G.

Definition 1.3.2: Directed Graph: A directed graph or digraph is a graph G (V, E) in which edges are ordered pairs (u,v), where $u,v \in V$. That is if there is an edge from *u* to *v*, there may or may not be an edge from *v* to *u*.

Example 1.3.1:

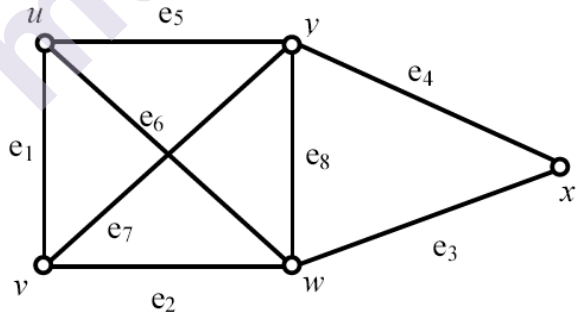


Fig. 1.3

Example 1.3.2

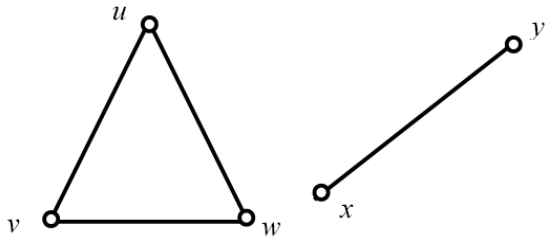


Fig. 1.4

In the example 1.3.1, graph G has, $V = \{u, v, w, x, y\}$ and $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$.

In the example 1.3.2, graph G has, $V = \{u, v, w, x, y\}$ and $E = \{uv, uw, vw, xy\}$ Note:

1. Typically number of vertices in a graph G is denoted by letter p and edges by q.
2. A graph in which both vertex set and edge set is finite is called as finite graph.
3. In this and the subsequent chapters, we shall mainly discuss finite graphs.
4. The graph with no vertices (and hence no edges) is termed as null graph.
5. A graph with just one vertex is termed as trivial graph.
6. We are discussing non-trivial and non-null graph.
7. In a graph, if an edge with identical end vertex is called as loop.
8. In a graph, two or more edges with same end vertices are termed as parallel edges.
9. A graph, in which loop and / or parallel edges are permitted, is termed as Multigraph.
10. Graph of Fig. 1.6 is a digraph or directed graph.

In the graph of Fig. 1.2, that is the graph of Konigsberg's bridge problem, we can observe parallel edges and the graph of Fig. 1.5, there is a loop at vertex u.

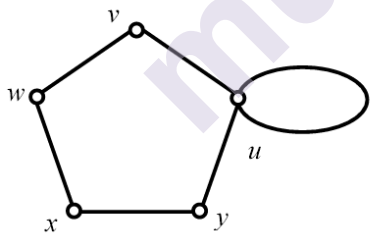


Fig. 1.5

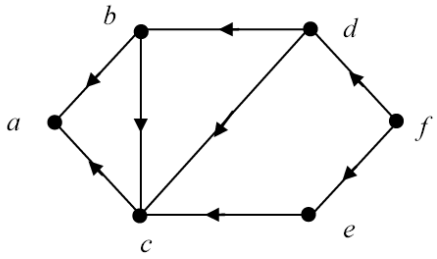


Fig. 1.6

Definition 1.3.3: Degree: The degree of a vertex of a graph is the number of edges incident to the vertex, with loops counted twice. We denote degree of vertex v in a graph G is denoted by deg G (v).

In Fig.1.3, degree of u is 3, where as in Fig.1.5, degree of u is 4.

Note: In case of directed graph, every vertex has two types of degrees, in-degree (that is number of edges entering the vertex) and out-degree (that is number of edges leaving the vertex). For the graph of Fig. 1.6, in-deg (a) = out-deg (a) = 1; in-deg (b) = 1, out-deg (b) = 2; in-deg (c) = 3, out-deg (c) = 1; in-deg (d) = 1, out-deg (d) = 2; in-deg (e) = out-deg (e) = 1; indeg (f) = 0, out-deg (f) = 2.

Definition 1.3.4: Walk: A walk consists of an alternating sequence of vertices and edges consecutive elements of which are incident, which begins and ends with a vertex.

Fig. 1.3, $ue_1ve_7ye_7ve_2w$ is a walk.

Definition 1.3.5: Trail: A trail is a walk in which no edges are repeated.

In Fig. 1.3, $ue_1ve_2we_6ue_5y$ is a trail.

Definition 1.3.6: Path: A path is a trail in which no vertices (except possibly the end vertices) are repeated.

In Fig 1.3, $ue_1ve_7ye_8we_3x$ is a path.

Definition 1.3.7: Circuit: A circuit is a closed trail (that is end vertices are same) with at least one edge is known as Circuit.

In Fig. 1.3, $ue_1ve_7ye_8we_3xe_4ye_5u$ is a circuit.

It can also be written, only in terms of vertices as: *uvywxyu*.

Definition 1.3.8: Cycle: A cycle is a circuit in which no edge is repeated.

In Fig. 1.3, $ue_1ve_2we_3xe_4ye_5u$ is a cycle.

It can also be written as *uvwxyu*, in terms of vertices alone.

Definition 1.3.9: Subgraph: A graph H = (H (V), H(E)) is called as subgraph of G = (G (V), G (E)), if $H(V) \subseteq G(V)$ and $H(E) \subseteq G(E)$.

Fig. 1.6 below is a subgraph of Fig. 1.3.

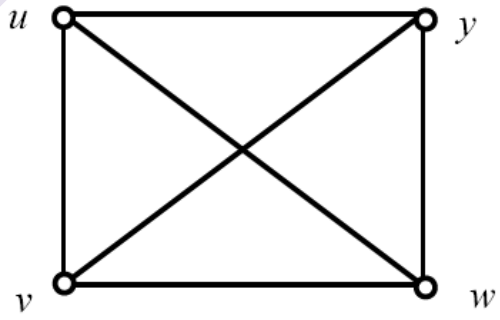


Fig. 1.6

Let G (V, E) be a graph. We can obtain a subgraph from a graph in any one of the following ways.

1. A subgraph H, can be obtained by deleting vertex subset U of V and by deleting all the edges from E which are incident with a vertex in U.

2. A subgraph H, can be obtained by deleting an edge set D that is subset of E and vertex set of H is same as vertex set of G. Such a subgraph is called as spanning subgraph of G.

Definition 1.3.10: Connected Graph: Graph G is connected if and only if there exists a walk between any pair of vertices.

Graph in Fig. 1.3 is connected.

Definition 1.3.11: Disconnected Graph: Graph G is disconnected if and only if there exists at least one pair of vertices which is not connected by a walk.

Graph if Fig. 1.4 is disconnected, as there is no walk between vertices u and x.

The distance *d(u, v)* between two vertices u and v of a graph G is the length of the shortest path (often termed as ***geodesic***) joining them if any; otherwise $d(u,v) = \infty$.

In a simple connected graph, distance is a metric; that is for all vertices *u*, *v*, and *w*,

1. $d(u,v) \geq 0$ and $d(u,v) = 0$ if and only if $u = v$.
2. $d(u, v) = d(v, u)$
3. $d(u,v) + d(v,w) \geq d(u,w)$

The diameter d (G) of a connected graph G is the length of any longest geodesic. In the graph of Fig. 1.3, d (G) is 2.

Definition 1.3.12: Complement of a Graph: Let G be a simple graph. The complement $\overline{G}$ of G is the simple graph whose vertex set is V (that is same the vertex set of G) and whose edges are the pairs of nonadjacent vertices of G.

Fig. 1.8 below is the complement of the graph of Fig. 1.3

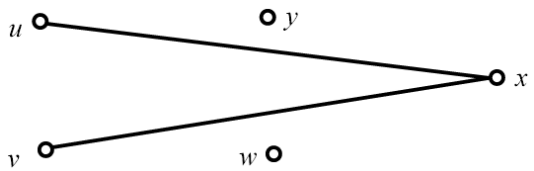


Fig. 1.8

Definition 1.3.13: Components: Connected component or Component of a graph is a subgraph in which any two vertices are connected to each other by paths, and which is connected to no additional vertices in the supergraph.

For example graph of Fig. 1.9 below is made up of three components.

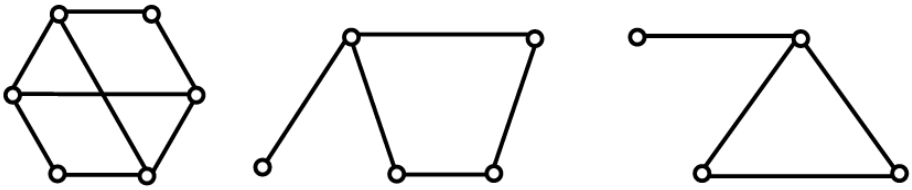


Fig. 1.9

Proof: Let G (V, E) be a disconnected and non-trivial graph and G^C be its complement. Let *u*, *v* be any two vertices in *V*. If there is no edge *uv* in G, then *uv* will be an edge in G^C. If the edge *uv* exists in G, then vertices *u* and *v* belong to same component (say H) of G. As G is disconnected it has at least two components. Let *w* be a vertex in *V* which belongs to a component other than H. Then, there are no edges *uw* and *wv* in G. Hence, *uw* and *wv* be edges in G^C, and hence we get a *u*-*v* path (*u*-*w*-*v*) in G^C.

Thus, any between two arbitrary vertices *u*, *v* of *V* there is a path in G^C. Hence, G^C is connected.

Note: The converse of the above result is not true. That is complement of a connected graph need not be connected. As an example consider the graphs in the Fig.1.10 below.

G G^C

Fig. 1.10

1.4 DIFFERENT TYPES OF GRAPHS

In this section we shall define and draw different types of graphs which will be useful for us in further discussion.

Definition 1.4.1: Complete Graph: A graph G is said to be complete, if every vertex of G is connected to every other vertex in the vertex set of G. 1.5 Mathematical Representation of Graph

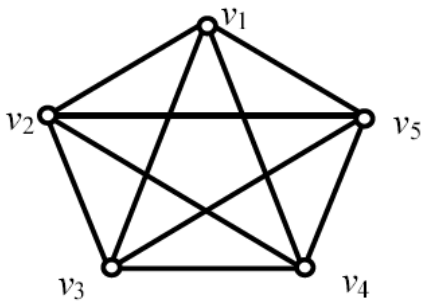
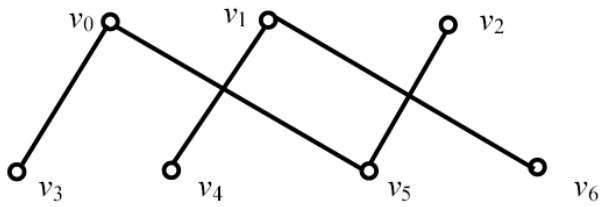


Fig. 1.10 (Complete Graph on 6 vertices)

Definition 1.4.2: Bipartite Graph: A graph G is said to be bipartite, if vertex set is divided into two disjoint sets such that no two vertices in the same set are connected.

Fig. 1.11 below is an example of a bipartite graph in which $V = V1 \cup V2$ and $V1 = \{v0, v1, v2\}$ and $V2 = \{v3, v4, v5, v6\}$.



Lemma 1.4.1: If G is a bipartite graph having partitions X and Y, then $\sum_{v \in X} \deg(v) = \sum_{v \in Y} \deg(v)$

Proof: We shall prove this lemma by induction on number of edges of G. Let $|X| = r$ and $|Y| = s$, for $r, s > 1$. (For if $r = s = 1$, then only one edge can be drawn and the lemma is trivially true.)

Take subgraph of G consisting of only vertices of G. Now, we shall start with an induction. Add one edge from any vertex of X and any vertex from Y. Then, $\sum_{v \in X} \deg(v) = \sum_{v \in Y} \deg(v)$. Now, suppose this is true for n – 1 edges, then on adding one more edge, exactly 1 is added to both $\sum_{v \in X} \deg(v)$ as well as $\sum_{v \in Y} \deg(v)$.

$K_{m,n}$: If a vertex set of a bipartite graph is partitioned into sets of sizes m and n, respectively and every vertex in the first set connected to every vertex in the set two then such a bipartite graph is known as complete bipartite graph and is denoted by $K_{m,n}$.

P_n is as a path on n vertices.
C_n is a cycle on n vertices.

It is very interesting to note the following theorems.

Theorem 1.4.1 Let G be a graph in which all vertices have degree at least two. Then G contains a cycle.

Proof: If G has a loop, it contains a cycle of length one, and if G has parallel edges, it contains a cycle of length two. So we may assume that G is simple.

Let $P := v_0 v_1 \ldots v_{k-1} v_k$ be a longest path in G. Because the degree of v_k is at least two, it has a neighbour v different from v_{k-1}. If v is not on P, the path $v_0 v_1 \ldots v_{k-1} v_k v$ is longer than P, which contradicts that P is a longest path. Therefore, $v = v_i$, for some i, $0 \leq i \leq k - 2$, and $v_i v_{i+1} \ldots v_k v_i$ is a cycle in G.

Theorem 1.4.2: Let G be undirected graph. G is bipartite if and only if it has no odd cycles.

Proof: Let G be bipartite graph. Let if possible it has an odd cycle. Let the cycle be $v_1v_2...v_{2k+1}v_1$. Let the two disjoint vertex sets of G be A and B. Then, we have $v_1 \in A, v_2 \in B, v_3 \in A, v_4 \in B$, and so on $v_{2k+1} \in A, v_1 \in B$, a contradiction that G is bipartite, as $v_1 \in A$ as well as $v_1 \in B$.

Thus, G has no odd cycle.

Conversely, let G has no odd cycles. We have to show that G is bipartite.

Without loss of generality, let G be connected, as the same logic can be applied to each of the components.
Choose any vertex v in the vertex set V of G.

Let A be the set of vertices such that the shortest path from v to each of the vertex in V is of odd length and B be the set of vertices such that the shortest path from v to each of the vertex in V is of even length. Then, $v \in B$. Also $A \cup B = V$ and $A \cap B = \phi$.

We shall prove that A, B is the partition of G.

For, if not, there exists two incident vertices x_1 and x_2, both in A or both in B. Without loss of generality, let both are in A. Then, x_1–v there is a path of odd length, v–x_2 there is a path of odd length and hence v–x_2 – x_1 – v is in an odd cycle in G, a contradiction.

Theorem 1.4.1 says that an even graph contains a cycle. Now let us prove even stronger result. That is an even graph can be partitioned into cycle and conversely.

Theorem 1.4.3: Let G (V, E) be an even graph. Then the edge set E of G can be partitioned into cycles such that no two cycles will share an edge.

Proof: Let G (V,E) be a graph whose vertices are all even. If there is more than one vertex in G, then each vertex must have degree greater than 0. Begin at any vertex u. Since the graph is connected (if the graph is not connected then the argument will be applied to separate components), there must be an edge {u, u_1} for some vertex $u_1 \neq u$. Since u_1 has even degree greater than 0, there is an edge {u1, u2}. These two edges make a trail from u to u_2. Continue this trail, leaving each vertex on an edge that was not previously used, until we reach a vertex v that we have met before. (Note: v may or may not be the same vertex as u. It does not matter either way.) The edges of the trail between the two occurrences of v must form a cycle. Call the cycle formed by this process C_1. If C_1 covers all the edges of G, the proof is complete. Otherwise, remove the edges forming C1 from the graph, leaving graph, say, G_1. All the vertices in G_1 are still even. So pick

some vertex u' in G1. Repeat the same process as before, starting with an edge $\{u', u'_1\}$. By the same argument, we can generate a new cycle C_2, which has no edges in common with C_1. If C_2 covers all the rest of the edges of G, then we are done. Otherwise, remove the edges forming C_2 from the graph, getting graph G_2, which again contains only even vertices. We continue in this way until we have used up all the edges of G. By this time we have a number of cycles, C_1, C_2,…, C_k which between them contain all the edges of G but no two of them have an edge in common.

The converse of Theorem 1.4.3 is also true and is obvious. The readers are encouraged to prove the same.

Definition 1.4.3: Regular Graph: A graph is said to be k – regular, if degree of every vertex v of G is k.

A complete graph on p vertices is $p–1$ regular. Fig. 1.12 below gives two examples of 3 – regular graphs.

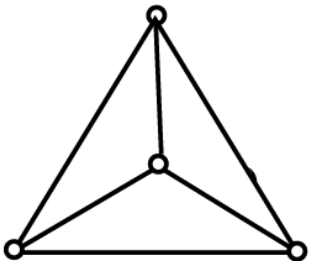


Fig. 1.12(a)

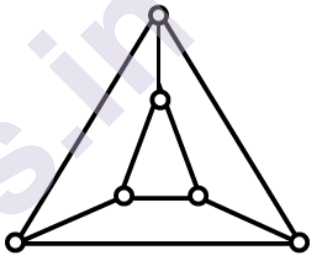


Fig. 1.13(b)

Definition 1.4.4: Tree: A connected graph having no cycle is called as a tree.

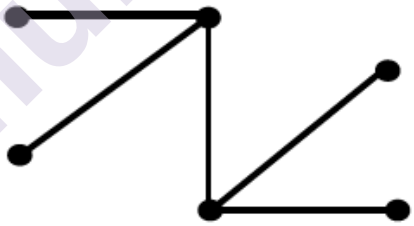


Fig. 1.13 (Tree)

Petersen's Graph : This graph on 10 vertices and 15 edges is very famous because it tends to be a counter-example to many generalizations of ideas that work for smaller graphs. As a rule of thumb, check any conjecture on the Petersen graph before trying to prove it.

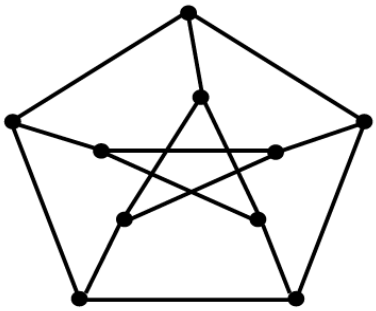


Fig. 1.14 (Petersen's Graph)

1.5 MATHEMATICAL REPRESENTATION OF GRAPH

Even though the pictorial representation of the Graph gives very good idea of the problem under consideration, to solve the problem mathematically as well as electronically, we need to have mathematical representation of the same.

The best way to represent a graph is using matrices. There are two types of matrices we shall discuss to represent graph, adjacency matrix and incidence matrix.

Definition 1.5.1: Incidence Matrix: Let G (V, E) be a graph having n vertices and m edges. The incidence matrix of G is an $n \times m$ matrix; M_G : = (m_{ve}), where m_{ve} = x where *x* is the number of times vertex v is incident with edge e

The incidence matrix of Fig. 1.3 is

$$M = \begin{bmatrix} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \\ u & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ v & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ w & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ x & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ y & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}$$

Definition 1.5.2: Adjacency Matrix: Let G (V, E) be a graph having n vertices. The adjacency matrix of G is an $n \times n$ matrix; A_G : = (m_{uv}), where

m_{uv} = 0 if there is no edge from u to v
= 1 if there an edge from u to v
= 2 if $u = v$ and there is a loop from u to itself.

The adjacency matrix of Fig. 1.3 is

$$A = \begin{bmatrix} & u & v & w & x & y \\ u & 0 & 1 & 1 & 0 & 1 \\ v & 1 & 0 & 1 & 0 & 1 \\ w & 1 & 1 & 0 & 1 & 1 \\ x & 0 & 0 & 1 & 0 & 1 \\ y & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Thus, we observe that in a simple graph, A is symmetric matrix and sum of the row (/column) elements is degree of the corresponding vertex.

Now let us prove some results based on the discussion above.

Theorem 1.5.1: The sum of the degrees of the vertices of a graph G is twice the number of edges. That is: $\sum deg\, v = 2q$, where q is number of edges in the graph.

Proof: Consider incidence matrix M of graph G. Sum of entries in every row is precisely, deg(v), and hence sum of all the entries in M is $\sum deg$ However, sum of every column is 2 as every edge has two end vertices and there are q columns corresponding to q edges and hence sum of all the entries in M is 2q. Thus, we get the required result.

Corollary 1.5.1: In any graph, the number of vertices of odd degree is even.
Proof: The proof is obvious, as if such vertices are odd in number then $\sum deg\, v$ will be odd, which contradicts Theorem 1.3.1.

If G (V, E) is a graph having no multiple edges, then it can be represented using Adjacency list, which specifies the list of adjacent vertices to every vertex in the graph. For example, the adjacency list for the graph of Fig. 1.3 is:
$u \to \{v, w, y\}; v \to \{u, w, y\}; w \to \{u, v, x, y\}; x \to \{w, y\}$ *and* $y \to \{u, v, w, x\}$

When a simple graph contains relatively few edges, that is, when it is sparse, it is usually preferable to use adjacency lists rather than an adjacency matrix to represent the graph.

Whereas if a simple graph is dense, that is, suppose that it contains many edges, say, more than half of all possible edges, then using an adjacency matrix to represent the graph is usually preferable over using adjacency lists.

Computationally speaking, adjacency matrices are more convenient than adjacency lists.

1.6 ISOMORPHISM

Let us have a look at the following graphs.

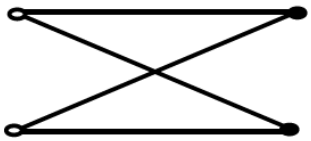


Fig. 1.15 (a)

Fig. 1.15 (b)

If the graph in Fig. 1.15(a) is made up of a string, we can changes the corners appropriately to get the graph of Fig. 1.15(b). The highlighted and non-highlighted vertices will correspond in these two graphs. Thus, we can say these graphs are similar though they do not appear to be the same. This gives us an intuitive idea about isomorphism.

Now, let us define graph isomorphism formally.

Definition 1.6.1: Isomorphism: Two graphs G1 and G2 are isomorphic (written as $G_1 \cong G_2$), if and only if there exists a one-to-one correspondence between their vertex sets, which preserves adjacency.

We make following observations from the definition above.

1. There exists a bijection *f*, from vertex set V_1 of G_1 to the vertex set V_2 of G_2.
2. Number of vertices and edges in both the graphs are same.
3. If *uv* is an edge in G_1 then *f(u) f(v)* is an edge in G_2.
4. For any vertex *v* in G_1, degree of *v* in G_1 is same as degree of *f(v)* in *G2*.

Once you see that graphs are isomorphic, it is easy to prove it. However, proving that they are not isomorphic can be sometimes very complex. It is not practically possible to check all possible correspondences. Hence, to show that two graphs are non-isomorphic, we try to find some intrinsic property that differs between the two graphs in question.

In the following examples we shall check whether given pair of graphs are isomorphic.

Example 1.6.1:

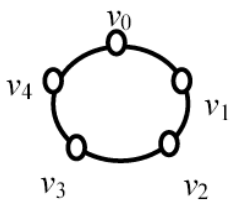


Fig. 1.16(a)

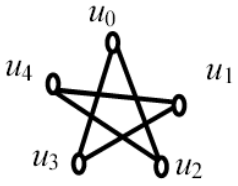


Fig. 1.16(b)

In the figures above, let us have correspondence as: $v0 \to u0, v1 \to u2, v2 \to u4, v3 \to u1$ and $v4 \to u3$. Then, this one to one correspondence defines isomorphism between these to graphs.

Example 1.6.2:

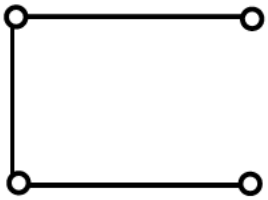


Fig. 1.17(a)

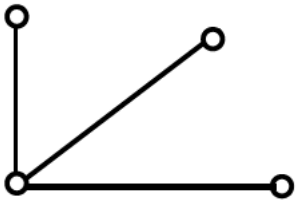


Fig. 1.17(b)

In the both the graphs of Fig. 1.17, there are 4 vertices and 4 edges each. However, in Fig. 1.17(a), 2 vertices are of degree 2 and 2 are of degree 1 and in Fig. 1.17(b), there are 3 vertices of degree 1 and one is of degree 3. Thus, these two graphs are not isomorphic.

Example 1.6.3:

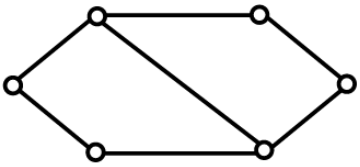


Fig. 1.18(a)

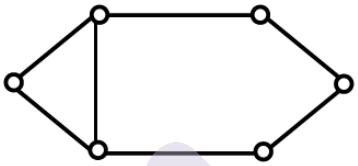


Fig. 1.18(b)

In the both the graphs of Fig. 1.18, there are 6 vertices and 7 edges each. However, in Fig. 1.15(a), there is one cycle of length 3 and in Fig. 1.15(b) has no cycle of length 3.

Thus, these two graphs are not isomorphic.

Example 1.6.4:

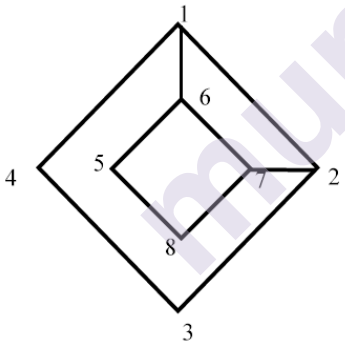


Fig. 1.19(a)

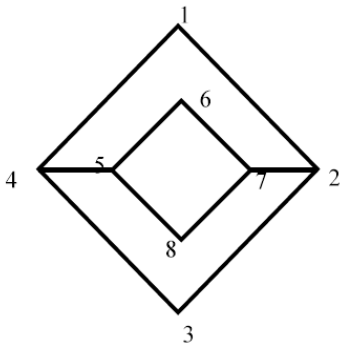


Fig. 1.19(b)

In the both the graphs of Fig. 1.19, there are 8 vertices and 10 edges each. Also there are 4 vertices of degree 3 and 4 of degree 2 in each of the graphs. However, in Fig. 1.19 (a) degree of vertices 1 and 3 is 2 and both are connected to vertices 2 and 4 of degree 3 and in Fig. 1.19 (b) vertices 3 and 4 are of degree 2 and these are connected to one vertex of degree 2 and one vertex of degree 3. Hence, we cannot find one-to-one correspondence between these two graphs and thus they are not isomorphic.

