1

NUMERICAL INTEGRATION – I

Chapter Structure

1.1 OBJECTIVE

* After going through this chapter you will be able to :
* Solve any definite integral in interval [a, b].
* Solve unknown function integral by numerical method.
* Learn different technique for solving definite integral by numerical method.
* Solve definite integral very efficient way.

1.2 INTRODUCTION

Differentiation and integration are basic Mathematical operation with a wide range of applications in many areas of science.

In this we are going to explose various ways for approximating the integral of a function over a given domain. These are various reasing as of why such approximations can be useful.

i) Not every function can be analytically integrated.

e.g. : $\int_0^1 \frac{1}{1+x^3}dx$ or $\int_0^1 \frac{\tan x}{x}dx$.

ii) Even if a closed integration formula exists it might still not be the most efficient way of calculating the integral.

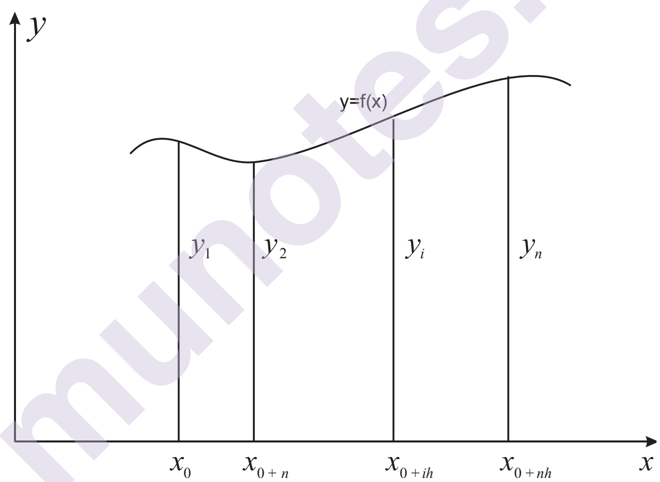
iii) It can happen that we need to integrate an unknown function in which only some examples of the functions are known.

A simpler approach for approximating the value of $\int_a^b f(x)dx$ would be to complete the product of the value of the function at one of the end points of the interval by the length of the interval.

In case we choose the end-point where the function is evaluated to be $x = a$ we obtained

$$\int_a^b f(x)dx \approx \mathrm{f}(a)(b-a)$$

This approximation is called rectangular method or the rectangular quadrature. The points $x_0, x_1, \ldots.., x_n$ that are used in the quadrature formula are called quadrature points.



We may approximate the integrate by a linear curve i.e. ($y = a + bx$) or by a second degree curve

i.e. by ($y = a + bx + cx^2$) …. and soon.

Here we will learn till three degree polynomial by numerical integration using :

i) Trapezoidal rule (linear curved)

ii) Simpson's one-third rule (second degree)

iii) Simpson's third-eight rule (third degree)

1.3 Newton – cote's quadrature formula :

The newton cote's is an extremely useful and straight forward family of numerical integration techniques using this we can derive the all three numerical integration formula which we need.

Let $I = \int_a^b f(x)\,dx$

Let us divide the interval (a, b) into n – sub intervals of width h. as shown in figure (1) so that $x_0 = a$, $x_1 = x_0 + h$, $x_2 = x_0 + 2h$ ….. + $x_n = x_0 + nh = b$.

Then $I = \int_{x_0}^{x_0+nh} f(x)\,dx$

taking $x = x_0 + ph \Rightarrow dx = hdp$

$I = \int_{x_0}^{x_0+nh} f(x_0 + ph)\,dph$

= By newton's forward interpolation formula.

$$= h\int_0^n \left[y_0 + p\Delta y_o + \frac{p(p-1)^{\Delta 2} y_w}{2!} + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots \right] dp$$

Integrating term by term we obtain

$$= nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{2h}\Delta^3 y_0 + \ldots.. \right]$$

$$\therefore \int_{x_0}^{x_0+nh} f(x)\,dx = nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{2n}\Delta^3 y_0 + \ldots. \right]$$

Which is known as the newton –cote's quadrature formula using this we can deduce the following rule taking $n = 1, 2, 3,$ …..

1.4 Trapezoidal Rule :

Taking n = 1 in newton cote's formula at the curve (x_0, y_0) to (x_1, y_1) as a straight line.

i.e. polynomial of order are so that higher order difference become zero we get,

$$\int_{x}^{x_0+h} f(x)dx = h\left(y_0 + \frac{1}{2}\Delta y_0\right)$$

$$= \frac{h}{2}(y_0 + y_1)$$

Similarly $\int_{x_0}^{x+2h} f(x)\,dx = \frac{h}{2}(y_1 + y_2)$

.

.

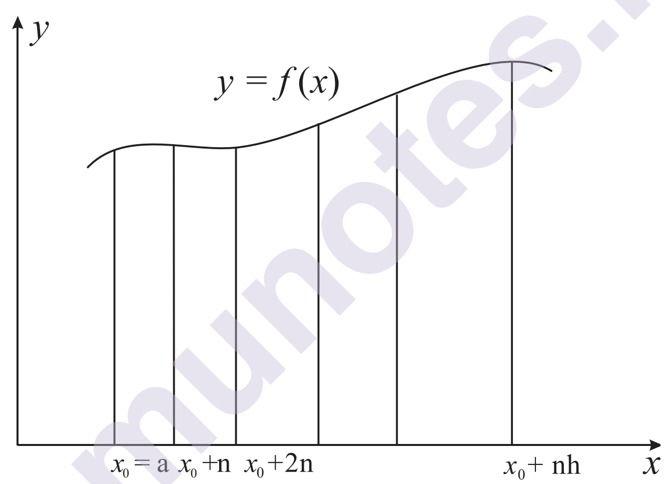
.

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}(y_{n-1} + y_n)$$

Adding these integrals we get

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + ... + y_{n-1})\right]$$

This is known as the trapezoidal rule.



Working of trapezoidal rule. For $\int_a^b f(x)\,dx$

Let $y = f(x)$ take n value of x in interval $[a, b]$

x	x_0	x_1	x_2	.....	x_n
y	y_0	y_1	y_2	.....	y_n

$$\int_a^b f(x)dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + ... + y_{n-1})\right]$$

where $h = \dfrac{b-a}{n}$

Example :

1) Evaluate $\int_0^6 \frac{1}{1+x^3}\,dx$ taking h = 1 by using trapezoidal rules.

Solution :

Divide the interval (0, 6) into six parts each of width h = 1. The value of $f(x)=\frac{1}{1+x^3}$ are given below :

x	0	1	2	3	4	5	6
y	1	0.5	0.11	0.0357	0.0154	0.0079	0.0046
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

by trapezoidal rule

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+.....+y_{n-1})\right]$$

$$=\frac{1}{2}\left[(1+0.0046)+2(0.5+0.11+0.0357+0.0154+0.0079)\right]$$

$$=\frac{1}{2}\left[(1.0046)+2(0.669)\right]$$

$$=\frac{1}{2}\left[1.0046+1.338\right]$$

$$=\frac{1}{2}\left[2.3426\right]$$

$$=1.1713$$

Ex.2 :

Evaluating $\int_0^1 \sqrt{\sin x+\cos x}\,dx$ taking 5 sub-intervals using trapezoidal rules.

Sol^n :

Here n = 5, a = 0, b = 1

$$h=\frac{b-a}{n}=\frac{1-0}{5}=0.2$$

x	0	0.2	0.4	0.6	0.8	1
y	1	1.0857	1.1448	1.1790	1.1891	1.1755

by trapezoidal rule

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+.....+y_{n-1})\right]$$

$$=\frac{0.2}{2}\left[(1+1.1755)+2(1.0857+1.1448+1.790+1.1891)\right]$$

$$=0.1\left[2.1755+2(4.5986)\right]$$

$$= 0.1\left[2.1755 + 9.1972\right]$$
$$= 1.13727$$

Ex.3 :

Evaluate $\int_0^1 \frac{dx}{1 + x^2}$ by trapezoidal rule.

6 – coordinate, also determine absolutely error.

Sol[n] :

Here a = 0, b = 1, n = 5

$$h = \frac{b-a}{n} = \frac{1-0}{5} = 0.2$$

$$f(x) = \frac{1}{1+x^2}$$

x	0	0.2	0.4	0.6	0.8	1
y	1	0.9615	0.8621	0.7353	0.6098	0.5

by trapezoidal rule

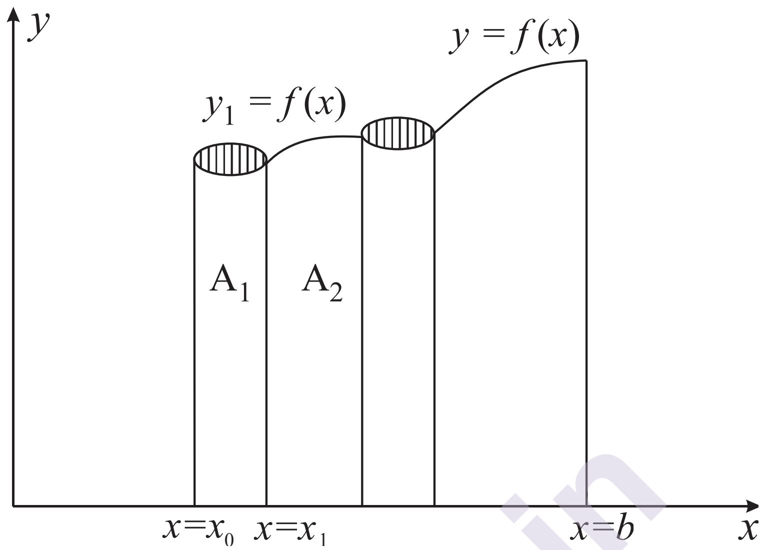
$$\int_a^b f(x)dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + ... + y_{n-1})\right]$$

$$= \frac{0.2}{2}\left[(1 + 0.5) + 2(0.9615 + 0.8621 + 0.7353 + 0.6098)\right]$$
$$= 0.1\left[1.5 + 2(3.1687)\right]$$
$$= 0.1\left[1.5 + 6.3374\right]$$
$$= 0.78374.$$

$$\int_0^1 f(x)dx = \int_0^1 \frac{1}{1 + x^2}\,dx = \left[\tan^{-1}(x)\right]_0^1 = \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4}$$

$$\text{Absolute error } = \frac{\pi}{4} - 0.78374$$
$$= 0.00165.$$

1.5 Error in Trapezoidal rule :



$\because$ The expression of $y = f(x)$ about $x = x_0$ by Taylor's theorem is given by

$$y = f(x) = f(x_0) + (x - x_0) f^1(x_0) + \frac{(x - x_0)^2}{2!} f^{11}(x_0) + ...$$

$$\int_{x_0}^{x_1} f(x)dx = \int_{x_0}^{x_1}\left[f(x_0) + (x - x_0) f^1(x_0) + \frac{(x - x_0)}{2!} f^{11}(x_1) + ...\right]$$

$$\int_{x_0}^{x_1} f(x)dx = \left[xf(x_0) + \frac{(x - x_0)^2}{2} f^1(x_0) + \frac{(x - x_0)^3}{6} f^{11}(x_0) + ...\right]_{x_0}^{x_1}$$

$$= (x - x_0) f(x_0) + \frac{(x_1 - x_0)^2}{2} f^1(x_0) + \frac{(x_1 - x_0)^3}{6} f^{11}(x_0) + ...$$

Put $x_1 - x_0 = h$

$$\int_{x_0}^{x_1} f(x)dx = hf(x) + \frac{h^2}{2} f^1(x_0) + \frac{h^3}{6} f^{11}(x_0) + ...$$

Area of trapezoidal in the interval $[x_0, x_1]$ is

$$A_1 = \frac{h}{2}[y_0 + y_1]$$

by trapezoidal rule

$$A_1 = \frac{h}{2}[f(x_0) + f(x_0 + h)]$$

by taylor's Thm

$$A_1 = \frac{h}{2}\left[f(x_0) + f(x_0) + hf^1(x_0) + \frac{h^2}{2!} f^{11}(x_0) + ...\right]$$

$$A_1 = \left[hf(x_0) + \frac{h^2}{2} f^1(x_0) + \frac{h^3}{4} f^{11}(x_0) + ...\right]$$

Let ε_1 be the error in the interval $[x_0, x_1]$

$$\varepsilon_1 = \int_{x_0}^{x_1} f(x)\,dx - A_1$$

$$= h^3 \left[\frac{1}{6} - \frac{1}{4}\right] f^{11}(x_0) + ...$$

$$\varepsilon_1 = \frac{-1}{12} h^3 f^{11}(x_0)$$

Similarly :

$$\varepsilon_1 = \int_{x_1}^{x_2} f(x)\,dx - A_2 = \frac{-1}{12} h^3 f^{11}(x_1)$$

$$\varepsilon_2 = \int_{x_2}^{x_3} f(x)\,dx - A_3 = \frac{-1}{12} h^3 f^{11}(x_2)$$

.

.

.

$$\varepsilon_n = \int_{x_{n-1}}^{x_n} f(x)\,dx - A_n = \frac{-1}{12} h^3 f^{11}(x_{n-1})$$

The total error

$$\varepsilon = \varepsilon_1 + ... + \varepsilon_n$$

$$\varepsilon = \frac{-h^3}{12}\left[f^{11}(x_0) + f^{11}(x_1) + f^{11}(x_2) + ... + f^{11}(x_{n-1})\right]$$

Choose $f^{11}\varepsilon_1 = \max\left\{ f^{11}(x_0), f^{11}(x_1), f^{11}(x_2), ...\right\}$

$$\varepsilon_1 \le \frac{-h^3}{12} f^{11}(\varepsilon_1) n$$

$$\varepsilon_1 \le \frac{-h^3}{12}\left(\frac{b-a}{h}\right) f^{11}(\varepsilon_1)$$

$$\varepsilon_1 \le -\frac{(b-a)}{12} h^2 f^{11}(\varepsilon_1)$$

Which is the required error in trapezoide rule.

Practice Problem :

Evaluate the following integral by using trapezoidal rule.

i) $\int_0^2 x^2\,dx$ with $n = 0$ Ans. : 2.68

ii) $\int_0^5 \frac{dx}{4x+5}$ with $n = 10$ Ans. : 0.4055

iii) $\int_0^{0.6} e^{-x^2}\, dx$ with $n = 6$ Ans. : 0.5357

1.6 Simpson's one third rule :

While taking n = 2 in newton cote's formula we get

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + ... + y_{n-1}) + 2(y_2 + y_4 + ... + y_{n-2})\right]$$

This is known as the Simpson's one third rule.

In Simpson's rule the given interval must be divided into even number of equal sub-interval since we find the area of two strips at 0 time.

Working of Simpson's one third rule for $\int_a^b f(x)\, dx$.

Let $y = f(x)$, taking n – value of x in [a, b].

x	x_0	x_1	x_2	x_3	x_n
y	y_0	y_1	y_2	y_3	y_n

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + ... + y_{n-1}) + 2(y_2 + y_4 + ... + y_{n-2})\right]$$

Where $h = b - a / n$

Ex. 1 :

Calculate upto 4 decimal places $\int_3^8 \frac{dx}{\sqrt{16x-x^2}}dx$ by using Simpson's $(1/3)^{th}$ rule taking n = 5.

Sol^n :

Here $f(x) = \frac{1}{\sqrt{16x - x^2}}$ a = 3, b = 8, n = 5,

$$h = \frac{b-a}{n} = \frac{8-3}{5} = 1$$

x	3	4	5	6	7	8
y	0.1601	0.1443	0.1348	0.1291	0.1259	0.125
	y_0	y_1	y_2	y_3	y_4	y_5

using Simpson's $(1/3)^{rd}$ rule.

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[\left(y_0 + y_1\right) + 4\left(y_1 + y_3 + \ldots.. + y_{n-1}\right) + 2\left(y_2 + y_4 + \ldots.. + y_{n+2}\right)\right]$$

$$= \frac{1}{3}\left[(0.160 + 0.125) + 4\,(0.1443 + 0.1291) + 2\,(0.1348 + 0.1259)\right]$$

$$= \frac{1}{3}\left[0.2851 + 1.0936 + 0.5286\right]$$

$$= \frac{1}{3}\left[1.9073\right]$$

$$= 0.6357$$

Ex. 2 :

Evaluate $\int_0^{\pi} \frac{\sin^2 x}{5 + 4\cos x}\,dx$ by taking 5 ordinates by Simpson's (1/3)rd rule.

Sol^{n} :

We divide the interval $(0, \pi)$ into 4 - equal part i.e. (5 ordinates).

$$h = \frac{b-a}{n} = \frac{\pi - 0}{4} = \pi/4$$

x	0	$\pi/4$	$\pi/2$	$3\pi/4$	π
y	0	0.0639	0.2	0.2302	0
	y_0	y_1	y_2	y_3	4

[using calculate in degree mode]

by Simpson's (1/3)rd rule.

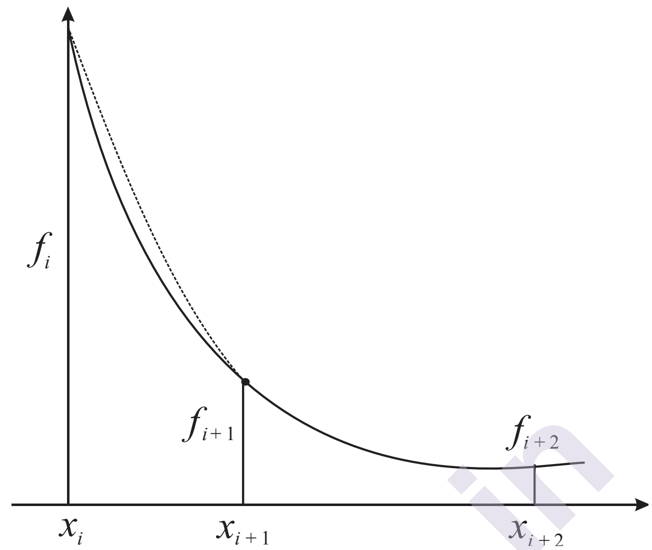
$$\int_a^b f(x)\,dx = \frac{h}{3}\left[\left(y_0 + y_n\right) + 4\left(y_1 + y_3 + \ldots..\right) + 2\left(y_2 + y_4 + \ldots..\right)\right]$$

$$= \frac{\pi/4}{3}\left[(0 + 0) + 4(0.0639 + 0.2302) + 2(0.2)\right]$$

$$= \frac{\pi}{12}\left[1.1764 + 0.4\right]$$

$$= 0.4127.$$

1.7 Error in Simpson's 1/3rd rule :



Let $y = f(x)$ be a continuous functions and continuous derivation of all orders in [a, b]. Divide the interval [a, b] into n equal sub-intervals by the points $a = x_0, x_1, \ldots, x_n = b$ & $x_1 = x_0 + ih$ $i = 1.2 \ldots n$.

The Taylor's expression of $y = f(x)$ atleast $x = x_0$ is

$$y = f(x) = f(x_0) + (x - x_0) f^1(x_0) + \frac{(x - x_0)^2}{2!} f^{11}(x_0) + \frac{(x - x_0)^3}{3!} f^{11}(x_0) \ldots$$

$$\int_{x_0}^{x_2} f(x)\,dx = \left[\int_{x_0}^{x_2} f(x_0) + (x - x_0) f^1(x_0) + \frac{(x - x_0)^2}{2!} f^{11}(x_0) + \frac{(x - x_0)^3}{3!} f^{111}(x_0) + \ldots\right] dx$$

$$= \left[xf(x_0) + \frac{(x - x_0)^2}{2!} f^1(x_0) + \frac{(x - x_0)^3}{3!} f^{11}(x_0) + \frac{(x - x_0)^4}{4!} f^{111}(x_0)\right]_{x_0}^{x_2}$$

$$= \left[(x_2 - x_0) f(x_0) + \frac{(x_2 - x_0)^2}{2!} f^1(x_0) + \frac{(x_2 - x_0)^3}{3!} f^{11}(x_0) + \ldots\right]$$

Put $x_2 - x_0 = 2h$

$$= 2hf(x_0) + 2h^2 f^1(x_0) + \frac{4}{3} h^3 f^{11}(x_0) + \frac{2}{3} h^4 f^{11}(x_2) \qquad \ldots\ldots\ldots\ldots (I)$$

by Simpson's 1/3rd rule.

$$A_1 = \frac{h}{3}\left[y_0 + 4y_1 + y_2\right]$$

$$= \frac{h}{3}\left[f(x_0) + 4f(x_0 + h) + f(x_0 + 2h)\right]$$

Using taylor's thm.

$$A_1 = \frac{h}{3}\left[f(x_0) + 4\left[f(x_0) + hf^1(x_0) + \frac{h^2}{2}f^{11}(x_0) + \frac{h^3}{6}f^{11}(x_0) + ...\right] + \right.$$

$$\left. f(x_0) + 2hf^1(x_0) + \frac{4h^2}{2}f^{11}(x_0) + \frac{8h^3}{6}f^{11}(x_0) + ...\right]^2$$

$$= \frac{h}{3}\left[6f(x_0) + 6hf^1(x_0) + \frac{8h^2}{2}f^{11}(x_0) + 2h^3 f^{11}(x_0)\right]$$

$$= 2hf(x_0) + 2h^2 f^1(x_0) + \frac{4}{3}h^3 f^{11}(x_0) + \frac{2}{3}h^4 f^{111}(x_0) + ... \text{ (I)}$$

Let ε_1 be the error in the interval $[x_0, x_1]$

$$E_1 = \int_{x_0}^{x_2} f(x)\,dx - A_1$$

$$= h^5 f^{\text{IV}}_{(x_0)}\left(\frac{4}{15} - \frac{5}{15}\right) + ...$$

$$= \frac{h^5}{90} f^{\text{IV}}_{(x_0)}$$

Similarly

$$E_3 = \int_{x_2}^{x_4} f(x)\,dx - A_2 = \frac{-h^5}{90} f^{IV}_{(x_2)}$$

.

.

.

$$E_{n-1} = \int_{x_{n-2}}^{x_n} f(x)\,dx - A_{n-2} = \frac{-h^5}{90} f^{IV}(x_{n-2})$$

$\therefore$ The total error

$$E = E_1 + E_2 + ... + E_{n-1}$$

$$= \frac{-h^5}{90}\left[f^{II}(x_0) + f^{IV}(x_2) + ... + f^{IV}(x_{n-2})\right]$$

Choose $f^{IV}_{\varepsilon_1} = M\,\text{ax}\left\{f^{IV}_{(x_0)}, f^{IV}_{(x_2)}, ..., f^{IV}_{(x_{n-2})}\right\}$

$$E \leq \frac{-h^5}{90} \frac{n}{2} f^{IV}(\varepsilon)$$

$$E \leq \frac{-h^5}{90} \frac{(b-a)}{h} f^{IV}(\varepsilon)$$

$$E \leq -\frac{(b-a)}{180} h^4 f^{IV}(\varepsilon) \qquad x_0 < \varepsilon < h_n$$

Which is the required even in composited Simpson's 1/3rd rule.