Definition 1.6.2: Automorphism: An automorphism of a graph is an isomorphism of a graph to itself.

In case of a simple graph, automorphism is just a permutation α of its vertex set which preserves adjacency. The automorphisms of a graph reflect its symmetries. For example, if *u* and *v* are two vertices of a simple graph, and if there is an automorphism α which maps *u* to *v*, then *u* and *v* are alike in the graph, and are referred to as similar vertices. Graphs in which all vertices are similar, such as the complete graph K_n, the complete bipartite graph $K_{n,n}$ are called vertex-transitive.

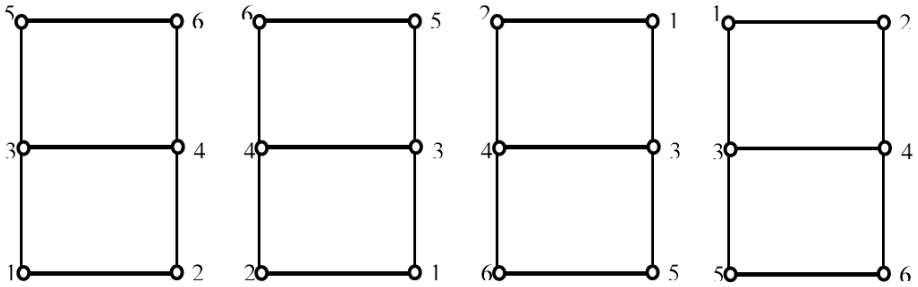


Fig. 1.20

The grid graphs of Fig. 1.20 have four automorphisms, (1, 2, 3, 4, 5, 6), (2, 1, 4, 3, 6, 5), (5, 6, 3, 4, 1, 2), and (6, 5, 4, 3, 2, 1). These correspond to the graph itself, the graph flipped leftto-right, the graph flipped up-down, and the graph flipped left-to-right and up-down, respectively, illustrated above.

1.7 SOLVED PROBLEMS

1. Let G be a graph which is isomorphic to its complement, then prove that G must have $4k$ or $4k+1$ vertices for some k.

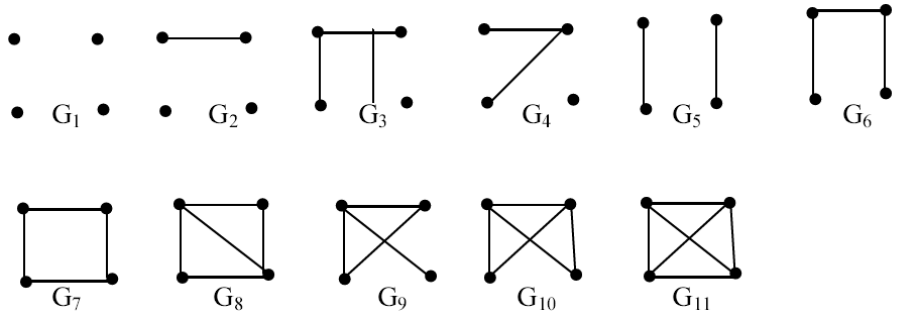
Solution:

Let $|V(G)| = p$ and $|E(G)| = q$. Since G is isomorphic to its complement, number of edges in G and G^C must be the same. Also, total number of edges in G and G^C together is $\frac{p(p-1)}{2}$. Hence, $\frac{p(p-1)}{2} = 2q$. Thus, $p(p-1)$ must be divisible by 4.

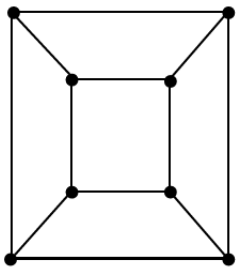
$\therefore p = 4k$ or $4k+1$ for some k.

2. Draw all non-isomorphic graphs on 4 vertices.

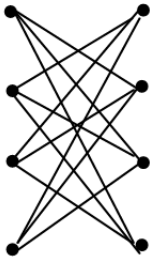
Solution: Following figures give 11 non-isomorphic graphs on 4 vertices.



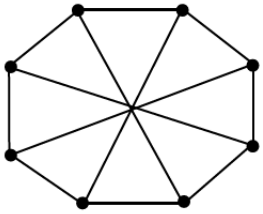
3. Which of the following graphs are isomorphic? Which of these are bipartite? Justify your answer.



G_1

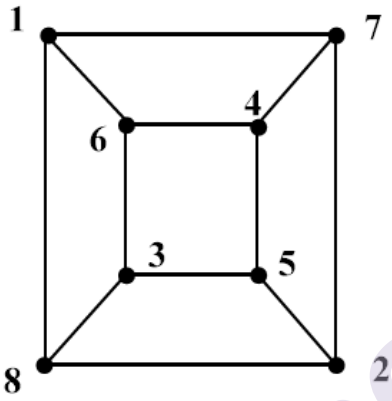


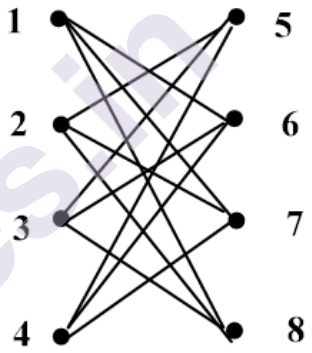
G_2



G_3

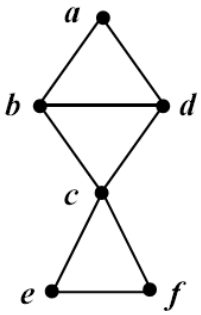
Solution: Graph G_1 has cycles are of length 4 and no cycle of length 5, whereas G_3 has cycles of length 5 and no cycle of length 4. Hence G_1 and G_3 are not isomorphic to each other. Now, we shall show that $G_1 \cong G_2$, by giving following vertex sequence.





Now, if f: $G_1 \to G_2$, by $f(i) = i$ for $1 \le i \le 8$. Then, f defines an isomorphism from G_1 to G_2. Since G_2 is isomorphic to G_1, it is not isomorphic to G_3. We further observe that G_1 (and hence G_2) have all even cycles, so G_1 and G_2 are bipartite, however, G_3 has odd cycles, hence it is not bipartite.

4. For each of the following sequences of vertices, state whether or not it represents a walk, trail, path, closed walk, closed trail, or cycle in the graph of the figure below.
(i) *abcefcbd* (ii) *abcefcd* (iii) *abcefcdba* (iv) *bcefcdb* (v) *bcdb*
(vi) *abefcd*



Solution: From the definitions of walk, trail and path, we can say the following:

(i) walk (ii) trail (iii) closed walk (iv) closed trail or circuit (v) closed path or cycle (vi) none

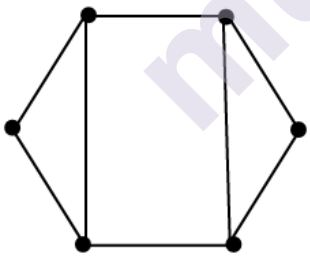
5. Let A be adjacency matrix of a simple graph G, where a_{ij} is relation between vertices v_i and v_j. Then (i, j)th term in A^k gives number of walks of lengths k from v_i to v_j.

Hence, prove that "A simple graph G with adjacency matrix A is bipartite if and only if, for each odd integer r, the diagonal entries of the matrix A^r are all 0".

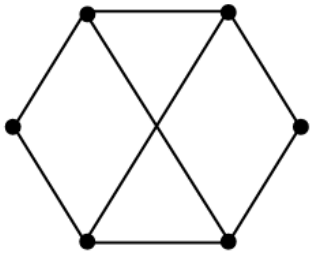
Solution : We use induction on k. When $k = 1$, a_{ij} counts the edges (walks of length 1) from v_i to v_j. When $k > 1$, every v_i- v_j walk of length k has a unique vertex v_r reached one step before the end at v_j. By the induction hypothesis, the number of v_i - v_r walks of length $k - 1$ is entry *(i, r)* in A^{k-1}. Thus, the number of v_i - v_j paths of length k that arrive via v_r on the last step is (*i, j*)th term in A_k. Thus, counting the *vi* - *vj* walks of length k via A_{k-1} by the definition of matrix multiplication, is the (i,j)th entry in A_k.

By the previous part, *(i, i)* th entry in A_r counts the closed walks of length r beginning at v_i. If this is always 0, then G has no closed walks of odd length through any vertex; in particular, G has no odd cycle and hence is bipartite. Conversely, if G is bipartite, then G has no odd cycle and hence no closed odd walk, since every closed odd walk contains an odd cycle.

6. Draw two non isomorphic graphs on 6 vertices such that each has 4 vertices of degree 3 and 2 vertices of degree 2.
Solution: Graph G_1 has triangles however graph G_2 has no triangle. Thus, these two graphs are required non isomorphic graphs.



G_1



G_2

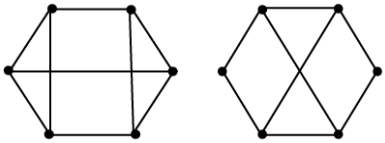
1.8 UNIT END EXERCISES:

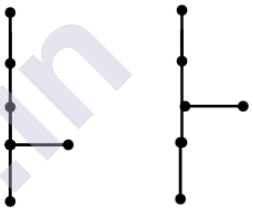
1. Prove that a complete graph on p vertices has $\frac{p(p-1)}{2}$ edges.
2. A graph G has 50 edges, four vertices of degree 2, six of degree 5, eight of degree 4 and the rest of degree 6. How many vertices does G have?
3. Prove that the complement of a complement of G is isomorphic to G.

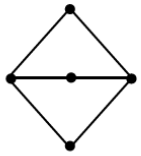
4. "If every vertex of G has degree 2 then it contains a cycle", prove or disprove.
5. Prove that, every graph has at least two vertices having same degree.
6. If G(V, E) is a k – regular bipartite graph having partitions A and B, then show that $|A| = |B|$.
7. Draw a graph having an adjacency matrix as given below.

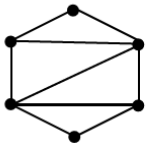
$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

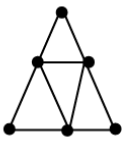
8. Which of the following pairs of graphs are isomorphic? Justify your answer for every pair.

(i) 

(ii) 

(iii) 

(iv) 



9. Given n and $0 \le r \le n-1$, where rn is even, define $H_{r,n}$ the Harary graph to be an r-regular graph on n vertices, as follows: label the vertices $0, 1, \cdots, n-1$ modulo n, and

If r is even, then join each vertex k to $k-\frac{1}{2}r$ to $k+\frac{1}{2}r$ (excluding self).

If r is odd, then join each vertex k to k to $k-\frac{1}{2}(r-1)$ to $k+\frac{1}{2}(r-1)$, then to $k+\frac{1}{2}n$.

Draw 4 – regular graph on 7 vertices and 3 regular graph on 8 vertices, using the definition above.

Will such a graph exist for rn odd? Justify your answer.

2

CONNECTIVITY

Unit Structure:

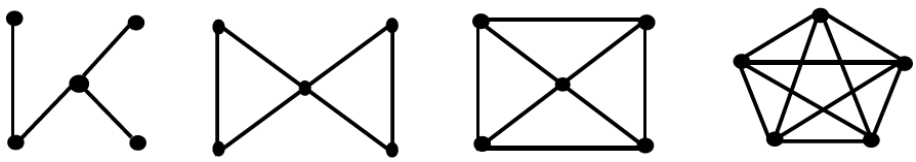
2.0 OBJECTIVES

1. Introduction
2. Vertex-edge connectivity
3. Shortest path-Dijkstra's algorithm
4. Construction of a reliable network
5. Menger's theorem

In the first chapter, we have defined connected graphs and component. In this chapter, we shall discuss graph connectivity in more details as it is of great importance in the practical applications. In the computer network, it will be crucial to know whether data can be transferred if one of the nodes or links fails. Some connected graphs can be disconnected by removing some vertices or edges. In this chapter, we shall understand the concept of connectivity further.

2.1 CUT VERTICES, BRIDGES AND BLOCKS

To understand the importance of connectivity intuitively, let us have a look at the following graphs of Fig2.1. All the graphs are on 5 vertices.



G_1 G_2 G_3 G_4

Fig. 2.1

G_1 is a minimal connected graph; deleting any edge disconnects it. G_2 cannot be disconnected by the deletion of a single edge, but can be disconnected by the deletion of one vertex, its cut vertex. There are no cut edges or cut vertices in G_3, but even so G_3 is clearly not as well connected as G_4, the complete graph on five vertices. Thus, intuitively, each successive graph is better connected than the previous one. We

now introduce two parameters of a graph, its connectivity and edge connectivity, which measure the extent to which it is connected.

For any graph G,

- δ : Minimum degree of the graph.
- Δ : Maximum degree of the graph.
- k: Connectivity of the graph that is minimum number of vertices that are to be removed to make the graph disconnected or a trivial.
- k': Edge connectivity of the graph that is minimum number of edges that are to be removed to make the graph disconnected or a trivial.

Thus, a connected graph is termed as k-connected, if we need to remove k vertices to disconnect the graph G.

In the graph G_1 of Fig. 2.1, δ = k = k' = 1.
In the graph G_2 of Fig. 2.1, δ = 2, k = 1, k' = 2.
In the graph G_3 of Fig. 2.1, δ = 3, k = 0, k' = 3.

Students are encouraged to find these parameters for the graph G4 of Fig. 2.1.

Before we proceed to prove some interesting results, let us define certain terms related to connectivity.

Definition 2.1.1: Cut vertex (Cut point): Let G (V, E) be a connected graph. Vertex $v \in V$, is called as cut vertex, if on removing v along with all its incident edges from the graph, resulting graph is disconnected.

Definition 2.1.2: Bridge: Let G (V, E) be a connected graph. An $e \in E$ is called as bridge, if on removing e from G, resulting graph is disconnected.

Definition 2.1.3: Non-separable Graph: A connected, non-trivial graph having no cut vertices is called as non-separable graph.

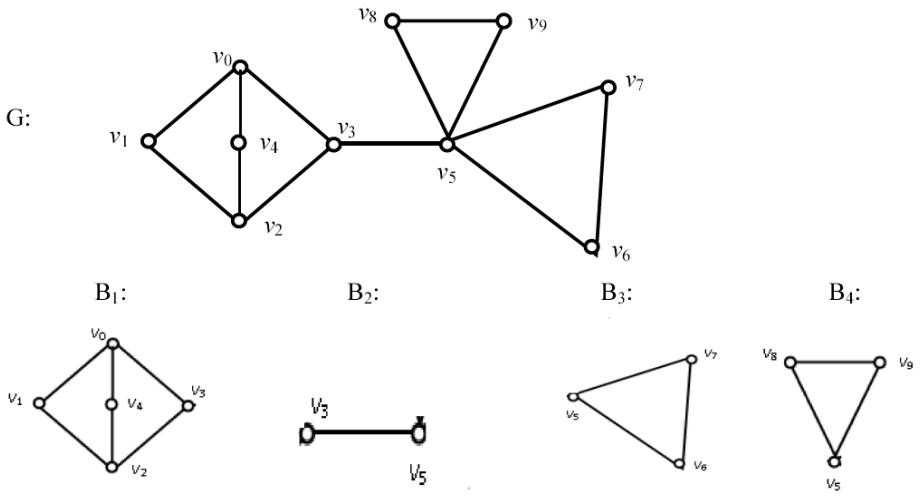


Fig. 2.2

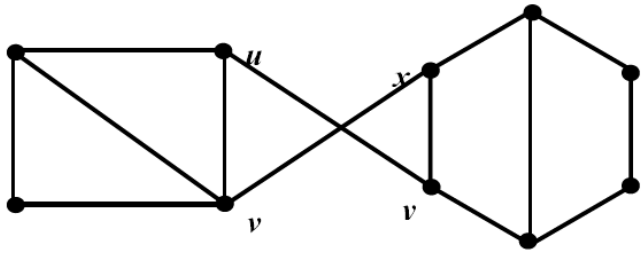


Fig. 2.3

The graph of Fig. 2.3 is 2-connected (i.e. k = 2, e.g., vertices u, v) and it has k' = 2 (i.e. on removing edges uy and vx, graph is disconnected)

Definition 2.1.4: Separation: A separation of G of order k is a pair of subgraphs (H, K) with H ∪ K = G and E(H ∩ K) = ∅ and |V (H) ∩ V (K)| = k. Such a separation is proper if V (H) \ V (K) and V (K) \ V (H) are nonempty.

E.g. Separation of graph in Fig. 2.3 is:

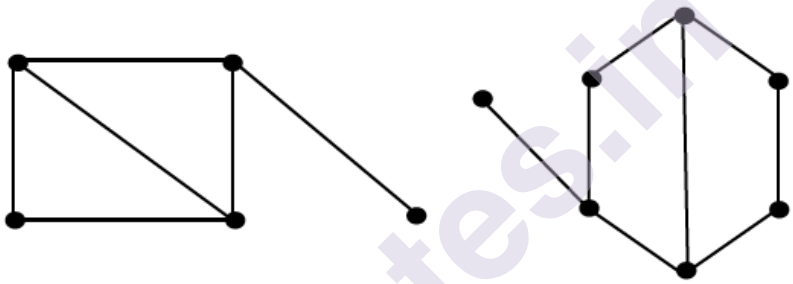


Fig. 2.4

Thus, from the definition of a separation, we observe that, a connected graph has a cut vertex if and only if it has 1 – separation. (i.e. $|V(H) \cap V(K)| = 1$).

Definition 2.1.4: Block: A block of a graph is a maximal non-separable subgraph. If G is non-separable, then G itself is a block.

In the connected graph G having vertex set $\{v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9\}$ of Fig.2.2, edge v_3v_5 is a bridge. The subgraphs B_1, B_2, B_3, and B_4 are blocks of the given graph.

We can observe that end vertices of a bridge are cut-vertices and an edge is a bridge if and only it is not on any cycle.

From the above discussion we can easily prove following theorems.

Theorem 2.1.1: Let *v* be a vertex of a connected graph G (V, E). The following statements are equivalent.

i. *v* is a cut vertex of *G*.
ii. There exist vertices *u* and *w*, distinct from *v* such that *v* is on every *u*-*w* path.

iii. There exists a partition of the set of vertices V-{v} into subsets U and W such that for any vertex $u \in U$ and $w \in W$, the point v is on every u-w path.

Theorem 2.1.2: Let x be an edge of a connected graph G (V, E). The following statements are equivalent.

i. x is a bridge of G.
ii. x is not on any cycle of G.
iii. There exist vertices u, v of G such that x in on every path joining u and v.
iv. There exists a partition of V into subsets U and W such that for any point $u \in U$ and $w \in W$, the edge x in on every path joining u and w.

Remarks:

- If a block B has at least three vertices, then B is 2-connected.
- If an edge is a block of G, then it is a cut-edge of G

Theorem 2.1.3: Two blocks in a graph share at most one vertex.

Proof: Let, if possible, B_1 and B_2 are two blocks of G, sharing two or more vertices. Then the deletion of any one of the vertices will not disconnect B_1 or B_2. Thus, B1 $\cap$ B2 is a subgraph of G having no cut vertex and a block of G, which contradicts maximality of B_1 and B_2.

- Blocks of G partition its edge set.
- If two blocks share a vertex, then it must be a cut-vertex of G.

Definition 2.1.5: Block Graph: Block graph of a graph G is a bipartite graph H in which one partition set consists of cut vertices of G and the other has a vertex b_i for every block B_i of G. We include (v, b_i) as an edge if cut vertex v is in block B_i.

Let us illustrate the definition with the help of following graph.

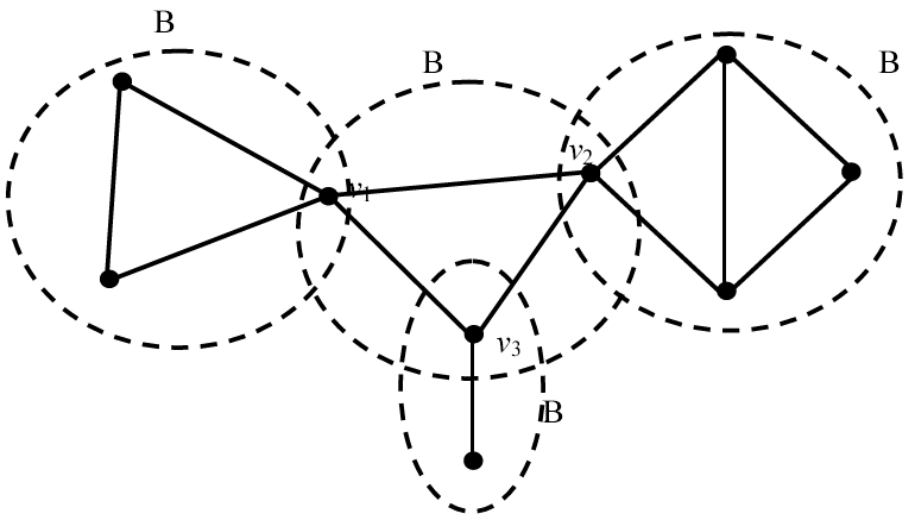


Fig. 2.4

The graph of above Fig.2.4 has four blocks B_1, B_2, B_3, B_4 and three cut vertices v_1, v_2, v_3. Hence, Block graph of the same is as below:

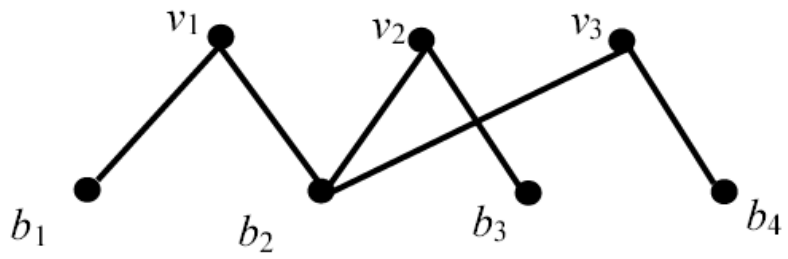


Fig. 2.5

Theorem 2.1.4: The block graph of a connected graph is a tree.

Proof: We have seen that a block graph of a connected graph is a bipartite graph. Let B(G) be a block graph of a connected graph G. By adding edges in a block of G, we do not change B(G), so let us assume that blocks are complete graphs. Since G is connected, B(G) is also connected. Now, we shall show that B(G) has no cycle.

Let if possible, it has a cycle C such that its alternate vertices correspond to cut vertices and blocks of G. It can be shown in the diagram of Fig. 2.6 below. White points represent vertices with respect to blocks and black points represent vertices with respect to cut vertices.

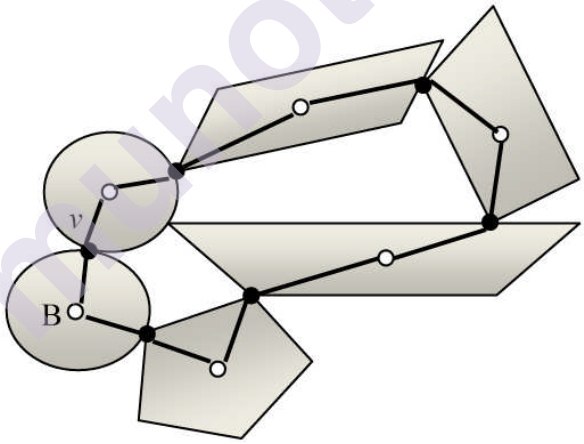


Fig. 2.6

Let cut vertices of G in order along C be $v_0, v_1, v_2, ..., v_k, v_0$, then $v_0v_1v_2...v_kv_0$ is a cycle $C \subseteq G$. If $B \in C$, then $B \cup C$ is a subgraph of G consisting of complete graph B together with a cycle C in which one edge is in B and atleast one edge not in B. Therefore, $B \cup C$ has no cut vertex, which contradicts maximality of B.

Now, we shall prove the relationship among k, k' and δ .

Theorem 2.1.5: For any graph G, $k \leq k' \leq \delta$.

Proof: We shall prove second part of the inequality first. If G has not edges, then $k' = 0$. Otherwise, if every edge incident to a vertex of

minimum degree is removed, the resulting graph is disconnected. In both these extreme cases, k' $\leq \delta$.

Now we shall prove k $\leq$ k'. Here we need to consider many cases. If G is disconnected or trivial, k = k' = 0. If G is connected and has a bridge at edge e, then k = 1, because either G has a cut-vertex which is an end vertex of e or G is K_2. Finally, let G has edge-connectivity k', such that k' $\geq$ 2. Then, removal of k' – 1 of these edges will produce a bridge, say, e = uv. For each of these k'–1 edges, select an incident vertex different from u or v. Removal of these k'–1 vertices also removes these k'–1 edges and possibly more. If the resulting graph is disconnected then k < k'. If not, e = uv is still a bridge in this resulting graph and removal of u or v will result into either trivial or disconnected graph, giving k $\leq$ k'.

Following graph in Fig. 2.3 has k = 2, k' = 3 and δ = 4.

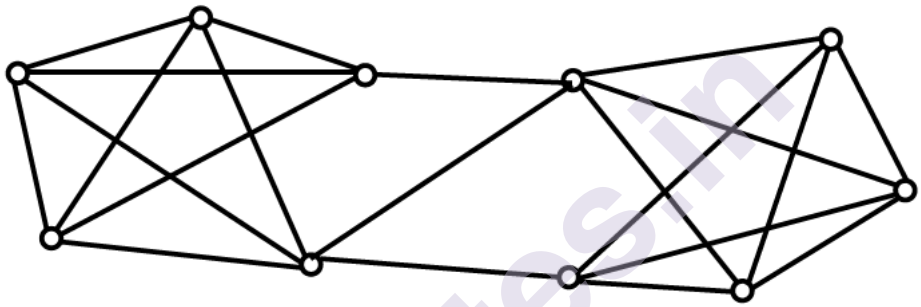


Fig. 2.6

2.2 MENGER'S THEOREMS

Menger's theorem gives characterisation of connectivity of finite, connected graphs. If *u* and *v* are any two vertices in a connected graph, then two paths connecting *u* and *v* are said to be disjoint or vertex disjoint, if they have no vertex (and hence no edge) in common. The paths are said to be edge disjoint, if there is no edge common.

Menger discusses minimum number of disjoint paths between any pair of vertices. Menger proved the results for vertex-connectivity as well as edge-connectivity.

Before we discuss Menger's theorem, let us have a look at the graph of Fig. 2.7 below:

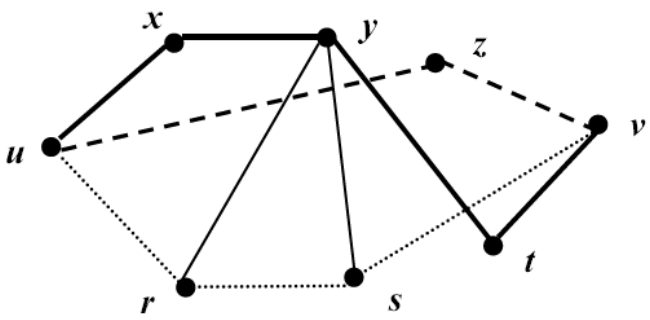


Fig. 2.7

In the Fig. 2.7, we observe that there are three vertices disjoint from u to *v, viz. u-x-y-t-v, u-zv* and *u-r-s-v*.

Vertex version of Menger's theorem discusses about the vertex disjoint path.

Theorem 2.2.1 (Menger's Theorem – Vertex Form): The minimum number of vertices separating two non-adjacent vertices s and t is the maximum number of number of disjoint paths connecting s and *t*.

Proof: If k vertices separate s and t, then obviously there can be no more than k disjoint paths. We shall show that these are exactly k. That is if there are k vertices separate s and t then exactly k internally disjoint paths separate s and t. This is true if $k = 1$. So let $k > 1$ and if possible the result is wrong. That is it takes less that k disjoint paths to disconnect s and t. Let h be the smallest such k, and let F be a graph with the minimum number of vertices for which the theorem fails for h. We remove edges from F until we obtain a graph G such that h vertices are required to separate s and t in G but for any edge x of G, only $h - 1$ vertices are required to separate s and t in $G - x$. We first investigate the properties of this graph G, and then complete the proof of the theorem.

By the definition of G, for any edge x of G, there exist a set S(x) of $h - 1$ vertices which separates s and t in $G - x$. Now $G - S(x)$ contains at least one s-t path, since it takes h points to separate s and t in G. Each such s-t path must contain the edge $x = uv$, since it is not a path in $G - x$. So $u, v \notin S(x)$ and if $u \neq s, t$, then $S(x) \cup \{u\}$ separates s and t in G. If there is a point w adjacent to both s and t in G, then $G - w$ requires $(h - 1)$ points to separate s and t and so it has $h - 1$ disjoint s-t paths. Replacing w, we have h disjoint s-t paths in G. So we have shown:

(I) No point is adjacent to both s and t in G.

Let W be any collection of h points separating s and t in G. An s –W path is a path joining s with some $w_i \in W$ and containing no other point of W, call the collections of all s-W paths and W-t paths P_s and P_t, respectively.

Then each s-t path begins with a member of P_s and ends with a member of P_t because every such path contains a point of W. Moreover, the paths in P_s and P_t, have the points of W and no others in common, since it is clear that each w_i is in at least one path in each collection and, if some other point were in both an s-W and a W-t path, then there would be an s-t path containing no point of W. Finally, either $P_s - W = \{s\}$ or $P_t - W = \{t\}$, since, if not, then both P_s plus the lines $\{w_1t, w_2t, ...\}$ and Pt plus the lines $\{sw_1, sw_2, ...\}$ are graphs with fewer points than G in which s and t are nonadjacent and h - connected, and therefore in each there are h disjoint s-t paths. Combining the s-W and W-t portions of these paths, we can construct h disjoint s-t paths in G, and thus have a contradiction. Therefore we have proved:

(II) Any collection W of h points separating s and t is adjacent either to s or to t.

Now we can complete the proof of the theorem. Let P = $\{s,u_1,u_2,...t\}$ be a shortest s-t path in G and let $u_1u_2 = x$. Note that by (I), $u_2 \neq$ t. Form $S(x) = \{v_1, v_2, ..., v_{k-1}\}$ as above, separating s and t in G – x. By (I), $u_1t \notin$ G and hence by II with $W = S(x) \cup \{u_1\}$, $sv_i \in$ G, for all i. Thus, by (I), $v_it \notin$ G, for every i. However, if we pick W = $S(x)$ $\cup$ $\{u_2\}$ instead, we have by (II) that $su_2 \in$ G, contradicting our choice of P as a shortest s-t path, and completing the proof of the theorem.

Definition 2.2.1: Line Graph: Let G be any graph. Line graph of G, denoted by L (G) is such that each vertex in L(G) represents an edge in G and if two edges are adjacent in G then there is an edge between two vertices of L(G) corresponding to these two edges.

Fig. 2.8 below gives an example of a graph and its line graph.

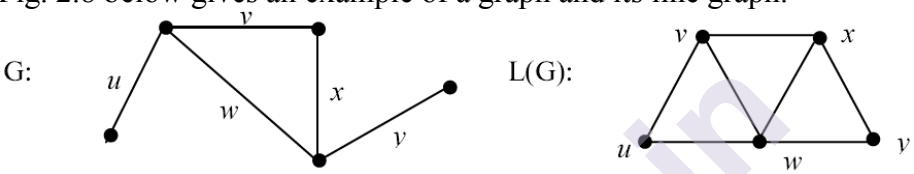


Fig. 2.8

Theorem 2.2.2 (Menger's Theorem – Edge Form): A graph G is k-edge-connected if and only if it contains k edge-disjoint paths between any two vertices.

Proof: Apply Menger's theorem-vertex form on the line graph of G.

2.3 CONSTRUCTION OF RELIABLE COMMUNICATION NETWORK

As we have seen in the first chapter, one of the applications of graph theory is to represent graph as a communication network. While constructing this network, it is necessary to make it sure that it does not get disconnected too often. Higher the connectivity and the edge connectivity of the network, the more reliable the network is. Minimum cost spanning tree constructed using Kruskal's algorithm (discussed in chapter 3), has connectivity 1 and hence it is not much reliable. Therefore, we try to generalise the problem.

Let k be given integer and let G be a weighted graph. Our aim is to find minnimum weight k – connected spanning subgraph. If $k = 1$, the problem can be solved using Kruskal's method. For $k > 1$, the problem is difficult and no solution is known. However, if G is a complete graph having unit weight, then it can be solved by the method given below.

If G is a complete graph on n vertices having unit weight, then the problem is simply to find a minimum m–connected ($m < n$) spanning subgraph, with as few edges as possible. We shall denote by $f(m, n)$, the least number of edges an m –connected graph on n vertices

can have and denote such a graph by $H_{m,n}$. Structure of $H_{m,n}$ depends on parities of m and n, and we have three cases as below.

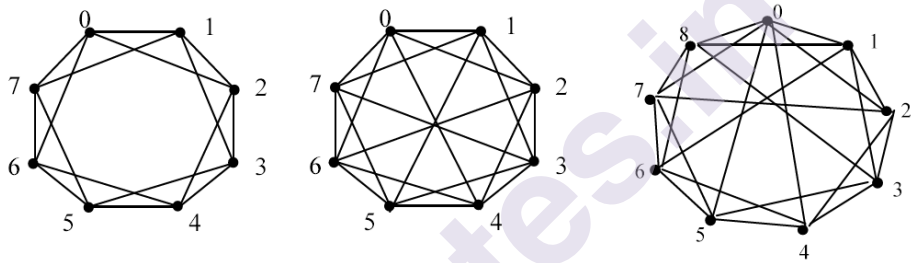
Case 1: Let m be even, $m=2r$. Number the vertices of graph on n vertices from 0 to $n-1$.

Connect vertices i and j if $j - i \equiv$ s (*mod n*) is such that $0 \leq s \leq r$.

Case 2: Let m be odd, $m = 2r + 1$ and n even. First draw Hm, n and then add edges from i to $i + (n/2)$.

Case 3: Let m be odd, $m = 2r + 1$ and n is also odd. Then $H2_{r+1, n}$ is constructed first by drawing $H_{2r,n}$ and then by adding edges joining vertex 0 to vertices $\frac{n-1}{2}$ and $\frac{n+1}{2}$ and vertex i to vertex $i+\frac{n+1}{2}$ for $1 \leq i < \frac{n-1}{2}$.

These three cases are shown in the Fig. 2.9(a), (b) and (c) below.



(a) (b) (c)

Fig. 2.9

2.4 DIJKSTRA'S ALGORITHM

If G (V, E) is any connected graph and u, v are any two arbitrary vertices in V. Then, there may exist multiple paths from u to v. One of the very important and practical applications of Graph theory is to find the shortest path from one vertex (node) to the other. The term "shortest" may refer to minimum distance, least cost or least time.

One of the most important algorithms which is of prime importance in Graph Theory is credited to a computer scientist Dijkstra. Main reason for the popularity of Dijkstra's algorithm is that it provides exact optimal solution to a large class of shortest path problems, and it is important theoretically, practically as well as educationally.

Before we proceed to discuss the example and algorithm, let us have a look at its salient features.

- Algorithm gives solution to single source shortest path problems.
- It works on both directed and undirected graphs.
- All edges must have non-negative weights.
- Graph must be connected.

Before we proceed to the algorithm, let us understand the same with the help of an example.

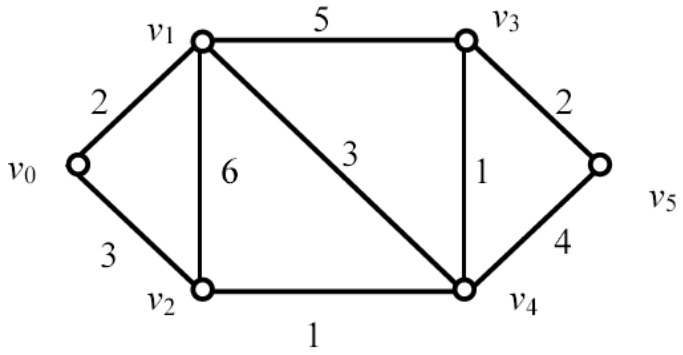


Fig. 2.7

Let us say, we have to find minimum distance from v_0 to every other vertex and the shortest path from v_0 to every other vertex.

We shall start with vertex v_0 and subsequently visit every vertex and once all the vertices are visited, we will terminate the procedure.

Let us introduce some parameters for the same.
L(i) : Shortest distance between v_0 and vertex i.
V : Set of visited vertices
U : Set of vertices not visited so far
w(i, j): Weight of edge connecting vertices i and j
P(i) : Shortest path from v_0 to i.

To begin with, we shall have $L(v_0) = 0$ and $L(v_1) = L(v_2) = L(v_3) = L(v_4) = L(v_5) = \square$. $V = \square$, $U = \{v_0, v_1, v_2, v_3, v_4, v_5\}$, $P(v_1) = P(v_2) = P(v_3) = P(v_0) = P(v_5) = \square$.

Initially, let v_0 be only visited vertex and hence V = { v_0}. We check all the edges having one end point in V and other in U for their distances and update V by picking up v_k for which L(v_k) is minimum. We keep on updating paths from v_0 to v_k every time. We shall continue the process till all the vertices of the graph are visited.

Step I:
V = { v_0 }, P(v_0) = { v_0 }. Now observe all the vertices adjacent to v0 and update these parameters as below:

L(v_1) = min {L(v_1), L(v_0) + w(v_0, v_1)} = min{ ∞, 0 + 2} = 2, P(v_1) = {v_0, v_1}
L(v_2) = min {L(v_2), L(v_0) + w(v_0, v_2)} = min{ ∞, 0 + 3} = 3, P(v_2) = {v_0, v2}

As L(v_1) is minimum, V = { v_0, v_1}.

Step II:
L(v_2) = min {L(v_2), L(v_1) + w (v_1, v_2)} = min {3, 2 + 6} = 3, P(v_2) = {v_0, v_2}

$L(v_3) = \min \{L(v_3), L(v_1) + w(v_1, v_3)\} = \min \{\infty, 2 + 5\} = 7$, $P(v_3) = \{v_0, v_1, v_3\}$
$L(v4) = \min \{L(v4), L(v1) + w(v_1, v_4)\} = \min \{\infty, 2 + 3\} = 5$, $P(v_4) = \{v_0, v_1, v_4\}$

As $L(v_2)$ is minimum, $V = \{v_0, v_1, v_2\}$

Step III:
$L(v_3) = \min \{L(v_3), L(v_2) + w(v_2, v_3)\} = \min \{7, 3+\square\} = 7$, $P(v_3) = \{v_0, v_1, v_3\}$
$L(v4) = \min \{L(v_4), L(v_2) + w(v_2, v_4)\} = \min \{5, 3+1\} = 4$, $P(v_4) = \{v_0, v_2, v_4\}$

As $L(v_4)$ is minimum, $V = \{ v_0, v_1, v_2, v_4\}$

Step IV:
$L(v_3) = \min \{L(v_3), L(v_4) + w(v_4, v_3)\} = \min\{7, 4+1\} = 5$, $P(v_3) = \{v_0, v_2, v_4, v_3\}$
$L(v_5) = \min\{L(v_5), L(v4) + w(v_4, v_5)\} = \min \{\square, 4+4\} = 8$, $P(v_5) = \{v_0, v_2, v_4, v_5\}$

As $L(v_3)$ is minimum, $V = \{v_0, v_1, v_2, v_4, v_3\}$

Step V:
$L(v_5) = \min \{L(v_5), L(v_3) + w(v_3, v_5)\} = \min \{8, 5+2\} = 7$, $P(v_5) = \{v_0, v_2, v_4, v_3, v_5\}$

Thus, $V = \{v_0, v_1, v_2, v_3, v_4, v_5\}$.

All the vertices are visited and we have found the shortest path and distance from v_0 to every other vertex.

In fact, what we have presented above is Dijkstra's shortest path method.

Now, let us present the steps of the algorithm.

Dijkstra's Shortest Path Algorithm:

Let G (V, E) be any connected, weighted graph, with $G_V = \{v_0, v_1, v_2, ..., v_n\}$.

Parameters:
$L(v_i)$: Shortest distance between source $s = v_0$ and vertex v_i.
$P(v_i)$: Set of vertices giving shortest path from s to v_i.
V : Set of visited vertices
U : Set of vertices not visited so far
$w(v_i, v_j)$: weight of an edge from v_i to v_j.
z : Destination vertex

Initial Values:
$V = \phi$, U = GV, $L(s = v_0) = 0$, $P(s = v_0) = \{v_0\}$
$L(v_i) = \infty$; for $1 \leq i \leq n$
$P(v_i) = \phi$; for $1 \leq i \leq n$

Procedure:

```
While z ∈ U
Begin
        u : vertex in U with L(u) minimum
        V := V ∪ {u}
        For every x ∈ U
                Begin
                        If (L(x) > L(u) + w (u, x)) then
                           (x) := L(u) + w(u, x)
                           P(x) := P(u) ∪ {x}
                End If
        End
End
```

Solved Problems:

1. If G is a bipartite k-regular graph such that $k \geq 2$, then prove that G has no bridge.

Solution: We will prove the result by contradiction. Assume G has a bridge $e = uv$.

Let's start with a couple of easy observations. Firstly, note that a bridge affects only the connected component it belongs to. Every connected component of a bipartite k regular graph is itself bipartite k-regular, so we can assume, without loss of generality, that G is a connected bipartite k-regular graph. Secondly, removal of an edge can split a connected graph into at most two connected components - to see why, observe that if we restore the edge, the graph should be connected, but three or more disjoint components cannot be linked by a single edge. Now assume G has classes A and B, where $u \in A$ and $v \in B$. Removal of e splits G into disjoint components G_1 and G_2. Let A_0 be the set of vertices of A in G_1 and A_{00} be those in G_2 - both these sets are non-empty. Similarly let B_0, B_{00} be the vertices of B in G_1 and G_2 respectively. Observe that the bridge e must be the only edge linking G_1 and G_2, and assume without loss of generality that $u \in A_0$ and $v \in B_{00}$. Now look at G_1, which is a bipartite graph with classes A_0 and B_0. Since e is the only edge linking G_1 and G_2, every other edge of G incident on A_0 or B_0 is retained in G_1. So every vertex in A_0 and B_0 still has degree k in G_1, except u which has degree $k - 1$. Let $a := |A_0|$ and $b := |B_0|$. Since no edge links two vertices in B_0 (bipartite property), the number of edges in G_1 is simply kb (every edge is incident to some vertex in B_0, so we can add up the degrees of the vertices in B_0). Similarly, adding up the degrees in A_0 instead, the number of edges is $k(a - 1) + k - 1$. Equating the two formulae, we have $k(a - 1) + k - 1 = kb$, $\Rightarrow k(a - b) = 1$. But this implies $k = 1$, which contradicts the given condition that $k \geq 2$. Hence the bridge cannot exist.

2. If G is a graph on p vertices such that $\delta > \frac{p-1}{2}$, then prove that G is connected.

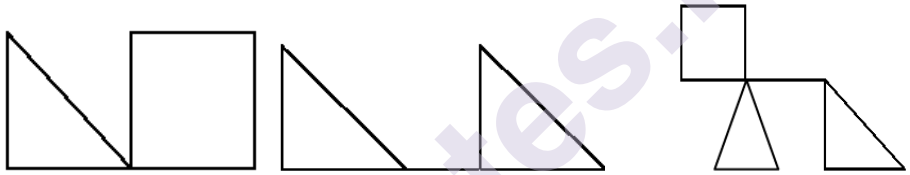
Solution: Let, if possible, G is not connected and has at two components G_1 and G_2 (same argument can be used, for more than two components). Let $|V(G_1)|$ = r and $|V(G_2)|$ = s, such that r ≤ s. Then, $\delta \le r-1 \le \frac{p}{2}-1 < \frac{p-1}{2} \Rightarrow \delta < \frac{p-1}{2}$. A contradiction to given hypothesis that $\delta < \frac{p-1}{2}$.

3. If G is k' edge-connected having q edges and p vertices, then $q \ge \frac{k'p}{2}$.

Solution : From Theorem 2.1.5, we have k' δ ≤ i.e. pk' ≤ pδ
As minimum degree of G is δ, we have $p\delta \le \sum \deg(v) = 2q$. (By Theorem 1.5.1)
Thus, $q \ge \frac{k'p}{2}$.

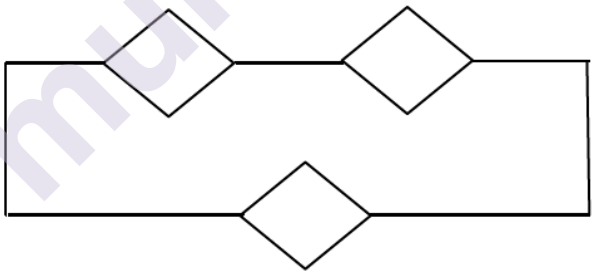
2.5 UNIT END EXERCISES:

1. Find cut vertices and cut edges of the following graphs.

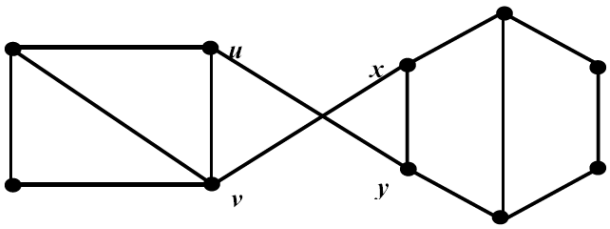


G_1 G_2 G_3

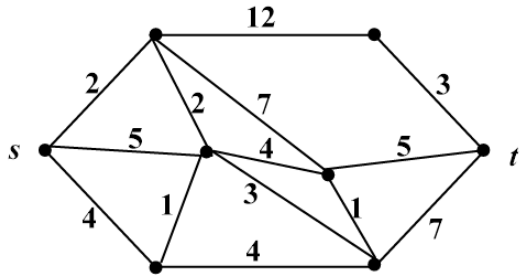
2. Find vertex connectivity (k) and edge connectivity (k') for the following graphs.



3. Give an example of a graph for which k < k' < δ.
4. Find blocks of the following graph and hence draw its block graph.



5. Use Dijkstra's algorithm to find the shortest path from s to t in the following graph.



6. If G is a connected graph having a bridge, then show that G has a cut vertex. Is the converse true? Justify your answer.

❖❖❖❖

3

TREES

Unit Structure:

3.0 OBJECTIVES

1. Characterization of trees.
2. Detailed discussion on spanning trees.
3. Kruskal's algorithm
4. Huffman Code
5. To know edge cuts, bonds, cut vertex and cut edge.
6. Construction of vector space and cycle subspace associated with a graph
7. Graph traversals

3.1 INTRODUCTION

In the chapter 2, we have defined trees and also its relationship with block graphs. In this chapter, we shall explore trees in details and also discuss a few practical applications of the same.

First, we will be enumerating all the trees on n vertices. Before that, let us draw trees on 7 vertices. A Few of such trees are shown in Fig. 3.1.

Following figure gives trees on 7 vertices.

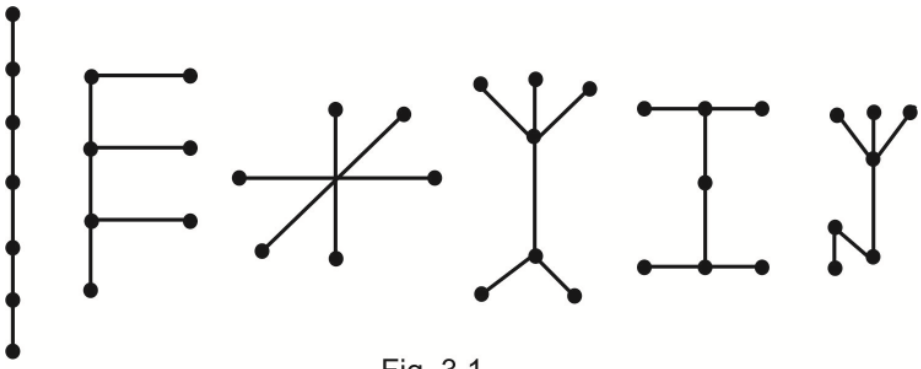


Fig. 3.1

Let us know a few more terms related to tree. As tree is an acyclic graph, acyclic graphs are called as Forest.

We have seen a spanning subgraph of a graph G in the chapter 1. If a spanning subgraph of a connected graph is a tree, it is called as spanning tree. Spanning trees are of utmost importance in the applications of graph theory.

Fig. 3.2(b) below is a spanning tree of the graph of Fig.3.2(a).

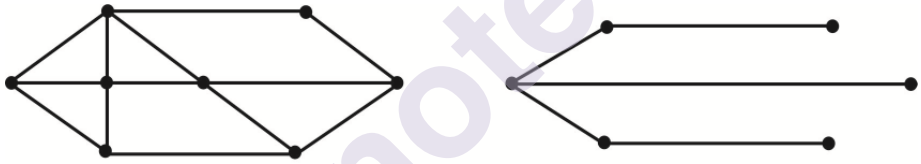


Fig. 3.2 (a) Fig. 3.2 (b)

We also observe the following facts about trees from the definition of tree and the discussion so far.

1. In a tree, any two vertices are connected by exactly one path. (For more than one path would result into a cycle.)
2. From Theorem 1.4.1, we know that if degree of every vertex is at least two, then the graph contains a cycle. And hence every tree has at least one vertex of degree at most one. In fact, if the tree is non-trivial then it has a vertex of degree exactly one. Vertex of degree one in a tree is termed as leaf.
3. Result: For a tree p = q + 1, or number of vertices is exactly one more than the number of edges.

Proof: Let T be a tree having p vertices and q edges. If T is trivial, then p = 1 and q = 0. Hence, the result is true. If T is K2, then p = 2 and q = 1. Again the result holds true. Let T be non-trivial and not K2. We shall prove the result by induction on number of vertices. Let the result be true for a tree having less than p vertices. Now, T has p vertices and it is non-trivial. Hence, it has at least one leaf, say at vertex v. Then, T – v is again a tree having p – 1 (< p) vertices and q –

1 edges. By induction hypothesis, p – 1 = (q – 1) + 1. Simplifying we get p = q + 1, as required.

4. Every non-trivial graph has at least two leaves. This follows easily from the theorem above, as $\sum di = 2q = 2(p - 1)$.

5. Let e = uv be any edge in tree T. Then, e has to be its cut edge, for if not, after removing e from T, T will be connected and hence, we will get a uv path in T – e, a contradiction to property 1 above.

6. Every non-leaf vertex of a tree is its cut vertex. For, let u be a vertex of a tree having degree at least 2 and v and w are its adjacent vertices. Then, uv and uw are the only paths from u to v and u to w respectively. Hence, on removing u from the tree, it becomes disconnected.

7. Every tree is bipartite. To see that, choose any vertex v from tree. Now divide vertices of the tree in two sets A and B such that, $u \in A$, if d(v, u) is even and $u \in B$, if d(v, u) is odd. By this choice, $v \in A$. Thus, $A \cup B = V$ (vertex set of the tree) and $A \cap B = \phi$.

8. A graph is connected if and only if it has a spanning tree. A spanning tree is a connected graph. From a connected graph, delete one edge at a time to remove cycles and we get a spanning tree.