1.8 Simpson's 3/8th rule :

While taking n = 3 in newton's take formula we set all differences uppers then 3rd forward difference will becomes zero.

$$\int_{x_0}^{x_3} f(x)\,dx = h\left[3y_0 + \frac{9}{2}(y_1 - y_0) + \frac{9}{h}(y_2 - 2y_1 + y_0) + \frac{3}{8}(y_3 - 3y_2 + 3y_1 - y_0) + ...\right]$$

$$= \frac{3h}{8}\left[8y_0 + 12(y_1 - y_0) + 6(y_2 - 2y_1 + y_0) + (y_3 - 3y_2 + 3y - y_0) + ...\right]$$

$$= \frac{3h}{8}\left[y_0 + 3y_1 + 3y_2 + y_3\right]$$

Similarly

$$\int_{x_3}^{x_6} f(x)\,dx = \frac{3h}{8}\left[y_3 + 3y_4 + 3y_5 + y_0\right]$$

.

.

.

$$\int_{x_{n-3}}^{x_n} f(x)\,dx = \frac{3h}{8}\left[y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n\right]$$

[where n is divisible by 3]

We add all above integral

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_3} f(x)\,dx + \int_{x_3}^{x_6} f(x)\,dx + ... \int_{x_{n-3}}^{x_n} f(x)\,dx$$

$$= \frac{3h}{8}\left[(y_0 + 3y_1 + 3y_2 + y_3) + (y_3 + 3y_4 + 3y_5 + y_6) + ... + (y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n\right])$$

$$= \frac{3h}{8}\left[(y_0 + y_n + 3(y_1 + y_2 + y_4 + y_5 + ... + y_{n-2} + y_{n-1}) + 2(y_3 + y_6 + y_9 + ... + y_{n-3}\right])$$

Working of Simpson's (3/8)th rule for $\int_a^b f(x)\,dx$

Let $y = f(x)$ take n- value of x in [a, b]

x	x_0	x_1	x_2	.....	.....	x_n
y	y_0	y_1	y_2	.....	.....	y_n

$$\int_a^b f(x)\,dx = \frac{3h}{8}\left[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + ...) + 2(y_3 + y_6 + ...)\right]$$

where $h = \dfrac{b-a}{n}$.

Ex 1 :

Evaluate $\int_4^{5.2} \log e^x \, dx$ by Simpson's (3/8)th rule taking n = 6 sub interval.

Sol$^{\mathbf{n}}$ **:**

Dividing the interval [4, 5.2] in six equal part taking h = 1 we get $f(x)$ value $f(x) = \log e^x$ are given below.

$$h = \frac{b-a}{n} = \frac{5.2-4}{6} = \frac{1.2}{6} = 0.2$$

x	4	4.2	4.4	4.6	4.8	5.0	5.2
y	1.3863	1.4351	1.4810	1.5260	1.5686	1.6094	1.6484

by using Simpson's (3/8)th rule.

$$\int_a^b f(x)dx = \frac{3h}{8}\left[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + ...) + 2(y_3 + y_6 + ...)\right]$$

$$= \frac{3 \times 0.2}{8}\left[(1.3863 + 1.6486) + 3(1.4351 + 1.4816 + 1.5886 + 1.6094) + 2(1.5260)\right]$$

$$= 0.075\left[3.0349 + 3(6.1147) + 2(1.5260)\right]$$
$$= 0.075\left[3.0349 + 18.3441 + 3.052\right]$$
$$= 0.075 \times 24.431$$
$$= 1.8323$$

Ex 2 :

Evaluate $\int_0^{0.6} e^{-x^2} dx$ taking n = 6 by Simpson's (3/8)th rule.

Sol^n :

Here $f(x) = e^{-x^2}$ $a = 0$ $b = 0.6$ $n = 6$

$$h = \frac{b-a}{n} = \frac{0.6-0}{6} = 0.1$$

x	0	0.1	0.2	0.3	0.4	0.5	0.6
y	1	0.99	0.9608	0.9139	0.8521	0.7788	0.6976
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

by using Simpson's (3/8)^th rule.

$$\int_a^b f(x)dx = \frac{3h}{8}\left[(y_0 + y_n) + 3(y_1 + y_2 + y_3 + y_4 + ...) + 2(y_3 + y_6 + ...)\right]$$

$$= \frac{3 \times 0.1}{8}\left[(1 + 0.6976) + 3\,(0.99 + 0.9608 + 0.8521 + 0.7788) + 2\,(0.9139)\right]$$

$$= 0.0375\,[1.6976 + 10.745 + 1.8278]$$

$$= 0.0375 \times 14.2705$$

$$= 0.5351.$$

1.9 Error in Simpson's (3/8)^th rule :

Let $y = f(x)$ be a continuous function and hence continuous derivatives of all under in [a, b]. Divide the interval [a, b] into n sub interval by the point $a = x_0, x_1, x_2, ..., \text{x}_{n=b}$ & $x_i = x_0 + ih$ $i = 1, 2, ..., n$

The taylor's expansion of $y = f(x)$ about $x = x_0$ is

$$y = f(x) = f(x_0) + (x - x_0)\,f^{I}(x_0) + \frac{(x-x_0)^2}{2!} f^{II}(x_0) + \frac{(x-x_0)^3}{3!} f^{III}(x_0) + \frac{(x-x_0)^4}{4!} f^{IV}(x_0) + ...$$

$$\therefore \int_{x_0}^{x_3} f(x)dx = \int_{x_0}^{x_3} [f(x_0) + (x - x_0)\,f^{I}(x_0) + \frac{(x-x_0)^2}{2!} f^{II}(x_0) + \frac{(x-x_0)^3}{3!} f^{III}(x_0) + \frac{(x-x_0)^4}{4!} f^{IV}(x) + ...]dx$$

$$= \left[xf(x_0) + \frac{(x-x_0)^2}{2} f^{I}(x_0) + \frac{(x-x_0)^3}{6} f^{II}(x_0) + \frac{(x-x_0)^4}{24} f^{III}(x_0) + \frac{(x-x_0)^5}{120} f^{IV}(x_0) + ...\right]_{x_0}^{x_3}$$

$$= (x_3 - x_0) f(x_0) + \frac{(x_3 - x_0)^2}{2} f^I(x_a) + \frac{(x_3 - x_0)^3}{6} f^{II}(x_0)$$

$$+ \frac{(x_3 - x_0)^4}{24} f^{III}(x_0) + \frac{(x_3 - x_0)^5}{120} f^{IV}(x_0) + ...II$$

Put $x_3 - x_0 = 3h$

$$= 3h\, f(x_0) + \frac{9}{2} h^2 f^I(x_0) + \frac{27}{6} h^3 f^{II}(x_0) + \frac{81}{24} h^4 f^{IV}(x_0) + \frac{243}{120} h^5 f^{IV}(x_0)$$

…. (I)

By Simpson's 3/8th rule

$$A_1 = \frac{3h}{8}\left[y_0 + 3y_1 + 3y_2 + y_3\right]$$

$$= \frac{3h}{8}\left[f(x_0) + 3f(x_0 + h) + 3f(x_0 + 2h) + f(x_0 + 3h)\right]$$

Using Taylor's thm

$$A_1 = \frac{3h}{8}\left[f(x_0) + 3\left[f(x_0) + hf^1(x_0) + \frac{h^2}{2} f^{11}(x_0) + \frac{h^3}{6} f^{111}(x_0) + \frac{h^4}{24} f^{iv}(x_0) + ...\right]\right.$$

$$+ 3\left(f(x_0) + 2hf^{II}(x_0) + \frac{4h^2}{2} f^{II}(x_0) + \frac{8h^3}{6} f^{III}(x_0) + \frac{16}{24} h^4 f^{IV}(x_0)^2 + ...\right)$$

$$\left. + \left(f(x_0) + 3hf^I(x_0) + \frac{9}{2} h^2 f^{II}(x_0) + \frac{27}{6} h^3 f^{III}(x_0) + \frac{81}{24} h^4 f^{IV}(x_0) + ...\right)\right]$$

$$= \frac{3h}{8}\left[8f(x_0) + 12hf^I(x_0) + 12h^2 f^{II}(x_0) + 9h^3 f^{II}(x_0) + \frac{132}{24} h^4 f^{IV}(x_0) + ...\right]$$

$$= 3hf(x_0) + \frac{9}{2} h^2 f^I(x_0) + \frac{9}{2} h^3 f^{II}(x_0) + \frac{27}{8} h^4 f^{III}(x_3) + \frac{33}{16} h^5 f^{IV}(x_0)$$

…… (II)

Let E_1 be the error in the interval $[x_0, x_3]$

$$I_2 = \frac{h}{2}\left[(y_0 + y_1) + 2(y_2)\right]$$

$$\frac{1}{2}\left[(0.5 + 0.1667) + 2(0.3333)\right]$$

$$I_2 = 0.6666$$

$$= \frac{-3h^5}{80} h^5 f^{IV}(x_0)$$

Similarly

$$\int_{x_3}^{x_6} f(x)dx - A_3 = \frac{-3}{80} h^5 f^{IV}(x_3)$$

.

.

.

$$\int_{x_{n-3}}^{x_n} f(x)dx - A_{n-3} = \frac{-3}{80} h^5 f^{IV}(x_{n-3})$$

$\therefore$ Total error = $\frac{-3h^5}{80}\left[f(x_0) + f^{IV}(x_3) + ... + f^{IV}(x_{n-3})\right]$

Choose $\text{Max } f^{IV}(\varepsilon_1) = M\text{ax}\left\{f(x_0), f^{IV}(x_0) ..... f^{IV}(x_{n-3})\right\}$

$$E \le \frac{h^5}{80} n\ f^{IV}(\varepsilon_1)$$

$$\le \frac{-h^5}{80} n f^{IV}(\varepsilon_1)$$

$$\le \frac{-(b-a)}{80} h^5\ f^{IV}(\varepsilon_1)$$

Which is required error in Simpson's 3/8th rule.

$$\int_{v_0}^{v_{0+nh}} f(v)dt = \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + y_5 + ... + y_{n-1}) + 2(y_2 + y_4 + ... + y_{n-2})\right]$$

where $h = \frac{b-a}{n}$ $\quad a = v_0 \quad b = v_{0+nh}$

In Simpson's (1/3)rd rule the given interval must be divide into even number of equal sub intervals.

Ex. 1 :

The velocity V km/min of a train which strats from rest is given at a fixed interval of time (min) as follows :

t:	0	2	4	6	8	10	12	14	16	18	20
v:	0	10	18	25	29	22	20	11	05	02	0

Estimate approximately the distance covered in 20 minutes by train.

Sol^n :

Let S (km) be the distance covered in t (min) by train

$$\therefore \frac{ds}{dt} = v$$

$$ds = vdt$$

$$\therefore S = \int_0^{20} v\,dt$$

by Simpson's (1/3)rd rule $h = b - a / n$

$$= \frac{20 - 0}{10} = 2$$

$$\therefore h = 2.$$

$$\therefore \int_0^{20} v\,dt = \frac{h}{3}\left[\left(y_0 + y_n\right) + 4\left(y_1 + y_3 + ...\right) + 2\left(y_2 + y_4 + ...\right)\right]$$

$$= \frac{2}{3}\left[\left(0 + 0\right) + 4\left(10 + 25 + 32 + 11 + 2\right) + 2\left(18 + 24 + 20 + 5\right)\right]$$

$$= \frac{2}{3}\left[0 + 4 \times 80 + 2 \times 72\right]$$

$$= 309.33\,\text{km}.$$

Hence in 20 mins train covered 309.33 kms distance.

* **Simpson's (3/8)th rule :**

This is 3rd child of newton cote's

$\therefore n = 3$ we get

$$\int_{v_0}^{v_{0+nh}} f\left(v\right)dt = \frac{3h}{8}\left[\left(y_0 + y_n\right) + 3\left(y_1 + y_2 + y_4 + y_5 + ...\right) + 2\left(y_3 + y_6 + ...\right)\right]$$

where $h = \dfrac{b - a}{n}$ $\quad a = v_0 \quad b = v_{0+nh}$

while applying the Simpson's (3/8)th rule the number of sub interval should be taken as multiple of 3.

Ex. a) :

The water under portion of a water tank is divided by horizontal plane's one meters apart into the following areas : 472, 392, 302, 198, 116, 60, 34, 12, 4 sq.m. use the Simpson's (3/8)th rule to find the volume in cubic meter between the two extreme ends.

Sol^n :

As given condition we can write :

Distance (S) :	1	2	3	4	5	6	7	8	9
Area (A) :	472	398	302	198	116	60	34	12	04

Here sub interval are multiple of 3

$$\therefore h = \frac{b - a}{n} = \frac{9 - 1}{8} = 1$$

$$\int_1^9 A\,ds = \frac{3h}{8}\left[\left(y_0 + y_n\right) + 3\left(y_1 + y_2 + y_4 + y_5\right) + 2\left(y_3 + y_6 + ...\right)\right]$$
$$= \frac{3(1)}{8}\left[\left(472 + 4\right) + 3\left(398 + 302 + 116 + 60 + 12\right) + 2\left(198 + 34\right)\right]$$
$$= \frac{3}{8}\left[476 + 2664 + 464\right]$$
$$= 1351.5 \text{ cube meters.}$$

Using these three numerical integration techniques. We can evaluate integration very easily.

Example :

Evaluate the following integral by using Simpson's 3/8th rule.

i) $\int_0^6 \frac{1}{(1+x)^2}\,dx$ with n = 6 — Ans. : 0.912

ii) $\int_0^{\pi} (4 + 2\sin x)\,dx$ with n = 6 — Ans. : 16.5679

iii) $\int_{0.2}^{14} \left(\sin x - \log e^{x+e^x}\right) dx$ with 12 — Ans. : 3.2561

1.10 Review :

1) Newton's cote's formula
2) Trapezoidal rule & error
3) Simpson's (1/3)rd rule & error
4) Simpson's (3/8)th rule & error

1.11 Unit End Exercise :

1) Obtained trapezoidal rule using newton cote's formula. Obtained error in composite Trapezoidal rule.

2) Compute the trapezoidal approximation for $\int_0^2 \sqrt{x}\,dx$ with n = 6. Compare the estimate with the exact value.

   Ans. : T = 1.81948, exact = $\frac{4\sqrt{2}}{3} = 1.8856$, error = 0.035076.

3) Use Simpson's (1/3)rd rule to approximate $\int_0^2 \sqrt{x}\,dx$ with n = 6. Compare the estimate with the exact value.

   Ans. : T = 1.8569, E.V. = 1.8856, Error = 0.01521.

4) Compare $\int_0^1 \left(x^2 + x - 1\right) dx$ with n = 10 using trapezoidal rule. Compute the error exactly and the error expression.

5) Obtain Simpson's $(1/3)^{rd}$ rule by from newton cote's formula.

6) Obtain Simpson's $(3/8)^{th}$ rule by from newton cote's formula.

7) Obtain error in composite of Simpson's $(1/3)^{rd}$ rule.

8) Obtain error in composite of Simpson's $(3/8)^{th}$ rule.

9) Compute $\int_0^3 \left(x^3 - 3x\right) dx$ with n = 6 by Simpson's $(3/8)^{th}$ rule.

10) Evaluate $\int_0^6 \frac{dx}{\sqrt{x+1}}$ with n = 6 using Simpson's $(1/3)^{rd}$ rule.

Ans. : 3.2961.

♦♦♦♦♦

2

Numerical Integration – II

Chapter Structure

2.1 Objective :

After going through this chapter you will able to :

* Solve integral equation which hose analytical solution.
* Solve multiple integral by using numerical method.
* Solve integral by different numerical integral methods.

2.2 Introduction :

Integral equation are of special application in applied mathematics. There are such integral who have no analytic solution. To solve that we can use numerical method of integration. Multiple integral equation are difficult to solve using numerical method. It is easy to solve using this methods we can comparing with the numerical expect value to find the error.

2.3 Romberg's Method :

Richardson extrapulation is used as the formation of a numerical integration called Romberg integration.

Consider the integral

$$I = \int_a^b f(x)\,dx$$

Let I_1, I_2 be approximated values of I obtained by using trapezoidal rule with different sub intervals of width h_1 & h_2 respectively.

Let E_1 & E_2 be the corresponding errors.
$\therefore$ The errors in trapezoidal rule is of order h^2.

$$I = I_1 + E_1 = I_1 + kh_1^2 \qquad \ldots\ldots\ldots \text{(I)}$$

$$I = I_2 + E_2 = I_2 + kh_2^2 \qquad \ldots\ldots\ldots \text{(II)}$$

from equations I & II

$$I_1 + kh_1^2 = I_2 + kh_2^2$$

$$I_1 - I_2 = k\left(h_2^2 - h_1^2\right)$$

$$k = \frac{I_1 - I_2}{\left(h_2^2 - h_1^2\right)}$$

Substituting value of k in equation (I) we get

$$I = I_1 + \frac{I_1 - I_2}{\left(h_2^2 - h_1^2\right)} h_1^2$$

$$= \left(h_2^2 - h_1^2\right) I_1 - \left(I_1 - I_2\right) h_1^2 \Big/ \left(h_2^2 - h_1^2\right)$$

$$I = \frac{h_2^2 I_1 - h_1^2 I_2}{\left(h_2^2 - h_1^2\right)}$$

Put $h_1 = h$ & $h_2 = h/2$ we get

$$I = \frac{\frac{h^2}{4} I_1 - h^2 I^2}{\frac{h^2}{4} - h^2}$$

$$I = \frac{4I_2 - I_1}{3}$$

This is known as Romberg's formula. Working of Romberg's integration method difference

h	$I(h) = I_1$			
$h/2$	$I(h/2) = I_2$	$I_{12} = \frac{4I_2 - I_1}{3}$	$I_{123} = \frac{4I_{23} - I_{12}}{3}$	$I_{1234} = \frac{4I_{234} - I_{123}}{3}$
$h/4$	$I(h/4) = I_3$	$I_{23} = \frac{4I_3 - I_2}{3}$	$I_{234} = \frac{4I_{234} - I_{23}}{3}$	
$h/8$	$I(h/8) = I_4$	$I_{34} = \frac{4I_4 - I_3}{3}$		

where $h = b - a$.

Use Romberg's formula to evaluate the integral $\int_0^1 \frac{dx}{1+x}$ correct upto 4 decimal place.

Sol^n :

Given : integral is $\int_0^1 \frac{dx}{1+x}$

Here $a = 0, b = 1 \quad \Rightarrow \quad h = b - a = 1 - 0 = 1$

$y = f(x) = \frac{1}{1+x}$

Taking $h = 1$

x	0	1
y	1	0.5

$\therefore$ by trapezoidal rule.

$$I_1 = \frac{h}{2}[y_0 + y_1] = \frac{1}{2}[1 + 0.5] = 0.75$$

$$\boxed{\therefore I_1 = 0.75}$$

Taking $h = 1/2$

x	0	1/2	1
y	1	2/3	0.5

By trapezoidal rule

$$I_2 = \frac{h}{2}[(y_0 + y_2) + 2(y_1)]$$

$$= \frac{1}{4}[(1 + 0.5) + 2(2/3)]$$

$$\boxed{I_2 = 0.7083}$$

Taking $h = \frac{1}{4}$

x	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
y	1	0.8	0.6667	0.5714	0.5

by trapezoidal rule

$$I_3 = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + y_3)]$$

$$= \frac{1}{8}[(1 + 0.5) + 2(0.8 + 0.6667 + 0.5714)]$$

$$\boxed{I_3 = 0.6970}$$

Taking

$$h = \frac{1}{8}$$

x	0	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1
y	1	0.8889	0.8	0.7273	0.6667	0.6154	0.5714	0.5333	0.5

$$I_4 = \frac{h}{2}\left[\left(y_0 + y_8\right) + 2\left(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7\right)\right]$$

$$= \frac{1}{16}\left[\left(1 + 0.5\right) + 2\left(0.8889 + 0.8 + 0.7273 + 0.6667 + 0.6154 + 0.5714 + 0.5333\right)\right]$$

$$\boxed{I_4 = 0.6914}$$

$\therefore$ by Romberg's integration method.

h	I			
1	0.75	$I_{12} = 4I_2 - I_1/3 = 0.6944$	$I_{123} = \frac{4I_{23} - I_{12}}{3} = 0.698$	$I_{1234} = \frac{4I_{(123)} - I_{234}}{3} = 0.6931$
$\frac{1}{2}$	0.7083	$I_{23} = \frac{4I_3 - I_2}{3} = 0.6932$	$I_{234} = \frac{4I_{34} - I_{23}}{3} = 0.6930$	
$\frac{1}{4}$	0.6970	$I_{34} = \frac{4I_4 - I_3}{3} = 0.6931$		
$\frac{1}{8}$	0.6941			

Ex. 2 :

Use Romberg's formula to evaluate the integral $\int_0^2 \frac{dx}{x^2+2}$ correct upto 4 decimal places.

Sol^n :

Given integral $\int_0^2 dx \big/ x^2 + 2$

Here $a = 0$ $\quad b = 2$ $\quad h = b - a = 2 - 0 = 2$

$f\left(x\right) = 1/x^2 + 2$

Taking $h = 2$

x	0	2
y	0.5	0.1667

$$I_1 = \frac{h}{2}\left(y_0 + y_1\right)$$

Taking $h = 1$

x	0	1	2
y	0.5	0.333	0.1667

$$I_2 = \frac{h}{2}\left[\left(y_0 + y_1\right) + 2\left(y_2\right)\right]$$

$$\frac{1}{2}\left[\left(0.5 + 0.1667\right) + 2\left(0.3333\right)\right]$$

$$I_2 = 0.6666$$

$$= \frac{2}{2}\left(0.5 + 0.1667\right) = 0.6667$$

$$\boxed{I_1 = 0.6667}$$

Taking $h = \frac{1}{2}$

x	0	0.5	1	1.5	2
y	0.5	0.4444	0.3333	0.2353	0.1667

$$I_3 = \frac{h}{2}\left[\left(y_0 + y_4\right) + 2\left(y_1 + y_2 + y_3\right)\right]$$

$$= \frac{1}{4}\left[\left(0.5 + 0.1667\right) + 2\left(0.4444 + 0.3333 + 0.2353\right)\right]$$

$$\boxed{I_3 = 0.6732}$$

Taking $h = \frac{1}{4}$

x	0	0.25	0.5	0.75	1	1.25	1.5	1.75	2
y	0.5	0.4848	0.4444	0.3902	0.3333	0.2807	0.2353	0.1975	0.16667

$$I_4 = \frac{h}{2}\left[\left(y_0 + y_8\right) + 2\left(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7\right)\right]$$

$$= \frac{1}{8}[\left(0.5 + 0.1667\right) + 2\ (0.4848 + 0.4444 + 0.3902 +$$

$$0.3333 + 0.2807 + 0.2353 + 0.1975$$

$$\boxed{I_4 = 0.6749}$$

Taking $h = \frac{1}{8}$

x	0	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1	1.125	1.25	1.375
y	0.5	0.4961	0.4848	0.4671	0.4444	0.4183	0.3902	0.3616	0.3333	0.3062	0.2807	0.2570

x	1.5	1.625	1.875	2
y	0.2353	0.1975	0.1813	0.1667

$$I_5 = \frac{h}{2}[(y_0 + y_{16}) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9 + y_{10} + y_{11} + y_{12} + y_{13}\ y_{14} + y_{15})]$$

$$= \frac{1}{16}\left[(0.5 + 0.1667) + 2\,(0.4961 + 0.4848 + 0.4671 + 0.4444 + 0.4183 + 0.3902 + 0.3616 + 0.3333 + 0.3062 + 0.2807 + 0.2570 + 0.2353 + 0.2154 + 0.1975 + 0.1813)\right.$$

$$\boxed{I_5 = 0.6753}$$

By Romberg's integration method $\left(I_{12} = \frac{hI_2 - I_1}{3}\right)$

h	I			
2	0.6667			
1	0.6666	$I_{12} = 0.6665$		
$1/2$	0.6732	$I_{123} = 0.6754$	$I_{123} = 0.6783$	
$1/4$	0.6749	$I_{34} = 0.6755$	$I_{234} = 0.6755$	$I_{1234} = 0.6745$
$1/8$	0.6753	$I_{45} = 0.6754$	$I_{345} = 0.6754$	$I_{2345} = 0.6754$

$$\boxed{I_{12345} = 0.6757}$$

Ex. 1 :

Evaluate the following integral by Romberg's method :

i) $\int_0^1 e^{-x^2} dx$ Ans. : 0.7468

2.4 Gaussian Quadrature :

In general Gaussian quadrature is of the form

$$\int_a^b w(x)\, f(x)\, dx = \sum_{k=0}^{n} a_k f(x_k)$$

where $w(x) > 0$ In [a, b] is called the weight function and a_k, x_k are called weights and nodes respectively.

Gaussian quadrature function for two points also known as Gauss Legendre integration method.

Consider the integral

$$I = \int_{-1}^{1} f(x)dx$$

Let $I = \sum_{k=1}^{2} a_k f(x_k)$

$$= a_1 f(x_1) + a_2 f(x_2) \qquad \text{.......... (I)}$$

Here x_1 & x_2 are nodes & a_1, a_2 are weights.

To find 4 unknowns a_1, a_2, x_1, x_2. Here we required h conditions. The condition that equation I is valid where $f(x)$ is a polynomial of degree ≤ 3.

In this case equation I is true for $f(x) = x^3, x^2, x, 1$.

where $f(x) = x^3$

$$\int_{-1}^{1} x^3\, dx = a_1 x_1^3 + a_2 x_2^3 \Rightarrow a_1 x_1^3 + a_2 x_2^3 = 0 \qquad \text{..... (II)}$$

where $f(x) = x^2$

$$\int_{-1}^{1} x^2\, dx = a_1 x_1^2 + a_2 x_2^2 \Rightarrow a_1 x_1^2 + a_2 x_2^2 = 3/2 \qquad \text{..... (III)}$$

where $f(x) = x$

$$\int_{-1}^{1} x^2\, dx = a_1 x_1 + a_2 x_2 \Rightarrow a_1 x_1 + a_2 x_2 = 0 \qquad \text{..... (IV)}$$

where $f(x) = 1$

$$\int_{-1}^{1} 1\, dx = a_1 + a_2 \Rightarrow a_1 + a_2 = 2 \qquad \text{..... (V)}$$

Multiply equation (IV) by x_1^2 & subtract from (II) we get,

$$a_2 x_2 \left(x_2^2 - x_1^2\right) = 0$$

$$\Rightarrow a_2 x_2 (x_2 - x_1)(x_2 + x_1) = 0$$

$$\therefore a_2 = 0 \text{ or } x_2 = 0 \text{ or } x_2 = x_1 \text{ or } x_1 = -x_2.$$

The cases $a_2 = 0,\ x_2 = 0$ & $x_2 = x_1$. Give result to invalid equations.

$\therefore$ we choose $x_1 = -x_2$

$\therefore$ from equation IV

$$a_1 x_1 - a_1 x_1 = 0 \qquad \text{..... (VI)}$$

From equation V & VI we get

$$a_1 = 1 \ \& \ a_2 = 1$$

From equation III we get

$$2x_1^2 = 2/3$$

$$x_1 = \pm 1/\sqrt{3}$$

We take $x_1 = 1/\sqrt{3}$ & $x_2 = -\sqrt{1/3}$ $(\because x_2 = -x_1)$

Substitute the value of $a_1 a_2$ & $x_1 x_2$ in equation (I)

$$\therefore \int_{-1}^{1} f(x)\,dx = f\left(1/\sqrt{3}\right) + f\left(-1/\sqrt{3}\right)$$

This is known as Gaussian quadrature two points functions.

Note :

Above answer gives solution of $\int_{-1}^{1} f(x)\,dx$

we need $\int_a^b f(x)\,dx$

There are, let $x = mt + c$

If $x = a \Rightarrow t = -1$

If $x = b \Rightarrow t = 1$

Such that $a = -m + c$

$b = m + c$

Solving this equation we get,

$m = \frac{b-a}{2}, \; c = \frac{b+a}{2}$

$\therefore x = \frac{b-a}{2}t + \frac{b+a}{2}$

$\therefore dx = \frac{b-a}{2}dt$

Substituting value of x & dx we get

$$\int_a^b f(x)\,dx = \int_{-1}^{1} f\left(\frac{b-a}{2}t + \frac{b+a}{2}\right)\left(\frac{b-a}{2}\right) dt$$

Ex. 1 :

Evaluate $\int_2^5 \left(2x^3 - 3x\right) dx$ by two points Gaussian quadrature formula

Sol^n :

Given integral $\int_2^5 \left(2x^3 - 3x\right) dx$

Gaussian two points quadrature formula is given by

$$\int_{-1}^{1} f(x)\,dx = f\left(\frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}\right)$$

Here $f(x) = \left(2x^2 - 3x\right)$ $\qquad a = 2, \qquad b = 5$

Put $\quad x = \frac{(b-a)t + (b+a)}{2}$

$x = \frac{3t+7}{2}$

$dx = \frac{3dt}{2}$

Where $x = 2 \Rightarrow t = -1$

$x = 5 \Rightarrow t = +1$

$$\int_2^5 \left(2x^2 - 3x\right) dx = \int_{-1}^{1} \left[2\left(\frac{3t+7}{2}\right)^2 - 3\left(\frac{3t+7}{2}\right)\right]\left(\frac{3}{2}\right) dt$$

$$= \int_{-1}^{1} \left[\frac{9t^2 + 42t + 49}{2} - \frac{9t + 21}{2}\right] \frac{3}{2}\, dt$$

$$= \frac{3}{4} \int_{-1}^{1} 9t^2 + 33t + 28\, dt$$

$$= \int_{-1}^{1} g(t)\, dt$$

Where $9(t) = \frac{3}{4}\left(9t^2 + 33t + 28\right)$

$\therefore$ by Gaussian two points quadrature formula

$$= \frac{3}{4}\left[g\left(1/\sqrt{3}\right) + g\left(-1/\sqrt{3}\right)\right]$$

$$= \frac{3}{4}\left(9\left(1/\sqrt{3}\right)^2 + 33\left(-1/\sqrt{3}\right) + 28\right)$$

$$+\left(9\left(1/\sqrt{3}\right)^2 + 33\left(-1/\sqrt{3}\right) + 28\right)$$

$$= \frac{3}{4}\left[31 + 31\right]$$

$$= 46.5$$

$$\therefore \int_2^5 \left(2x^2 - 3x\right) dx = 46.5$$

2.5 Gaussian Quadrature for 3 points formula :

Consider the integral

$$I = \int_{-1}^{1} f(x)\, dx$$

Let $I = \sum_{k=1}^{3} a_k f\left(x_k\right)$

i.e. $I = a_1 f\left(x_1\right) + a_2 f\left(x_2\right) + a_3 f\left(x_3\right)$ ……….. (I)

Where the nodes $x_1\ x_2\ x_3$ & weights $a_1\ a_2\ a_3$ are to be determine.

To determine we use some as we had done in Gaussian two point's quadrature formula & we get,

$$\int_{-1}^{1} f(x)\, dx = \frac{1}{9}\left[5f\left(\frac{\sqrt{3}}{5}\right) + 8f(0) + 5f\left(-\frac{\sqrt{3}}{5}\right)\right]$$

[Few are left for exercise]

Ex. 2 :

Evaluate $\int_0^1 \frac{dx}{1+x}$ by Gaussian 3 points quadrature formula.

Sol^n :

Given integral $\int_0^1 \frac{dx}{1+x}$

$$f(x) = \frac{1}{1+x}, \qquad a = 0, \qquad b = 1$$

Put $\quad x = \frac{(b-a)+(b+a)}{2}$

$$x = \frac{t+1}{2}$$

$$dx = {dt}/{2}$$

Where $x = 0 \qquad t = -1$

$x = 1 \qquad t = 1$

$$\int_0^1 \frac{dx}{1+x} = \int_{-1}^1 \frac{dt/2}{1+\left(\frac{t+1}{2}\right)} = \int_{-1}^1 \frac{dt}{t+3}$$

$$= \int_{-1}^1 g(t)\, dt$$

$$\because g(t) = \frac{1}{t+3}$$

By Gaussian 3 points quadrature formula

$$\int_0^1 f(x)\,dx = \frac{1}{9}\left[5g\sqrt{3/5} + 8g(0) + 5g\left(\sqrt{3}/5\right)\right]$$

$$= \frac{1}{9}\left[5\,\frac{1}{\sqrt{3}/5+3} + 8\left(\frac{1}{0+3}\right) + 5\left(\frac{1}{-\sqrt{3}/5+3}\right)\right]$$

$$= \frac{1}{9}\left[\frac{5\sqrt{5}}{\sqrt{3}+3\sqrt{5}} + \frac{8}{3} + \frac{5\sqrt{5}}{\sqrt{3}+3\sqrt{5}}\right]$$

$$= \frac{1}{9}\left[\frac{75}{42} + \frac{8}{3}\right]$$

$$\int_0^1 f(x)\,dx = 0.4947\,.$$

Ex. 2 :

Evaluate the following example by two point Gaussian quadrature formula or three points.

i) $\int_0^1 te^{2t}\ dt$ Ans. : 2 points = 3477.5439

3 points = 4967.1067

2.6 Numerical evaluation of double integrals :

If $f(x,\ y)$ is a continuous function & defined on a closed rectangle.

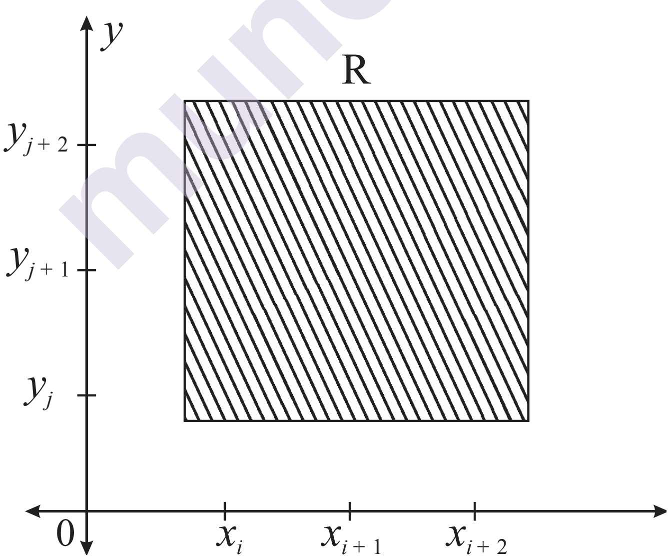
$R\left\{(x,\ y)/\ a \le x \le b,\ c \le y \le d\right\}$ then

$$\iint_R f(x,\ y)\ dx\,dy = \int_c^d\int_a^b f(x,\ y)\,dx\ dy$$

Hence we are going to double integral using trapezoidal rule & Simpson's 1/3rd rule.

Trapezoidal rule of double integral :

Consider $I = \int_{y_j}^{y_{i+z}}\int_{x_i}^{x_{i+z}} f(x,y)\,dx\,dy$ where $h = x_{i+1} - x_i$ & $k = y_{j+1} - y_j$.



$$I = \int_{y_j}^{y_{j+2}}\left[\int_{x_i}^{x_{i+2}} f(x,\ y)\ dx\right] dy$$

Apply trapezoidal rules inner integration.

$$I = \int_{y_j}^{y_{j+2}} \frac{h}{2}\left[f(x_i, y) + 2f(x_{i+1},\ y) + f(x_{i+2},\ y)\right] dy$$

$\therefore$ again applying trapezoidal rule we get,

$$I = \frac{hk}{4}[f\left(x_i,\, y_i\right) + 2f\left(x_i,\, y_{j+1}\right) + f\left(x_i,\, y_{j+2}\right)$$
$$+ 2\left[f\left(x_{i+1},\, y_j\right) + 2\ f\left(x_{i+1},\, y_{j+1}\right)\right] + f\left(x_{i+1},\, y_{j+2}\right)$$
$$+ f\left(x_{i+2},\, y_j\right) + 2f\left(x_{i+2},\, y_{j+1}\right) + f\left(x_{i+2},\, y_{j+2}\right)]$$

Continuously working we can solve triple integral also but it is more complex to solve.

Ex. 1 :

Evaluate $\int_3^7\int_3^7 \frac{1}{x^2 + y^2}\, dx\, dy$

using trapezoidal rule with $h = 2$ & $k = 2$.

Sol[n] :

Given integral $\int_3^7\int_3^7 \frac{1}{x^2 + y^2}\, dx\, dy$

Here $h = 2$ $\quad x : 3, 5, 7$

$k = 2$ $\quad y : 3, 5, 7$

x/y		3 (x_1)	5 (x_2)	7 (x_3)
y_1	3	18	34	58
y_2	5	34	50	74
y_3	7	58	74	98

$\therefore$ by trapezoidal rule form multiple integral

$$I = \frac{hk}{4}\left[\left(f_{(ij)} + 2\left(f_{(i,j+1)}\right) + f_{(i,j+2)}\right)\right.$$
$$+ 2\left(f_{(i+1,j)}\right) + 2f_{(i+1,j+1)} + f_{(i+1,j+2)}$$
$$\left. + \left(f_{(i+2,\ j)} + 2f_{(i+2,j+1)} + f_{(i+2,\ j+2)}\right)\right]$$

$$= \frac{2\times 2}{4}\left[f(3,3) + 2f(3,5) + f(3,7) + 2\left(f(8,3) + 2f(3,5) + f(5,7)\right)\right]$$
$$+ \left(f(7,3) + 2f(7,5) + f(7,7)\right)]$$
$$= 1\left[\left(18 + 2(34) + 58\right) + 2\left(34 + 2(50) + 74\right) + \left(58 + 2(74) + 98\right)\right]$$
$$= 864.$$

2.7 Simpson's rule for double integrals :

It is same as trapezoidal we had done we need to apply Simpson's 1/3rd rule twice we get,

$$I = \frac{hk}{9}\left[\left(f_{(x_i, y_j)} + 4f\left(x_i, y_{j+1}\right) + f\left(x_i, y_{j+2}\right)\ \right) + 4\left(f\left(x_{i+1}, \mathrm{y}_j\right)\right.\right.$$

$$\left.\left. + 4\left(f_{(x_{i+1}, y_{j+1})}\right) + f_{(x_{i+1}, \mathrm{y}_{j+2})}\right) + \left(f_{(x_{i+2}, y_j)} + 4\left(f_{(x_{i+2}, y_{j+1})}\right) + f_{(x_{i+2}, y_{j+2})}\right)\right]$$

Ex. :

Evaluate $\int_0^1 \int_0^2 xy\ dx\ dy$ by Simpson's rule taking $h = 0.5,\ k = 0.5$.

Sol^n :

Given integral $I = \int_0^1 \int_0^2 xy\ dx\ dy$

Here $h = 0.5$ $x = 0, 0.5, 1$

$k = 1$ $y = 0, 0.5, 1, 1.5, 2$

x/y		x_1	x_2	x_3
		0	0.5	1
y_1	0	0	0	0
y_2	0.5	0	0.25	5
y_3	1	0	0.5	1
y_4	1.5	0	0.75	1.5
y_5	2	0	1	2

By Simpson's 1/3rd rule for double integration integral.

$$I = \frac{hk}{9}\left[\left(f\left(x_1, y_1\right) + f\left(x_1, y_5\right)\right) + 4\left(f\left(x_1, y_2\right) + f\left(x_1, y_4\right)\right) + 2f\left(x_1, y_3\right)\right]$$

$$+ 4\left[f\left(x_2, y_1\right) + f\left(x_2, y_5\right) + 4\left(f\left(x_2, y_2\right) + f\left(x_2, y_4\right)\right) + 2f\left(x_2, y_3\right)\right]$$

$$\left. + f\left(x_3, y_1\right) + f\left(x_3, y_5\right) + 4\left(f\left(x_3, y_2\right) + f\left(x_3, y_4\right) + 2f\left(x_3, y_3\right)\right)\right]$$

$$= \frac{0.5 + 0.5}{9}\left[\left(0 + 0 + 4\left(0 + 0\right) + 0\right) + 4\left[0 + 1 + 4\left(0.25 + 0.75\right) + 2\left(0.5\right)\right]\right.$$

$$\left. + \left[\left(0 + 2\right) + 4\left(0.5 + 1.5\right) + 2\left(11\right)\right]\right]$$

$$= \frac{0.25}{9}\left[0 + 24 + 12\right]$$

$= 0.25 \times 36/9$

$= 1.$

2.8 Review :

* Romberg's method of integration.
* Gaussian quadrature formula for 2 & 3 points.
* Multiple integral by trapezoidal rule & Simpson's $1/3^{rd}$ rule.

2.9 Unit & Exercise :

1) Evaluate $\int_0^{\pi} \sin x\, dx \quad h = 2$ by Romberg's integration method

Ans. : 1.99857

2) Evaluate $\int_1^5 \frac{dx}{x}$ by Romberg's integration method

Ans. : 1.6289

3) $\int_0^{\pi/2} \left(x^2 + x + 1\right)\cos x dx$ by Romberg's integration method

Ans. : 2.032

4) For an integral $\int_a^b f(x)\, dx$, derive the two point Gaussian quadrature formula.

5) For an integral $\int_a^b f(x)\, dx$, derive the Romberg's integration method using trapezoidal rule.

6) For an integral $\int_a^b f(x)\, dx$, derive the three point Gaussian quadrature formula.

7) Evaluate $\int_0^2\int_0^2 x + y\ dx\ dy$ by trapezoidal rule by taken $h = k = 1$.

8) Evaluate $\int_0^2\int_0^3 \frac{1}{\sqrt{x^2 + y^2}}\ dx\ dy$ by Simpson's $1/3^{rd}$ rule by taking $h = k = 1$.

9) Evaluate $\int_0^1 dx/2x + 3$ by two points Gaussian quadrature.

10) Evaluate $\int_2^3 \frac{\cos 2x}{1 + \sin x}\, dx\, dy$ Gaussian 2 & 3 points formula.

11) Evaluate $\int_4^5 \frac{dx}{1 + \sin x}$ by Gaussian 2 & 3 points quadrature formula.

12) Evaluate $\int_0^{0.6}\int_0^{0.4} \sin x \cos y\ dx\, dy$ by Trapezoidal & Simpson's $1/3^{rd}$ rule, by taking $h = k = 2$.

13) Evaluate $\int_{1}^{5}\int_{1}^{5}\frac{d.1}{\sqrt{x^2+y^2}}\,dx\,dy$ using Trapezoidal rule, by taking $h=k=2$.

Ans. : 4.1345

14) Evaluate $\int_{0}^{\pi/2}\int_{0}^{\pi/2}\sin(x+y)\,dx\,dy$ by Simpson's 1/3rd rule, by taking $h=k=\pi/4$.

Ans. : 2.0091

15) Evaluate $\int_{0}^{1.5}\int_{0}^{1}e^{x+y}\,dx\,dy$ by Trapezoidal & Simpson's 1/3rd rule, by taking $h=k=0.5$

Ans. : Trapezoidal = 6.2334
Simpson's = 5.0171.

♦♦♦♦♦

3

Approximation of function

Chapter Structure

3.1 Objective :

After going through this chapter you will be able to :

* Fit a straight line by least square.
* Fit a second degree polynomial by least square method.
* Fitting a polynomial of degree M by least square method.
* Fit a curve for exponential function.

3.2 Introduction :

The function approximations needs for many branches of applied mathematics and computer science. In mathematics, least squares can be applied to approximating a given functions. The best approximation can be defined as that which minimizes the difference between the original function and the approximation we also going to used weighted approximation.

The process of finding the equations of curve of best fit which may be most useful for predicting the unknown n values is known as curve fitting. Here we get two value are by observation and other by the predicating value. The difference between this values is called residual an error.

Thus, the principle of least square's status that "The sum of the residuals squares of is minimum".

3.3 General least squares method :

Let (x_i, y_i) where $i = 1, 2, ..., n$ be given n-points.
To fit the curve

$$y = p(x, a_0, a_1, a_2, ..., a_n) \qquad \text{.......} \ (I)$$

To the given n-points where $a_0, a_1, a_2, ..., a_n$ are unknowns parameters whose values are to be determined. Taking $x = x_i$ the value of y obtained from equation (I) we can write as

$$y_i = P(x_i, a_0, a_1, ..., a_k)$$

The quantity y_i is called the predicated or expected values of y at point $x = x_0$ and y_i is called the observed values of y.

The difference $(y_i - y_i)$ is called the residual or error corresponding to point $x = x_i$.

Let $E = E(a_0, a_1, ..., a_k)$ be the sum of square of the residual then

$$E = \sum_{i=1}^{n} (y_i - Y_i)^2$$

$$= \sum_{i=1}^{n} \left[y_i - P(x_i; a_0, a_1, ..., a_k) \right]^2$$

by the principle of least square the value of E will be minimum i.e.

$$\frac{\partial E}{\partial a_0} = 0, \frac{\partial E}{\partial a_1} = 0, ..., \frac{\partial E}{\partial a_k} = 0$$

These equation are called normal equations.
To solve these normal equations with help of given points and we obtain the values of a_i where $i = 1, 2, ..., k$.

Suppose the values of a_i are $a_0 = a_0^*, a_1 = a_1^*, ..., a_k = a_k^*$ substitute all these values in equation I.

$$y_i = P(x_i, a_0^*, a_1^*, ..., a_k^*)$$

Which is required best fitting curve.

3.4 Fitting of a straight line :

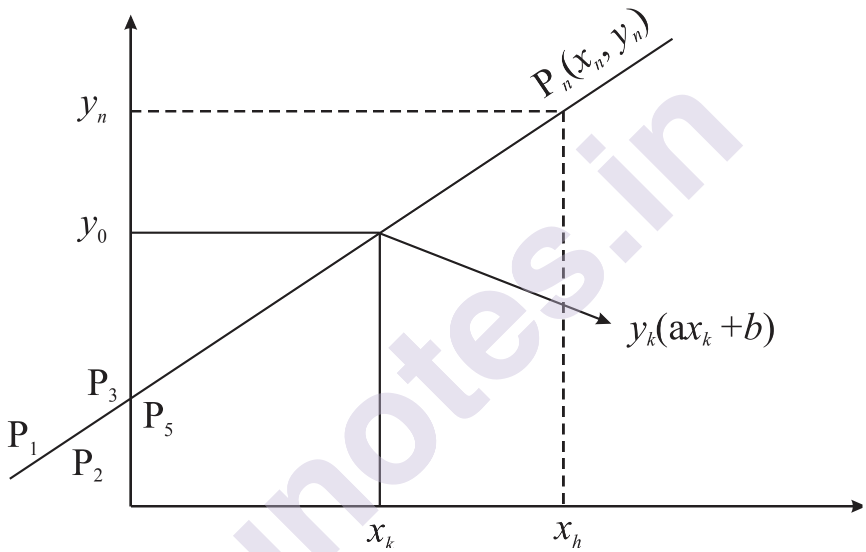
Given the general form of a straight line

$y = ax + b$

Where a, b are unknown parameters whose values are to be determine.

Let $P_1(x_1, y_1),\ P_2(x_2, y_2),\ .....,\ P_k(x_k, y_k),\ ......,\ P_n(x_n, y_n)$

are the given n-points.