3.2 CHARACTERISATION OF TREES

Theorem 3.2.1: Let G(V, E) be a graph having p vertices and q edges. The following statements are equivalent for G.

i. G is a tree.
ii. Every two points of G are joined by a unique path.
iii. G is connected and p = q + 1.
iv. G is acyclic and p = q + 1.
v. G is acyclic and any two nonadjacent vertices of G are joined by an edge e, then G + e has exactly one cycle.
vi. G is connected and every edge of G is in cut edge.

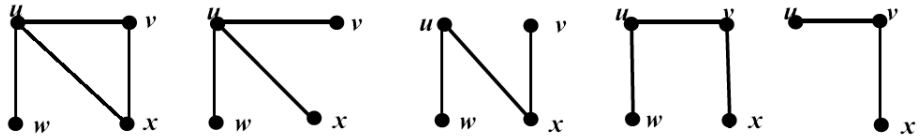
This characterisation of tree can be proved from the discussion so far.

3.3 EDGE CUTS AND BONDS

First, we shall start with edge cuts.

Let G(V, E) be a graph and X, Y by subsets of V, not necessarily distinct. We denote E[X, Y] to be the set of all edges of G having one end in X and the other in Y and by e(X, Y) their number. If X = Y, we simply write as E(X) and e(X) for E[X, X] and e(X, X). If Y = V – X, the set E[X, Y] is called as the edge cut associated with X and is denoted by ∂ (X). Obviously, ∂ (X) = ∂ (V – X) and ∂ (V) = ϕ.

Let us illustrate edge cuts with an example.



G ∂ (u) ∂ (u, v) ∂ (u, x) ∂ (u, w, x)

Fig. 3.3

The minimal edge cut of a graph is called as bond, thus bond is an edge cut such that none of its edge-subset is an edge cut. To illustrate, let us have a look at the following figure.

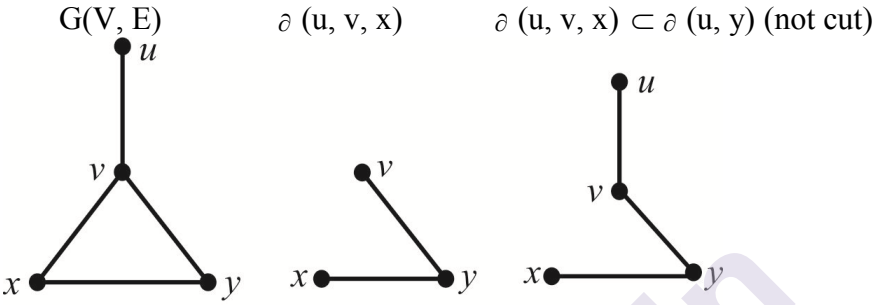


Fig. 3.4

In the Fig. 3.4, ∂ (u, v, x) is a bond whereas ∂ (u, y) is not as it has a subset which is an edge cut.

Now we shall discus two important results associated with edge cuts.

3.4 GRAPHS AND VECTOR SPACE

Before, we define a vector space on graph, let us first define a binary operation - symmetric difference (denoted by Δ), on graphs. Let G (V, E) be a graph, having p vertices and q edges.

Let $E = \{e_1, e_2, ..., e_q\}$. Let E_1 and E_2 be two subsets of E. We define, $E_1 \Delta E_2$ as:

$E_1 \Delta E_2 = E_1 \cup E_2 - E_1 \cap E_2$. Thus, for a graph in Fig. 3.5 below, we shall illustrate Δ .

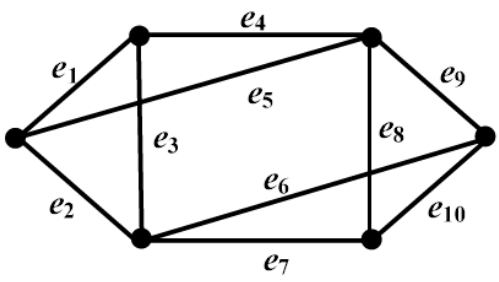


Fig. 3.5

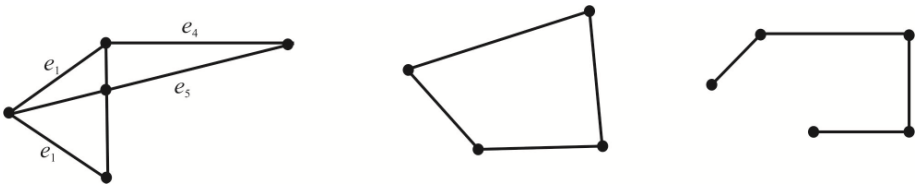


Fig. 3.6

Fig. 3.5 shows a graph on 8 vertices and 10 edges. E_1 and E_2 are two subsets of edge set E of the graph. Then the Fig. 3.6 shows $E_1 \Delta E_2$.

$E_1 = \{e_1, e_2, e_4, e_5\}$, $E_2 = \{e_2, e_5, e_7, e_8\}$, then $E_1 \Delta E_2 = \{e_1, e_4, e_7, e_8\}$

Notations : If X and Y are two subsets of edge set E of a graph G(V, E), having p vertices and q edges, we associate vectors $X = (x_1, x_2, ..., x_q)$ and $Y = (y_1, y_2, ..., y_q)$, such that, $x_i = 1$, $y_j = 1$, if edge $x_i \in X$, edge $y_j \in Y$, else $x_i = 0$, $y_j = 0$. We define operation Δ on the elements of X, Y with a rule: $1 \Delta 1 = 0$, $0 \Delta 0 = 0$, $1 \Delta 0 = 1$ and $0 \Delta 1 = 1$. Thus, for the edge sets E_1 and E_2 in the example above, we have, vector $E_1 = (1, 1, 0, 1, 1, 0, 0, 0, 0, 0)$ and vector $E_2 = (0, 1, 0, 0, 1, 0, 1, 1, 0, 0)$ and hence vector $E1 \Delta E2 = (1, 0, 0, 1, 0, 0, 1, 1, 0, 0)$. Thus, $E1 \Delta E2 = \{e_1, e_4, e_7, e_8\}$, as we have obtained earlier.

Thus, the q-vector representing the symmetric difference of q-vectors E_1 and E_2 is in fact the q-vector of symmetric difference of E_1 and E_2. But, E_1 and E_2 are two subgraphs of G. Hence, we have defined binary operation on the subgraphs and also represented these subgraphs in the form of a vector. The set of all 2 q q-vectors, (all zeros indicate null graph), is a set of all edge induced subgraphs of G. We denote it by ε (G). This set forms a vector space on the field GF(2) or Z_2. This can be easily verified.

1. ε (G) is an a belian group under Δ.
a. Let X, Y be any two vectors (edge sets) in ε (G). Then, $X \Delta Y$ is also a vector in ε (G), and hence ε (G) is closed under ε (G).
b. Vector associated with ϕ is (0, 0, ..., 0) and $X \Delta (0, 0, ..., 0) = X = (0, 0, ..., 0)$. Thus, identity exists.
c. If X is any edge vector, then, $X \Delta X = 0$ (zero vector). This shows the existence of inverse with respect to Δ.
d. The operation Δ is clearly commutative.
2. Scalar multiplication on vectors is distributive.
3. Scalar multiplication is associative.

Also observe that vectors (1, 0, 0, ..., 0) associated with edge set $\{e_1\}$, (0, 1, 0, ..., 0) associated with edge set $\{e_2\}$, ..., (0, 0, ..., 0, 1) associated with edge set {eq}, forms a basis for the vector space. Thus, the dimension of the vector space ε (G) is q.

Definition 3.3.1: Fundamental Cycle: Let T be any spanning tree of a connected graph G.

Adding just one edge to a spanning tree will create a cycle; such a cycle is called a fundamental cycle with respect to a spanning tree T.

There is a distinct fundamental cycle for each edge; thus, there is a one-to-one correspondence between fundamental cycles and edges not in the spanning tree. For a connected graph with p vertices, any spanning tree will have p − 1 edges, and thus, for a graph of q edges and any one of its spanning trees will have q − p + 1 fundamental cycles.

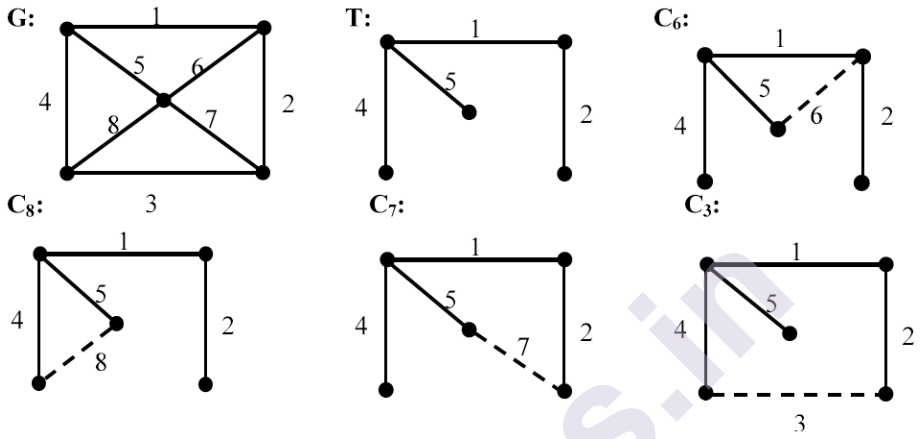


Fig. 3.7

Note: Number on the edge indicate just an edge number, so 3 means e_3.

In the above Fig. 3.7, T is a spanning tree of graph G and C_3, C_6, C_7 and C_8 are fundamental cycles of G with respect to T. (C_i is obtained from T by adding an edge e_i).

For any given spanning tree the set of all q − p + 1 fundamental cycles forms a cycle basis, a basis for the cycle subspace of ε (G).

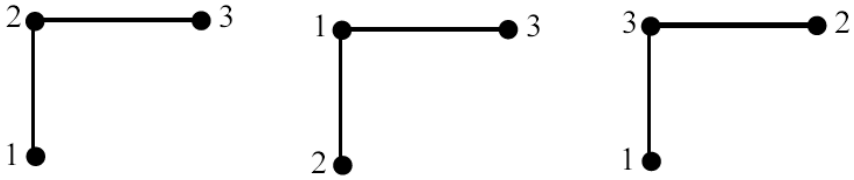
3.5 CAYLEY'S FORMULA

Cayley's formula counts the number of labelled trees on n vertices. In other words, it counts the number of spanning trees of a complete graph K_n. However, it does not count the number of non-isomorphic trees on *n* vertices.

Before we proceed to the formula, let us find number of labelled trees for small values of *n* such as 2, 3, 4 and then we shall generalise using Cayley's formula.

For $n = 2$,  $\mathbf{T_2 = 1}$

For $n = 3$, $\mathbf{T_2 = 3}$



For $n = 4$, $T_4 = 16$

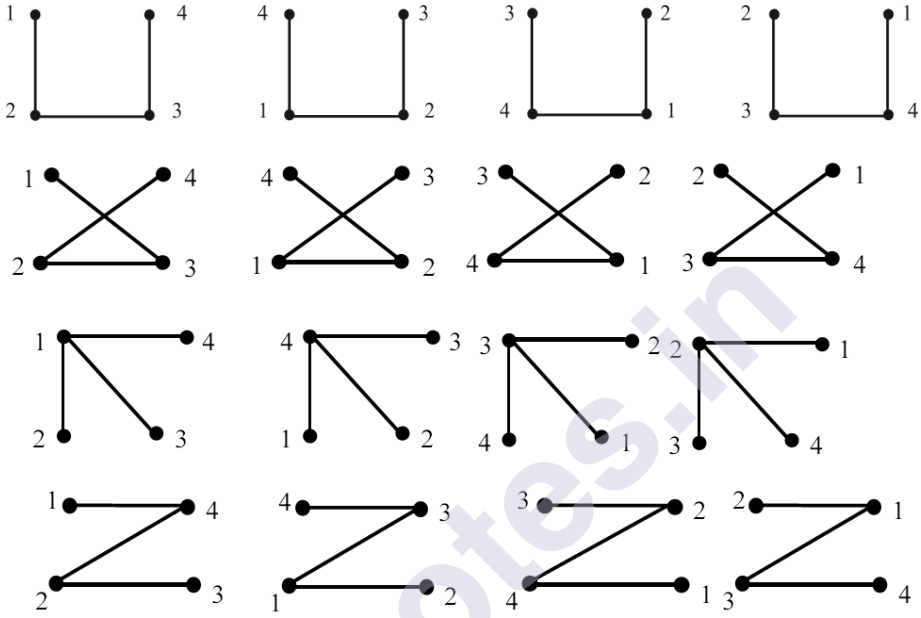


Fig. 3.8

Let Tn denote number of labelled trees on n vertices. Then, Cayley's Formula states that:
$T_n = n^{n-2}$.

Now let us count number of labelled trees on n vertices. In fact, Cayley's formula gives this count and many proofs of the formula are available. We shall use the simplest algorithm for counting that is "Prüfer Encoding". Before we proceed to the proof of the formula, let us understand "Prüfer Encoding".

Prüfer Encoding:

The most straight forward method of showing that a set has a certain number of elements is to find a bijection between that set and some other set with a known number of elements. In this case, we are going to find a bijection between the set of Prüfer sequences and the set of spanning trees.

A Prüfer sequence is a sequence of $n - 2$ numbers, each being one of the numbers from 1 to n. We observe that, there are n^{n-2} Prüfer sequences for any given n, where we allow repetitions. The following is an algorithm that can be used to encode any tree into a Prüfer sequence. Let T_n be a set of all trees on n vertices.

Algorithm (Coding):

1. Take any tree, t $\in$ T_n, whose vertices are labelled from 1 to n in any manner.
2. Let i = 1, t_1 = t.
3. From t_i, choose vertex v with the smallest label whose degree is equal to 1, and write down the value of its only neighbour, say ai ($1 \leq i \leq n-1$) (We have already shown that any tree must have at least two leaf vertices).
4. Construct tree t_{i+1} from t_i by removing vertex v and edge va_i.
5. Update i to $i + 1$. Repeat from step 3 for the new, smaller tree. Continue until only one vertex remains.
6. Drop last neighbour from the list to get a sequence of $n - 2$ vertices. (In fact, we observe that the last in the least is always the vertex with the highest label.)

Now, we shall apply this algorithm to a tree on 8 vertices below.

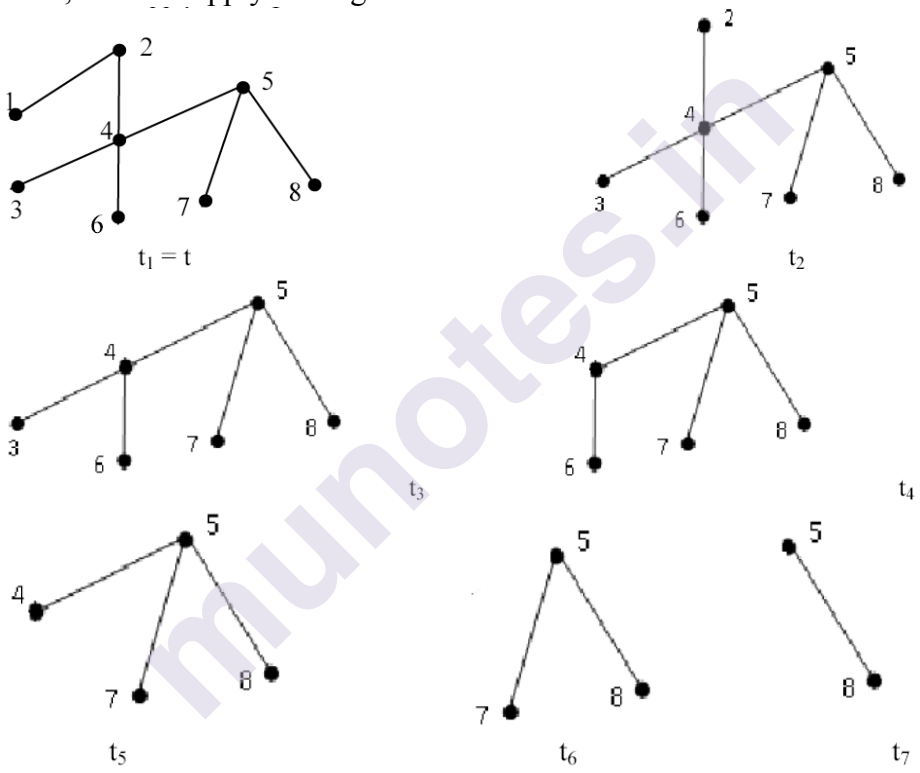


Fig. 3.9

Step 1: Remove vertex v = 1 and $a_1 = 2$
Step 2: Remove vertex v = 2 and $a_2 = 4$
Step 3: Remove vertex v = 3 and $a_3 = 4$
Step 4: Remove vertex v = 6 and $a_4 = 4$
Step 5: Remove vertex v = 4 and $a_5 = 5$
Step 6: Remove vertex v = 7 and $a_6 = 5$
Step 7: Remove vertex v = 5 and $a_7 = 8$

Now, as per the algorithm, we have to drop a7 = 8 and hence the Prüfer sequence is: {2, 4, 4, 4, 5, 5}.

We observe that the vertex label frequency in the sequence is deg (vertex) – 1. Also a vertex with degree one never appears in the sequence.

Thus, we have seen how to construct a Prüfer Sequence from a tree.
Now is the time to construct a tree from a given Prüfer sequence P.
We shall first discuss the algorithm for the same.
Algorithm (Decoding):

1. P is given Prüfer sequence and L = {1, 2, ..., n–1, n}.
2. Let *j*: First label in P and *k*: the least number in L that does not occur in P.
3. Connect an edge between *j* and *k*.
4. Remove *j* and *k* from P and L respectively.
5. Perform steps 2 to 4, till P is non-empty.
6. There will be exactly two numbers in L remaining. Join an edge between these two.

Thus, we get a tree corresponding to given Prüfer sequence.

Now, let us apply it on the Prüfer sequence P = {2, 4, 4, 4, 5, 5}.
Step 1: P = {2, 4, 4, 4, 5, 5}; L = {1, 2, 3, 4, 5, 6, 7, 8}; j = 2; k = 1
Step 3: P = {4, 4, 4, 5, 5}; L = {2, 3, 4, 5, 6, 7, 8}; j = 4; k = 2
Step 4: P = {4, 4, 5, 5}; L = {3, 4, 5, 6, 7, 8}; j = 4; k = 3
Step 5: P = {4, 5, 5}; L = {4, 5, 6, 7, 8}; j = 4; k = 6
Step 6: P = {5, 5}; L = {4, 5, 7, 8}; j = 5; k = 4
Step 7: P = {5}; L={5, 7, 8}; j = 5; k = 7
Step 8: P = ϕ; L = {5, 8}
Step 9: Connect edge 5-8.

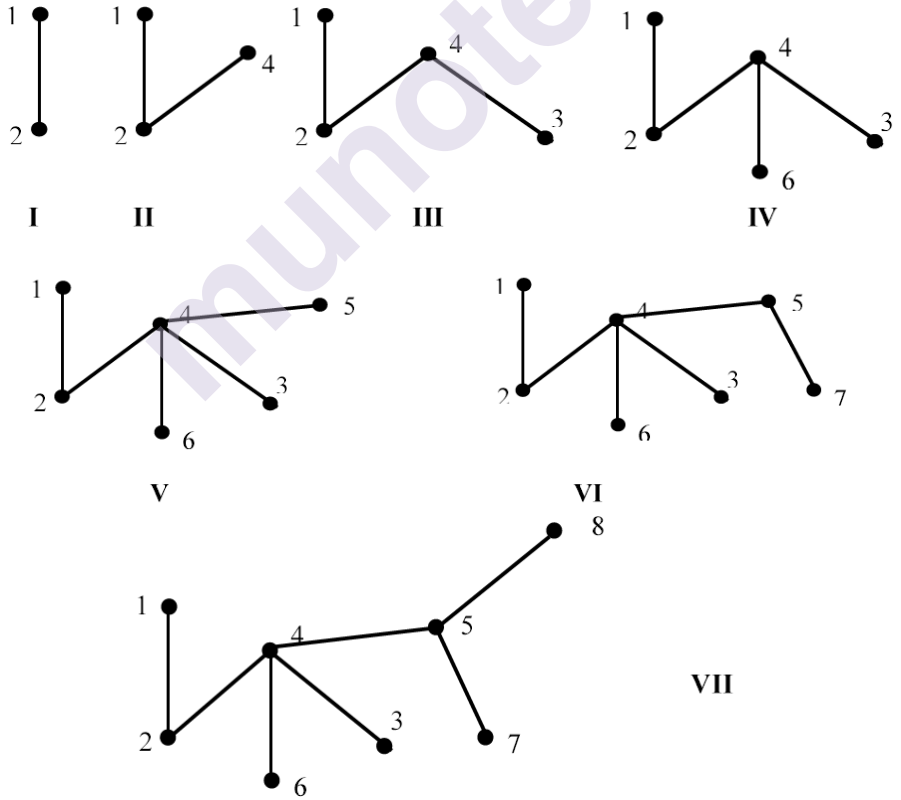


Fig. 3.10

Following the above steps, we have now reconstructed our original tree on 8 vertices, as in Fig. 3.9. It may be oriented differently, but all of the vertices are adjacent to their correct neighbours, and so we have the correct tree back. Since there were no ambiguities on how

to encode the tree or decode the sequence, we can see that for every tree there is exactly one corresponding Prüfer Sequence, and for each Prüfer Sequence there is exactly one corresponding tree. More formally, the encoding function can be thought of as taking a member of the set of spanning trees on n vertices, Tn, to the set of Prüfer Sequences with n−2 terms, Pn . Decoding would then be the inverse of the encoding function, and we have seen that composing these two functions results in the identity map. If we let f be the encoding function, then the above statements can be summarized as follows:
$f: T_n \rightarrow P_n$, $f^{-1} : P_n \rightarrow T_n$, and $f^{-1} \circ f = I$.

Since we have found a bijective function between Tn and Pn , we know that they must have the same number of elements. We know that $|Pn| = n^{n-2}$, and so $|Tn| = n^{n-2}$.

3.6 MINIMAL COST SPANNING TREE AND KRUSKAL'S ALGORITHM

In many real life problems in Graph Theory, such as, networking, travelling salesman, it is necessary to find the minimum cost spanning tree of a graph, where, weight is some value associated with the graph, which may represent distance, cost etc.

One of the popular algorithms is credited to Kruskal, which finds minimum cost spanning tree for a given connected graph. We shall look at the algorithm for a weighted, connected graph G.

Like Dijkstra's algorithm, it also follows greedy approach. The fundamental idea behind Kruskal's algorithm is very simple. Start with a null graph. At each step, choose an edge which is not added so far and has the least weight. If this edge forms a cycle then look for the other edge else add this chosen edge.

Algorithm: The Kruskal Algorithm
Input: a weighted connected graph G = (G, w)
Output: an optimal tree T = (V,T) of G, and its weight w(T)

1: set T := ϕ , w(T) := 0 (T denotes the edge set of the current graph)
2: while there is an edge e $\in$ E \ T such that T $\cup$ {e} is the edge set of a graph do
3: choose such an edge e of minimum weight
4: replace T by T $\cup$ {e} and w (T) by w (T) + w(e)
5: end while
6: return ((V, T), w(T))

Let us apply Kruskal's algorithm to get the minimum cost spanning tree as in Fig.3.11.

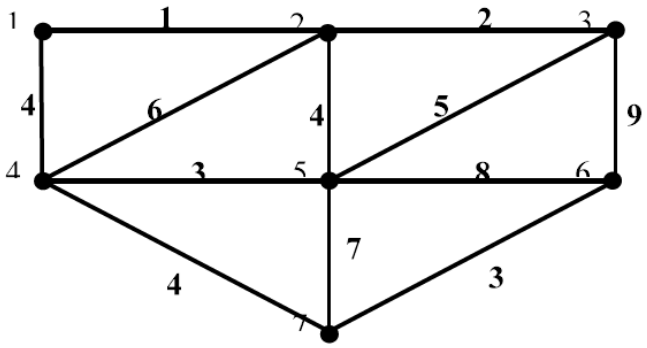


Fig. 3.11

1. T = ϕ, w(T) = 0
2. T = {12}, w(T) = 1
3. T = {12, 23}, w(T) = 3
4. T = {12, 23, 67}, w(T) = 6
5. T = {12, 23, 67, 45}, w(T) = 9
6. T = {12, 23, 67, 45, 14}, w(T) = 13
7. Note: Though wt(edge2-5) is 4, it is not chosen as it forms a cycle
8. T = {12, 23, 67, 45, 14, 47}, w(T) = 17
9. Done

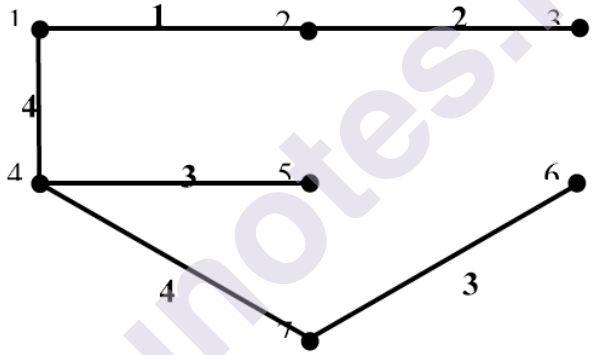


Fig. 3.12

Following theorem proves correctness of Kruskal's algorithm.

Theorem 3.5.1: Minimal cost spanning tree, generated by Kruskal's algorithm is optimal.

Proof: Let T be the spanning tree for G generated by Kruskal's algorithm. Let T' be a
minimum cost spanning tree for G. Show that both T and T' have the same cost. If edges of T and T' are the same, we are done. Let E(T) ≠ E(T').

Let e be a minimum cost edge such that $e \in E(T')$ and $e \in E(T)$. On including e in E(T'), cycle is created. Let $ee_1e_2...e_k$ be the cycle created. This cycle should have an edge, say e_j, which does not belong to T. Now, $w(e_j) \geq w(e)$, else Kruskal's algorithm would select this edge. Consider tree T'' = E(T') $\cup$ {e} – {ej}. (Here cycle created by e is broken by ej to get another spanning tree T''). Also, w(T'') ≤ w(T'), which does not have e. Repeat this process for all such edges in T – T' to eventually get T from T' without changing the weight. Thus, T is optimal.