Taking $x = x_k$ the observed value of y is y_k and corresponding value on then fitting of a straight line is $y_k = ax_k + b$.

If E_k is the residual of approximation at $x = x_k$ then

$$E_k = y_k - Y_k$$

Let $E = E(a, b)$ be the sum of squares of the residuals then

$$E = \sum_{i=1}^{n} (y_i - Y_i)^2$$

$$= \sum_{i=1}^{n} \left[y_i - (ax_i + b) \right]^2 \qquad \text{.......... (II)}$$

by principle of least square the values E will be minimum if

$$\frac{\partial E}{\partial a} = 0,\ \frac{\partial E}{\partial b} = 0,\ .....,$$

$$..............\frac{\partial}{\partial a} \sum_{i=1}^{n} \left[y_i - (ax_i + b) \right]^2 = 0$$

$$\sum_{i=1}^{n} 2\left[y_i - \left(ax_i + b\right)\right]\left(-x_i\right) = 0$$

$$\sum_{i=1}^{n}\left[x_i y_i - ax_1^2 + bx_i\right] = 0$$

$$\sum_{i=1}^{n} x_i y_i = \sum_{i=1}^{n} ax_1^2 + bx_i$$

$$\therefore \sum_{i=1}^{n} x_i y_i = a\sum_{i=1}^{n} x_i^2 + b\sum_{i=1}^{n} x_i \qquad \text{..........(III)}$$

Similarly

$$\frac{\partial}{\partial b}\sum_{i=1}^{n}\left[y_i - \left(ax_i + b\right)\right]^2 = 0$$

$$\sum_{i=1}^{n} 2\left[y_i - \left(ax_i + b\right)\right]\left(-1\right) = 0$$

$$\sum_{i=1}^{n}\left[y_i - \left(b + ax_i\right)\right] = 0$$

$$\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} ax_i + b$$

$$\sum_{i=1}^{n} y_i = a\sum_{i=1}^{n} x_i + nb \qquad \text{..........(IV)}$$

From the equation (III) & (IV) are called normal equation of a straight line.

Solving these normal equations with the help of given points and we obtained the values of $a = a^*$ & $b = b^*$

$$\therefore y = a^* x + b^*$$

Which is required best fit of a straight line.

Ex :

Use least squares method to fit a straight line for the following data :

x	3	2	1.5	–2	0.5
y	2	–1	–0.5	1	2

Also estimate the total error.

Sol^n :

Equation of straight line

$$y = ax + b \quad \text{....... (I)}$$

Normal equation of a straight line are

$$\Sigma y = a\Sigma x + bn \quad \text{...... (II)}$$

$$\Sigma xy = a\Sigma x^2 + b\Sigma x \quad \text{...... (III)}$$

x	y	x^2	xy
3	2	9	6
2	–1	4	–2
1.5	–0.5	2.25	–0.75
–2	1	4	–2
0.5	2	0.25	1
$\Sigma x = 5$	$\Sigma y = 3.5$	$\Sigma x^2 = 19.5$	$\Sigma xy = 2.25$

N = 5

We get two equation

$$3.5 = 5a + 5b \quad \text{...... (IV)}$$

$$2.25 = 19.5a + 5b \quad \text{..... (V)}$$

Solving equation (IV) and (V) we get

$a = 0.0862$ $\qquad$ $b = 0.7862$

Put value of 'a' and 'b' in equation (I)

$$y = -0.0862x + 0.7862$$

This is equation straight line

x	y (obseved value)	Y_i (expected value)	$E_i = (y_i - Y_i)$	E_i^2
3	2	0.5276	1.4724	2.1679
2	–1	0.6136	–1.6138	2.6043
1.5	–0.5	0.6569	–1.1569	1.3384
–2	1	0.9586	0.0414	0.00071
0.5	2	0.7431	1.2569	1.5798
				7.6921

$\therefore$ The total error

$$E = \sum_{i=1}^{5}(y_i - Y_i)^2 = 7.6921$$

Case i) :

Fitting a straight line for odd numbers i.e. (n = odd number)

The procedure of fitting a straight line by the method of least square for odd numbers may be simplified as follows :

$$x = \frac{x - M}{h}$$

Where M is the middle term h is difference between any two successive value.

Here $\Sigma x = 0$ normal equation can be written as

$$\Sigma y = nb \Rightarrow b = \frac{\Sigma y}{n}$$

$$\Sigma xy = \Sigma x^2 \Rightarrow a = \frac{\Sigma\, x\, y}{\Sigma\, x^2}$$

Case ii) :

Fitting a straight line for even numbers of term if n is even them

$$x = 2\left[\frac{x - M}{h}\right]$$

Where M is the arithmetic mean of two middle values.
h is difference between any two successive value.

Here $\Sigma x = 0$ normal equation given by

$$\Sigma y = nb \quad \Rightarrow b = \Sigma y / n$$

$$\Sigma xy = a\Sigma x^2 \quad \Rightarrow a = \Sigma\, xy / \Sigma\, x^2$$

Ex :

Fit a straight line $y = ax + b$ to the following data :.

x	2010	2011	2012	2013	2014	2015	2016
y	12	16	21	24	28	32	38

Sol^n :

The value of x is to high

$\therefore$ we use change of origin.

Here n = 7 which odd number

middle value M = 2013, h = 1.

x	y	$x = \dfrac{x-M}{h}$	x^2	$\Sigma \times y$
2010	12	–3	9	–36
2011	16	–2	4	–32
2012	21	–1	1	–21
2013	24	0	0	0
2014	28	1	1	28
2015	32	2	4	64
2016	38	3	9	114
	$\Sigma y = 171$	$\Sigma x = 0$	$\Sigma y^2 = 28$	$\Sigma xy = 117$

Normal equation are

$$a = \frac{\Sigma\, xy}{\Sigma\, x^2} = \frac{117}{28} = 4.1786$$

$$b = \frac{\Sigma y}{h} = \frac{171}{7} = 24.4286$$

$\therefore$ The remained best

$\therefore$ The fit of straight line is

$$y = ax + b$$

$$y = 4.1786x + 24.4286$$

$$y = 4.1786\left(\frac{x - 2013}{1}\right) + 24.4286$$

$$y = 4.1786x - 8411.5218 + 24.4286$$

$$y = 4.1786x - 8387.0932$$

Ex. :

Fit a straight line by method of least squares also find an estimated value for the year 2020.

Year	2011	2012	2013	2014	2015	2016
(Production) termunits	30	35	42	48	53	60

Sol^n :

Magnitude of years are high
$\therefore$ we use charge of origin
Here $n = 6$ which even

M = Arithmetic mean of middle year

$$= \frac{2013 + 2014}{2} = 2013.5$$

$h = 1$year

$\therefore x = 2(x - 2013.5)/1$

Year x	Production y	$x = 2(x - 2013.5)$	x^2	xy
2011	30	–5	25	–150
2012	35	–3	9	–105
2013	42	–1	1	–42
2014	48	1	1	48
2015	53	3	9	159
2016	60	5	25	300
	$\Sigma y = 268$	$\Sigma x = 0$	$\Sigma x^2 = 90$	$\Sigma xy = 210$

$\therefore$ Normal equation for even numbers of terms

$$a = \frac{\Sigma \times y}{\Sigma x^2} = \frac{210}{70} = 3$$

$$b = \frac{\Sigma y}{h} = \frac{268}{6} = 44.67$$

$\therefore$ The best fit of a straight line is

$$y = ax + b$$

$$y = 3x + 44.67$$

$$y = 3\left[2(x - 2013.5)\right] + 44.67$$

$$y = 6x - 12081 + 44.67$$

$y = 6x - 12036.33$

Where $x = 2020$ then

$y = 6(2020) - 12036.33$

$y = 83.67$ tones.

3.5 Fitting of a curve of second degree polynomial to fit the second degree polynomial OR (parabola) :

$$P(x) = ax^2 + bx + c \qquad \text{…. (I)}$$

Where a, b, c are unknown parameters where values are to be determine.

Taking $x = x_k$ the observed value of y is y_k and corresponding value on the fitting of a second degree polynomial is

$$y_k = ax^2{}_k + bx_k + c$$

If E_k is the error of approximation at the points $x = x_k$ then,

$$E_k = y_k - Y_k$$

This error may be negative or +ve zero.

Let $E = E(a, b, c)$ be the sum of squares of residuals then

$$E = \sum_{i=1}^{n} (y_i - Y_i)^2$$

$$= \sum_{i=1}^{n} \left[y_i - \left(ax_i^2 + bx_i + c \right) \right]^2$$

By the principle of least squares. The value of E will be minimum.

i.e. $\frac{\partial E}{\partial a} = 0, \frac{\partial E}{\partial b} = 0 \ \& \ \frac{\partial E}{\partial c} = 0$

Now,

$$\frac{\partial E}{\partial a} = 0$$

$$\sum_{i=1}^{n} 2\left[y_i - \left(ax_i^2 + bx_i + c \right) \right]\left(-x_i^2 \right) = 0$$

$$\sum_{i=1}^{n} x_i^2, y_i = a\Sigma x_i^4 + b\Sigma x_i^3 + c\Sigma x_i^2 \qquad \text{………. (II)}$$

$$\frac{\partial E}{\partial b} = 0$$

$$2\sum_{i=1}^{n}\left[y_i - \left(ax_i^2 + bx_i + c\right)\right]\left(-x_i\right) = 0$$

$$\sum_{i=1}^{n} x_i\, y_i = a\sum_{i=1}^{n} x_i^3 + b\sum_{i=1}^{n} x_i^2 + c\sum_{i=1}^{n} x_i \qquad \text{....... (III)}$$

$$\frac{\partial E}{\partial c} = 0$$

$$2\sum_{i=1}^{n}\left[y_i - \left(ax_i^2 + bx_i + c\right)\right]\left(-1\right) = 0$$

$$\Sigma y_i = a\sum_{i=1}^{n} x_i^2 + b\sum_{i=1}^{n} x_i + n\,c \qquad \text{....... (IV)}$$

The equation (II) (III) (IV) are called normal equation as solve the normal equations with the helps of given data and we obtain $a = a^*, b = b^*$ & $c = c^*$

Put these values in equation .... (I)

$$y = P(x) = a^* x^2 + b^* x + c^*$$

Which is required best fit for second degree polynomial.

Ex. :

Fit a second degree parabola using the method of least square.

x	–3	–1	0	2	4
y	–8	–3	3	8	11

Sol^n :

Let equation of 2^{nd} degree polynomial.

$$y = ax^2 + bx + c \qquad \text{.......... (I)}$$

and normal equation are

$$\Sigma y = a\Sigma x^2 + b\Sigma x + n\,c \qquad \text{.......... (II)}$$

$$\Sigma xy = a\Sigma x^3 + b\Sigma x^2 + c\Sigma x \qquad \text{.......... (III)}$$

$$\Sigma x^2 y = a\Sigma x^4 + b\Sigma x^3 + c\Sigma x^2 \qquad \text{.......... (IV)}$$

x	y	x^2	x^3	x^4	xy	x^2y
–3	–8	9	–27	31	24	–72
–1	–3	1	–1	01	03	–03
0	3	0	0	00	00	00
2	8	4	8	16	16	24
4	11	16	64	256	44	176
$\Sigma x = 2$	$\Sigma y = 1$	$\Sigma x^2 = 30$	$\Sigma x^3 = 44$	$\Sigma x^4 = 354$	$\Sigma xy = 87$	$\Sigma x^2 y = 125$

∴ Normal equation are

$$30a + 2b + 5c = 11 \quad \text{..........(V)}$$

$$44a + 30b + 2c + 87 \quad \text{..........(VI)}$$

$$354a + 44b + 30c = 125 \quad \text{..........(VII)}$$

Solving equation (V), (VI), (VII) we get

$a = -0.2269,$ $b = 3.0774,$ $c = 2.3303$

Put value of a, b, c in equation (I)

$$y = -0.2269x^2 + 3.0774x + 2.3303$$

Which is the required best fit of second degree polynomial.

3.6 Fitting of a polynomial of degree M :

To fit the m[th] degree polynomial.

$$y = P(x) = a_m x^m + a_{m-1} x^{m-1} + a_{m-2} x^{m-2} + ... + a_1 x + a_0$$

Where $a_m, a_{m-1}, a_{m-2}, ..., a_1, a_0$ unknown parameters are whose values to be determine. Taking $x = x_k$ the observed value of y is y_k and corresponding values as the fitting of a m[th] degree polynomial is

$$y_k = a_m x_k^m + a_{m-1} x_k^{m-1} + a_{m-2} x_k^{m-2} + ... + a_1 x^1{}_k + a_0$$

If E_k is the residual of approximation at the point $x = x_k$ then,

$$E_k = y_k - Y_k$$

This error may be negative an +ve an zero.

Let $E = E_1\left(a_m, a_{m-1}, a_{m-2}, \ldots.., a_0\right)$ be the sum of squares residual then,

$$E = \sum_{i=1}^{n}\left[y_i - Y_i\right]^2$$

$$= \sum_{i=1}^{n}\left[y_i - \left(a_m x_i^m + a_{m-1}x_i^{m-1} + \ldots.. + a_1 x + a_0\right)\right]^2$$

By principle of least squares the value of E will be minimum,

If $\frac{\partial E}{\partial a_m} = 0, \frac{\partial E}{\partial a_{m-1}} = 0, \ldots.., \frac{\partial E}{\partial a_0} = 0$

Now $\frac{\partial E}{\partial a_0} = 2\sum_{i=1}^{n}\left[y_i - \left(a_m x_i^m + a_{m-1}x_i^{m-1} + \ldots.. + a_1 x + a_0\right)(-1) = 0\right]$

$$\sum_{i=1}^{n} y_i = na_0 + a_1\sum_{i=1}^{n} x_i + a_2\Sigma x_i^2 + \ldots.. + a_m\Sigma x_i^m$$

Similarly

$$\frac{\partial E}{\partial a_1} = 0$$

$$\sum_{i=1}^{n} x_i\, y_i = a_0\Sigma x_i + a_1\Sigma x_i^2 + \ldots.. + a_m\Sigma x_i^{m-1}$$

.

.

.

$\frac{\partial E}{\partial a_m} = 0$ we get

$$\sum_{i=1}^{n} x_i^m\, y_i = a_0\Sigma x_i^m + a_1\Sigma x_i^{m+1} + \ldots.. + a_m\Sigma x_i^{2m}$$

The above equation is called normal equation.

Solve the normal equation with the help of given data and we obtain $a_m = a_m{}^*, a_{m-1} = a_{m-1}{}^*, a_{m-2} = a_{m-2}{}^*, \ldots..., a_1 = a^*, a_0 = a_0{}^*$.

Put all these values in equation ……... (I)

$$y = P(x) = a_m^*\, x^m + a_{m-1}^* x^{m-1} + \ldots.. + a_1^* x + a_0^*$$

Which is required best fit for m^th degree polynomial.

3.7 Fitting of curve for exponential function :

Let the exponential curve

$y = a_0 e^{bx}$

taking log on both side we get

$\log\ y = \log a + bx$

We can write this as

$y = A + Bx$ ……. (I)

Where $y = \log y$, A $= \log a_0$ and $B = a$.

Thus equation is straight line.

$\therefore$ we can use fitting of curve of straight line by least square method.
Similarly we can curve for any exponential curve like
$y = ab^x$, $y = ax^b$.

Ex. :

Fit a curve of the type $y = ae^{bx}$ to the following data :

x	2	6	10	14
y	2.5	3.2	4.7	5.9

Sol^n :

Given curve

$y = ab^x$

taking log both sides we get

$\log y = \log a + x \log b$

Put $\log y = Y$, $A = \log\ a$, $x = \log x$, $Y = A + bx$ ……. (II)

This is straight line least square method.

$\therefore$ Normal equation are

$\Sigma y = nA + b\Sigma x$

$\Sigma xy = A\Sigma x + b\Sigma x^2$

x	y	$x = \log x$	$y = \log y$	x^2	xy
2	2.5	0.6931	0.9163	0.4804	0.6351
6	3.2	1.7918	1.1632	3.2105	2.0892
10	4.7	2.3026	1.5476	5.3020	3.5635
14	5.9	2.6391	1.7750	6.9648	4.6544
		$\Sigma x = 7.4266$	$\Sigma y = 5.4021$	15.9577	10.9672

$\therefore$ Normal equation

$$5.4021 = 4A + 7.4266b$$
$$10.9672 = 7.4266A + 15.9577b$$

$A = 0.5482$ $\qquad$ $b = 0.4322$

Equation become

$$Y = 0.5482 + 0.4322X$$

But $A = \log e^a$

$$\therefore 0.5482 = \log e^a$$
$$a = e^{0.5482}$$
$$a = 1.7301$$

Equation (I) becomes

$\therefore y = 1.7301;$ $\qquad$ $x = 0.4322$

Which is the required least fit of curve.

3.8 Review :

* We have learn to fit a straight line by least squares method.
* We have fit quadratic curve fit by least squares method (second degree).
* We have fitting of a polynomial of degree M.
* We have fitting of a curve for exponential function.

3.9 Unit End Exercise :

1) Fit a straight time $y = a + bx$ for the following data by using least square method.

i)

x	4	3	8	7	9
y	11	10	14	12	18

ii)

Year (x)	2009	2010	2011	2012	2013	2014	2015
Production	2.5	3	4.2	4.8	5.3	6.4	7.3

iii)

x	0.2	0.4	0.8	1	1.2	1.4
y	3	4	6	9	13	10

iv)

x	2	$1/6$	-1.5	$-1/3$
y	1	$-5/6$	-2	$-2/3$

2) Fitting of a second degree curve using the method of least square :

i)

x	-5	-3	0	8	5
y	-11	-8	-2	0	4

ii)

x	1941	1951	1961	1971	1981	1991	2001
y	1.1	1.3	1.6	2	2.7	3.4	4.1

3) Fit the curve $X = a + bx^2$ to the following data :

$x =$	10	20	30	40	50
$y =$	8	10	15	21	30

4) Fit a curve of the type $Y = ax^b$ to the following data :

x	10	20	30	40	50
y	3.5	6.2	9.5	15.3	20.4

5) The pressure & volume of the gas are relation by the equation $pv^{\lambda} = k$ where λ & k are constant fit this relation for the following data using principle of least square :

x	0.5	10	1.5	2.0	2.5	3.0
y	1.62	1	0.75	0.62	0.52	0.46

6) Using 'Principle of least squares' fit a straight line $y = a + bx$ for n-points.

Using principle of least squares fit a second degree polynomial $y = ax^2 + bx + c$ for n-points.

7) Using principle of least squares fit a polynomial of degree M for n-points.

8) Using principle of least squares fit a straight line for relation $y = ab^x$.

9) Using principle of least squares for a straight line for relation $y = ae^{bx}$.

♦♦♦♦♦

4

Least squares approximation

Chapter Structure

4.1 Objective :

After going through this chapter you will be able to :

* Fit polynomial for continuous function by least squares method.
* Orthogonalization approximation by Gram – Schmidt process.
* Fit a curve for chebyshev polynomial by least square method.
* Compute different point of sequence by DFT & FFT.

4.2 Introduction :

A more general least squares problems is the weighted least squares approximation problem. We consider a weight function *w* (*x*) to be continuous an (a, b) with positive mass.

i.e. $\int_a^b w(x) > 0$

Given set of points $(x_1, y_1)\,(x_2, y_2)\,.....\,(x_n, y_n)$ to given importance to the points we assigning weights. If all the data points have amount of weights then the weights as $w(x) = 1$.

Here $w(x)$ and *y* are known functions.

$\therefore$ The above $(k+1)$ equations is a system of line equation with $(k+1)$ unknowns.

$\therefore$ This system has a unique solution.
Let the solution is

$$a_0 = a_0^*,\ a_1 = a_1^*, ..., a_k = a_k^*$$

Put all values in equation (I) we get

$$P(x) = a_0^* + a_1^* x^1 + a_2^* x^2 + ... + a_k^* x^k + ...a_k^* x^k$$

Which is required best approximation.

Ex. :

Construct a least squares linear approximation to the function $y = x^3$ on the interval [0, 1] with respect to the weight function $w(x) = 1$.

Sol^n :

Let the linear approximation be
$p(x) = a_0 + a_1 x$ .... (I)

Where $a_0,\ a_1$ are unknown parameter which are to be determine.
Let $E = E(a_0,\ a_1)$ be the sum of squares of residuals, then,

$$E = \int_a^b w(x)\left[y - p(x)\right]^2 dx$$
$$= \int_0^1 1\left[x^2 - (a_0 + a_1 x)\right]^2 dx$$

By principle of least squares the values of E will be minimum.

$$\frac{\partial E}{\partial a_0} = 0 \text{ and } \frac{\partial E}{\partial a_1} = 0$$

$\therefore$ For $\frac{\partial E}{\partial a_0} = 0$

$$\int_0^1 2\left[x^3 - (a_0 + a_1 x)\right](-1)\, dx = 0$$

$$\therefore \int_0^1 (a_0 + a_1 x)\, dx = \int_0^1 x^3\, dx$$

$$\left[a_0 + a_1 \frac{x^2}{2}\right]_0^1 = \left[\frac{x^4}{4}\right]_0^1$$

$$a_0 + \frac{a_1}{2} = \frac{1}{4}$$

$$4a_0 + 2a_1 = 1 \qquad \text{.......... (II)}$$

For $\frac{\partial E}{\partial a_1} = 0$

$$\int_0^1 2\left[x^3 - \left(a_0 + a_1 x\right)\right]\left(-x\right)\, dx$$

$$\int_0^1 \left(a_0 x + a_1 x^2\right)\, dx = \int_0^1 x^4\, dx$$

$$\left. a_0 \frac{x^2}{2} + a_1 \frac{x^3}{3}\right]_0^1 = \left[\frac{x^5}{5}\right]_0^1$$

$$\frac{a_0}{2} + \frac{a_1}{3} = \frac{1}{5}$$

$$15a_0 + 10a_1 = 1 \qquad \text{.... (III)}$$

Solving equation (II) & (III) we get

$a_0 = 0.8 \qquad a_1 = -1.1$

Put $a_0 = 0.8$ & $a_1 = -1.1$ in I we get

$p\left(x\right) = 0.8 - 1.1x$

Which is the remind least square linear or approximation.

4.3 Orthogonal Function :

The set of function $\left\{f_0\left(x\right), f_1\left(x\right), \ldots\ldots, f_n\left(x\right)\right\}$ in [a, b] is called a set of orthogonal functions, with respect to a weight function $w\left(x\right)$, if

$$\int_a^b w\left(x\right)\, f_j\left(x\right), f_i\left(x\right) dx = \begin{cases} 0 & if \quad i \neq j \\ c_i & if \quad i = j \end{cases}$$

Where c_j is a real positive number further more,

if $c_j = 1,\ j = 0, 1, \ldots\ldots, n$ than the orthogonal set is called an orthonormal set.

Least squares approximation of a function using orthogonal polynomial :

Let $y = f(x)$ be continuous function an [a, b] and it is approximated by k^{th} degree polynomial given by,

$$p(x) = a_0 p_0(x) + a_1 p_1(x) + ... + a_k p_k(x) \qquad \text{..........} (I)$$

Which orthogonal polynomial and $a_0, a_1, ..., a_k$ are unknown parameters which are to be determine.

Let $E = E(a_0, a_1, ..., a_k)$ by the sum of squares of residual then,

$$E = \int_a^b w(x)\left[y - P(x)\right]^2 \, dx$$
$$E = \int_a^b w(x)\left[y - a_0P_0(x) + a_1P_1(x) + ... + a_kP_k(x)\right]^2 \, dx$$

$\therefore$ By principle of least squares the value of E will be minimum if,

$$\frac{\partial E_1}{\partial a_0} = 0, \frac{\partial E}{\partial a_1} = 0, ..., \frac{\partial E}{\partial a_k} = 0$$

Now,

$$\frac{\partial E}{\partial a_0} = 2\, w(x) \int_a^b P_0(x)\left[y - a_0P_0(x) - a_1P_1(x), ..., a_kP_k(x)\right] dx = 0$$
$$\int_a^b P_0(x)\, y\, dx = w(x) \int_a^b \left[a_0P_0(x) + ... + a_kP_k(x)\right] P_0(x)\, dx$$

Since $\left[P_n(x)\right]_{k=0}^{n=k}$ is an orthogonal set we have

$$\int_a^b w(x)\, P_0^2(x) dx = c_0$$
$$\& \int_a^b P_0(x)\, P_i(x) dx = 0 \qquad i \neq 0$$

Applying the above orthogonal property we get

$$\int_a^b w(x) P_0(x)\, y\, dx = c_a a_0$$
$$\therefore a_0 = \frac{1}{c_0} \int_a^b w(x) P_0(x)\, Y.dx$$

Similarly

$$\frac{\partial E}{\partial a_1} = -2w(x) \int_a^b P_1(x)\left[y - a_0P_0(x) - a_1P_1(x) ..... a_kP_k(x)\right] dx = 0$$

$$a_1 = \frac{1}{c_1}\int_a^b w(x)P_1(x) \bullet Y\,dx$$

In general we have

$$a_n = \frac{1}{c_n}\int_a^b w(x)P_n(x)\; y \cdot dx \qquad n = 0, 1, ..., k$$

Where $c_n = \int_a^b w(x)\; P_n^2(x)\; dx$

Let the values of a_n be $a_0 = a_0^*, a_1 = a_1^*, ..., a_k = a_k^*$ substitute all these value in equation I, we get $p(x) = a_0^*,\; p_0(x) + a_1^*\; p_1(x) + ... + a_k^* p_k(x)$

Which is the required least squares approximation.

4.4 Gram – Schmidt or thogonalizing process :

The classical Gram – Schmidt process can be used for this p starting with the set of monomials purpose $\{c, x, ..., x^n\}$ we can generate the orthonormal set of polynomials $\{P_k(x)\}$ with respect to weight function $w(x)$ can be generated from the relation.

$$P_k(x) = x^k - \sum_{r=0}^{k-1} a_{kr}\,\Pr^*(x)\; k = 1, 2, ...$$

where

$$a_{kr} = \int_a^b \frac{w(x)\; x^k g_r^*(x)dx}{w(x)\; x^k g_r^2(x)dx} \;\; \& \;\; g_0^*(x) = 1$$

Ex. :

Using Gram – Schmidt orthogonal rizing process first three orthogonal polynomial $P_0(x), P_1(x)$ and $P_2(x)$ on [0, 1] with respect to weight function $w(x) = 1$ using this polynomial obtain the least squares approximation second degree for the function $y = \sqrt{x}$ on [0, 1].

Sol^n :

Use Gram – Schmidt orthogonalizing process to determine $P_0(x), P_1(x), P_2(x)$.

$$f_k^*(x) = x^k \; - \sum_{r=0}^{k-1} a_{kr}\; f_r^*(x) \qquad k = 1, 2, ...$$

where $a_{kr} = \int_a^b w(x)\; x^k f_r^*(x)\; dx \Big/ \int_a^b w(x) f_r^{*2}(x) dx$ &

$f_0^*(x) = 1$

here $f_0^*(x) = 1 = P_0(x)$

now $f_1^*(x) = x - \left[a_{10}\ f_0^*(x)\right]$

where $a_{10} = \int_0^1 (1)x(1)dx \Big/ \int_0^1 (1)(1)^2\,dx = \left[\frac{x^2}{2}\right]_0^1 \Big/ [x]_0^1$

$a_{10} = 1/2$

$\therefore f_1^*(x) = x - \left(\frac{1}{2}(1)\right) = x - \frac{1}{2} = P_1(x)$

Now,

$f_2^*(x) = x - \left[a_{20} f_0^*(x) + a_{21} f_1^*(x)\right]$

Where $a_{20} = \int_0^1 \frac{(1)\ x^2\,(1)\ dx}{(1)\,(1)^2\ dx} = \frac{(x^3/3)}{(x)!} = 1/3$

$$a_{21} = \int_0^1 \frac{(1)x^2\,(x-1/2)\ dx}{(1)\left(x-\frac{1}{2}\right)^2 dx} = \frac{\int_0^1 \left(x^3 - \frac{1}{2}x^2\right) dx}{\int_0^1 \left(x^2 - x + 1/4\right) dx} = \frac{\frac{1}{4} - \frac{1}{6}}{\frac{1}{3} + \frac{1}{2} + \frac{1}{4}} = 1$$

$\therefore a_{21} = 1$

$f_2^*(x) = x^2 - \left[1/3(1) + (1)\,(x-1/2)\right]$

$= x^2 - \left[1/3 + x - 1/2\right]$

$f_2^*(x) = x^2 - x - 1/6 = P_2(x)$

Let $P(x) = a_0 P_0(x) + a_1 P_1(x) + a_2 P_2(x)$ ………. (I)

Where a_0, a_1, a_2 are unknown parameter whose values are to be determine.

Using the condition of orthogonality

$$a_i = \frac{\int_0^1 w(x)\cdot y \cdot P_i(x)\ dx}{\int_a^b w(x)\cdot P_i^2(x)\ dx}$$

$$a_0 = \frac{\int_0^1 (1)\ \sqrt{x}\ (1)\ dx}{\int_a^b (1)\,(1)^2\ dx} = \frac{\left[x^{3/2} \big/ 3/2\right]_0^1}{[x]_0^1} = \frac{2}{3}$$

$$= \frac{\int_0^1 (1)\sqrt{x}\left(x - 1/2\right)dx}{(1)\left(x - 1/2\right)^2 dx} = \frac{\left[\frac{x^{5/2}}{5/2} - \frac{x^{3/2}}{2(3/2)}\right]_0^1}{\left[\frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{4}\right]_0^1} = \frac{\left[\frac{2}{5} - \frac{1}{3}\right]}{1/3 - 1/2 + 1/4} = 4/5$$

$$\therefore a_1 = 4/5$$

$$a_2 = \frac{\int_0^1 (1)\sqrt{x}\left(x^2 - x + 1/6\right)dx}{\int_a^b (1)\left(x^2 - x - 1/6\right)^2 dx}$$

$$= \frac{\int_0^1 \left(x^{5/2} - x^{3/2} + \sqrt{x}/6\right)dx}{\int_0^1 \left(x^4 - 2x^3 + x^2 + \frac{2}{3}x^2 - \frac{1}{3}x + \frac{1}{36}\right)dx}$$

$$= \frac{\left(\frac{x^{7/2}}{7/2} - \frac{x^{5/2}}{5/2} + \frac{1}{6}\frac{x^{3/2}}{3/2}\right)_0^1}{\left(\frac{x^5}{5} - \frac{2x^4}{4} + \frac{x^3}{3} + \frac{1}{3}\frac{x^3}{3} - \frac{1}{3}\frac{x^2}{2} + \frac{1}{36}x\right)_0^1}$$

$$= \frac{\left[\frac{2}{7} - \frac{2}{5} + \frac{1}{9}\right]}{\left[\frac{1}{5} - \frac{1}{2} + \frac{1}{3} + \frac{1}{9} - \frac{1}{6} + \frac{1}{36}\right]}$$

$$= \frac{-1/315}{\frac{2.9}{45} - \frac{23}{36}} = -\frac{4}{7}$$

$$\therefore a_2 = -4/7$$

Put all values in equation (I) we get

$$p(x) = \frac{2}{3}(1) + \frac{4}{5}\left(x - 1/2\right) - \frac{4}{7}\left(x^2 - x + 1/6\right)$$

$$p(x) = 0.1714 + 1.3714x - 0.5714x^2$$

4.5 Chebyshev polynomials :

The set of polynomials defined by $T_n(x) = \cos\left[n \cdot \cos^{-1}(x)\right]$ $n \geq 0$ on [-1, 1] are called the chebyshev polynomial of degrees.

It can be written as

$$T_n(x) = \cos\left[n \cdot \cos^{-1}(x)\right]$$
$$= \cos n\theta \qquad n = 0, 1, ...$$

Where $\theta = \cos^{-1} x$ on $x = \cos\theta$

we derive a recursive relation by noting that

taking $n = 0$ $\qquad T_0(x) = 1$

taking $n = 1$ $\qquad T_1(x) = x$

$$T_{n+1}(x) = \cos(n+1)\theta$$
$$= \cos n\theta \cos\theta - \sin n\,\theta \sin\theta$$
$$T_{n+1}(x) = \cos(n-1)\theta = \cos n\theta \cos\theta + \sin n\theta \sin\theta$$
$$\therefore T_{n+1}(x) + T_{n-1}(x) = 2\cos n\theta \cos\theta$$

$\therefore$ But we know that $\cos\theta = x$, so,

$$T_{n+1}(x) = 2\cos\theta \cos\theta - T_{n-1}(x)$$
$$T_{n+1}(x) = 2x \cdot T_n(x) - T_{n-1}(x) \qquad n \geq 1$$

This above is a three term recurrence relation to generate the chebyshev polynomials.

11.5.1 The orthogonal polynomial of the chebyshev polynomial :

1) $T_n(x)$ is a chebyshev polynomial of degree n, if n is even than $T_n(x)$ is even. If n is odd then $T_n(x)$ is odd.

2) $|T_n(x)| \leq 1$ for $x \in [-1, 1]$.

3) The chebyshev polynomial $T_n(x)$ are orthogonal with respect to the weight function $w(x) = \dfrac{1}{\sqrt{1 - x^2}}$ & in the interval $[-1, 1]$.

$$\int_{+1}^{-1} \frac{T_m(x)\, T_n(x)}{\sqrt{1 - x^2}}\, dx = \begin{cases} 0 & if \quad m \neq n \\ \pi/2 & if \quad m = n \neq 0 \\ \pi & if \quad n = m = 0 \end{cases}$$

The first seven chebyshev polynomial are :

$$T_0(x) = 1$$
$$T_1(x) = x$$
$$T_2(x) = 2x^2 - 1$$
$$T_3(x) = 4x^3 - 3x$$
$$T_4(x) = 8x^4 - 8x^2 + 1$$
$$T_5(x) = 16x^5 - 20x^3 + 5x$$
$$T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$$

It is also possible to express powers of x in terms of chebyshev polynomials we find

$$1 = T_0(x)$$
$$x = T_1(x)$$
$$x^2 = \frac{1}{2}\left[T_2(x) + T_0(x)\right]$$
$$x^3 = \frac{1}{4}\left[T_3(x) + 3T_1(x)\right]$$
$$x^4 = \frac{1}{8}\left[T_4(x) + 4T_2(x) + 3T_0(x)\right]$$
$$x^5 = \frac{1}{16}\left[T_5(x) + 5T_3(x) + 10T_1(x)\right]$$
$$x^6 = \frac{1}{32}\left[T_6(x) + 6T_4(x) + 15T_1(x) + 10T_0(x)\right]$$

and so on …..

Ex. :

Use chebyshev polynomial to obtain the least squares approximation of second degree for the function $f(x) = x^3$ on the interval $[-1, 1]$ with respect to the weight function $w(x) = \frac{1}{\sqrt{1 - x^2}}$.

Sol^n :

The least square approximation of second degree be

$$p(x) = a_0T_0(x) + a_1T_1(x) + a_2T_2(x) \qquad \text{……. (I)}$$

Where T_0, T_1, T_2 and chebyshev polynomial & a_0, a_1, a_2 are unknown parameter whose values are to be determine.

Using the condition of orthogonalizing we obtain

$$a_i = \frac{\int_a^b w(x)\ yT_i(x)\ dx}{\int_a^b w(x)\ T_i^2(x)\,dx} \qquad i = 0, 1, 2, ...$$

$$a_0 = \frac{\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\ x^3(-1)\ dx}{\int_{-1}^{1} \frac{(1)^2}{\sqrt{1-x^2}}\ dx}$$

$$= \frac{\int_{-1}^{1} \frac{1}{4}\left[T_3(x) + 3T_1(x)\right]T_1(x)\,dx \Big/ \sqrt{1-x^2}}{\int_{-1}^{1} \frac{T_0(x)\ T_0(x)}{\sqrt{1-x^2}}\ dx}$$

$$= \frac{\frac{1}{4}\left[0 + 0 + 0\right]}{\pi}$$

[$\therefore$ By property of chebyshev polynomial]

$$= 0$$

$$a_0 = 0.$$

$$a_1 = \frac{\int_{-1}^{1} \frac{x^3 T_1(x)}{\sqrt{1-x^2}}\ dx}{\int_{-1}^{1} \frac{\left(T_1(x)\right)^2}{\sqrt{1-x^2}}\ dx} = \frac{\int_{-1}^{1} \frac{1}{4}\left[T_3(x) + 3T_1(x)\right]T_1(x)\,dx}{\int_{-1}^{1} \frac{T(x)\ T(x)}{\sqrt{1-x^2}}\ dx}$$

$$a_1 = \frac{\left[\frac{1}{4}\left[0 + \frac{3\pi}{2}\right]\right]}{\pi/2} = \frac{3}{4}$$

$$a_2 = \frac{\int_{-1}^{1} x^3\ T_2(x)\ w(x)\ dx \Big/ \sqrt{1-x^2}}{\int_{-1}^{1} \frac{\left[T_2(x)\right]}{\sqrt{1-x^2}}\,dx}$$

$$= \frac{\dfrac{\int_{-1}^{1} \frac{1}{4}\left[T_3(x) + 3T_1(x)\right]\ T_2(x)\ dx}{\sqrt{1-x^2}}}{\int_{-1}^{1} T_2(x)T_2(x) \Big/ \sqrt{1-x^2}\ dx}$$

$$= \frac{\frac{1}{4}[0+0]}{\pi/2}$$

$$= 0$$

$\therefore$ Put a_0, a_1 and a_2 in equation (I)

$$P(x) = 0T_0(x) + \frac{3}{4}T_1(x) + 0T_2(x)$$

$$\therefore P(x) = \frac{3}{4}\,T_1(x)$$

Which is the required least square approximation.

4.6 Discrete Fourier Transforms (DFT) :

A finite sequence of n real a number is given by

$$x = \{x(0),\, x(1),\, ...,\, x(n-1)\}$$

The n-points discrete Fourier transform of sequence x is defined by

$$F_U x(k) = X(k) = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} x(j)\, e^{2\pi jk/n}$$

Where $k = 0, 1, 2, ..., n-1$

$\therefore$ The discrete Fourier transformation of n-points sequence 'r' of complex numbers.

$$\mathrm{X}(k) = \{x(0),\, x(1),\, ...,\, x(n-1)\}$$

The n-points inverse discrete Fourier transform of sequence X is defined by

$$F_N^{-1}\,\mathrm{X}(k) = x(k) = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} x(j)\, e^{-2\pi ijk/\mathrm{n}}$$

Ex. :

Compute the 4 points DFT of sequence $x = (3, 4, 5, 6)$.

Sol^n :

Given real sequence

$\therefore x(0) = 3,\ x(1) = 4,\ x(2) = 5,\ x(3) = 6$

$\therefore$ The 4 points DFT of sequences is given by

$$F_4 x(k) = x(k) = \frac{1}{\sqrt{4}} \sum_{j=0}^{3} x(j)\, e^{2\pi ijk/4}$$

where k = 0, 1, 2, 3

$$F_4 x(0) = \frac{1}{2}\left[x(0)e^0 + x(1)\, e^{i\pi k/2} + x(2)e^{i\pi k} + x(3)e^{3i\pi k/2}\right]$$

$$F_4(x)(0) = \frac{1}{2}\left[3 + 4e^{i\pi k/2} + 5e^{i\pi k} + 6e^{3i\pi k/2}\right]$$

$$F_4 x(0) = \frac{1}{2}\left[3 + 4e^{0} + 5e^{0} + 6e^{0}\right] = \frac{1}{2}[18] = 9$$

$$F_4 x(0) = 9$$

$$F_4 x(1) = \frac{1}{2}\left[3 + 4e^{i\pi/2} + 5e^{i\pi} + 6e^{3i\pi/2}\right]$$

$$F_4 x(1) = \frac{1}{2}\left[3 + 4\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) + 5(\cos\pi + i\sin\pi) + 6(\cos 3\pi/2 + i\sin 3\pi/2)\right]$$

$$F_4(x)(1) = \frac{1}{2}\left[3 + 4i + 5(-1) + 6(-i)\right]$$

$$F_4(x)(1) = \frac{1}{2}[-2 - 2i] = 1 - i$$

$$F_4 x(1) = -i - j$$

$$F_4 x(2) = \frac{1}{2}\left[3 + 4e^{i\pi} + 5e^{z\pi i} + 6e^{3\pi i}\right]$$

$$F_4 x(2) = \frac{1}{2}\left[3 + 4(\cos\pi + i\sin\pi) + 5(\cos 2\pi + i\sin 2\pi) + 6(\cos 3\pi + i\sin 3\pi)\right]$$

$$F_4 x(2) = \frac{1}{2}\left[3 + 4(1) + 5(1) + 6(-1)\right]$$

$$F_4 x(2) = -1$$

$$F_4 x(3) = \frac{1}{2}\left[3 + 4e^{3i\pi/2} + e_5^{3i\pi} + 63^{9i\pi/2}\right]$$

$$F_4 x(3) = \frac{1}{2}\left[3 + 4(-i) + 5(-1) + 6(-i)\right]$$

$$F_4 x(3) = \frac{1}{2}[-2 + 2i]$$

$$F_4 x(3) = i - 1$$

The 4 points DFT of sequence '*x*' is

Ex. :2

Compute the 4 points inverse DFT of sequences

$$x = \left(\tfrac{1}{2}, \left(i - \frac{1}{2}\right), \tfrac{1}{2}, -i - \tfrac{1}{2}\right)$$

Sol^n :

Given complex sequence is

$$x = \left(1/2, \left(i - \frac{1}{2} \right), 1/2, -i - 1/2 \right)$$

$$\therefore x(0) = 1/2,\ x(1) = i - 1/2,\ x(2) = 1/2,\ x(3) = -i - 1/2$$

$\therefore$ The 4 points inverse DFT of sequence ‘x’ is defined by

$$F_4^{-1}x(k) = x(k) = \frac{1}{\sqrt{4}} \sum_{i=0}^{3} x(j)\, e^{-2\pi ji\, k/4}$$

$$F_4^{-1}x(k) = \frac{1}{2}\left[x(0)e^0 + x(1)\, e^{\pi ik/2} + x(2)e^{i\pi k} + x(3)\, e^{3\pi k/2} \right]$$

$$F_4^{-1}x(k) = \frac{1}{2}\left[\frac{1}{2} + \left(i - \frac{1}{2} \right) e^{-i\pi k/2} + \frac{1}{2} e^{\pi ik} + \left(-i - \frac{1}{2} \right) e^{-3\pi k/2} \right]$$

where k = 0, 1, 2, 3

$$F_4^{-1}x(0) = \frac{1}{2}\left[\frac{1}{2} + (i - 1/2)\, O^0 + \frac{1}{2}\, e^0 + (-i - 1/2)e^0 \right]$$

$$F_4^{-1}x(0) = \frac{1}{2}\left[\frac{1}{2} + i - \frac{1}{2} + \frac{1}{2} - i - \frac{1}{2} \right]$$

$$F_4^{-1}x(0) = 0$$

$$F_4^{-1}x(0) = 0$$

$$F_4^{-1}x(1) = \frac{1}{2}\left[\frac{1}{2} + (i - 1/2)\, e^{-i\pi/2} + \frac{1}{2}\, e^{i\pi} + (-i - 1/2)e^{-3\pi/2} \right]$$

$$F_4^{-1}x(1) = \frac{1}{2}\left[\frac{1}{2} + \left(i - \frac{1}{2} \right)(-i) + \frac{1}{2}(-1) + \left(-i - \frac{1}{2} \right)(i) \right]$$

$$F_4^{-1}x(1) = 1$$

$$F_4^{-1}x(1) = 1$$

$$F_4^{-1}x(2) = \frac{1}{2}\left[\frac{1}{2} + (i - 1/2)\, e^{-i\pi} + \frac{1}{2}\, e^{-2i\pi} + (-i - 1/2)e^{-3\pi i}\right]$$

$$F_4^{-1}x(2) = \frac{1}{2}\left[\frac{1}{2} + \left(i - \frac{1}{2}\right)(-1) + \frac{1}{2}(1) + \left(-i - \frac{1}{2}\right)(-1)\right]$$

$$F_4^{-1}x(2) = \frac{1}{2}\left[\frac{1}{2} - i + \frac{1}{2} + \frac{1}{2} + i + \frac{1}{2}\right]$$

$$F_4^{-1}x(2) = 1$$

$$F_4^{-1}x(2) = 1$$

$$F_4^{-1}x(3) = \frac{1}{2}\left[\frac{1}{2} + (i - 1/2)\, e^{3i\pi/2} + \frac{1}{2}\, e^{3i\pi} + (-i - 1/2)e^{3i\pi/2}\right]$$

$$F_4^{-1}x(3) = \frac{1}{2}\left[\frac{1}{2} + \left(1 - \frac{1}{2}\right) i + \frac{1}{2}(-1) + \left(-i - \frac{1}{2}\right)(-i)\right]$$

$$F_4^{-1}x(3) = \frac{1}{2}\left[\frac{1}{2} - 1 - \frac{1}{2} - \frac{1}{2} - 1 + \frac{1}{2}\right]$$

$$F_4^{-1}x(3) = -1$$

$$F_4^{-1}x(3) = -1$$

$\therefore$ The 4 points inverse DFT of sequence 'x' is $x(k) = (0, 1, 1, -1)$.

4.7 Fast Fourier Transforms (FFT) :

The Fast Fourier Transforms (FFT) is the efficient implementation of the Discrete Fourier Transforms (DFT).

The FFT was discovered by curvely & Tukar in 1965. The FFT is based on the following observation.

$*$ Let data (x_k, y_k), $x_k \frac{k\pi}{n}$, $k = 0, 1, \ldots, 2_{n-1}$ be given and that p & q are exponential polynomial of degree at must $(n-1)$ which interpole part of the data according to $p(x_{2j}) = f(x_{2j})$, $q(x_{2j}) = f(x_{2j}+1)$, $j = 0, 1, \ldots, n-1$ then the exponential polynomial of degree at most $2n-1$ which interpolates all the given data is

$$P(x_k) = f(x_k) \qquad\qquad K = 0,1......2n-1$$

By assumption we have

$$p(x) = \frac{1}{2}\left(1 + e^{inx}\right) p(x) + \frac{1}{2}\left(1 - e^{ink}\right) q\left(x - \frac{\pi}{n}\right)$$

Proof : Since $e^{inx} = \left(e^{ix}\right)^n$ has degree n & p and q have degree atmost $n-1$. It is clear that phase degree at most $2n-1$.