3.7 ROOTED AND BINARY TREES

A rooted tree T(x) is a tree T with a specified vertex x, called the root of T. An orientation of a rooted tree in which every vertex but the root has in-degree one is called a branching. We refer to a rooted tree or branching with root x as an x-tree or x-branching, respectively.

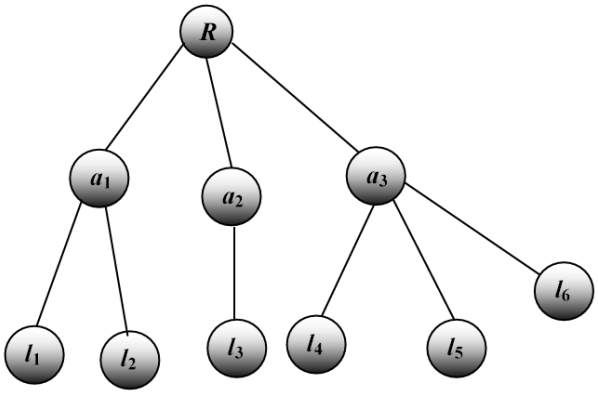


Fig. 3.13

In the Fig. 3.13 above, R is root having in-degree 0 and out-degree 3. The vertices a1, a2 and a3 have in-degree 1 and out-degrees 2, 1 and 3 respectively. Vertices l1, l2, l3, l4, l5 and l6 have in-degree 1 each and out-degree 0. Vertices having out degree 0 are termed as leaves.

Vertices a1, a2 and a3 are called as children of R and R is the parent of these vertices. Vertices a1, a2 and a3 are siblings, as they have same parent.

Thus, rooted tree is a directed graph.

Binary trees are special kind of rooted trees. In binary trees there are maximum 2 children to any vertex. Vertex having no child is as before termed as leaf.

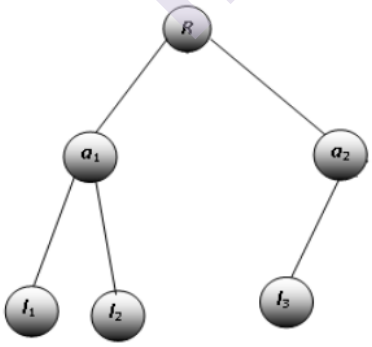


Fig. 3.14(a)

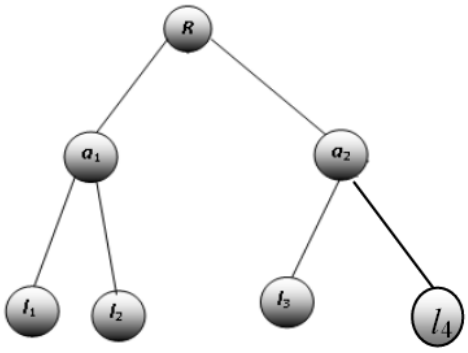


Fig. 3.14(b)

Rooted trees and branching are effective tools in designing of efficient algorithms for the purpose of reachability. There are certain terminologies exclusively associated with rooted binary trees.

- The depth of a node is the number of edges from the root to the node.

- The height of a node is the number of edges from the node to the deepest leaf.
- The height of a tree is a height of the root. (Thus, height of tree in Fig. 3.14(a) is 2)
- A complete binary tree is a binary tree, which is completely filled, with the possible exception of the bottom level, which is filled from left to right. (Fig. 3.14(b))

As graphs and trees have many real life applications, the vertices are usually used to store some data and we may need to search or traverse to the node to access the data. Hence, traversal is of utmost importance in graphs and trees. Now, shall look at some algorithms for the tree and graph traversals.

3.7.1 Breadth First Search:

In most types of tree-search, the criterion for selecting a vertex to be added to the tree depends on the order in which the vertices already in the tree T were added. A tree-search in which the adjacency lists of the vertices of T are considered on a first-come-first-served basis, that is, in increasing order of their time of incorporation into T, is known as breadth-first search. In order to implement this algorithm efficiently, vertices in the tree are kept in a queue; this is just a list Q which is updated either by adding a new element to one end (the tail/ rear of Q) or removing an element from the other end (the head/front of Q). At any moment, the queue Q comprises all vertices from which the current tree could potentially be grown.

Initially, at time t = 0, the queue Q is empty. Whenever a new vertex is added to the tree, it joins Q. At each stage, the adjacency list of the vertex at the head of Q is scanned for a neighbour to add to the tree. If every neighbour is already in the tree, this vertex is removed from Q. The algorithm terminates when Q is once more empty. It returns not only the tree (given by its predecessor function p), but also records the level of each vertex in the tree and, more importantly, their distances from r in G. It also returns a function t which records the time of incorporation of each vertex into the tree T. We keep track of the vertices in T by colouring them black. The notation G(x) signifies a graph G with a specified vertex (or root) x. Recall that an x-tree is a tree rooted at vertex x.

Algorithm: Breadth-First Search (BFS)
Input: a connected graph G(r)
Output: an r-tree T in G with predecessor function p, a level function l, such that $l(v) = d_G(r,v)$ for all $v \in V$, and a time function t
1: set i := 0 and Q := ϕ
2: increment i by 1
3: colour r black
4: set l(r) := 0 and t(r) := i
5: append r to Q
6: while Q is nonempty do
7: consider the head x of Q
8: if x has an uncoloured neighbour y then
9: increment i by 1

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10: colour y black
11: set p(y) := x, (y) := (x) + 1 and t(y) := i
12: append y to Q
13: else
14: remove x from Q
15: end if
16: end while
17: return (p,l, t).

Before we discuss the algorithm in detail, let us first implement it on the following graph.

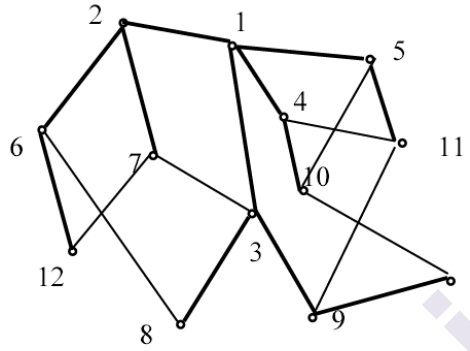


Fig. 3.15

Now, let us apply BFS on the Fig. 3.15.

i	Colour	Length (*l*)	Predecessor (*p*)	Time (*t*)
0	1 ***B***	$l(1) = 0$	-	-
1	2 ***B***	$l(2) = 1$	1	1
2	3 ***B***	$l(3) = 1$	1	2
3	4 ***B***	$l(4) = 1$	1	3
4	5 ***B***	$l(5) = 1$	1	4
	Vertex 1 removed from Q			
5	6 ***B***	$l(6) = 2$	2	5
6	7 ***B***	$l(7) = 2$	2	6
	Vertex 2 removed from Q			
7	8 ***B***	$l(8) = 2$	3	7
8	9 ***B***	$l(9) = 2$	3	8
	Vertex 3 removed from Q			
9	10 ***B***	$l(10) = 2$	4	9
	Vertex 4 removed from Q			
10	11 ***B***	$l(11) = 2$	5	10
	Vertex 5 removed from Q			
11	12 ***B***	$l(12) = 3$	6	11
	Vertex 6 removed from Q			
	Vertex 7 removed from Q			
	Vertex 8 removed from Q			
12	13 ***B***	$l(13) = 3$	9	12
	Vertex 9 removed from Q			
	Vertex 10 removed from Q			
	Vertex 11 removed from Q			
	Vertex 12 removed from Q			
	Vertex 13 removed from Q			

Q: $\phi \rightarrow 1 \rightarrow 12 \rightarrow 123 \rightarrow 1234 \rightarrow 12345 \rightarrow 2345 \rightarrow 23456 \rightarrow 234567 \rightarrow 34567 \rightarrow 345678 \rightarrow 3456789 \rightarrow 456789 \rightarrow 45678910 \rightarrow 5678910 \rightarrow 567891011 \rightarrow 67891011 \rightarrow 6789101112 \rightarrow 789101112 \rightarrow 89101112 \rightarrow 9101112 \rightarrow 910111213 \rightarrow 10111213 \rightarrow 111213 \rightarrow 1213 \rightarrow 13 \rightarrow \phi$.

Based on this, the highlighted edges in the Fig. 3.15 show the BFS tree.

3.7.2 Depth-First Search

Depth-first search is a tree-search in which the vertex added to the tree T at each stage is one which is a neighbour of as recent addition to T as possible. In other words, we first scan the adjacency list of the most recently added vertex x for a neighbour not in T. If there is such a neighbour, we add it to T. If not, we backtrack to the vertex which was added to T just before x and examine its neighbours, and so on. The resulting spanning tree is called a depth-first search tree or DFS-tree.

This algorithm may be implemented efficiently by maintaining the vertices of T whose adjacency lists have yet to be fully scanned, not in a queue as we did for breadth-first search, but in a stack. A stack is simply a list, one end of which is identified as its top; it may be updated either by adding a new element as its top or else by removing its top element. In depth-first search, the stack S is initially empty. Whenever a new vertex is added to the tree T, it is added to S. At each stage, the adjacency list of the top vertex is scanned for a neighbour to add to T. If all of its neighbours are found to be already in T, this vertex is removed from S.

The algorithm terminates when S is once again empty. As in breadth-first search, we keep track of the vertices in T by colouring them black. Associated with each vertex v of G are two times: the time f(v) when v is incorporated into T (that is, added to the stack S), and the time l(v) when all the neighbours of v are found to be already in T, the vertex v is removed from S, and the algorithm backtracks to p(v), the predecessor of v in T. The time increments by one with each change in the stack S. In particular, f(r) = 1, l(v) = f(v) + 1 for every leaf v of T, and l(r) = 2n.

Algorithm: Depth-First Search
Input: a connected graph G
Output: a rooted spanning tree of G with predecessor function p, and two time functions f and l.
1: set i := 0 and S := ϕ
2: choose any vertex r (as root)
3: increment i by 1
4: colour r black
5: set f(r) := i
6: add r to S (that is push r on stack S)
7: while S is nonempty do
8: consider the top vertex x of S
9: increment i by 1

10: if x has an uncoloured neighbour y then
11: colour y black
12: set p(y) := x and f(y) := i
13: add y to the top of S (that is push y on stack S)
14: else
15: set l(x) := i
16: remove x from S (that is pop x from the stack S)
17: end if
18: end while
19: return (p, f, l)

Now, let us apply DFS above for the graph below:
Highlighted edges in the graph indicate DFS tree.

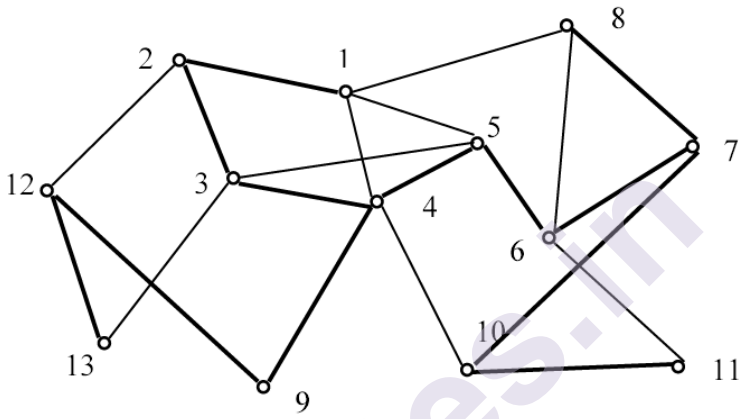


Fig. 3.16

i	Colour	*f*	*L*	*P*
0	-	-	-	-
1	1 B	1	**Push(1)**	-
2	2 B	2	**Push(2)**	1
3	3 B	3	**Push(3)**	2
4	4 B	4	**Push(4)**	3
5	5 B	5	**Push(5)**	4
6	6 B	6	**Push(6)**	5
7	7 B	7	**Push(7)**	6
8	8 B	8	**Push(8)**	7
9	***Pop 8***	-	*l*(8)=9	-
10	10 B	10	**Push(10)**	7
11	11 B	10	**Push(11)**	10
12	***Pop 11***	-	*l*(11)=12	-
13	***Pop 10***	-	*l*(10)=13	-
14	***Pop 7***	-	*l*(7)=14	-
15	***Pop 6***	-	*l*(6)=15	-
16	***Pop 5***	-	*l*(5)=16	-
17	9 B	17	**Push(9)**	4
18	12 B	18	**Push(12)**	9
19	13 B	19	**Push(13)**	12
20	***Pop 13***	-	*l*(19)=20	-
21	***Pop 12***	-	*l*(12)=21	-
22	***Pop 9***	-	*l*(9)=22	-
23	***Pop 4***	-	*l*(4)=23	-
24	***Pop 3***	-	*l*(3)=24	-
25	***Pop 2***	-	*l*(2)=25	-
26	***Pop 1***	-	*l*(1)=26	-

Note: Unused columns are used to indicate push and pop.

3.8 HUFFMAN CODES

In Computer Science, it is required to encode the text into a bit-string. Suppose, we want to encode all the alphabets in English (where no distinction is made between lowercase and uppercase letters). We can represent each letter with a bit string of length five, because there are only 26 letters and there are 32 bit strings of length five (E.g. 00000, 00001, 00010, so on, the count is 2^5). The total number of bits used to encode data is five times the number of characters in the text when each character is encoded with five bits. Is it possible to find a coding scheme of these letters such that, when data are coded, fewer bits are used? We can save memory and reduce transmittal time if this can be done.

We now introduce an algorithm that takes as input the frequencies (which are the probabilities of occurrences) of symbols in a string and produces as output a prefix code that encodes the string using the fewest possible bits, among all possible binary prefix codes for these symbols. This algorithm, known as Huffman coding, was developed by David Huffman. This algorithm assumes that we already know how many times each symbol occurs in the string, so we can compute the frequency of each symbol by dividing the number of times this symbol occurs by the length of the string. Huffman coding is a fundamental algorithm in data compression, the subject devoted to reducing the number of bits required to represent information. Huffman coding is extensively used to compress bit strings representing text and it also plays an important role in compressing audio and image files. Given symbols and their frequencies, our aim is to construct a rooted binary tree where the symbols are the labels of the leaves. The algorithm begins with a forest of trees each consisting of one vertex, where each vertex has a symbol as its label and where the weight of this vertex equals the frequency of the symbol that is its label. At each step, we combine two trees having the least total weight into a single tree by introducing a new root and placing the tree with larger weight as its left subtree and the tree with smaller weight as its right subtree. Furthermore, we assign the sum of the weights of the two subtrees of this tree as the total weight of the tree. The algorithm is finished when it has constructed a tree, that is, when the forest is reduced to a single tree.

Algorithm: Huffman Code:
Input: Huffman(C: symbols a_i with frequencies w_i , i = 1, . . . , n)
F := forest of *n* rooted trees, each consisting of the single vertex a_i and assigned weight w_i
Output: Huffman code of every symbol
Procedure:
while F is not a tree

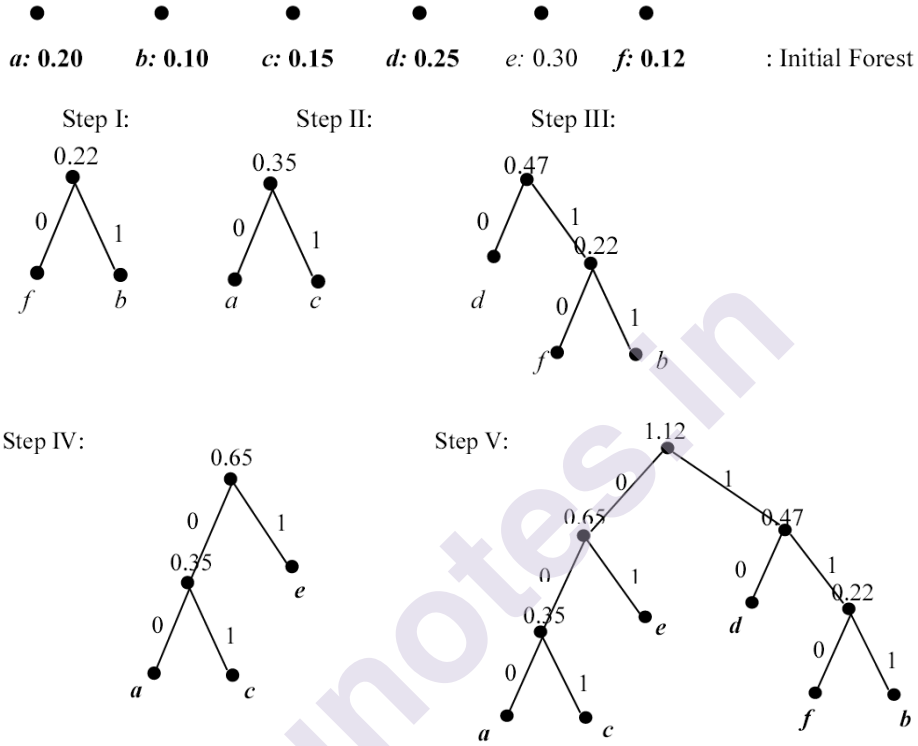
Replace the rooted trees T and T' of least weights from F with w(T) ≥ w(T') with a tree having a new root that has T as its left subtree and T' as its right subtree.

Label the new edge to T with 0 and the new edge to T' with 1.
Assign w(T) + w(T') as the weight of the new tree.

End while;
Assign the Huffman code for the symbol ai as the concatenation of the labels of the edges in the unique path from the root to the vertex ai.

Let us understand the algorithm better with the help of an example.

Use Huffman coding to encode these symbols with given frequencies: a: 0.20, b: 0.10, c: 0.15, d: 0.25, e: 0.30, f: 0.12. What is the average number of bits required to encode a character?



In the Step V, we get the necessary coding tree. The codes are as below:
a: 000; b: 111; c: 001; d: 10; e: 01; f: 110

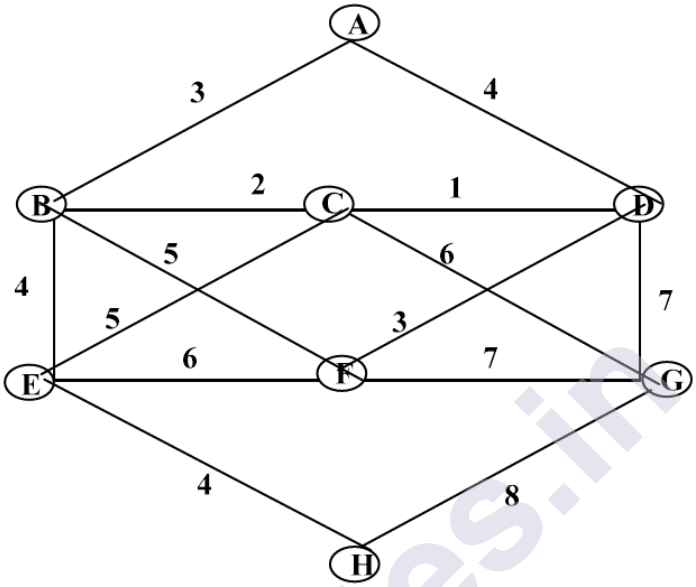
Average number of bits required is:
3 x 0.2 + 3 x 0.1 + 3 x 0.15 + 2 x 0.25 + 2 x 0.3 + 3 x 0.12 = 2.81

Note that Huffman coding is a greedy algorithm. Replacing the two subtrees with the smallest weight at each step leads to an optimal code for these symbols can encode these symbols using fewer bits.

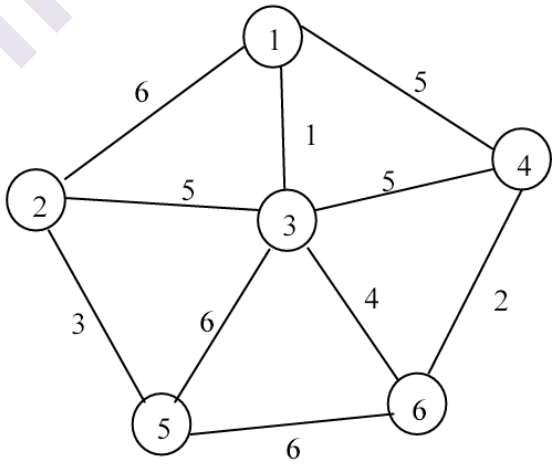
Solved Problems:
1. The complete bipartite graphs K1,n, known as the star graphs. Prove that the star graphs are the only complete bipartite graphs which are trees.
Solution: The number of edges in a star graphs is n – 1. Also one partite has 1 vertex and the other has n vertices, no two vertices in the second partition can be connected and hence star graph is acyclic and connected. Thus, it is a tree. N Now, in Km, n, with m, n > 0 degree of every vertex is at least two. Hence it has a cycle. So it cannot be a tree.

3.9 UNIT END EXERCISES:

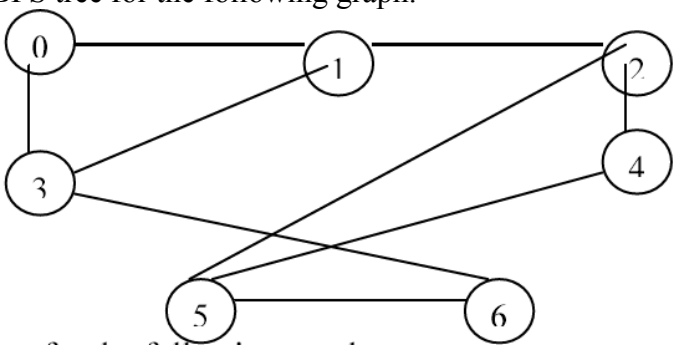
1. Let G be a connected graph which is not a tree and C be a cycle in G. Prove that complement of any spanning tree of G contains an edge of C.
2. Prove Theorem 3.2.1.
3. Find minimum cost spanning tree for the following graph using Kruskal's algorithm.



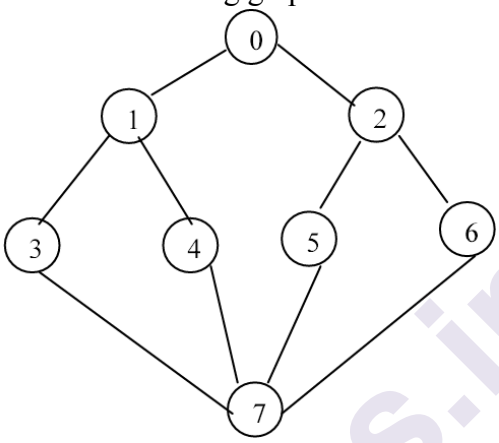
4. Like Kruskal's algorithm, Prim's algorithm too find minimum cost spanning tree. However, in Prim's algorithm, at every step, we add an edge having least cost provided when an edge is added cycles is not formed. (E.g. in the above graph, we start with edge CD, followed by CB, BA, DF, however, AD is not chosen as a cycle is formed.) Find minimum cost spanning tree for the following graph using Prim's algorithm.



5. Draw BFS tree for the following graph.

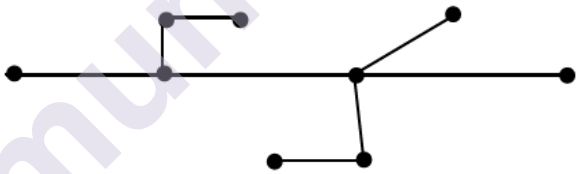


6. Draw DFS tree for the following graph.



7. Use Huffman coding to encode these symbols with given frequencies: A: 0.10, B: 0.25, C: 0.05, D: 0.15, E: 0.30, F: 0.07, G: 0.08. What is the average number of bits required to encode a symbol?

8. Find Prüfer sequence for the tree below.



9. Decode the Prüfer sequence (1, 1, 1, 1, 6, 6, 5)

4

EULERIAN AND HAMILTONIAN GRAPHS

Unit Structure:

4.0 OBJECTIVES

1. Definition and characterisation of Eulerian graph
2. Fleury's algorithm
3. Application of Eulerian graph: Chinese postman's problem
4. Definition of Hamiltonian graph
5. Sufficient and necessary conditions for Hamiltonian graphs
6. Degree majorisation
7. Application of Hamiltonian graph: Travelling salesman's problem

4.1 INTRODUCTION

In chapter one, we have seen that the Konigsberg's bridge problem, devised by Euler, gave birth to Graph Theory. While proving non-existence of the necessary path, Euler defined a class of graphs, which are now termed as Eulerian graphs. In this chapter, we shall discuss Eulerian graphs in detail.

Speaking plainly, in a graph, if you start from any vertex and without lifting your pencil and traversing all the edges exactly once, if you can come back to the starting vertex, you have come across an Eulerian graph and you have traversed an Eulerian circuit. To explain further, let us have a look at the graph if Fig. 4.1 below.

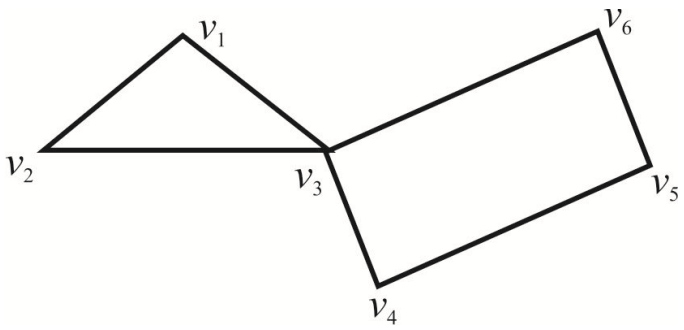


Fig. 4.1

In the Fig. 4.1 above, if we start at vertex v_1, we can take the route as $v_1v_2v_3v_4v_5v_6v_3v_1$. Thus, the graph of Fig. 1.4 is Eulerian. Now, let us define Eulerian graphs formally.