We need to verify the interpolation

$$p(x_k) = \frac{1}{2}\left(1 + e^{inx_k}\right) p(x_k) + \frac{1}{2}\left(1 + e^{inx_k}\right) q\left(x_k - \frac{\pi}{n}\right)$$

Where $e^{inx_k} = e^{ink\pi/n} = \left(e^{\pi}\right)^k = (-1)^k$

$\therefore P(x_k) = P(x_k)$ if k is even

$= q(x_k - \pi / n)$ if k is odd

Let k be even i.e. k = 2j then by assumption on P.

$$P(x_{2j}) = P(x_{2j}) = f(x_{2j})$$

$$x_{2j+1} - \frac{\pi}{n} = \frac{(2_{j+1})\pi}{n} - \frac{\pi}{n} = \frac{2j\pi}{n} = x_{2j}$$

- To compute for curve vectors.

$x = [x_0, x_1, x_2, ..., x_{n-1}]$ of discrete data by DFT $\hat{a}$ of a is compute component wise as

$$\hat{a}_k = \frac{1}{N}\sum_{j=0}^{n-1} a_j e^{-ik2\pi j/N} \qquad k = 0, 1, ..., N-1$$

of course. The FFT can be used to compute these co-efficient. There is an analogs formula for the inverse DFT

$$a_j = \sum_{k=0}^{n-1} \hat{a}_k = e^{ik2\pi j/N} \qquad \text{J} = 0, 1, ..., \text{N-1}$$

4.8 Review :

* Least square's method for continuous functions.
* Approximation using orthogonal function using Gram – Schmidt orthogonalizing process.
* Chebyshev polynomials & it's properties.
* Discrete Fourier Transform & Fast Fourier Transform.
* Computing sequences of points by DFT & FFT.

4.9 Unit End Exercise :

1) Construct a least squares linear approximation to the function $y = x^2$ on the interval [0, 1] with respect to the weight function $w(x) = 1$.

2) Construct a least squares quadratic approximation to the function $y = e^x$ on the interval [0, 1] with respect to the weight function $w(x) = 1$.

3) Obtain the least squares polynomial approximation of degree 2 for the function $y = \sqrt{x}$ on it interval [0, 2] with respect to the weight function $w(x) = 1$.

4) Using the Gram – Schmidt orthogonalizing process compute the first three orthogonal polynomial $p_0(x), p_1(x)$ & $p_0(x), p_1(x)$ which an orthogonal an the interval $[-1, 1]$ with respect to the weight function $w(x) = \frac{1}{\sqrt{1 - x^2}}$.

5) Use chebyshev polynomial to obtain the least square approximation of second degree for the function. $F(x) = x^4$ on the interval $[-1, 1]$ with respect to the weight function $w(x) = \frac{1}{\sqrt{1 - x^2}}$.

6) Use chebyshev polynomial to obtain the least squares approximation for the function $f(x) = 3x^4 + 2x^3 + x + 2$ function $w(x) = \frac{1}{\sqrt{1 - x^2}}$.

7) Compute the 4 points DFT of sequences $x = (1, 2, 3, 4)$.

8) Use chebyhev polynomial to obtain the least squares approximation for the function $f(x) = 5x^3 + 6x^2 - 5x + 3$ on the interval $[-1, 1]$ with respect to the weight function $w(x) = \frac{1}{\sqrt{1 - x^2}}$.

9) Compute the 4 points DFT of sequence $x = (0, 1, 1, -1)$.

10) Compute the 4 points inverse 0 DFT of sequence

$x = (5, -(1+i), -1, i-1)$.

11) Determine the normal equation if the cubie polynomial $y = a_0 + a_x + a_2x^2 + a_3x^3$ is fitted to the data (x_i, y_i), $i = 0, 1, 2, \ldots.., m$.

12) Prove that $x^2 = \frac{1}{2}\left[T_0(x) + T_2(x)\right]$.

13) Prove that $T_n(x)$ is a polynomial in x of degree A.

14) Express the following polynomial as sums of chebyshev polynomials.

a) $1 + x - x^2 + x^3$

b) $1 - x^2 + 2x^4$

15) Determine the least square method for continuous function in $[a, b]$.

♦♦♦♦♦

5

Numerical Solution of Differential Equation – I

Chapter Structure

5.1 Objective :

After going through this chapter you will be able to :

* Solve non-linear equation of two variables by numerical method.
* Use iterative technique which gives approximate value of thereat.
* Solution of initial value problems of ordinary differential equations by various method.

    1) Taylor's series method
    2) Picard's method
    3) Euler's method
    4) R – K method.

5.2 Introduction :

In this chapter we are going to study initial value problem that is the solution to a differential equation that satisfies a given initial conditions.

The initial value problem of an ordinary first order differential equation has the form $\frac{dx}{dy} = f(x, y)$ with initial conditions when $x = x_0$ and $y = y_0$.

Such differential equations are used to model problems in science and engineering such problems are too complicated to solve exactly. Thus in such case we solve that problems using numerical method.

5.3 Taylor Series Method :

Consider initial value problem of an first order differential equation.

$$\frac{dy}{dx} = f(x, y) \qquad \text{.......... (I)}$$

with initial condition $y(x_0) = y_0$.

If $y(x)$ is the exact solution of differential equation (I) Talyor's Series of $y(x)$ about $x = x_0$ is given by,

$$y(x) = y(x_0) + (x - x_0)\, y^{I}(x_0) + \frac{(x - x_0)^2}{2!}\, y^{II}(x_0) + \frac{(x - x_0)^3}{3!}\, y^{III}(x_0) \qquad \text{.......... (II)}$$

To find $y(x)$ of the first order differential equation.

$$\frac{dy}{dx} = f(x, y)$$

$$\therefore y^{I} = f(x, y)$$

$$\therefore y^{II} = \frac{d}{dx} y^{I} = \frac{\partial}{\partial x}\left[f(x, y)\right] + \frac{\partial}{\partial y}\left[f(x, y)\right]\frac{\partial y}{\partial x} = f_x + f_y \cdot f$$

$$y^{III} = \frac{d}{dx}\left(y^{II}\right) = \frac{d}{dx}\left[fx + fy \cdot f\right]$$

$$\therefore y^{III} = \frac{d}{dx} fx + \frac{d}{dx}\left(fy \cdot f\right)$$

$$y^{III} = fxx + 2f \cdot fxy + f^2\, fyy + fx \cdot fy + f \cdot f^2 y.$$

Use initial condition $y(x_0) = y_0 + 0$. Obtain $y_0^{I}, y_0^{II}, y_0^{III}$ substitute all these values in equation (II) use get

$$y(x) = y_0 + (x - x_0)\, y_0^{I} + \frac{(x - x_0)^2}{2!}\, y_0^{II} + \frac{(x - x_0)^3}{3!}\, y_0^{III} + ...$$

Which is the solution of given differential equation (I).

Ex. 1 :

Using Taylor's series method, obtain the solution of $\frac{dy}{dx} = 3x + y^2$ and $y = 1$ & $x = 0$. Find the value of $y(0.1)$ compute upto four places of decimals.

Sol^n :

Given equation $\frac{dy}{dx} = 3x + y^2$ with initial condition $y(0) = 1$.

$\frac{dy}{dx} = 0 + (1)^2 = 1$

$\frac{d^2y}{dx^2} = 3 + 2y\frac{dy}{dx}$ put $y(0) = 1$

$= 3 + 2(1)(1) = 5$

$\frac{d^3y}{dx^3} = 2y\,\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2$

$= 2(1)(5) + 2(1)^2 = 12$

$\frac{d^4y}{dx^4} = 2y\,\frac{d^3y}{dx^3} + 2\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right) + 4\frac{dy}{dx}\,\frac{d^2y}{dx^2}$

$= 2(1)(12) + 2(1)(5) + 4(1)(5) = 54$

$\therefore$ By Talyor's series method.

$$y(x) = y_0 + (x - x_0)\,y_0^{I} + \frac{(x - x_0)^2}{2!}\,y_0^{II} + \frac{(x - x_0)^3}{3!}\,y_0^{III} + \frac{(x - x_0)^4}{4!}\,y_0^{IV} + ...$$

$$y(x) = 1 + (x - 0)1 + \frac{(x - 0)^2}{2!}5 + \frac{(x - 0)^3}{3!}12 + \frac{(x - 0)^4}{4!}\,54 + ...$$

$$y(x) = 1 + x + \frac{5x^2}{2} + 2x^3 + \frac{9}{4}x^4 + ...$$

For $y(0.1)$ put $x = 0.1$

$$y(0.1) = 1 + 0.1 + \frac{5}{2}(0.1)^2 + 2(0.1)^3 + \frac{9}{4}(0.1)^4 + ...$$

$$= 1 + 0.1 + 0.025 + 0.002 + 0.000225$$

$$y(0.1) = 1.127225\,.$$

5.3.1 Improving Accuracy for Taylor Series Method :

The error in Taylor method is in the order of $(x - x_0)^{n+!}$. The accuracy can be improved by dividing the entire interval into subintervals $(x_0, x_1)(x_1, x_2)(x_2, x_3)$..... of equal length and computing $y(x_i)$, $i = 1, 2, ..., n$ successively using the Taylor series expansion using $y(x_i)$ as initial condition we compute $y(x_i + 1)$ it is given by,

$$y(x_i + 1) = y(x_i) + \frac{y(x_i)}{1!}(x_{i+1} - x_i) + \frac{y^{II}(x_i)}{2!}(x_{i+1} - x_i)^2 + ... + \frac{y^m(x_i)}{m!}(x_{i+1} - x_i)^m$$

In above expression put $x_{i+1} - x_i = h$ where $i = 0, 1, ..., n-1$

$\therefore$ we get

$$y(x_{i+1}) = y(x_i) + \frac{h}{1!} y^I(x_i) + \frac{h^2}{2!} y^{II}(x_i) + ... + \frac{h^m}{m!} y^m(x_i)$$

Now denote $y(x_{i+1}) = y(x_i) = y_{i+1}$ $y(x_i) = y_i$, above expression becomes

$y_{i+1} = y_i + \frac{h}{1!} y_i^I + \frac{h^2}{2!} y_i^{II} + ... + \frac{h^m}{m!} y_i^m$. This formula can be used recursively to obtain y_i values.

5.4 Picards Method :

Consider the differential equation $\frac{dy}{dx} = f(x, y)$ .... (I)

with initial condition $x = x_0$, $y = y_0$.

Integrating the above equation in the interval (x_0, x) we get,

$$\int_{x_0}^{x} dy = \int_{x_0}^{x} f(x, y)\, dx$$

$$y(x) - y(x_0) = \int_{x_0}^{x} f(x, y)\, dx$$

$$y(x) = y(x_0) + \int_{x_0}^{x} f(x, y)\, dx \qquad \text{.......... (II)}$$

Since y appears under the integral. Since on the eight the integration can not formed. Thus we shall replace the variable y by constant or a function of 'x'. since we know that the initial value of y at $x = x_0$.

This value we assue as first approximation to the solution.

$$y^{(1)} = y_0 + \int_{x_0}^{x} f\left(x,\, y_0\right) dx$$

To obtain second approximation $x = x_0$ put on right hand side of equation (II) we get,

$$y^{(2)} = y_0 + \int_{x_0}^{x} f\left(x,\, y^{(1)}\right) dx$$

In this we get $y^{(3)},\, y^{(4)},\, ...$

$\therefore$ In general

$$y^{(n+1)} = y_0 + \int_{x_0}^{x} f\left(x,\, y^{n}\right) dx$$

This equation is known as Picard's method.

Ex. :

Solve the differential equation $\frac{dy}{dx} = xe^{y}$ with $y(0) = 0$ by Picard method find $y(0.2)$. Check the error with analytically.

Sol^{n}. :

Given differential equation

$$\frac{dy}{dx} = xe^{y} \qquad \text{.... (I)}$$

with initial condition $y(0) = 0$ i.e. $x_0 = 0,\, y_0 = 0$

by Picard's method

$$y^{\text{II}} = y_0 + \int_{x_0}^{x} f\left(x,\, y_0\right) dx$$

$$= y_0 + \int_{x_0}^{x} xe^{y_0}\, dx$$

$$= 0 + \int_{0}^{x} xe^{0}\, dx = \int_{0}^{x} x dx = \frac{x^2}{2}$$

$$y^{(2)} = y_0 + \int_{x_0}^{x} f\left(x,\, y^{(1)}\right) dx$$

$$= 0 + \int_{0}^{x} xe^{y x^2/2}\, dx = \int_{0}^{x} xe^{x^2/2}\, dx$$

Put $\frac{x^2}{2} = t \Rightarrow xdx = dt$

$$\therefore \int_0^x e^t dt = \left[e^t\right]_0^x = \left[e^{x^2/2}\right]_0^x = e^{x^2/2} - 1$$

$$y^{(2)} = e^{x^2/2} - 1$$

$$y^{(3)} = y_0 + \int_{x_0}^x f\left(x, y^{(2)}\right) dx$$

$$= 0 + \int_0^x xe^{\left(y^{(2)}\right)} dx = \int_0^x x\left(e^{x^2/2} - 1\right) dx.$$

Now this integration is more difficult therefore we stop at $y^{(2)}$

$\therefore y(x) = y^{(2)} = e^{x^2/2} - 1$

$\therefore$ To find $y(0.2)$ taking $x = 0.2$ we get,

$$y(0.2) = e^{(0.2)^2/2} - 1 = 0.0202013.$$

We shall solve the given equation analytically

$$\frac{dy}{dx} = xe^y$$

$$e^{-y} dy = xdx$$

Integrating both side we get,

$$\int e^{-y} dy = \int x dx$$

$$-e^{-y} = \frac{x^2}{2} + c$$

Put $x = 0, y = 0$ we get, $c = -1$

$$\therefore e^{-y} = 1 - x^2/2$$

$$y = -\log\left(1 - x^2/2\right)$$

Which is particular solution of d.E.

$$\therefore y(0.2) = -\log\left(1 - (0.2)^2/2\right)$$

$= 0.0202027$

Error $= 0.0202027 - 0.0202013$

$= 1.4 \times 10^{-6}$

5.5 Euler's Method :

Euler's method is the one step method and has limited application because of it's law accuracy. Consider the differential equation $\frac{dy}{dx} = f(x, y)$ ………. (I) with initial value $y(x_0) = y_0$.

Let, $x_i = x_0 + ih \qquad i = 1, 2, 3, ...$

Integration equation (I) by the limit x_0 to x_1, we get,

$$\int_{x_0}^{x_1} dy = \int_{x_0}^{x_1} f(x, y)\, dx$$

$$y(x_1) - y(x_0) = \int_{x_0}^{x_1} f(x, y)\, dx$$

$$y(x_1) = y(x_0) + \int_{x_0}^{x_1} f(x, y)\, dx$$

$$y_1 = y_0 + \int_{x_0}^{x_1} f(x, y)\, dx$$

Assuming that $f(x, y) = f(x_0, y_0)$ in $x_0 \le x \le x_1$

$$\therefore y_1 = y_0 + \int_{x_0}^{x_1} f(x_0, y_0)\, dx$$

$$y_1 = y_0 + f(x_0, x_0)(x_1 = x_0)$$

$$y_1 = y_0 + h\, f(x_0, y_0)$$

Again integrating equation (I) between x_1 and x_2, we get,

$$\int_{x_1}^{x_2} dy = \int_{x_1}^{x_2} f(x, y)\, dx$$

$$y(x_2) - y(x_1) = \int_{x_1}^{x_2} f(x, y) dx$$

$$\therefore y_2 = y_1 + \int_{x_1}^{x_2} f(x, y)\, dx$$

$$\therefore y_2 = y_1 + h\, f(x_1, y_1)$$

Similarly we can obtain $y_3, y_4, ...$

In general $y_{n+1} = y_n + h\, f(x_n, y_n)$

This is known as Euler's formula.

Ex. : Using Euler's method find an approximate value y corresponding to $x = 2$ given that $\frac{dy}{dx} = 3x^2 + 1$ with $y(1) = 2$ taking interval $h = 0.2$ also find error in it.

Sol^{n}. :

Given differentiable equation is

$$\frac{dy}{dx} = 3x^2 + 1$$

$$\therefore f(x, y) = 3x^2 + 1 \text{ with initial } x_0 = 1,\ y_0 = 2$$

By Euler's method

$$y_{n+1} = y_n + h\,f(x_n,\ y_n) \qquad \text{……….}(I)$$

Taking $n = 0,\ y_1 = y_0 + h\,f(x_0,\ y_0)$

Put $x_0 = 1,\ y_0 = 2$

$$f(x_0,\ y_0) = f(1,\ 2) = 3(1)^2 + 1 = 4$$

$$\therefore y_1 = y_0 + h\,f(x_0,\ y_0)$$

$$y_1 = 2 + (0.2)(4) = 2.8$$

Taking $n = 1$ in equation (1) we get $x_1 = 1 + 0.2 = 1.2$

$$y_2 = y_1 + h\,f(x_1,\ y_1)$$
$$= 2.8 + 0.2\ f(1.2,\ 2.8)$$
$$= 2.8 + 0.2\left(3(1.2)^2 + 1\right)$$
$$= 3.864.$$

Taking $n = 2$ in equation (1) with $x_2 = 1.2 + 0.2 = 1.4$

$$y_3 = y_2 + h\,f(x_2,\ y_2)$$
$$= 3.864 + 0.2\ f(1.4,\ 3.864)$$
$$y_3 = 5.24.$$

Taking $n = 3$ in equation (1) with $x_3 = 1.4 + 0.2 = 1.6$

$$y_4 = y_3 + h\,f(x_3,\ y_3)$$
$$y_4 = 5.24 + 0.2\ f(1.6,\ 5.24)$$
$$= 6.976.$$

Taking $n = 4$ in equation (1) with $x_5 = 1.6 + 0.2 = 1.8$

$$y_5 = y_4 + h\,f(x_4,\ y_4)$$
$$= 6.976 + 0.2\ f(1.8,\ 6.976).$$
$$= 9.12.$$

$\therefore$ Thus the required approximate value of $y(2) = 11.72$.

We shall solve the given equation analytically

$$\frac{dy}{dx} = 3x^2 + 1$$

$$dy = 3x^2 + 1dx$$

Integrating both side

$$\int dy = \int 3x^2 + 1dx$$

$$y = \frac{3x^3}{3} + x + c$$

$$y = x^3 + x + c.$$

Taking initial value $x_0 = 1,\ y_0 = 2$ we get value of c

$$2 = (1)^3 + 1 + c$$

$$2 - 2 = c$$

$$c = 0.$$

$$y = x^3 + x$$

$$y(2) = (2)^3 + 2$$

$$= 8 + 2$$

$$= 10$$

Error = Exact Value – Approximate Value

$$= 10 - 9.12$$

$$= 0.88.$$

5.5.1 Accuracy of Euler's Method :

Since Euler's method use Taylor's series iteratively, the truncation error causes loss of accuracy. The truncation introduced by the step itself is known as the local truncation error and the sum of the propagated error and local error is called Global Truncation Error.

Consider Taylor's expansion

$$y_{n+1} = y_n + h\,y_n^{\mathrm{I}} + \frac{h^2}{2!}\,y_n^{\mathrm{II}} + \frac{h^3}{3!}\,y_n^{\mathrm{III}} + ...$$

Since only first two terms are used in Euler's formula the local truncation error is given by,

$$E_{t,\,n+1} = \frac{h^2}{2!}\,y_n^{\mathrm{II}} + \frac{h^3}{3!}\,y_n^{\mathrm{III}} + ...$$

The local truncation error of Euler's method is of the order h^2. If the final estimation required n steps, the global truncation error at the target point b will be

$$1E_{tg}l = \sum_{i=1}^{n} c_i h^2 = (c_1 + c_2 + ... c_n)\, h^2$$

$$\therefore 1E_{tg}l = nch^2$$

Where $c = c_1 + c_2 + ... {c_n}/{n}$

But $n = {(b - x_0)}/{h}$

$$\therefore 1E_{tg}l = (b - x_0)\, ch.$$

5.6 Euler's Modified Formula :

Consider the differential equation $\frac{dy}{dx} = f(x, y)$ .... (I)

with initial condition $y(x_0) = y_0$.

Integrating both side between limit x_0 to we get,

$$\int_{x_0}^{x_1} dy = \int_{x_0}^{x_1} f(x, y)\, dx$$

$$y_1 = y_0 + \int_{x_0}^{x_1} f(x, y)\, dx$$

$\therefore$ By trapezoidal rule, we get,

$$y_1 = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1)\right] \quad .... \text{(II)}$$

But $f(x_1, y_1)$ which occurs on the right hand side of equation (II), cannot be calculate since y_1 is unknown so first we calculate y_1 from Euler's formula.

$\therefore$ By Euler's formula

$y_1 = y_0 + hf(x_0, x_0)$

Put this value in equation (II) we set first approximation y_1^1.

$$y_1^1 = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_0 + hf(x_0, y_0))\right]$$

To obtain 2nd approximation to y_1 i. e. $y_1^{(2)}$ put $y_1 = y_1^{(1)}$ in right hand side of equation (II)

$$y_1^2 = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_0 + hf(x_0, y_0))\right]$$

Continuous in this process until $y_1^{(k+1)} \approx y_1^{(1)}$ the $(k+1)^{th}$ approximation to y_1 is

$$y_1^{k+1} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^k)\right]$$

In general

$$y_{n+1}^{k+1} = y_n + \frac{h}{2}\left[f\left(x_n, y_n\right) + f\left(x_{n-1}, y_{n-1}\right)\right]$$

Where $y_{n+1}^{k} = y_n + h\,f\left(x_n, y_n\right)$ [by Euler formula]

Ex. :

Using Euler's modified formula. Find approximation value of y when $x = 0.3$ given that $\frac{dy}{dx} = x + y$ and $y(0) = 1$ with $h = 0.1$.

Sol^{n}. :

Given differential equation is $\frac{dy}{dx} = x + y$ with initial value condition $x_0 = 0$ and $y_0 = 1$ & $h_0 = 0.1$.

$\therefore f(x, y) = x + y$

For 1st approximation :

$$f\left(x_0, y_0\right) = 0 + 1 = 1$$

$$y_1 = y_0 + h\,f\left(x_0, y_0\right)$$

$$= 1 + 0.1(1) = 1.1$$

$$f\left(x_1, y_1\right) = f(0.1, 1.1) = 1.1 + 0.1 = 1.2$$

$$y_1^1 = y_0 + \frac{h}{2}\left[f\left(x_0, y_0\right) + f\left(x_1, y_1\right)\right]$$

$$= 1 + 0.05[1 + 1.2]$$

$$= 1.11.$$

For 2nd approximation :

$$f\left(x_1, y_1\right) = f(0.1, 1.11) = 1.21$$

$$y_2 = y_1 + h\,f\left(x_1, y_1\right)$$

$$= 1.11 + 0.1(1.21) = 1.231$$

$$f\left(x_2, y_2\right) = f(0.2, 1.231) = 0.2 + 1.231 = 1.431$$

$$y_2^1 = y_1 + \frac{h}{2}\left[f\left(x_1, y_1\right) + f\left(x_2, y_2\right)\right]$$

$$= 1.11 + 0.05\,[1.21 + 1.431]$$

$$y_2^1 = 1.242.$$

For 3rd approximation :

$$f\left(x_2, y_2\right) = f(0.2, 1.242) = 0.2 + 1.242 = 1.442$$

$$y_3 = y_2 + h\,f\left(x_2, y_2\right)$$

$$= 1.242 + 0.1(1.442)$$

$$y_3 = 1.3862.$$

$$f(x_3, y_3) = (0.3, 1.3862)$$
$$= 0.3 + 1.3862$$
$$= 1.6862.$$

$$y_3^1 = y_2 + h/2\left[f(x_2, y_2) + f(x_3, y_3)\right]$$
$$= 1.242 + 0.05\left[1.442 + 1.6862\right]$$
$$= 1.3984.$$

$\therefore$ The approximate value of $y(0.3) = 1.3984$.