Definition 4.1.1: Eulerian Trail: In a connected graph G, a path that visits every edge exactly once is termed as an Eulerian Trail.

Definition 4.1.2: Eulerian Circuit: In a connected graph G, circuit in which every edge is traversed exactly once is termed as Eulerian circuit.

Definition 4.1.3: Eulerian Graph: A connected graph having Eulerian circuit is called as Eulerian graph.

The graph in Fig. 4.1 is Eulerian.

A traversable graph is one that can be drawn without taking a pen from the paper and ithout retracing the same edge. In such a case the graph is said to have an Eulerian trail.

Characterisation of Eulerian graph:

While showing that there no path that crosses all the bridges exactly once, in the Konigsberg's bridge problem, Euler represented the bridges in the form of a graph as we have seen in Fig. 1.2. He stated that any graph to be traversed in this manner has to be even. Thus, we have a very important property of Eulerian graph as below:

Theorem 4.1: A graph is Eulerian if and only if it is connected and even.

Proof: Suppose a graph G is Eulerian. Since G is Eulerian, it has Eulerian circuit and Eulerian circuits are connected. Hence, G is connected. Now, let v be any vertex in G. Let us traverse Eulerian circuit from v. For traversing all the edges, a vertex is entered and left, contributing two to the degree of any vertex. This is true whenever we traverse a vertex (note that a vertex may occur multiple times in the Eulerian circuit). At the end, we return to v, contributing two to degree of v as well. Thus, degree of every vertex is even.

Conversely, let G be even and connected graph. We shall prove that it is Eulerian, that is it has an Eulerian cicuit. Suppose that graph G is connected and its all the vertices are even. If there is more than one vertex in G, then each vertex must have degree greater than 0. Begin at

a vertex v. As we know, a graph with even vertices partitions into cycles (Theorem 1.4.3), we know that v will be on at least one cycle. Since G is connected, there must be an edge {v, v1} for some vertex v1≠v. Since v1 has even degree greater than 0, there is an edge {v1, v2} where v2 ≠ v1. These two edges make a trail from v to v2. Continue this trail, leaving each vertex on an edge that was not previously used, until returning to v. This is always possible, because v is on a cycle. Call the circuit (cycle is a circuit) formed by this process C1. If C1 covers all the edges of G, then we are done. Otherwise, remove all the edges that contribute to C1 from G, leaving the graph G0. The remaining vertices are still even, and since G is connected there is some vertex u in both G0 and C1. Repeat the same process as before, beginning at u. The new circuit, C2, can be added to C1 by starting at v, moving along C1 to u, travelling around C2 back to u and then along the remainder of C1 back to v. Repeat this process, adding each new circuit found to create a larger circuit. Since G is finite, this process must end at some point, and the resulting circuit will be an Eulerian circuit.

If an Eulerian graph is small then in general it is straight forward to find an Eulerian circuit. But that need not be the case always. In such situation Fleury's algorithm comes to help us. It finds Eulerian circuit, given any Eulerian graph.

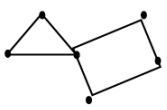
The basic idea behind Fleury's algorithm is very simple. Start with any vertex u, the next vertex v is chosen in such way that uv is not a bridge unless there is no other option. Let us first write down the steps of the algorithm and then apply on the graph of Fig. 4.1.

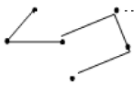
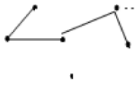
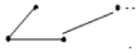
Fleury's Algorithm:

Input: A connected even graph G and a vertex u of G

Output: An Eulerian circuit C of G starting (and ending) at u
1: set C := u, x := u, F := G
2: while $\partial F(x) \neq \phi$ do
3: choose an edge e := xy $\in \partial F(x)$, where e is not a cut edge of F unless there is no alternative
4: replace uCx by uCxey, x: = y, and F:= F – e
5: end while
6: return C

Let us apply the algorithm on Fig, 4.1.

C	F	x	$\partial_F(x)$ (Edges)	y
v_1		v_1	v_1v_2, v_2v_3	v_3

v_1v_3		v_3	v_1v_3, v_3v_4, v_3v_6 v_1v_3 is bridge	v_4
$v_1v_3v_4$		v_4	v_4v_5	v_5
$v_1v_3v_4v_5$		v_5	v_5v_6	v_6
$v_1v_3v_4v_5v_6$		v_6	v_6v_3	v_3
$v_1v_3v_4v_5v_6v_3$		v_3	v_3v_2	v_2
$v_1v_3v_4v_5v_6v_3v_2$		v_2	v_2v_1	v_1
$v_1v_3v_4v_5v_6v_3v_2v_1$	ϕ	-	-	-

In the last step we get the necessary Eulerian circuit: $v_1v_3v_4v_5v_6v_3v_2v_1$.

Following theorem gives the proof of correctness of Fleury's algorithm.

Theorem 4.2: If G is an even and connected graph, then the circuit C, returned by Fleury's algorithm is an Eulerian circuit.

Proof: The sequence C is initially a trail, and remains one throughout the procedure because Fleury's Algorithm always selects an edge of F (that is, an as yet unchosen edge) which is incident to the terminal vertex x of C. Moreover, the algorithm terminates when $\partial_F(x) = \phi$, that is, when all the edges incident to the terminal vertex x of C have already been selected. Because G is even, we deduce that x = u; in other words, the trail C returned by the algorithm is an Eulerian circuit.

Now, we shall show that C has all the edges of G. Let us assume contrary.

Let $C = \{v_0v_1v_2....v_kv_{k+1}....v_nv_0\}$. Let F = G – E(C).

$\therefore$ $E(F) \neq \phi$. Hence there are vertices of positive degree in F. In fact every vertex in F has even degree, since F is obtained from G by deleting edges of a circuit. Let $S = \{v \in F: \deg_F(v) > 0\}$. Let H be a subgraph of G induced by the vertex set S. Then, $v_0 \in V–S$, as we terminate the algorithm once $\partial_F(v_0) = \phi$.

Graph induced by S Graph induced by V – S

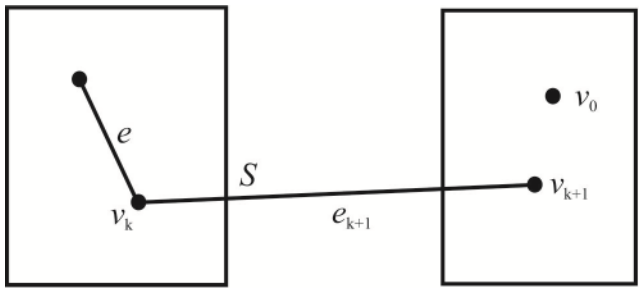


Fig. 4.2

Let vk be the last vertex in C such that $v_k \in S$. Then $v_{k-1} \in V-S$ and $e_{k+1} = v_k v_{k+1}$ is the only edge joining S induced subgraph and V–S induced subgraph. Therefore, i. e_{k+1} is a bridge in F.

Next, since every vertex of H has positive degree, there exists an edge e, incident with vk in H. It is not a bridge of H since every vertex of H has even degree. Hence e is not a bridge of F either as $H \subseteq F$. Now, while executing (k+1)th iteration, we preferred to choose e_{k+1} to e, thus ii. e_{k+1} is not a bridge in F (by Fleury's rule).

(i) and (ii) contradict. (Refer Fig. 4.2)

4.2 RANDOMLY EULERIAN GRAPHS

We have seen the traceability of Eulerian graph. In fact, Fleury's algorithm gives a systematic way of traversing an Eulerian graph to get an Eulerian circuit. Now, let us look at a very interesting class of Eulerian graph and that is randomly Eulerian graph.

An eulerian graph G is randomly Eulerian from a vertex v of G if the following procedure always results in an eulerian circuit of G:

Begin a trail at v by choosing any edge incident with v. Next (and at each step thereafter), the trail is continued by selecting any edge not already chosen which is adjacent with the edge most recently selected. The process terminates when no such edge is available. Equivalently, a graph G is randomly Eulerian from v if every trail of G beginning at v can be extended to an Eulerian circuit of G.

Before, we proceed to prove the results about randomly Eulerian graphs, let us look at the examples of Eulerian graphs which are (or are not) randomly Eulerian.

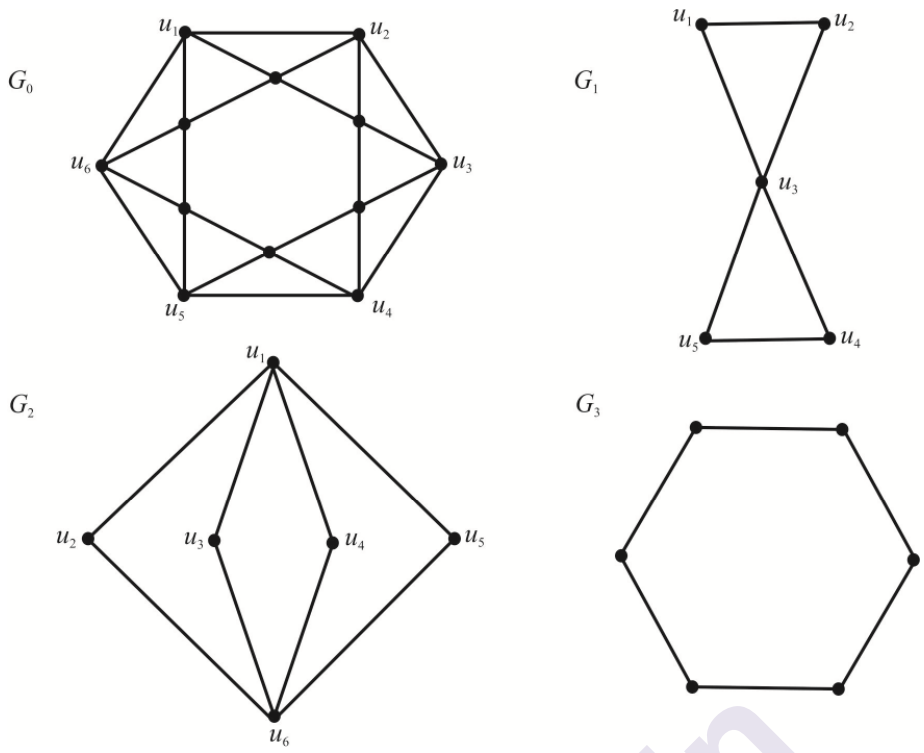


Fig. 4.3

In the graphs of Fig. 4.3, we observe that, G0 is not randomly Eulerian, G1 is randomly Eulerian only from u3, G2 is randomly Eulerian from u1 and u6 and G3 is randomly Eulerian from all its vertices.

Explanation:
1. In G0, trail $u_1u_2u_3u_5u_4u_3u_1u_5u_6u_4u_2$ cannot be extended to Eulerian circuit.
2. In G1, trail u1u2u3u1 cannot be extended to Eulerian circuit.
3. In G2, trail u2u6u3u1u2 cannot be extended to Eulerian circuit.

(Note: The trick for verification is very simple. While applying Fleury's algorithm to find Eulerian circuit, if bridge is encountered then traverse the bridge instead of other non-bridge option.)

Theorem 4.3: An Eulerian graph G is randomly Eulerian for a vertex v if and only if v is on every cycle of G.

Proof: Since Eulerian graph is even, it can be partitioned into cycles (Theorem 1.4.3).

Let G be randomly Eulerian for a vertex v. Let, if possible, there is a cycle C of which does not contain v. Let G1 = G – C. Since G and C are Euler graphs, so is G1 (as it too will have all vertices of even degree). G1 need not be connected but it should have a maximal connected component G(v) that includes vertex v. Then the sum, G2 = G(v) + C has to be connected Eulerian graph.

With this background, we shall observe the following properties.

1. An Eulerian graph G is randomly Eulerian for a vertex v if and only if v is on every cycle of G.

2. If a graph G has exactly two odd vertices u and v, then G is randomly traversable from u to v if and only v is on every cycle of G.

4.3 CHINESE POSTMAN PROBLEM

A Chinese mathematician called Kuan Mei-Ko was interested in a postman delivering mail to a number of streets such that the total distance walked by the postman was as short as possible. How could the postman ensure that the distance walked was a minimum?

The problem defined above is termed as Chinese Postman Problem.

In this type of problems, we can represent the route with a graph and check if is traversable.

If it is an Eulerian graph then we know we can traverse easily and the minimum distance will be same as sum of weights of all the edges.

If the graph is not an Eulerian graph, it will have odd vertices. We know that a graph has even number of odd vertices. So in that case we pair up the vertices, which contribute the least distance and add up to the total weight of the graph to get the minimum distance.

This is an idea behind the algorithm to solve the Chinese Postman Problem.

In the following example a postman has to start at A, walk along all 13 streets and return to A. The numbers on each edge represent the length, in metres, of each street. The problem is to find a trail that uses all the edges of a graph with minimum length.

Before we proceed to solve the problem, let us have a look at the pre-requirements for the algorithms and then the steps of the algorithm.

We have seen that any graph as even number of odd vertices. Let us find out how many pairs are possible when there are 2*n* odd vertices.

1. Suppose there are 2 odd vertices a_1 and a_2. Then only one pair is possible.
2. Suppose there are 4 odd vertices a1 a_2, a_3 and a_4. Then three pairs are possible. E.g. $\{a_1a_2, a_3a_4\}$, $\{a_1a_3, a_2a_4\}$ and $\{a_1a_4, a_2a_3\}$ (i.e. 3 x 1)
3. Suppose there are 6 odd vertices a_1, a_2, a_3, a_4, a_5 and a_6. Then, 15 pairs are possible, as a1 can be paired with 5 other vertices and remaining four vertices can be paired in 3 x 1 pairs, so total pairs are 5 x 3 x 1 = 15.
4. We continue this way, for 8 odd vertices, 7x5x3x1 = 105 pairs are possible.
5. For 2*n* vertices, (2*n* – 1) x (2n – 3) x ... x 3 x 1 pairs are possible.

Now that we know how to count number of odd pairs, let us understand the algorithm.

Algorithm:
1. List all odd vertices.
2. List all possible pairings of odd vertices.
3. For each pairing find the edges that connect the vertices with the minimum weight.
4. Find the pairings such that the sum of the weights is minimised.
5. On the original graph add the edges that have been found in Step 4.
6. The length of an optimal Chinese postman route is the sum of all the edges added to the total found in Step 4.
7. A route corresponding to this minimum weight can be easily found.

Let us implement the same on Fig. 4.4.

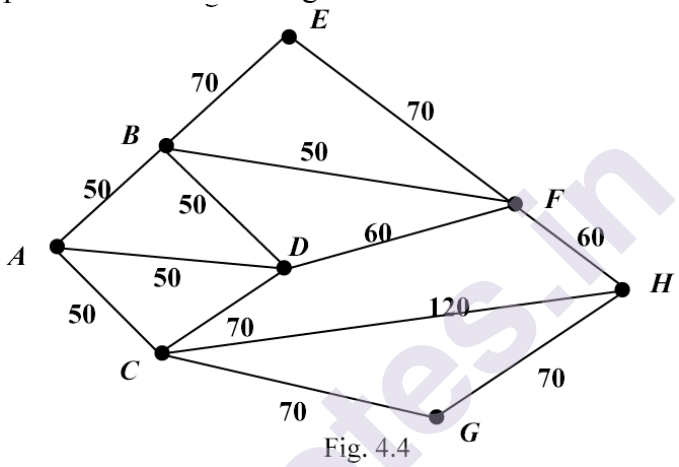


Fig. 4.4

1. There are only two odd vertices, viz. A and H. So just one pair is possible.
2. Minimum cost from A to H is 160 (A – B – F – H).
3. Draw the additional edges on the original graph along this path. (Fig. 4.5)
4. The minimum cost for this problem is sum of all weights of original graph and 160, which is 840 + 160 = 1000.
5. One possible path is ABEFBDACGHCDFH-FBA.

To find the route of the postman, one has to simply add the extra edges with reference to the minimum distance so that all the vertices have even degree and find an Eulerian circuit from the starting (odd) vertex.

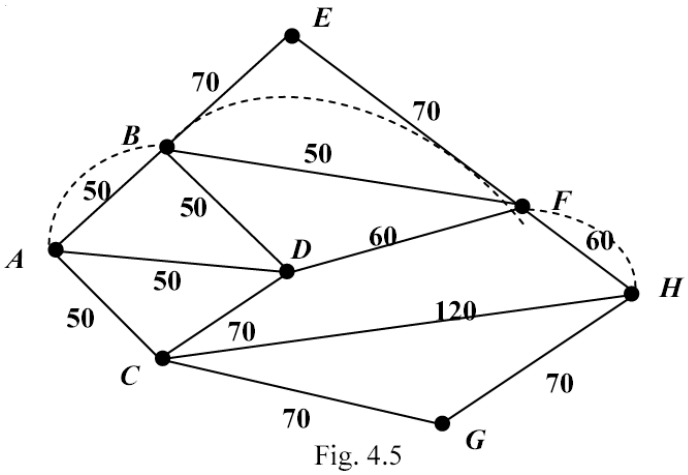


Fig. 4.5

4.4 HAMILTONIAN GRAPH

A path having connecting all the vertices of a connected graph is called as Hamiltonian path and a cycle through all the vertices of a graph is termed as Hamiltonian cycle. Hamilton described this graph, in the form of a puzzle. He drew a dodecahedron and put pins at five consecutive vertices of the same. Other player has to put the pins to complete a spanning cycle. A dodecahedron with its spanning cycle (highlighted) is shown in Fig. 4.6 below.

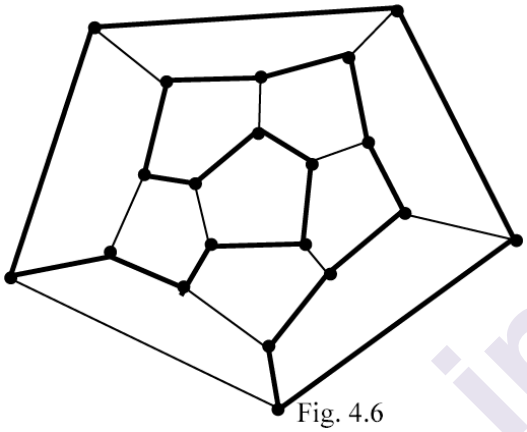


Fig. 4.6

We can make very interesting observations of Hamiltonian graphs.

1. A complete graph on n vertices, where $n \geq 3$
2. A cycle on n vertices is Hamiltonian.
3. Though every Eulerian graph need not be Hamiltonian, line graph of a every Eulerian graph is Hamiltonian.
4. If G is Hamiltonian then the graph obtained by adding edges to non-adjacent vertices of G, remains Hamiltonian.
5. Every non-Hamiltonian graph can be converted into Hamiltonian by adding edges into it. (This is true because complete graph is Hamiltonian)

Definition: A graph G is said to be maximal non-Hamiltonian, if it not Hamiltonian and on adding an edge between any pair of its non-adjacent vertices, it becomes Hamiltonian.
The graph in Fig. 4.7 below is maximal non-Hamiltonian.

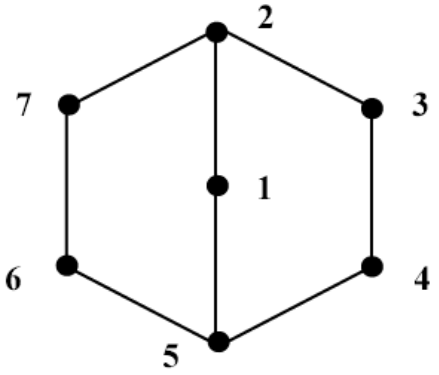


Fig. 4.7

Like Eulerian graph, Hamiltonian graph has no elegant characterisation. However, there are some necessary and some sufficient conditions to determine whether given graph is Hamiltonian.

Now, we shall prove a necessary condition for a graph to be Hamiltonian.

Theorem: If G is Hamiltonian, then for every non-empty subset $S \subseteq V(G)$, $c(G - S) \leq |S|$. ($c(G - S)$ is number of distinct components obtained after removing S from G).

Proof: Let G be Hamiltonian. Let $\phi \neq S \in V(G)$. Let $w \in S$. Let C be Hamiltonian cycle of G beginning at w. Let $c(G - S) = k$.

If $k = 1$, then S being non-empty set we readily see that $|S| \geq 1 = k$.

Now let $k > 1$ and G1, G2, ..., Gk be the components of G – S.

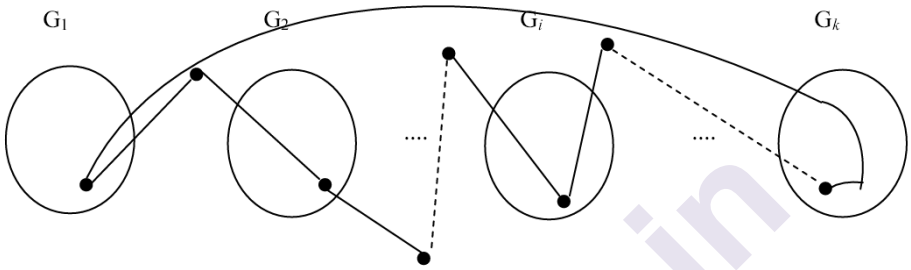


Fig. 4.8

Before we proceed, let us give some orientation to C.

(Fig. 4.9). Let u_i be the last vertex of C that belongs to G_i.

Fig. 4.9

Let v_i be the next vertex of C that comes after u_i (this is possible due to the orientation given to C). Obviously, v_i can not belong to G_i. This is because u_i is the last vertex in G_i. Also v_i cannot belong to any other G_j, for $1 \leq j \leq k$. For else components G_i and G_j will be connected; however all the components are mutually disjoint. Hence, $v_i \in S$. Also, $v_i \neq vj$ for $i \neq j$. (Fig. 4.8 and Fig. 4.10). This is because $u_i v_i \in E(G)$ for every i.

Therefore, for each i, $v_i \in S \Rightarrow |S| \geq k$.

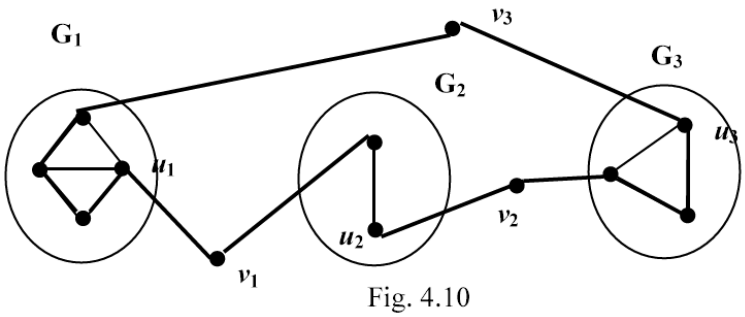


Fig. 4.10

Thus, we have proved a necessary condition for a graph to be Hamiltonian. As a result, the contra positive of this condition, "Let G be a connected graph. If there exists a subset S of V, such that c(G – S) > | S |, then G is non-Hamiltonian", can be useful to show that a graph is not Hamiltonian. We shall check this with the help of an example below. (Fig. 4.11)

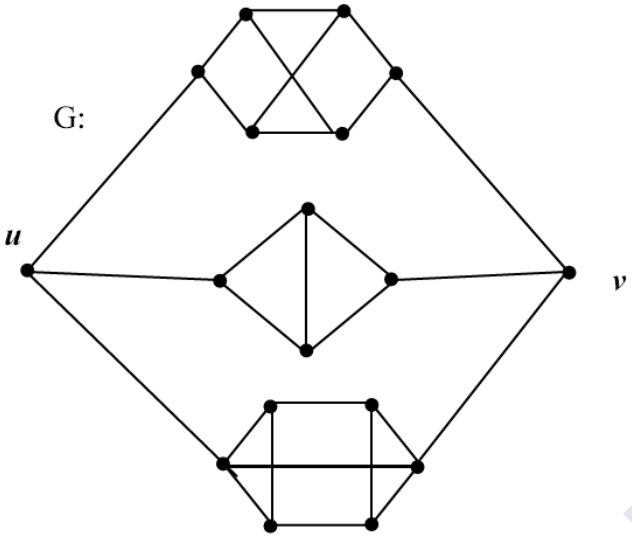


Fig. 4.11

However, the condition that we have proved is necessary condition for a graph to be Hamiltonian and not sufficient. The reader can verify that Petersen's graph satisfies the necessary condition of Theorem 4, however the graph is not Hamiltonian.

Now, we shall prove the sufficient conditions for a graph to be Hamiltonian.

Dirac's theorem gives a sufficient condition for a graph to be Hamiltonian,

Theorem: (Dirac's Theorem): If a connected graph G has $n \geq 3$ vertices and degree of every vertex is at least , then it has Hamiltonian cycle.

Proof: We shall prove the result by contradiction. Let, if possible, G be a graph having $n \geq 3$ vertices such that degree of every vertex is at least $\frac{n}{2}$, but it has no Hamiltonian cycle.

Let $P \equiv p_1 p_2 ... p_k$ be the longest path in G. If p_1 is connected to a vertex v not on path P, then $vp_1p_2...p_k$ will be longer than P, hence p_1 has all its adjacent vertices in path P. Same argument holds true for p_k as well. Thus, p_1 and p_k are adjacent to vertices on P itself and not outside the path. As $\deg(p_1) \geq \frac{n}{2}$ and p_1 cannot be adjacent to itself $k \geq \frac{n}{2} + 1$.

So we claim that there is some value of j $(1 \leq j \leq k)$ such that pk is adjacent to p_j and p_1 is adjacent to p_{j+1} (Fig. 4.8). Suppose the claim is not true. Then since all the adjacent vertices of p_1 lie on P, there must be at least $\deg(p_1)$ vertices that are not adjacent to p_k and similarly there must be at least $\deg(p_k)$ vertices that are not adjacent to

p1. Thus the path P must have at least deg(p1) + deg(pk) + 1 = n + 1 vertices, which contradict that there are n vertices.
This gives us cycle C : $p_j+1p_j+2...p_k–1p_kp_jp_j–1p_j–2...p_1\ p_{j+1}$.
Now, we shall show that C is the Hamiltonian cycle of G.

Let, if possible, G – C is non-empty. Then, there is a vertex v in G – C such that v is connected to C at some vertex pi (this is because G is connected) and hence path starting from v and through pi around the cycle C is longer than P which contradicts our choice of P.

Hence C is the necessary Hamiltonian cycle.

Thus, from the above theorem, we observe that Dirac's theorem gives sufficient condition for a graph to be Hamiltonian. However, this condition is not necessary, as reader can readily see that a cycle Cn ($n \geq 5$) is Hamiltonian but $\delta(C_n) = 2 < \frac{n}{2}$.

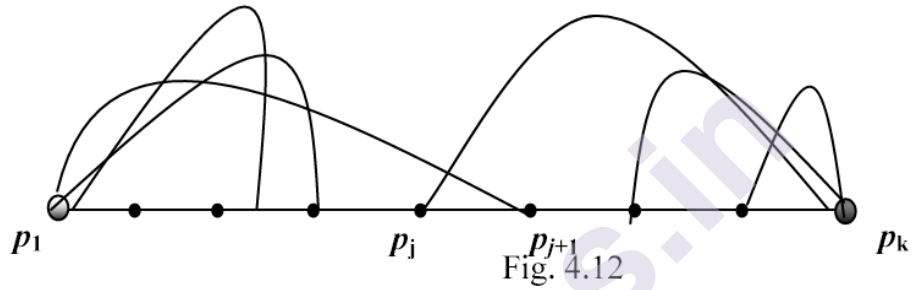


Fig. 4.12

Definition: Closure of a Graph: Let G be a graph on n vertices. If there is a pair of nonadjacent vertices u_1 and v_1 such that deg(u1) + deg(v1) $\geq$n, then join u_1 and v_1 by an edge to form a super graph G_1 of G. In G_1, if there is a pair of non-adjacent vertices u_2 and v_2 such that deg(u2) + deg(v2)$\geq$n, then join u_2 and v_2 by an edge to form a super graph G_2 of G_1.

Continue this process till no such pair exists. Then the graph so formed is called as closure of G and is denoted by c(G).

We shall understand the definition with the help of an example in Fig. 4.9 below.

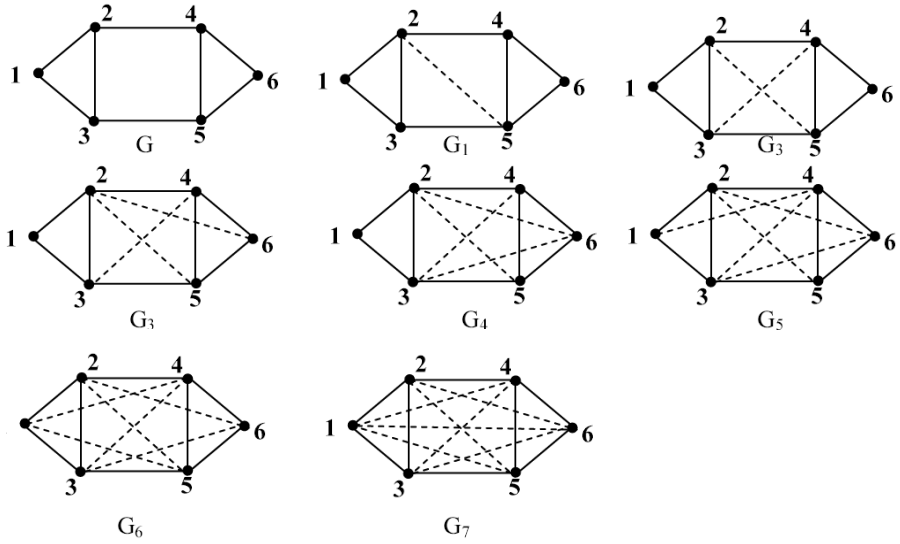


Fig. 4.13

Bondy-Chvatal Theorem: Let G be a simple graph on n vertices and u and v be nonadjacent vertices in G such that $d(u) + d(v) \geq n$, then G is Hamiltonian if and only if G + uv is Hamiltonian.

Proof: If G is Hamiltonian, then G + uv is Hamiltonian.

Conversely, let G + uv is Hamiltonian. Let, if possible, G is not Hamiltonian.

Since G + uv is Hamiltonian and G is not, every Hamiltonian cycle of G + uv contains edge uv. Thus, there is a Hamiltonian path $u = v_1\text{-}v_2\text{-}v_3\text{-}...\text{-}v_n = v$ in G. Define sets S and T as:
$S = \{v_i : uv_i + 1 \in E(G)\}$ and $T = \{v_i : v_iv \in E(G)\}$.

Since $v \notin S \cup T$, $|S \cup T| < n$. (i.e. number of vertices in S U T is less than n).

Also $|S \cap T| = 0$, for if there is a vertex $v_i \in S \cap T$, the there will be a Hamiltonian cycle $v_1v_2...v_iv_nv_{n-1}...v_i+1v_1$ which belongs to G, which is not possible by our assumption.

$\therefore \deg(u) + \deg(v) = |S| + |T| = |S \cup T| - |S \cap T| < n$.

This contradicts given condition that $d(u) + d(v) \geq n$.

Thus, our assumption is wrong. Therefore, G is Hamiltonian.

Theorem: A graph G is Hamiltonian if and only if its closure is Hamiltonian.

Proof: If G is Hamiltonian, then obviously closure of G is Hamiltonian.

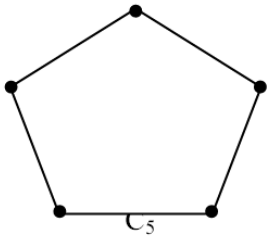
Let closure of G is Hamiltonian. Let closure of G is made by successively adding edges uv (where $d(u) + d(v) \geq n$) and we obtain a sequence $G_1, G_2, ..., G_k$. Then, by successively applying Theorem on Gk, Gk–1, ..., G1, we can prove that G is Hamiltonian.

4.5 DEGREE MAJORISATION

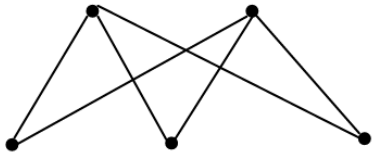
A sequence of real numbers $(p_1, p_2, ..., p_n)$ is said to be majorised by another such sequence $(q_1, q_2, ..., q_n)$ if $p_i \leq q_i$ for $1 \leq i \leq n$.

A graph G is said to be majorised by other graph H, if |V(G)| = |V(H)| and a non-decreasing degree sequence of G is majorised by that of H.

As an example, observe the graphs C_5 and K_2, 3 of Fig. 4.14 below.



C_5
(2, 2, 2, 2, 2)



$K_{2,3}$
(2, 2, 2, 3, 3)

Fig.4.14

Theorem: Let G be a simple graph having degree sequence (d_1, d_2, ..., d_n), where $d_1 \leq d_2 \leq ... d_n$ and $n \leq 3$. Suppose there is no value of m for which $d_m \leq m$ and $d_n - m < n - m$. Then G is Hamiltonian.

Proof: Let G be a simple graph that follows the hypothesis given in the theorem statement.

We shall show that its closure is complete and hence by Theorem we can say that G is Hamiltonian.

Let, if possible, closure of G, (say G') is not complete. We shall denote degree of a vertex v in G' by d'(v). Let u and v be two non-adjacent vertices in G', such that, d'(u) $\leq$ d'(v) (A)

We choose these two such that d'(u) + d'(v) is as large as possible. Since G' is not Hamiltonian, d'(u) + d'(v) < n (B)

Now, let S be set of vertices in V – {v} which are not adjacent to v and T be set of vertices in V – {u} which are not adjacent to u in G'. Then we have | S | = n – 1 – d'(v) and | T | = n – 1 – d'(u) (C)

Further, by the choice of u and v, each vertex in S has degree at most d'(u) and each vertex of T $\cup$ {u} has degree at most d'(v). Set m = d'(u).

Then, by (A), d'(v) < m – n.

By (B) and (C) S and hence G' has at least m vertices of degree at most m.

Also by (B) and (C), T $\cup$ {u} and hence G' has n – m vertices of degree less than n – m.

Because G is a spanning subgraph of G', the same is true for G.

Thus, $d_m \leq m$ and $d_{n-m} \leq n - m$. But this is contrary to the hypothesis since by (A) and (B), m < n/2. Hence, G' is complete and G is Hamiltonian.

Now we shall prove a very important theorem about Hamiltonian graphs by Chvátal.

Before we proceed, let us first introduce the notion of "join" of two graphs.

Let G and H be two disjoint graphs. Join of G and H, denoted by G □ H, is obtained by connecting every vertex of G to each and every vertex of H.

Now, for $1 \le m < n/2$, define $C_{m, n}$ as $K_m \vee \left(K_m^C + K_{n-2m}\right)$. For example $C_{1,5} = K_1 \vee \left(K_1^C + K_3\right)$ and $C_{2,5} = K_2 \vee \left(K_2^C + K_1\right)$ are shown in Fig.4.15.

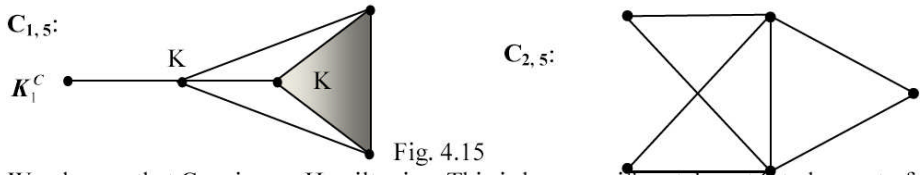


Fig. 4.15

We observe that Cm, n is non-Hamiltonian. This is because, if we take set S to be a set of m vertices having degree n – 1, (e.g. K_m), then on removing S from $C_{m,n}$, we get m + 1 components. Thus, by contra positive of Theorem (necessary condition for a graph to be Hamiltonian), we see that $C_{m,n}$ is non-Hamiltonian.

Theorem: (Chvátal' theorem): If G is non Hamiltonian simple graph with $n \ge 3$ then G is degree majorised by some $C_{m,n}$.

Proof: Let G be a non Hamiltonian simple graph with degree sequence $(d_1, d_2, ..., d_n)$, where $d_1 \le d_2 \le ... d_n$ and $n \ge 3$. Then by Theorem, there exists $m < n/2$ such that $d_m \le m$ and $d_{n-m} \le n - m$. Therefore, $(d_1, d_2, ..., d_n)$ is majorised by the sequence (m, ...,m, n – m – 1, ..., n – m – 1, n – 1, ..., n – 1) with m terms equal to m (contribution from K_m^C), n – 2m terms equal to n – m (Contribution from K_{n-2m}) and m terms equal to n – 1 (Contribution from K_m), which is a degree sequence of $C_{m, n}$.

Theorem: If G is a simple graph with $n \ge 3$, such that $E(G) > \left(\frac{n}{2}\right) + 1$, then G is Hamiltonian. In fact, only non Hamiltonian simple graphs having n vertices and $\left(\frac{n}{2}\right) + 1$ edges are $C_{1, n}$ and for n = 5, $C_{2, 5}$.

Proof: Let G be a non Hamiltonian graph. Then we shall show that E(G) is at most $\left(\frac{n}{2}\right) + 1$. Since G is non Hamiltonian, it degree majorised by some $C_{m, n}$ by Chvátal's theorem. Hence,

$$
\begin{aligned}
E(G) &\le E(C_{m, n}) \\
&= \binom{m}{2} + m^2 + m(n-2m) + \binom{n-2m}{2} \\
&= \frac{1}{2}\left[m^2 + (n-2m)(n-m-1) + m(n-1)\right] \\
&= \binom{n-1}{2} + 1 - \frac{1}{2}(m-1)(m-2) - (m-1)(n-2m-1) \qquad (A) \\
&\le \binom{n-1}{2} + 1
\end{aligned}
$$

If we observe (A), equality holds if m = 1 and for m = 2, n = 5 and hence G has same degree sequence as that of $C_{1,\ n}$ and $C_{2,5}$ respectively. In both the cases, $G \cong C_{1,n}$ or $G \cong C_{2,5}$.

Thus, from the above result, we say that maximum number of edges in a simple non Hamiltonian graphs having n vertices, is $\left(\frac{n-1}{2}\right)+1$.

4.6 TRAVELLING SALESMAN'S PROBLEM:

Now, we shall look at an important application of Hamiltonian graph that is Travelling salesman's problem.

A travelling salesman wishes to visit a number of towns and return to his starting point.

Given the travelling times between two towns, how should he plan his itinerary so that he visits each town exactly once and travels for the least time? In graphical terms, we have to find the minimum weight Hamiltonian cycle in a weighted complete graph.

In contrast to the Chinese Postman Problem, there is no efficient algorithm to solve travelling salesman's problem.

The approach can be start with any vertex and find all possible Hamiltonian cycles from that vertex and find total cost for each of the cycle. However, this approach needs (n – 1)! computations which is very large as value of n gets moderately big. Hence the approach is not practicable.

In the light of above discussion, what we try to find is a reasonably good (not necessarily an optimal) solution.

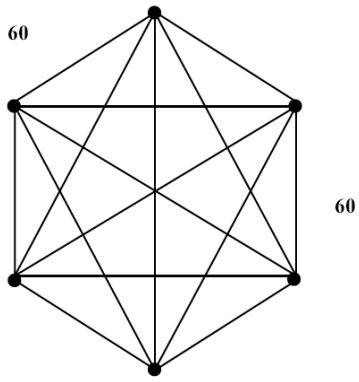


Fig. 4.16

Let us represent the above graph in the form of matrix.

	A	*B*	*C*	*D*	*E*	*F*
A	–	56	35	2	51	60
B	56	–	21	57	78	70
C	35	21	–	36	68	78
D	2	57	36	–	51	61
E	51	78	68	51	–	13
F	60	70	78	61	13	–

Now, let Hamiltonian cycle begins at A. So from A we choose a vertex which is at the minimum distance, so it D, in this problem. Now scan D's row for the least weight. It is 36 for C (we cannot choose 2, as A is chosen). We keep on scanning next row for the least possible weight value, avoiding earlier chosen vertex or formation of cycle that has less number of vertices than a complete cycle.

A cycle that is obtained for the problem of Fig.4.16 is A-D-C-B-F-E-A having reasonably optimal weight as 191.

Please note that this may not be an actual optimal weight.

4.7 UNIT END EXERCISES:

1. Ore's Theorem states that "If a connected graph G has $n \geq 3$ vertices and for every pair u, v of non-adjacent vertices, $\deg(u) + \deg(v) \geq n$, then G is Hamiltonian". Prove this theorem.
2. Give two examples of maximal non-Hamiltonian graphs.
3. Draw a graph corresponding to the following degree sequences or if no graph exists, justify:
   (i) (2, 2, 2, 2)
   (ii) (1, 2, 3, 4)
   (iii) (2, 2,3, 3)
   (iv) (1, 1, 1, 2)
4. Prove or disprove: Two graphs with the same degree sequence are isomorphic.

5

Matching and Ramsey Theory

Unit Structure :
5.1 Introduction
5.2 Matching In Bipartite Graphs
5.3 Independent sets and covering
5.4 The Personnel Assignment Problem
5.5 Ramsey Number
5.6 Unit End Exercise

5.1 INTRODUCTION

Matching theory is used to find the similarity between the graphs. It is an important tool in the fields like computer vision and pattern recognition. Matching has applications in flow networks, scheduling and planning, modeling bonds in chemistry, graph coloring, the stable marriage problem, neural networks in artificial intelligence and more. In image recognition applications, the results of image segmentation in image processing typically produces data graphs with number of vertices much larger than in the model graphs data expected to match against. Perfect Matching theory is also known as graph isomorphism problem. Many graph matching algorithm exist in order to optimize for the parameters.
First we have to go through some definitions.

Definition 1. Matching in Graph: A matching in graph G is a set $M = \{e_1, e_2, e_3, \cdots, e_k\}$ of edges such that each vertex $v \in V(G)$ appears in at most one edge of M i.e $e_i \cap e_j = \phi$, for all i, j.
The size of matching is the number of edges that appears in the matching.

Definition 2. M- Saturated vertex: A vertex v is called as M-saturated if for some $e = \{xy\}$, $e \in M$.
If x is not saturated then it is unsaturated. $2 \mid M \mid$ =number of M-saturated vertices.

Definition 3. Perfect Matching: A perfect matching in a graph G is a matching in which every vertex of G appears exactly once, i.e Which saturates every vertex or a matching of size exactly $\frac{n}{2}$.

Definition 4. Maximum Matching: A matching m is called maximum if no other matching in G has a larger size.

Definition 5. Maximal Matching: A matching m is called maximal if $M \cup \{e\}$ is not a matching for any $e \in E(G)$.

Definition 6. M- alternating Path: Let M be a matching in a graph G . A path P in G is said to be M-alternating if every other edge in P appear in M.

Definition 7. M- augmenting Path: An M-augmenting path is an M-alternating path $P = \{v_1, v_2, v_3, \cdots, v_k\}$ such that both v_1, v_k are not vertices in M.

Theorem 5.1.1. ***Berg theorem****: Let M be matching in a graph G Then M is a maximum matching if and only if there does not exist any M- augmenting path in G.*

Proof. Suppose that M is a matching in G, such that there exist an M-augmenting path say P. Notice that P must have odd length. Since its edges alternate between edges in M and edges in $G \setminus M$, and further both begins and ends with edges from $G \setminus M$. Let M' be the set of edges in P that are not in M. Then notice that $\mid M' \mid > \mid M \mid$ and M' is a matching in G. so M is not a maximum matching. Hence if M is a maximum matching, there can not be any M-augmenting path in G.
Conversely assume that M is a matching having no M-augmented path in G. Let M_2 be a maximum matching in G. Note that by above argument , there is no M_2 augmenting path in G.
Let H be the sub graph of G having $E(H) = \{e \in M \text{ but } e \notin M_2\} \cup \{e \in M_2 \text{ but } e \notin M\}$. Let us consider the possible components of H. Note that every vertex has degree 0, 1, or 2 in H. Vertices of degree 0 are disregarded as their components are trivial. If a component has all vertices of degree 2. It must be an even cycle alternating between edges of M and edges of M_2. If a components has a vertex degree 1 it must be a path, alternating between edges of M and edges of M_2. Note that neither M nor M_2 has an augmented path G, we must have that such a components begins with an edge from one matching and ends with an edge from other matching.
In either cases every non trivial component of H has exactly half of its

edges from M and exactly half of its edges from M_2. Hence we must have that $\mid M \setminus M_2 \mid = \mid M_2 \setminus M \mid$. Hence
$\mid M \mid = \mid M \cap M_2 \mid + \mid M \setminus M_2 \mid = \mid M \cap M_2 \mid + \mid M_2 \setminus M \mid = \mid M_2 \mid$.
and thus M is also a maximum matching in G. □

5.2 MATCHING IN BIPARTITE GRAPHS

Let G be a graph with vertex set partitioned into two subsets A and B such that every edges in G has one end points in A and the other in B. Such graph is called as Bipartite graph. We will use the notation G(A, B) for bipartite graph.
Suppose that A and B are subsets of G. We can say that A can match with B if there exist a Matching $M = \{e_1, e_2, e_3, \cdots, e_k\}$ such that each e_i has one vertex in A and the other in B and every vertex in A and B appears in the matching.

5.2.1 Neighbour set of S in Graph G:

N(v)=$\{u \in V(G) \mid u$ *is adjacent to* $v\}$ is called as the set of neighbour of v.
Given a set $S \subset V(G)$, we write $\mathbf{N(S)} = \cup_{v \in S} N(v)$ is the set of vertices that are adjacent to atleast one vertex in S.

Theorem 5.2.1. ***Hall's Theorem*** *Let G(A, B) be a bipartite graph. Suppose that* $\mid A \mid \leq \mid B \mid$. *Then there exists a matching M of size* $\mid A \mid$ *in G if and only if for every subset* $S \subset A$, *we have that* $\mid N(S) \mid \geq \mid S \mid$. *In particular, If* $\mid A \mid = \mid B \mid$, *then G has a perfect matching under this condition.*

Proof. Let $A = \{u_1, u_2, \cdots, u_k\}$ and $B = \{v_1, v_2, \cdots, v_l\}$ with $k \leq l$.
First suppose that there exist a matching M of size $\mid A \mid$ in G. Since G is bipartite, every edges of M includes one vertex of A. By possible relabeling of B, suppose that $M = \{e_1, e_2, \cdots, e_k\}$ where $e_i = \{u_i, v_i\}$ for each i.

Let $S \subset A$ with $S = \{u_{s_1}, u_{s_2}, \cdots, u_{s_t}\}$. Then $v_{s_j} \in N(S)$ for all $1 \leq j \leq t$. and hence $\mid N(S) \mid \geq t = \mid S \mid$.
Converse we prove by induction on $\mid A \mid$. First , note that if $\mid A \mid = 1$, the theorem is trivial.
Let us assume that the theorem holds for $\mid A \mid = k - 1$.
Let G(A, B) be a bipartite graph with $\mid A \mid = k$ satisfying Hall's condition. we consider two cases.

Case 1 For every proper subset S of A, $\mid N(S) \mid \geq \mid S \mid +1$.

Consider u_1, without loss of generality suppose that u_1 is adjacent to v_1 (and possibly some other vertices also). Consider the subgraph H of G on $A \setminus \{u_1\}$, $B \setminus \{v_1\}$, i.e remove both vertices u_1 and v_1 from consideration. Let S be a subset of $A \setminus \{u_1\}$. Then note that $N_H(S)$ is either equal to $N_G(S)$ or has been reduced in size by 1 due to removal of v_1. In either case, $\mid N_H(S) \mid \geq \mid N_G(S) \mid -1 \geq \mid S \mid$. And hence H satisfies Hall's condition. We can therefore obtain a matching in H of size $\mid A \mid -1$ by the induction hypothesis. adding the edge $\{u_1, v_1\}$ to such a matching produces a matching in G of size $\mid A \mid$.

Case 2: A contains a proper subset S having $\mid S \mid = \mid N(S) \mid$.

Note that since S is a proper subset of A, by induction hypothesis we have that the subgraph of G on (S, N(S)) satisfies Hall's condition and hence we may find a matching on this subgraph of size $\mid S \mid$. Let H be the subgraph of G on $(A \setminus S, B \setminus N(S))$: Note that we have removed the same number of vertices from both A and B.
Let T be a subset of $A \setminus S$. Then notice that $N_G(S \cup T) = N_G(S) \cup N_H(T)$ and these two neighborhoods are disjoint. Furthermore by Hall's condition we have $\mid N_G(S \cup T) \mid \geq \mid S \cup T \mid = \mid S \mid + \mid T \mid$.
Therefore $\mid N_H(T) \mid = \mid N_G(S \cup T) \mid - \mid N_G(S) \mid \geq \mid S \mid + \mid T \mid - \mid S \mid = \mid T \mid$. and hence H satisfies Hall's condition. We thus can find a matching in H of size $\mid A \setminus S \mid$. Taking the union of this matching with the matching on S gives a matching in G of size $\mid A \mid$ as desired. □

Some obvious features of perfect Matching:
(1) If G has an odd number of vertices, then it has no perfect matching.
(2) If G has any isolated vertices then it has no perfect matching.
(3) If G has a component of odd size, then it has no perfect matching.

Notation: For any given graph G, $o(G)$ denote the number of odd components of G.

Theorem 5.2.2. ***Tutte's Theorem****: Let G be a graph . Then G contains a perfect matching if and only if for every proper subset $S \subset V(G)$ we have $o(G \setminus S) \leq \mid S \mid$.*

Proof. We first consider the forward implication. Suppose that G contains a perfect matching M. Let $S = \{v_1, v_2, \cdots, v_k\} \subset V(G)$ be a proper subset of V(G). let G_1, G_2, $\cdots$, G_n be the odd components of $G \setminus S$. Since G_i is odd, some vertex u_i of G_i must be matched under M with a vertex v_i of S.
$\therefore$, $o(G \setminus S) = n \leq \mid S \mid$

Let us now consider the backward implication. Let G be a graph on n vertices. We shall prove this by induction on n. Note the base case is when n=2. Then G is K_2. It satisfies Tutte's condition and has a perfect matching. Thus theorem hold for n=2.
We assume that if graph has n-2 or fewer vertices (note , we may delete two vertices since no odd graph has a perfect matching and hence n is even.) then the theorem holds.
Now we have a graph G on n vertices in which Tutte's condition is satisfied. We consider two case:
case 1: For every proper subset S of V(G), $O(G \setminus S) \leq \mid S \mid -1$.
Note that as n is even, we must have that $o(G \setminus S)$ and $\mid S \mid$ are of the same parity. so infact we have $o(G \setminus S) \leq \mid S \mid -2$ for every proper subset of V(G).
Fix an edge uv and consider $H = G \setminus \{u, v\}$ having n-2 vertices. Let $T \subset V(H)$. Note that $o(H \setminus T) = O(G \setminus (T \cup \{u, v\})) \leq \mid t \cup \{u, v\} \mid -2 = \mid T \mid$. Hence H satisfies tutte's criterion and thus H has a perfect matching M'. Taking $M = M' \cup \{uv\}$ yields a perfect matching in G.
case 2: There exist a proper subset S of V(G) with $o(G \setminus S) = \mid S \mid$.
Let S be the largest proper subset of $V(G)$ having $o(G \setminus S) = \mid S \mid = k$ and let $S = \{v_1, v_2, \cdots, v_k\}$. We first claim that $G \setminus S$ contains only odd components. Let if H be an even component of $G \setminus S$. Then fix any vertex $u_0 \in V(H)$. Note that $H \setminus \{u_0\}$ must have atleast one odd component as it has an odd number of vertices and hence $\mid S \cup \{u_0\} \mid \geq o(G \setminus (S \cup \{u_0\})) \geq \mid S \cup \{u_0\} \mid$. yielding a strictly larger set satisfying the hypothesis of the case. Hence no even component exist.
Let $G_1, G_2, \cdots, G_k$ be the k-components of $G \setminus S$ and each of these is odd. Create a bipartite graph B as follows.