5.7 Runge – Kutta Method :

Runge-Kutta method is also called as RK-method it is the generalization of the concept used in modified Euler's method.

The Runge-Kutta method do on required the calculation of higher order derivatives their designer to give greater accuracy.

* **First order Runge-Kutta Method :**

Consider a differential equation $\frac{dy}{dx} = f(x, y)$ …. (I)

with initial condition $y(x_0) = y_0$

$\therefore$ By Euler's formula

$$y_1 = y_0 + h f(x_0, y_0)$$
$$y_1(x_1) = y_0 + h f(x_0, y_0)$$

Taking $x_1 = x_0 + h$ we get,

By Taylor's series

$$y(x_1) = y_0 + h y_0^{\mathrm{I}} + \frac{h^2}{2!} y_0^{\mathrm{II}} + ...$$

$\therefore$ Euler's method agrees with Taylor's series upto the first 2 term's.

Hence Euler's formula is the first order Runge-Kutta method.

* **Second Order Runge-Kutta Method :**

Consider a differential equation $\frac{dy}{dx} = f(x, y)$ …. (I)

with initial condition $y(x_0) = y_0$.

$\therefore$ By Euler's modified formula

$$y_1 = y_0 + h/2$$
$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right]$$

i.e. $y_{n+1} = y_n + \frac{h}{2}\left[f\left(x_n, y_n\right) + f\left(x_n, h_n, y_n + h\left(f_n\right)\right)\right]$

Let $k = h\ f\left(x_n, y_n\right)$

$$k_2 = h f\left[x_{n+h}, y_{n+h}\ f\left(x_n, y_n\right)\right]$$

$$k_2 = h f\left[x_{n+h}, y_1 + k_1\right]$$

Putting the values of k_1 and k_2 in (II) we get,

$$y_{n+1} = y_n + \frac{1}{2}\left(k_1 + k_2\right)$$

This is known as Runge-Kutta formula of order.

* **3^{rd} Order Runge-Kutta Method :**

Consider a differential equation $\frac{dy}{dx} = f\left(x, y\right)$ …. (I)

with initial condition $y\left(x_0\right) = y_0$.

To determine y_1 the 3^{rd} order R-K is given by

$$y_1 = y_0 + \frac{1}{6}\left(k_1 + 4k_4 + k_3\right)$$

Where $k_1 = h f\left(x_0, y_0\right)$

$$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = h f\left(x_0 + h, y_0 + k_2\right)$$

Where $k_2 = h f\left(x_0 + h, y_0 + k_1\right)$.

5.6.1 4^{th} Order Runge-Kutta Method or Runge-Kutta Method :

Consider a differentiable equation

$$\frac{dy}{dx} = f\left(x, y\right) \text{ with } y\left(x_0\right) = y_0.$$

To determine y_1 the 4^{th} order R-K formula is given by

$$y_1 = y_0 + \frac{1}{6}\left[k_1 + 2k_2 + 2k_2 + 2k_3 + k_4\right]$$

Where $k_1 = h f\left(x_0, y_0\right)$

$$k_2 = h f\left(x_0 + h/2, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = h f\left(x_0 + h/2, y_0 + k_2/2\right)$$

$$k_4 = h f\left(x_0 + h, y_0 + k_3\right).$$

Ex. :

Using R-K method of order 2 approximate value of y where $x = 1$ given that $\frac{dy}{dx} = 3x + y^2$ with initial condition $y(1) = 1.2$.

Sol^n. :

Given differential equation $\frac{dy}{dx} = 3x + y^2$ …… (I)

Where $f(x, y) = 3x + y^2$ with $x_0 = 1,\ y_0 = 1.2$ and $h = 0.1$.

$$\therefore f(x_0, y_0) = f(1, 1.2) = 3(1) + (1.2)^2 = 4.44$$

$$\therefore k_1 = h\,f(x_0, y_0) = 0.1 \times 4.44 = 0.444$$

$$f(x_0 + \text{h}, y_0 + k_1) = f(1.1, 1.2 + 0.444)$$

$$= f(1.1, 1.644)$$

$$= 3(1.1) + (1.644)^2$$

$$= 6.$$

$$k_2 = h\,f(x_0 + \text{h},\ y_0 + k_1) = 0.1 \times 6 = 0.6$$

$\therefore$ By R-K method of 2nd order

$$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2)$$

$$y_1 = 1.2 + \frac{1}{2}(0.444 + 0.6)$$

$$= 1.722$$

Ex. 2 :

Using R-K method of order 3, approximate value of y when $x = 0.2$ given that $\frac{dy}{dx} = x^2 - y$ with initial condition $y(1) = 1$ and $h_0 = 0.1$.

Sol^n. :

Given differential equation

$$\frac{dy}{dx} = x^2 - y$$

$$\therefore f(x, y) = x^2 - y \text{ with initial } x_0 = 1,\ y_0 = 1$$

1st approximation and $h_0 = 0.1$

$f(x_0, y_0) = f(1, 1) = (1)^2 - 1 = 0$

$\therefore k_1 = h\,f(x_0, y_0) = 0.1 \times 0 = 0$

$f(x_{0+h}, y_{0+k}) = f(1.1 + 1) = (1.1)^2 - 1 = 0.21$

$k_2 = h\,f(x_0 + \text{h}, y_0 + k_2) = f(1.1, 1.021)$

$= (1.1)^2 - (1.02)$

$= 0.189.$

$k_3 = h\,f(x_0 + \text{h}, y_0 + k_2) = 0.1 \times 0.189 = 0.0189$

$f\left(x_0 + h/2, y_0 + \frac{k_1}{2}\right) = f(1.05, 1) = (1.05)^2 - 1 = 0.1.25$

$k_4 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1 \times 0.1025 = 0.01025$

$\therefore$ R-K method of 3rd order

$$y_1 = y_0 + \frac{1}{6}[k_1 + 4k_4 + k_3]$$

$$y_{(0.1)} = 1 + \frac{1}{6}[0 + 4 \times 0.01025 + 0.0189]$$

$$= 1.00998.$$

2nd approximation

$x_1 = 1.1,\ y_1 = 1.00998,\ h = 0.1$

$f(x_1, y_1) = (1.1)^2 - 1.00998 = 0.2$

$k_1 = h\,f(x_0, y_0) = 0.1 \times 0.2 = 0.02$

$f(x_1 + \text{h}, y_1 + k_1) = f(1.2, 1.02998) = 0.41002$

$k_2 = h\,f(x_1 + \text{h}, y_1 + k_1) = 0.1 \times 0.41002 = 0.041$

$f(x_1 + \text{h}, y_1 + k_2) = f(1.2, 1.05098)$

$= 0.38902$

$k_3 = h\,f(x_1 + \text{h}, y_1 + k_2) = 0.1 \times 0.38902 = 0.0389$

$f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = f(1.5, 1.01998) = 0.30252$

$k_4 = h\,f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.1 \times 0.30252 = 0.030252$

$$\therefore y(0.2) = y_1 + \frac{1}{6}[k_1 + h k_4 + k_3]$$

$$= 1.00998 + \frac{1}{6}[0.02 + 4 \times 0.030252 + 0.0389]$$

$$= 1.03996$$

$y(0.2) = 1.03996$

Ex.3:

Using R-K method of 4 th order + 0 Find approximate value of y when $x = 0.2$ given that $\frac{dy}{dx} = x^3 + y$ with initial condition
y(0) = 1 $ h = 0.1

Sol^n: Given differential equation

$$\frac{dy}{dx} = x^3 + y$$

$$\therefore f(x,y) = x^3 + y \ with\, x_o = 0, y_0 = 1, h_0 = 0.1$$

1st approximation

$$f(x_0, y_0) = f(0,1) = (0)^3 + 1 = 1$$

$$k_1 = hf(x_0, y_0) = 0.1 \times 1 = 0.1$$

$$k_2 = hf\left(x_0 + h/2, y_0 + \frac{k_1}{2}\right) = 0.1f(0.05 + 1.05)$$

$$0.1\left[(0.05)^3 + 1.05\right]$$

$$= 0.105$$

$$k_3 = hf\left(x_0 + h/2, y_0 + \frac{k_2}{2}\right) = 0.1f(0.05 + 1.0525)$$

$$0.1\left[(0.05)^3 + 1.0525\right]$$

$$= 0.10526$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1f(0.1 + 1.0526)$$

$$0.1\left[(0.1)^3 + 1.10526\right]$$

$$= 0.110626$$

According to 4th order R-K method

$$y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= 1 + \frac{1}{6}[0.1 + 2 \times 0.105 + 2 \times 0.10526 + 0.110626]$$

$$= 1.10591.$$

2nd approximation

$$x_1 = 0.1,\ y_1 = 1.105191,\ h = 0.1$$

$$k_1 = h\,f(x_1,\ y_1) = 0.1\,f(0.1,\ 1.10591)$$

$$= 0.1\left[(0.1)^3 + 1.105191\right]$$

$$= 0.1106191.$$

$$k_2 = h\,f\left(x_1 + h/2,\ y_1 + \frac{k_1}{2}\right) = h\,f(0.15,\ 1.1605)$$

$$= 0.1\left[(0.15)^3 + 1.1605\right]$$
$$= 0.1163875.$$

$$k_3 = h\,f\left(x_1 + h/2,\, y_1 + \frac{k_2}{2}\right) = h\,f(0.15, 1.6338)$$
$$= 0.1\left[(0.1)^3 + 1.16338\right]$$
$$= 0.1166755.$$

$$k_4 = h\,f\left(x_{0+n}, y_{0+k_3}\right) = h\,f(0.2, 1.2218665)$$
$$= 0.1\left[(0.2) + 1.2218665\right]$$
$$= 0.12298.$$

$$\therefore y_2 = y_1 + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

$$= 1.105191 + \frac{1}{6}\left[0.1106191 + 2 \times 0.1163875 + 2 \times 0.116675 + 0.12298\right]$$
$$= 1.2218$$
$$\therefore Y(0.2) = 1.2218.$$

5.8 Review :

In this chapter we learn

* Definition of initial value problem of an ordinary first order differential equations.
* Solution of initial value problems of ordinary first order differential equation by various method.

One – steps method :

i) Taylor's series method
ii) Picard's method
iii) Euler's method
iv) R-K method of 2nd order
v) R-K method of 3rd order
vi) R-K method of 4th order

5.9 Unit End Exercise :

1) Use Taylor's series method to solve the equations $\frac{dy}{dx} = -xy$ with $y(0) = 1$ estimate $y(0.4)$.

2) Use Taylor's series method to sole the equation $\frac{dy}{dx} = 2y + 3e^x$ with initial $y(0) = 0$ estimate $y(0.2)$ check the error with exact value.

3) Using Taylor's series method find $y(0.2)$ by solving differential equation $\frac{dy}{dx} = 1 + xy$ where $y(0) = 2$ taking $h = 0.1$.

4) Find $y(0.2)$ by Picard's method given that $\frac{dy}{dx} = xy$ with initial condition $y(0) = 1$ check the error with correct answer.

5) Find $y(0.5)$ solve the differential $y^1(x) = xe^y$ with $y(0) = 0$ by Picard's method.

6) Using Picard's method solve the differential $\frac{dy}{dx} = \frac{y - x}{y + x}$ with $y(0) = 1$ approximate value of y where $x = 0.1$.

7) Using Euler's method find an approximate value of y corresponding to $x = 1.6$ given $\frac{dy}{dx} = y^2 - \frac{y}{x}$ with $y(1) = 1$ and $h = 0.2$.

8) Using Euler's modified formula solve $\frac{dy}{dx} = 1 - 2xy$ given $y = 0$, at $x = 0$ from $x = 0 + 00.6$ taking the interval $h = 0.2$.

9) The initial value problem $y^1 = xy + x^2 - 2$ with $y(1) = 2$ find the value of $y(1.2)$ with $h = 0.2$.

i) Using R-K method 2nd order
ii) Using R-K method 3rd order
iii) Using R-K method 4th order

10) Use the Range-Kutta 4th order method to find $y(0.2)$ with $h = 0.1$ for the initial value problem $\frac{dy}{dx} = \sqrt{x + y}$ with $y(0) = 1$.

♦♦♦♦♦

6

Numerical Solution of Differential Equation – II

Unit Structure

6.1 Objective :

After studying this chapter you will be able to :

* Solve simultaneous first order differential equations.
* Solve higher order differential equation.
* Find solution of initial value problem of ordinary first order differential equation by multi-step method by :
    1) Milne – Simpson method
    2) Adams Bashfarth maulton method
* Accuracy of multi-step method.
* Stability of numerical solution.

6.2 Introduction :

In previous chapter we have solve 1^{st} order differential equation by different method. Here we are going to solve simultaneous differential equation with some method also solve higher under different equation take

$\frac{d^2y}{dx^2} = f(x, y, z)$ with given initial condition to prove the efficiency of estimations in one step method we need to use multi-step method by predication and correction formula.

6.3 Simultaneous First Order Differential Equations :

Both Euler's Method :

Taylor's method, Picard's method, Euler method and Runge-Kutta method can be used to find the approximate solution to the system of first order differential equations.

Consider the first order differential equations.

$$\frac{dx}{dt} = f(t, x, y)$$

$$\frac{dy}{dt} = g(t, x, y)$$

with initial condition $x = x_0,\ y = y_0$ when $t = t_0$ or $x(t_0) = x_0$ and $y(t_0) = y_0$.

Taking small change, assuming that $\Delta t = h,\ \Delta x = k$ and $\Delta y = \ell$.

The 4th order R-K method given by

$$k_1 = h f(t_0, x_0, y_0)$$

$$\ell_1 = h g(t_0, x_0, y_0)$$

$$k_2 = h f\left(t_0 + h/2,\ x_0 + \frac{k_1}{2},\ y_0 + + \ell_1/2\right)$$

$$\ell_2 = h f\left(t_0 + h/2,\ x_0 + \frac{k_1}{2},\ y_0 + + \ell_1/2\right)$$

$$k_3 = h f\left(t_0 + h/2,\ x_0 + \frac{k_2}{2},\ y_0 + + \ell_2/2\right)$$

$$\ell_3 = h f\left(t_0 + h/2,\ x_0 + \frac{k_2}{2},\ y_0 + + \ell_2/2\right)$$

$$k_4 = h f(t_0 + h,\ x_0 + k_3,\ y_0 + + \ell_3)$$

$$\ell_4 = h f(t_0 + h,\ x_0 + k_3,\ y_0 + + \ell_3)$$

$$\therefore x_1 = x_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$y_1 = y_0 + \frac{1}{6}(\ell_1 + 2\ell_2 + 2\ell_3 + \ell_4)$$

The extension of the R-K method to a system of n equation is quite straight forward.

Similarly we solving 1st order differential equation by Taylor's series method.

6.3.1 Taylor's method for simultaneous 1 st order differential equation :

Consider the 1st order differential equation

$$\frac{dx}{dt} = f(t, x, y)$$

$$\frac{dy}{dt} = g(t, x, y)$$

with initial condition $x = x_0$, $y = y_0$ and $t = t_0$. Let h be the small change then by Taylor's series expansion,

$$x_1 = x(t_1) = x(t_0 + h)$$

$$= x(t_0) + h\,x^{\mathrm{I}}(t_0) + \frac{h^2}{2!}x^{\mathrm{II}}(t_0) + \frac{h^3}{3!}x^{\mathrm{III}}(t_0) + \ldots..$$

$$= x_0 + h\,x_0^{\mathrm{I}} + \frac{h^2}{2!}x_0^{\mathrm{II}} + \frac{h^3}{3!}x_0^{\mathrm{III}} + \ldots..$$

$$y_1 = y(t_1) = y(t_0 + h)$$

$$= y_0 + h\,y_0^{\mathrm{I}} + \frac{h^2}{2!}y_0^{\mathrm{II}} + \frac{h^3}{3!}y_0^{\mathrm{III}} + \ldots..$$

* Using Taylor series method evaluate $x(0.3)$ and $y(0.3)$ given that $\frac{dx}{dt} = y + \log t$ and $\frac{dy}{dt} = \cos t - x$ with initial condition $x(1) = 2$ and $y(1) = 1$.

Sol^n. :

By Taylor series method

$$x_1 = x(t_1) = x_0 + h\,x_0^{\mathrm{I}} + \frac{h^2}{2!}x_0^{\mathrm{II}} + \ldots.. \qquad \ldots\ldots\ldots. \text{(I)}$$

$$y_1 = y(t_1) = y_0 + h\,y_0^{\mathrm{I}} + \frac{h^2}{2!}y_0^{\mathrm{II}} + \ldots.. \qquad \ldots\ldots\ldots. \text{(II)}$$

Given differential equation

$$\frac{dx}{dt} = y + \log t,\ \frac{dy}{dt} = \cos t - x$$

$$f(t, x, y) = y + \log t,\ g(t, x, y) = \cos t - x$$

initial value $x_0 = 2,\ y_0 = 1,\ t_0 = 1$

$$\frac{dx}{dt} = y + \log t \quad \left.\frac{dx}{dt}\right/_{t = t_0} = x^{\mathrm{I}}(1) = 1 + \log 1 = 0$$

$$x_0^{\mathrm{I}} = 1.$$

$$\frac{dy}{dt} = \cos t - x \quad \left.\frac{dx}{dt}\right/_{t = t_0} = \cos(1) - 2$$

$$y_0^{\mathrm{I}} = 1.4597.$$

$$\frac{d^2x}{dt^2} = y^{\mathrm{I}} + \frac{1}{t} \quad \left.\frac{d^2x}{dt^2}\right/_{t = t_0} = 1.4597 + 1$$

$$x_0^{\mathrm{II}} = 2.4597.$$

$$\frac{d^2y}{dt^2} = \text{-}\sin t - x^{1} \quad \left.\frac{d^2y}{dt^2}\right/_{t = t_0} = -\sin 1 - 1$$

$$y_0^{\mathrm{II}} = -1.8415.$$

$$\frac{d^3x}{dt^3} = y^{\mathrm{II}} - \frac{1}{t^2} \quad \left.\frac{d^3x}{dt^3}\right/_{t = t_0} = y_0^{\mathrm{II}} + \frac{1}{(1)^2} = -1.8415 + 1$$

$$= -0.8415.$$

$$\frac{d^3y}{dt^3} = -\cos t - x^{\mathrm{II}} \quad \left.\frac{d^3x}{dt^3}\right/_{t = t_0} = \sin(1) - 2.4592$$

$$= -3.3006.$$

$$\frac{d^4x}{dt^4} = y^{\text{III}} + \frac{1}{t^3} \quad \left.\frac{d^4x}{dt^4}\right/_{t = t_0} = -\,3.3006 + \left.1\right/(1)^3$$
$$= -2.3006\,.$$

$$\frac{d^4y}{dt^4} = \sin\,t - x^{\text{IV}} \quad \left.\frac{d^4y}{dt^4}\right/_{t = t_0} = \sin 1 + 0.8415$$
$$= 1.68297\,.$$

Put all these value in equation (I) and (II)

$$x_1 = x(0.2) = 2 + (0.2)\,(1) + \frac{(0.2)^2}{2!}\,(2.4597) + \frac{(0.2)^3}{3!}$$
$$(-1.8415) + \frac{(0.2)^4}{4!}\,(-2.3006)$$
$$= 2.2466\,.$$

$$y_1 = y(0.2) = 1 + (0.2)\,(1.4597) + \frac{(0.2)^2}{2!}\,(-1.8415) + \frac{(0.2)^3}{3!}\,(-3.3006)$$
$$+ \frac{(0.2)^4}{4!}\,(1.68297)$$
$$= 1.25082\,.$$

* Using R-K method of 4th order find approximate value of x & y at $t = 0.1$ the following system $\frac{dx}{dt} = x^2 + y,\ \frac{dy}{dt} = 2x + y^2$ with initial condition $x_0 = 1,\ y_0 = 1,\ t_0 = 0$.

Sol^n^. :

To compute $x_1 = x(0.1)$ and $y_1 = y(0.1)$ use R-K method of 4th order.

$$x_1 = x_0 + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right] \qquad \ldots\ldots\ldots\ (\text{I})$$

$$y_1 = y_0 + \frac{1}{6}\left[\ell_1 + 2\ell_2 + 2\ell_3 + \ell_4\right] \qquad \ldots\ldots\ldots\ (\text{II})$$

Given $dx/dt = x^2 + y, \qquad dy/dt = x + y^2$

$f(t_0, x_0, y_0) = x^2 + y, \qquad y(t_0, x_0, y_0) = x + y^2,$

with $x_0 = 1,\ y_0 = 1,\ t_0 = 0$

$$k_1 = h\,f\left(t_0,\ x_0,\ y_0\right) = (0.1)\left(x_0^2 + y_0\right) = 0.1\left[(1)^2 + 1\right] = 0.2$$

$\operatorname{Im}(\lambda, h)$

$$k_1 = h\,f\left(t_0 + h/2,\ x_0 + k2/2,\ y_0 + \ell_1/2\right)$$

$$= 0.1g\left(0.05, 1.118, 1.176\right) = 0.1\left[2\left(1.18 + (1.761)^2\right)\right] = 0.3619$$

$$= 0.1\ f\left[0.05,\ 1.1,\ 1.15\right]$$

$$= 0.1\left[(1.1)^2 + 1.15\right]$$

$$= 0.236.$$

$$= \left(0.1 \times g\right)\left(0.05; 1.118, 1.176\right)0.1\left[2\left(1.118 + (1.1781)\right)\right] = 0.3619$$

$$= 0.1\ g\left[0.05,\ 1.1,\ 1.15\right]$$

$$= 0.1\left[2(1.1) + (1.15)^2\right]$$

$$= 0.35225.$$

$$k_3 = h\,f\left(t_0 + h/2,\ x_0 + \frac{k_2}{2},\ y_0 + \ell_2/2\right)$$

$$= \left(0.1 \times 9\right)\left(0.05; 1.118, 1.176\right)0.1\left[2\left(1.118 + (1.1781)^2\right]\right. = 0.3619$$

$$= 0.1\ f\left[0.05,\ 1.118,\ 1.761\right]$$

$$= 0.1\left[(1.118)^2 + 1.1761\right]$$

$$= 0.2426.$$

$$\ell_3 = h\,g\left(t_0 + h/2,\ x_0 + k_1/2,\ y_0 + \ell_2/2\right)$$

$$k_4 = h\,f\left(t_0 + h,\ x_0 + k_3,\ y_0 + \ell_3\right)$$

$$= 0.1\ f\left[0.1,\ 1.2426,\ 1.3619\right]$$

$$= 0.1\left[(1.2426)^2 + 1.3619\right]$$

$$= 0.2906.$$

$$\ell_4 = h\,g\left(t_0 + h,\, x_0 + k_3,\, y_0 + \ell_3\right)$$
$$= 0.1g\left[0.1, 1.2426, 1.3619\right]$$
$$= 0.1g\left[2\left(1.2426\right) + \left(1.3619\right)^2\right]$$
$$= 0.4339\,.$$

Put all value in equation (I) and (II) we get,

$$x_1 = 1 + \frac{1}{6}\left[0.2 + 2\left(0.236\right) + 2\left(0.2426\right) + 0.2926\right]$$
$$= 1.2416\,.$$

$$y_1 = 1 + \frac{1}{6}\left[0.3 + 2\left(0.35225\right) + 2\left(0.3619\right) + 0.4339\right]$$
$$= 1.3604\,.$$
$$\therefore x\left(0.1\right) = 1.2416, \qquad \mathrm{y}\left(0.1\right) = 1.3604\,.$$

6.4 Second Order Differential Equation :

Consider the second order differential equation

$$\frac{d^2y}{dt^2} = f\left(x,\, y,\, \frac{dy}{dx}\right)$$
$$y\left(x_0\right) = y_0, \quad \left(\frac{dy}{dx}\right)_{x = x_0} = y_0^I$$

Substituting $\frac{dy}{dx} = z$, we get,

$$\frac{dy}{dx} = f\left(x,\, y,\, z\right)$$

with initial condition $y\left(x_0\right) = y,\, z\left(x_0\right) = y_0^{\mathrm{I}}$.