Let V=S and let $U = \{u_1, u_2, \cdots, u_k\}$. Put an edges $u_i v_j$ in B if v_j is adjacent to some vertex in G_i.

Claim: B satisfies Hall's criterion.

Suppose T is a proper subset of U with $\mid T \mid = t$. Suppose that $\mid N(T) \mid < \mid T \mid$. Let $S' = \{v_{i_1}, v_{i_2}, \cdots, v_{i_r}\} = N(T) \subset S$, with $r < t$. Note that for each $u_i \in T$, we have that G_i is a component of $G \setminus S'$. since it is adjacent to no other vertices in S. Hence $o(G \setminus S') \geq t > r = \mid S' \mid$. This is a contradiction of Tutte's condition. Hence B satisfies Hall's criterion.
Therefore , there exist a perfect matching M in B. Without loss of generality by relabel the components of $G \setminus S$ so that we have, for every $v_i \in S$. That v_i is adjacent to some vertex of G_i call this vertex as u_i.

Let us first show that we can find a perfect matching in $G_j \setminus u_j$ for all

j. If $| V(G_j) | = 1$ then we are done. if not , suppose that $| V(G_j) | \geq 3$. Let $H = G_j \setminus \{u_j\}$. Let $W \subset V(H)$ be a proper subset. Note that $O(H \setminus W) = o(G \setminus (S \cup \{u_j\} \cup W)) - (k-1)$. Since we have all the odd components G_1, G_2, $\cdots$, G_k. Excepting G_j counted in the right hand side, which can be rectified by simply subtracting k-1. Moreover due to maximality of S. We have $o(G \setminus (S \cup \{u_j\} \cup W)) <| S \cup \{u_j\} \cup W |=| W | +k+1$ and hence $o(H \setminus W) <| W | +k+1-(k-1) =| W | +2$. As $o(H \setminus W)$ and $| W |$ must have same parity. This implies $o(H \setminus W) \leq| W |$. Hence H satisfies Tutte's criterian. Thus by induction we can form a perfect matching in H.

By performing the above procedure. We thus can form a perfect matching M_j in $G_j \setminus \{u_j\}$ for all j. Taking $M_1 \cup M_2 \cup \cdots \cup M_j \cup \{u_1v_1, u_2v_2, \cdots, u_kv_k\}$ yields a perfect matching in G. □

5.3 Independent sets and covering

Definition 8. Independent set(Stable set): Let G(V, E) be a graph. A independent set is a subset C of V such that no two vertices of C are adjacent in G.
An independent set is maximum if G has no independent set C' with $| C' |>| C |$.

Definition 9. Vertex cover:A vertex cover is a set W of V such that every edge of G has atleast one end in W.

Definition 10. Edge cover:A edge cover is a subset F of E such that for each vertex v there exist $e \in F$ satisfying $v \in e$.
Note: An edge cover can exist only if G has no isolated vertices.

Notation:
$\alpha(G) = max\{| C | / C \text{ is a independent set.}\}$

$\tau(G) = min\{| W | / W \text{ is a vertex cover}\}$

$v(G) = max\{| M | / M \text{ is a Matching}\}$

$\rho(G) = min\{| F | / F \text{ is an edge cover}\}$
Note: $\alpha(G) \leq \rho(G)$ and $v(G) \leq \tau(G)$

Theorem 5.3.1. *A set $C \subset V$ is an independent set of G if and only if $V \setminus C$ is a vertex covering of G.*

Proof. By definition , C is an independent set of G if and only if no edge of G has both ends in Cor equivalently if and only if each edge has atleast one end in $V \setminus C$. But this is so if and only if $V \setminus C$ is a vertex covering of G. □

Corollary 5.3.1. $\alpha(G) + \tau(G) = n$.

Proof. Let C be a maximum independent set of G and W be a minimum vertex covering of G. Then by above theorem $V \setminus W$ is an independent set and $V \setminus C$ is a vertex covering.
Therefore $n - \tau(G) =\mid V \setminus W \mid \leq \alpha(G) \cdots (1)$.

$n - \alpha(G) =\mid V \setminus C \mid \geq \tau(G) \cdots (2)$.
From (1) and (2) we have $\alpha(G) + \tau(G) = n$. □

Theorem 5.3.2. ***(Gallai's theorem)****: If G=(V, E) is a graph without isolated vertices then* $v(G) + \rho(G) =\mid V \mid$.

Proof. Let M be a matching of size v(G). Let U be the set of M- unsaturated vertices (vertices which are not end point of any edge in M). Since G has no isolated vertex and M is maximum, there exist a set E' of $\mid U \mid$ edges, one incident with each vertex in U. Clearly, $M \cup E'$ is an edge covering of G, and so
$\rho(G) \leq\mid M \cup E' \mid = v(G) + (n - 2v(G)) = n - v(G)$
$\rho(G) + v(G) \leq n \cdots (1)$
Now let L be a minimum edge covering of G, set H=G[L] and let M be a maximum matching in H. Denote the set of M-unsaturated vertices in H by U. Since M is maximum, H[U] has no links and therefore
$\mid L \mid -v(G) =\mid L \setminus M \mid \geq\mid U \mid = n - 2v(G)$
$\mid L \mid + \mid M \mid \geq n$
Because H is a subgraph of G, M is a matching in G and so

$\rho(G) + v(G) \geq\mid L \mid + \mid M \mid \geq n$
$\rho(G) + v(G) \geq n \cdots (2)$

from (1) and (2) we get

$\rho(G) + v(G) = n$ □

Let M be a matching and K be a covering such that $\mid M \mid =\mid K \mid$ then, M is a maximum matching and K is a minimum covering.

Proof. If M' is a maximum matching and K' is minimum covering then $\mid M \mid \leq\mid M' \mid \leq\mid K' \mid \leq\mid K \mid$

Since $\mid M \mid =\mid K \mid$, it follows that $\mid M \mid =\mid M' \mid$ and $\mid K \mid =\mid K' \mid$. □

Theorem 5.3.3. ***Koings matching Theorem****: In a bipartite graph, the number of edges in a maximum matching is equal to the number of vertices in a minimum covering.*

Proof. Let G be a bipartite graph with bipartition (X, Y) and let M' be a maximum matching of G. Let U be the set of M'- unsaturated vertices in X and Let Z be the set of all vertices connected by M'-alternating paths to vertices of U. Let set $S = Z \cap X$ and $T = Z \cap Y$.
Then by Hall's theorem, we have that every vertex in T is M'-saturated and N(S)=T. Define $K' = (X \setminus S) \cup T$. Every edge of G must have atleast one of its ends in K' or otherwise, there would be an edge with one end in S and one end in $Y \setminus T$, contradicting N(S)=T. Thus K' is a covering of G and Clearly $\mid M' \mid = \mid K' \mid$. By above lemma K' is a minimum covering. The theorem follows. □

Theorem 5.3.4. ***Konigs's edge covering theorem****: In a bipartite graph G with no isolated vertex, the number of vertices in a maximum independent set is equal to the number of edge in a minimum edge covering.*

Proof. let G be a bipartite graph with no isolated vertex, By Gallai's theorem, we have $\alpha(G) + \tau(G) = v(G) + \rho(G)$ and since G is bipartite , it follows from Konig's matching theorem $v(G) = \tau(G)$. Thus $\alpha(G) = \rho(G)$. □

5.4 The Personnel Assignment problem:

In a certain company , n workers X_1, X_2, $\cdots$, X_n are available for n jobs y_1, y_2, $\cdots$, y_n, each worker being qualified for one or more of these jobs. Can all the men be assigned, one man per job, two jobs for which they are qualified? This is the personnel assignment problem.

We construct a bipartite graph G with bipartition (X, Y), where $X = \{x_1, x_2, \cdots, x_n\}$, $Y = \{y_1, y_2, \cdots, y_n\}$ and x_i is joined to y_j if and only if worker x_i is qualified for job y_j. The problem becomes one of determining whether or not G has a perfect matching. According to Hall's theorem either G has such a matching or there is a subset S of X such that $\mid N(S) \mid < \mid S \mid$. Now we present an algorithm to solve the personnel assignment problem.

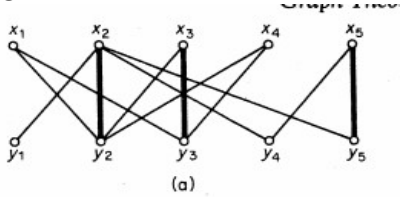
Algorithm: Start with an arbitrary matching M.

(1) If M saturates every vertex in X then stop otherwise, Let u be an M-unsaturated vertex in X. Set $S = \{u\}$ and $T = \phi$.

(2) If N(S)=T then $| N(S) <| S$, Since $| T |=| S | -1$ then stop. since by Hall's theorem there is no matching that saturates every vertex in X. Otherwise , let $y \in N(S) \setminus T$.

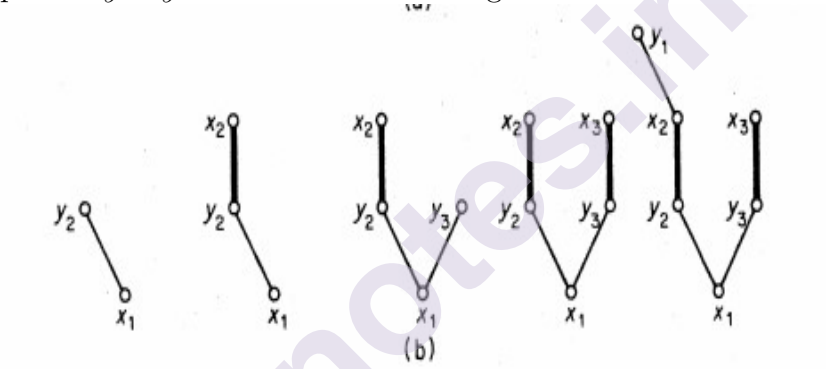
(3) If y is M-saturated, let $yz \in M$. Replace S by $S \cup \{z\}$ and T by $T \cup \{y\}$ and go to step2. (observe that $| T |=| S | -1$ is maintained after this replacement.) Otherwise, let P be an M-augmenting (u-y) path. Replace M by $M' = M \Delta E(P)$ and go to step 1.

Consider a graph given below with initial matching $M = \{x_2y_2, x_3y_3, x_5y_5\}$.



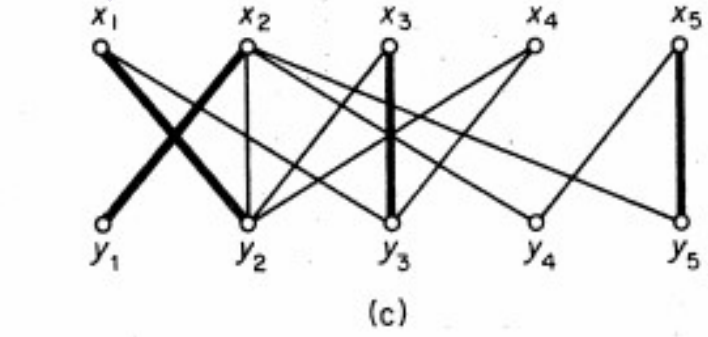
assignment 1.jpg

An M-alternating tree is grown, starting with x_1 and the M-augmenting path $x_1y_2x_2y_1$ found. as shown in figure below.

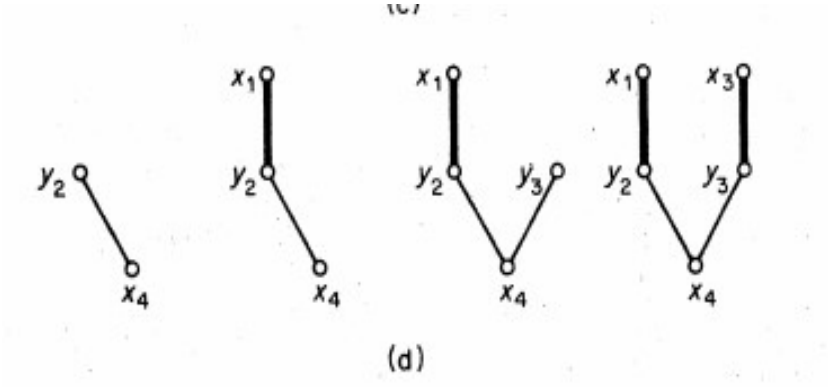


This result in a new matching $M' = \{x_1y_2, x_2y_1, x_3y_3, x_5y_5\}$

as shown in figure below.



An M'-alternating tree is now grown from x_4. Since there is no M'-augmenting path with origin x_4, the algorithm terminates.



The set $S = \{x_1, x_3, x_4\}$ with neighbour set $N(S) = \{y_2, y_3\}$ shows that G has no perfect matching.

Flow Chart of Hungarian Method:

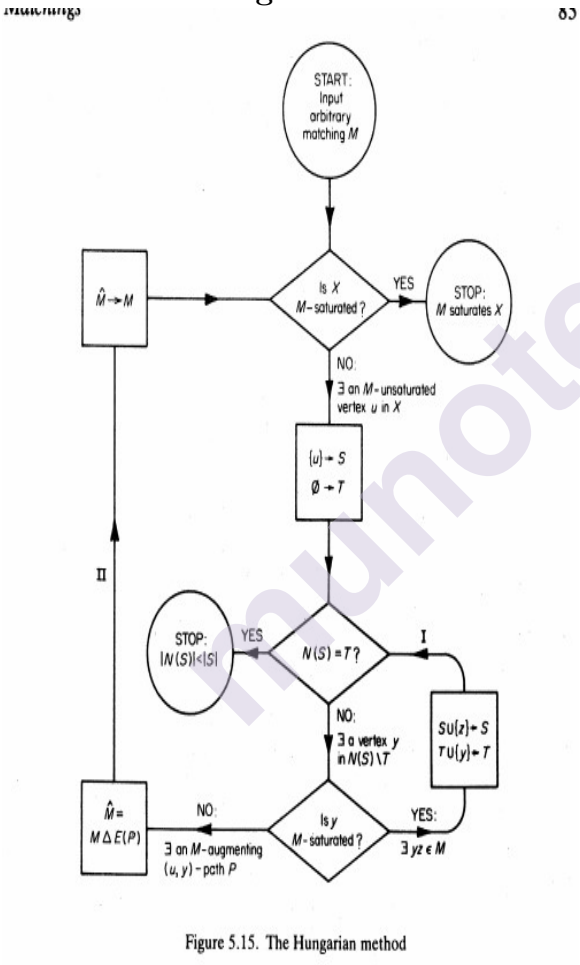


Figure 5.15. The Hungarian method

5.5 Ramsey Number

Definition 11. Clique: A clique of a simple graph G is a subset S of

V(G) such that any two vertices of S are adjacent.

S is a clique of G if and only if S is an independent set of G^c .
If G has no large cliques, then G has a large independent set.
The above remark was first proved by Ramsey (1930).

Question:

Among 6 people , there are either 3 who know each other or 3 who do not know each other.
Proof: Let 1,2,3,4,5,6 be the 6 people. Consider these 6 people as vertices of graph. i vertex is connected to j vertex by an edge then it means i and j know each other. otherwise they do not know each other. Let us select vertex 1 and join the with some of remaining vertices. By pigenhole principle, either 1 knows 3 people or 1 does not know 3 people i.e 1 is connected to either 3 vertices or not connected to 3 vertices by an edge.

Case1: If 1 is adjacent to three vertices .
If 1 is adjacent to three vertices say a, b, c. If two of a, b, c are adjacent then 1 with those two vertices form K_3. This means 1 and other two people know each other.

Otherwise if none of a, b, c are adjacent. Then we get three isolated vertices, i.e we get three people they do not know each other.
Case2: 1 is not adjacent to three vertices.
If 1 is not adjacent to three vertices a, b, c. If two of a, b, c are not adjacent then we get three vertices isolated. means Three people do not know each other.

Definition 12. If we color the edges of K_n with red or blue, then there is ethier a set of r vertices such that edges among them are colored red or a set of b vertices such that edges among them are colored blue.

Definition 13. Ramsey Numbers: For any positive integers k and l ≥ 2 there exist a smallest integer t=r(k, l) such that any graph on t vertices contains either a clique of k vertices or an independent set of l vertices.

r(1,l)=r(k,1)=1
r(2, l)=l, r(k, 2)=k, r(k, l)=r(l, k)

Theorem 5.5.1. *For any two integers $k \geq 2$ and $l \geq 2$ then prove that $r(k,l) \leq r(k,l-1) + r(k-1,l)$. Furthermore, if $r(k,l-1)$ and $r(k-1,l)$ are both even, then strict inequality holds .*

Proof. Let G be a graph on $r(k,l-1) + r(k-1,l)$ vertices, and let $v \in V$. We distinguish two cases:

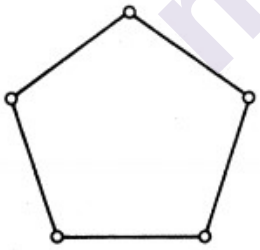
Case1: v is non adjacent to a set of atleast $r(k, l-1)$ vertices.
$G[S]$ contains either a clique of k vertices or an independent set of $l-1$ vertices and therefore $G[S \cup \{v\}]$ contains either a clique of k vertices or an independent set of l vertices.

Case2: v is adjacent to a set T of atleast $r(k-1, l)$ vertices.
G[T] contains either a clique of k-1 vertices or an independent ser of l vertices.
Therefore $G[T \cup \{v\}]$ contains either a cilque of k vertices of an independent set of l vertices.
Since one of case(1) and case(2) must hold because the number of vertices to which v is non adjacent plus the number of vertices to which v is adjacent is equal to $r(k, l-1) + r(k-1, l) - 1$. □

Proof. Thus $r(k, l) \leq r(k, l-1) + r(k-1, l)$
Now suppose that r(k, l-1) and r(k-1, l) are both even and let G be a graph on $r(k, l-1) + r(k-1, l) - 1$ vertices. Since G has an odd number of vertices, then some vertices v is of even degree; in particular , v cannot be adjacent to precisely $r(k-1, l)-1$ vertices. Consequently, either case1 or case2 of above holds and therefore G contains either a clique of k vertices or an independent set of l vertices.
Thus $r(k, l) \leq r(k, l-1) + r(k-1, l) - 1$ as stated. □

Theorem 5.5.2. *Prove that* $r(3,3) = 6$, $r(3,4) = 9$, $r(3,5) = 14$, $r(4,4) = 18$.

Proof. (1): From Theorem above we can get, $r(3,3) \leq r(3,2)+r(2,3) = 3+3 = 6 \cdots (1)$
From figure below , The 5 cycle contains no clique of three vertices and no independent set of three vertices.

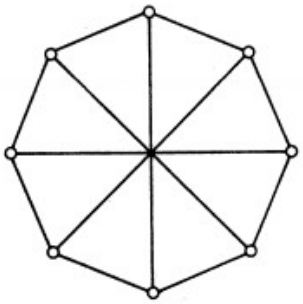


(a)

It shows that $r(3,3) \geq 6 \cdots (2)$.
from (1) and (2) $r(3,3) = 6$ □

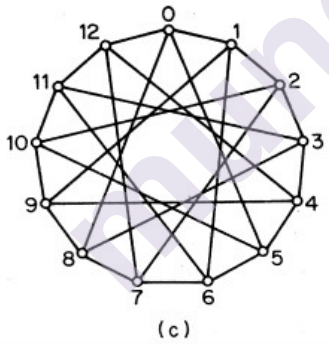
Proof. (ii) From Theorem above we can get, $r(3,4) \leq r(3,3)+r(2,4)-1 = 6+4-1 = 9 \cdots (1)$

The graph of figure below , contains no clique of three vertices and no independent set of four vertices.



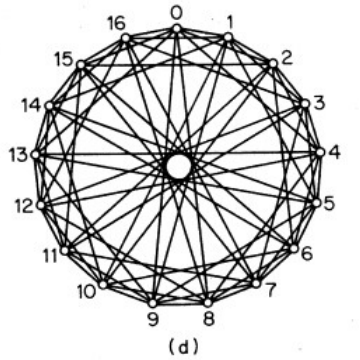
It shows that $r(3,4) \geq 9 \cdots (2)$.
from (1) and (2)
$r(3,4) = 9$ □

Proof. (iii) From Theorem above we can get
$r(3,5) \leq r(3,4) + r(2,5) = 9 + 5 = 14 \cdots (1)$
The graph of figure below , contains no clique of three vertices and no independent set of five vertices.



It shows that $r(3,5) \geq 14 \cdots (2)$.
from (1) and (2)
$r(3,5) = 14$ □

Proof. (iv) From Theorem above we can get
$r(4,4) \leq r(4,3) + r(3,4) = 9 + 9 = 18 \cdots (1)$
The graph of figure below , contains no clique of four vertices and no independent set of four vertices.



It shows that $r(4,4) \geq 18 \cdots (2)$.
from (1) and (2)
$r(4,4) = 18$ □

Definition 14. Ramsey graph:A (k, l)- Ramsey graph is a graph on $r(k,l)-1$ vertices that contains neither a clique of k vertices nor an independent set of l vertices.
Such a graph exist for all $k \geq 2$ and $l \geq 2$. All the graph shown in the above theorem represent Ramsey- graph.

Theorem 5.5.3. $r(k,l) \leq \binom{k+l-2}{k-1}$.

Proof. We will prove this by induction on k, l. First consider the base case as k=l=2. $r(2,2) = 2 \leq \binom{2+2-2}{2-1}$.
Now assume the relation holds for all k=x-1, l=y and k=x , l=y-1 cases.
We now prove result holds for k=x, l=y.
By using above theorem and Pascal's rule
$r(k,l) \leq r(k-1,l) + r(k,l-1) \leq \binom{(k-1)+l-2}{(k-1)-1} + \binom{k+(l-1)-2}{k-1} = \binom{k+l-2}{k-1}$
$r(k,l) \leq \binom{k+l-2}{k-1}$ □

Theorem 5.5.4. *Prove that* $r(k,k) \geq 2^{\frac{k}{2}}$.

Proof. Since $r(1,1) = 1$ and $r(2,2) = 2$, we may assume that $k \geq 3$. Let S_n the set of simple graphs with vertex set $\{v_1, v_2, \cdots, v_n\}$ and ${S_n}^k$ the set of those graphs in S_n that have a clique of k vertices.
As each subset of the $\binom{n}{2}$ possible edges $v_i v_j$ determines a graph in S_n.
Therefore, $| S_n | = 2^{\binom{n}{2}} \cdots (1)$
Similarly, the number of graphs in S_n having a particular set of k vertices as a clique is $2^{\binom{n}{2}-\binom{k}{2}}$.Since there are $\binom{n}{k}$ distinct k-elements subsets of $\{v_1, v_2, v_3, \cdots, v_n\}$. we have,
$| {S_n}^k | \leq \binom{n}{k}.2^{\binom{n}{2}-\binom{k}{2}} \cdots (2)$
Dividing (2) by (1) we get,
$\frac{|{S_n}^k|}{|S_n|} \leq \binom{n}{k}.2^{-\binom{k}{2}} < \frac{n^k.2^{-\binom{k}{2}}}{k!}$ □

Proof. Suppose , now that $n < 2^{\frac{k}{2}}$
$\frac{|{S_n}^k|}{|S_n|} \leq \frac{2^{\frac{k^2}{2}}.2^{-\binom{n}{2}}}{k!} = \frac{2^{\frac{k}{2}}}{k!} < \frac{1}{2}$

Therefore, fewrer than half of the graph in S_n contains a clique of k-vertices. Also, fewer than half of the graphs in S_n contains an independent set of k - vertices. Hence some graph in S_n contains neither a clique of k- vertices nor an independent set of k vertices. Because this holds for any $n < 2^{\frac{k}{2}}$, we have $r(k,k) \geq 2^{\frac{k}{2}}$ □

Corollary 5.5.1. *If $m = min\{k,l\}$, then $r(k,l) \geq 2^{\frac{m}{2}}$.*

Theorem 5.5.5. $r(k_1, k_2, \cdots, k_m) \leq r(k_1 - 1, k_2, \cdots, k_m) + r(k_1, k_2 - 1, \cdots, k_m) + \cdots + r(k_1, k_2, \cdots, k_m - 1) - m + 2$

Theorem 5.5.6. $r(k_1 + 1, k_2 + 2, \cdots, k_m + 1) \leq \frac{(k_1+k_2+\cdots+k_m)!}{k_1!.k_2!.k_3!.....k_m!}$

5.6 Chapter End Exercise

1. Show that every k-cube has a perfect matching ($k \geq 2$).
2. Find the number of different perfect matchings in K_{2n} and $K_{n,n}$.
3. Show that a tree has at most one perfect matching.
4. Deduce Hall's theorem from Konig's theorem.
5. Derive Hall's theorem from Tutte's theorem.
6. Show that a tree G has a perfect matching if and only if $o(G \backslash v) = 1$ for all $v \in V$.
7. Describe how the Hungarian method can be used to find a maximum matching in a bipartite graph.
8. Show that G is bipartite if and only if $\alpha(H) \geq \frac{1}{2}v(H)$ for every subgraph H of G.
9. Show that G is bipartite if and only if $\alpha(H) = \rho(H)$ for every subgraph H of G such that $\delta(H) > 0$.
10. Show that, for all k and l, $r(k,l) = r(l,k)$.
11. Let r_n denote the Ramsey number $r(k_1, k_2, \cdots, k_n)$ with $k_i = 3$ for all i. (a) Show that $r_n \leq n(r_{n-1} - 1) + 2$. (b) Noting that $r_2 = 6$, use (a) to show that $r_n \geq [n!e] + 1$. (c) Deduce that $r_3 \geq 17$.
12. Deduce that $r(2^n + 1, 2^n + 1) \geq 5^n + 1$ for all $n \geq 0$.