These constitute system of simultaneous equations.

Ex. :

Use Runge-Kutta method to find $y\left(0.2\right)$ for the equation $\frac{d^2y}{dx^2} = x\frac{dy}{dx} - y$ given that $y = 1, \frac{dy}{dx} = 0$ when $x = 0$ with $h = 0.2$.

Sol[n]. :

Given 2[nd] order differential equation

$$\frac{d^2y}{dx^2} = x\,\frac{dy}{dx} - y$$

Taking $\frac{dy}{dx} = z = f(x, y, z)$ we get,

$$\frac{d^2y}{dx^2} = xz - y = g(x, y, z)$$

with initial condition $x = 0,\ y = 1,\ z = 0$ with $h = 0.2$.

$$k_1 = h\,f(x, y, z) = h\,z = 0.2 \times 0 = 0$$

$$\ell_1 = h\,g(x, y, z) = h(x, z - y) = 0.2\,(0 \times 0 - 1) = 0.2$$

$$k_2 = h\,f\left(x + h/2,\ y + k_1/2,\ z + \ell_1/2\right) = h\left(z_1 + M_1/2\right) = 0.2(0 - 0.1) = -0.02$$

$$\ell_2 = h\,g\left(x + h/2,\ y + k_1/2,\ z + \ell_1/2\right) = h\left[x + h/2\left(z + M_1/2\right) - \left(y + h/2\right)\right]$$

$$= 0.2\left[(0 + 0.1)\,(0 - 0.01) - 1\right] = 0.202$$

$$k_3 = h\,f\left(x + h/2,\ y + k_2/2,\ z + \ell_2/2\right) = 0.2\left[0 - \frac{0.202}{2}\right] = -0.202$$

$$\ell_3 = h\,g\left(x + h/2,\ y + k_2/2,\ z + \ell_2/2\right) = 0.2\,[0.1 - 0.101,\ 0.99]$$

$$= 0.2[-0.0101 - 0.99] = -0.20002$$

$$k_4 = h\,f(x_0 + h,\ y_0 + 9,\ z_0 + \ell_3) = h(z + M_3) = 0.2[0 - 0.20002] = -0.040004$$

$$\ell_4 = h\,g(x_0 + h,\ y_0 + k_3,\ z + \ell_3) = 0.2\,\left[(0.2)\,(-0.20002) - (1 - 0.202)\right]$$

$$= 0.2\,[-0.040004 - 0.9798] = -0.2039608$$

Put all these value in R-K method of 4th order

$$y(0.2) = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$
$$= 1 + \frac{1}{6}[0 + 2(-0.02) + 2(-0.0202) + (-0.04004)]$$
$$= 1 + \frac{1}{6}(-0.120404)$$
$$= 0.9799.$$

$$z(0.2) = z_0 + \frac{1}{6}[\ell_1 + 2\ell_2 + 2\ell_3 + \ell_4]$$
$$= 1 + \frac{1}{6}[-0.2 + 2(-0.0202) + 2(-0.20002) - 0.20396]$$
$$= \frac{1}{6}[-1.2080008]$$
$$= -0.20133.$$

6.5 Multi-Step Methods (Prediction – Correction Method):

A pair of multi-step methods are used in conjunction with each other, are for predicting the value of y_{i+1} and the other for correcting the predicted value of y_{i+1} such method are called prediction – correction method.

6.5.1 Milne – Simpson Method :

Consider a differential equation

$$\frac{dy}{dx} = f(x, y) \qquad \text{……….(I)}$$

with initial value $y(x_0) = y_0$

Integrating equation I with the limit x_0 and x_n

$$\int_{x_0}^{x_4} dy = \int_{x_0}^{x_4} f(x, y)\, dy$$

$$y_n = y_0 + \int_{x_0}^{x_4} f(x, y)\, dx \qquad \text{……….(II)}$$

We use newton forward difference interpolation formula in the form

$$f(x, y) = f(x_0, y_0) + n\Delta f(x_0, y_0) + \frac{n(n-1)}{2!}\Delta^2 f(x_0, y_0)$$
$$+ \frac{n(n-1)(n-2)}{h} \Rightarrow \Delta^3 f(x_0, y_0) + ...$$

Taking $f(x_0, y_0) = f_0$ from equation (II)

$$y_n = y_0 + \int_{x_0}^{x_4}\left[f_0 + n\Delta f_0 + \frac{n(n-1)}{2!}\Delta^2 f_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 f_0 + ...\right]dx$$

where	x	n
$\therefore x = x_0 + nh$	x_0	0
$dx = hdn$	x_n	n

$$\therefore y_4 = y_0 + h\int_0^4\left[f_0 + n\Delta f_0 + \frac{n^2 - n}{2}\Delta^2 f_0 + \frac{n^3 - 3n^2 + 3n}{6}\Delta^3 f_0 + ...\right]dn$$

$$y_4 = y_0 + h\left[nf_0 + \frac{n^2}{2}\Delta f_0 + \frac{1}{2}\left(\frac{n^3}{3} - \frac{n^2}{2}\right)\Delta^2 f_0 + \frac{1}{6}\left(\frac{n^4}{4} - n^3 + n^2\right)\Delta^3 f_0 + ...\right]_0^4$$

neglecting 4th and higher power of Δ

$$\therefore y_4 = y_0 + h\left[4f_0 + 8\Delta f_0 + \frac{20}{3}\Delta^2 f_0 + \frac{8}{3}\Delta^3 f_0\right]$$

Put $\Delta = E - 1$

$$y_4 = y_0 + \frac{h}{3}\left[12f_0 + 24(E-1)f_0 + 20(E-1)^2 f_0 + 8(E-1)^3 f_0\right]$$

$$y_4 = y_0 + \frac{h}{3}\left[12f_0 + 24(E-1)f_0 + 20(E^2 - 2E + 1)\right.$$
$$\left. f_0 + 8(E^3 - 3E^2 + 2E - 1)f_0\right]$$

We know that $E[f(x)] = f(x+h)$

$$E[f(x_0)] = f(x_1),\ E(x, y) = f$$

$$y_4 = y_0 + \frac{h}{3}\left[12f_0 + 2h(f_1 - f_0) + 20(f_2 - 2f_1 + f_0) + 8(f_3 - 2f_2 + 3f_1 - f_0)\right]$$

$$y_4 = y_0 + \frac{h}{3}\left[8f_1 - 4f_2 + 8f_3\right]$$

$$y_4 = y_0 + \frac{4h}{3}\left[2f_1 - f_2 + 2f_3\right]$$

$$\left(y_4\right)_p = y_0 + \frac{4h}{3}\left[2y_1^1 - y_2^1 + 2y_3^1\right]$$

In general

$$\left(y_{n+1}\right)_p = y_{n-3} + \frac{4h}{3}\left[2y_{n-2}^1 - y_{n-1}^1 + 2y_n^1\right]$$

This is known as Milne's predication method.

6.5.2 Simpson's Method :

Consider a differential equation

$$\frac{dy}{dx} = f\left(x, y\right) \quad \text{...........}$$

(I)

with initial condition $y\left(x_0\right) = y_0$

Integrating equation (I) between the limit x_0 to x_2 we get,

$$y_2 = y_0 + \int_{x_0}^{x_2} f\left(x, y\right) dx$$

By Newton's forward difference interpolation formula.

$$f\left(x, y\right) = f_0 + n\Delta f_0 + \frac{n\left(n-1\right)}{2!}\Delta^2 f_0 + \frac{n\left(n-1\right)\left(n-2\right)}{3!}\Delta^3 f_0 + ...$$

$$y_2 = y_0 + \int_{x_0}^{x_2} f_0 + \Delta f_0 + \frac{n^2 - n}{2}\Delta^2 f_0 + \frac{1}{6}\left(n^3 - 3n^2 + 2n\right)\Delta^3 f_0 + ... dx$$

$\because x = x_0 + nh \Rightarrow dx = ndh$ limit change

x	h
x_0	0
x_2	2

$$y_2 = y_0 + h\int_0^2\left[f_0 + n\Delta f_0 + \frac{n^2 - n}{2}\Delta^2 f_0 + \frac{\left(n^3 - 3n^2 + 2n\right)}{6}\Delta^3 f_0 + ...\right] dn$$

$$= y_0 + h\left[nf_0 + \frac{n^2}{2}\Delta f_0 + \frac{1}{2}\left(\frac{n^3}{3} - \frac{n^2}{2}\right)\Delta^2 f_0 + \frac{1}{6}\left(\frac{n^4}{4} - n^3 - n^2\right)\Delta^3 f_0 + ...\right]_0^2$$

Neglecting 3rd and higher order difference

Put $\Delta = E - 1$

$$y_4 = y_0 + h\left[2f_0 + 2(E-1)\ f_0 + \frac{1}{3}(E-1)^2\ f_0\right]$$

$$= y_0 + \frac{h}{3}\left[6f_0 + 6(E-1)\ f_0 + \left(E^2 - 2E + 1\right) f_0\right]$$

$$= y_0 + \frac{h}{3}\left[f_0 + 4E_1 + f_2\right]$$

$$y_2 = y_0 + \frac{h}{3}\left[y_0^1 + 4y_1^1 + y_2^1\right]$$

$$(y_4) = y_2 + \frac{h}{3}\left[y_2^1 + 4y_3^1 + y_4^1\right]$$

In general

$$\left(y_{n+1}\right)_c = y_{n-1} + \frac{h}{3}\left[y_{n-1}^1 + 4y_n^1 + y_{n+1}^1\right]$$

This is known as Simpson's correction formula.

Ex. :

Given differential equation

$$\frac{dy}{dx} = 1 + y^2$$

Where $y(0) = 0$ estimate $y(0.8)$ using the Milne Simpson predication correction method taking $h = 0.2$.

Sol^{n}.:

Given differential equation

$$\frac{dy}{dx} = 1 + y^2$$

$f(x, y) = 1 + y^2$ $\qquad h = 0.2$

$x_0 = 0 \qquad x_1 = 0.2 \qquad x_2 = 0.4 \qquad x_3 = 0.6$

By R-K method of 4th order

$y_0 = 0 \qquad y_1 = 0.2027 \qquad y_2 = 0.4228 \qquad y_3 = 0.6841$

$$y_1^1 = f(x_1, y_1) = f(0.2, 0.2027) = 1 + (0.42027)^2$$
$$= 1.0411.$$

$$y_2^1 = f(x_2, y_2) = f(0.4, 0.4228) = 1 + (0.4228)^2$$
$$= 1.1787.$$

$$y_3^1 = f(x_3, y_3) = f(0.6, 0.6841) = 1 + (0.6841)^2$$
$$= 1.4679.$$

Milne predication formula is

$$(y_4) = y_0 + \frac{4 \times h}{3}\left[2y_1^1 - y_2^1 + 2y_3^1\right]$$
$$= 0 + \frac{4 \times 0.2}{3}\left[2(1.0411) - 1.1787 + 2(1.4679)\right]$$
$$= 1.0238.$$

Using Simpson's correction formulas is given by

$$(y_4)_C = y_2 + \frac{h}{3}\left[y_2^1 + 4(y_3^1) + y_4^1\right] \qquad \text{.......... (II)}$$

$$y_4^1 = f(x_4, y_4) = f(0.8, 1.0238)$$
$$= 1 + (1.0238)^2$$
$$= 2.0482.$$

$$(y_4)_C = 1.1787 + \frac{0.2}{3}\left[1.787 + 4(1.4679) + 2.0482\right]$$
$$= 1.7853.$$

We can again use the correction formula II to refine the estimate.

$$y_4^1 = f(x_4, y_4) = f(0.8, 1.7853)$$
$$= 1 + (1.7853)^2$$
$$= 4.1873.$$

$$(y_4)_C = 1.787 + \frac{0.2}{3}\left[1.1787 + 4(1.4679) + 4.1873\right]$$
$$= 1.9278.$$

Which is not same again we use

$$y_4^1 = f(x_4, y_4) = f(0.8, 1.9278)$$
$$= 1 + (1.9278)^2$$
$$= 4.7164.$$

$$(y_4)_C = 1.787 + \frac{0.2}{3}\left[1.1787 + 4(1.4679) + 4.7164\right]$$
$$= 1.9631.$$

Note :

Milne formula is used to predict the value of y_{i+1}, evaluation of f_{i+1}, correction of y_{i+1}, then to improved value of f_{i+1}.

It is also possible to use the correction formula repeatedly redefine the estimate value of y_{i+1}, before moving on to the next stage.

6.6 Adams Baeshforth Moulton Method :

Consider a differential equation

$$\frac{dy}{dx} = f(x, y) = I$$

with initial condition $y(x_0) = y_0$

Integrating equation 'I' between the limit x_0 to x_1 we get,

$$y_1 = y_0 + \int_{x_2}^{x_1} f(x, y)\, dx \qquad \text{......... (II)}$$

Using Newton Backward difference interpolation formula in the term.

$$f(x, y) = f(x_0, y_0) + n\nabla f(x_0, y_0) + \frac{n(n+1)}{2!}\nabla^2 f(x_0, y_0) + ...$$

$$y_1 = y_0 + \int_{x_2}^{x_1}\left[f_0 + n\nabla f_0 + \frac{n^2 + n}{2!}\nabla^2 f_0 + \frac{n^3 + 3n^2 + 2n}{6}\nabla^2 f_0 + ...\right] dx$$

$$x = x_0 + nh$$
$$dx = hdn$$

where

x	h
x_0	0
x_1	1

$$y_1 = y_0 + \left[h \int_0^1 f_0 + n\nabla f_0 + \frac{n^2 + n}{2} \nabla^2 f_0 + ... \right] dx$$

$$y_1 = y_0 + h\left[nf_0 + \frac{n^2}{2}\nabla f_0 + \frac{1}{2}\left(\frac{n^3}{3} + \frac{h^2}{2}\right)\nabla^2 f_0 + \left(\frac{n^4}{4} + n^3 + n^2\right)\nabla^3 f_0 + ...\right]_0^1$$

Neglecting 4th and higher order and put $\nabla = 1 - E^{-1}$ we get,

$$y_1 = y_0 + h\left[f_0 + \frac{1}{2}\left(1 - E^{-1}\right) f_0 + \frac{5}{12}\left(1 - E^{-1}\right)^2 f_0 + \frac{3}{8}\left(1 - E^{-1}\right) f_0 + ...\right]$$

$$= y_0 + h\left[f_0 + \frac{1}{2}\left(1 - E^{-1}\right) f_0 + \frac{5}{12}\left(1 - 2E^{-1} + E^{-2}\right) + \frac{3}{8}\cdot\frac{3}{8}\left[1 - 3E^1 + 3E_0^2 - E^3\right] f \right]$$

$$= y_0 + \frac{h}{24}\left[24 f_0 + 12\left(f_0 - f_1\right) + 10\left(f_0 - 2f_{-1} + f_{-2}\right) + 9\left(f_0 - 3f_{-1} + 3f_{-2} + f_{-3}\right)\right]$$

$$y_1 = y_0 + \frac{h}{24}\left[55 f_0 - 59 f_{-1} + 37 f_{-2} - 9 f_{-3}\right]$$

$$y_1 = y_0 + \frac{h}{24}\left[55 y_0^1 - 59 y_{-1}^{-1} + 37 y_{-2}^{-1} + 9 y_{-3}^1\right]$$

$$\left(y_4\right)_p = y_3 + \frac{h}{24}\left[55 y_3^{\mathrm{I}} - 59 y_2^{\mathrm{II}} + 37 y_1^{\mathrm{I}} - 9 y_0^1\right]$$

In general

$$\left(y_{n+1}\right)_p = y_n + \frac{h}{24}\left[55 y_n^1 - 59 y_{n-1}^1 + 37 y_{n-2}^1 - 9 y_{n-3}^1\right]$$

This is known as adam's basehforths predication formula for correction formula

Given differential equation

$$\frac{dy}{dx} = f(x, y)$$

with initial condition $y(x) = y_0$ .... (I)

Integration between limit $x_0 + x_1$ we get,

$$y_1 = y_0 + \int_{x_0}^{x_1} f(x, y)\, dx \qquad \text{.......... (II)}$$

Using Newton Backward difference inter pollution formula

$$f(x, y) = f_1 + n\nabla f_1 + \frac{n(n+1)}{2!}\nabla^2 f_1 + \frac{n(n+1)(n+2)}{3!}\nabla^3 f_1 + \ldots..$$

where $n = x - x_1/h$

$$y_1 = y_0 + \int_{x_0}^{x_1}\left(f_1 + n\nabla f_1 + \frac{n(n+1)}{2!}\nabla^2 f_1 + \frac{n(n+1)(n+2)}{3!}\nabla^2 f_1 + \ldots..\right)dx$$

$x = x_1 + nh$ where

$dx = hdn$

x	n
x_0	-1
x_1	0

$$y_1 = y_0 + h\int_{-1}^{0}\left[f_1 + n\nabla f_1 + \frac{1}{2}(n^2 + n)\nabla^2 f_1 + \frac{1}{6}(n^3 + 3n^2 + 2n)\nabla^3 f_1 + \ldots..\right]dn$$

Neglecting 4th and higher order, we get and put $\nabla = 1 - E^{-1}$

$$y_1 = y_0 + h\left[f_1 - \frac{1}{2}(1 - E^{-1})f_1 - \frac{1}{2}(1 - 2E^{-1} + E2) - \frac{1}{24}(1 - 3E^{-1} + 3E^{-2} - E^{-3})\right]$$

$$y_1 = y_0 + \frac{h}{24}\left[24f_1 - 12(f_1 - f_0) - 2(f_1 - 2f_0 + f_1) - 1(f_1 - 3f_0 + 3f_1 - f_2)\right]$$

$$y_1 = y_0 + \frac{h}{24}\left[9f_1 + 19f_0 - 5f_1 - f_2\right]$$

$$y_1 = y_0 + \frac{h}{24}\left[9y_1^1 + 19y_0^1 - 5y_1^1 - y_{-2}^1\right]$$

$$(y_n)_C = y_3 + \frac{h}{24}\left[9y_4^1 + 19y_3^1 - 5y_2^1 + y_1^1\right]$$

In general

$$(y_{n+1})_C = y_n + \frac{h}{24}\left[9y_{n+1}^1 + 19y_n^1 - 5y_{n-1}^1 + y_{n-2}^1\right]$$

This is known as Adams – Moultan correction formula.

Ex. :

Using Adam – Baeshforth Moultan method

$$\frac{dy}{dx} = \frac{2y}{x}$$

with initial $y(1) = 2$ estimate $y(2)$ assuming $h = 0.25$.

Sol[n]. :

Given differential equations

$$\frac{dy}{dx} = \frac{2y}{x}$$

$$\therefore f(x, y) = {}^{2y}\!/_{x}$$

with initial condition $y(1) = 2$

$$\therefore x_0 = 1; \qquad y_0 = 2$$

By using R-K method of 4th order we get,

$$x_0 = 1, \quad x_1 = 1.25, \quad x_2 = 1.5, \quad x_3 = 1.75$$

$$y_0 = 2, \quad y_1 = 3.13, \quad y_2 = 4.5, \quad y_3 = 6.13$$

$$y_0^1 = f(x_0, y_0) = \frac{2 \times 2}{1} = 4$$

$$y_1^1 = f(x_1, y_1) = f(1.25, 3.13) = \frac{2(3.13)}{1.25} = 5.008$$

$$y_2^1 = f(x_2, y_2) = f(1.5, 4.5) = \frac{2 \times 4.5}{1.5} = 6$$

$$y_3^1 = f(x_3, y_3) = f(1.75, 6.13) = \frac{2 \times 6.13}{1.75} = 7.0057$$

By Adams – Bushforth predication formula

$$(y_4)_p = y_3 + \frac{h}{24}\left[55y_3^1 - 59y_2^1 + 37y_1^1 - 9y_0^1\right]$$

$$(y_4)_p = 6.13 + \frac{0.25}{24}\left[55 \times 1.0057 - 59(6) + 37(5.008) - 9(4)\right]$$

$$= 8.0113.$$

$$y_4^1 = f(x_4, y_4) = f(2, 8.0113) = \frac{2 \times 8.0113}{2} = 8.0113$$

Adoms – Bashfourth correction formula

$$(y_4)_C = y_3 + \frac{h}{24}\left[9y_4^1 + 19y_3^1 - 5y_2^1 + y_1^1\right]$$

$$(y_4)_C = 6.13 + \frac{0.25}{24}\left[9(8.0113) + 9(7.0057) - 5(6) + (5.008)\right]$$
$$= 8.0073.$$

Hence, $y(2) = 8.0073$.

6.7 Accuracy of Multi-step Method :

We know that for each differential equation there is an optimum steps size h. if his also large, accuracy diminished and if it too small round off error would dominate and reduce accuracy.

By computing the predicated used corrected values of y_{i+1}, we can estimates the size and sign of error.

Let's denote the predicated value $(y_{n+1})_p$ and corrected value $(y_{n+1})_c$. Similarly denote the truncation error in predicated value by E_{tp} and corrected value by E_{tc}.

$$\therefore E_{tp} = y - (y_{n+1})_p$$
$$E_{tc} = y - (y_{n+1})_c$$

Where y denotes the exact value of y of x_{n+1} then differences between the error is

$$E_{tp} - E_{tc} = (y_{n+1})_c - (y_{n+1})_p$$

A large difference indicates that step size is too large. In such cases we must reduce the size of h.

Both the Milne and Simpson formula are of order 4^4 and their error terms are of order h^5.

The truncation error in Mile's formula is

$$E_{tp} = \left(\frac{28}{90}\right) y^5(\theta_1)\, h^5$$

The truncation error in Simpson's formula's

$$E_{tc} = \frac{1}{10}(y^5)(\theta_2)\, h^5$$

Let's assume that

$$(\theta_1)(y^5) = (y^5)(\theta_2)$$
$$\frac{E_{tp}}{E_{tc}} = -28$$
$$E_{tp} = -28\, E_{tc}$$

$$\therefore E_{tp} - E_{tc} = \left(y_{n+1}\right)_c - \left(y_{n+1}\right)_p \text{ becomes}$$

$$-E_{tc} - 28E_{tc} = \left(y_{n+1}\right)_c - \left(y_{n+1}\right)_p$$

$$-29E_{tc} = \left(y_{n+1}\right)_c - \left(y_{n+1}\right)_p$$

$$E_{tc} = \frac{-\left(y_{n+1}\right)_c - \left(y_{n+1}\right)_p}{29}$$

If the answer is required to a precision of a decimal digits then

$$\left|E_{tc}\right| = \left|\frac{\left(y_{n+1}\right)_c - \left(y_{n+1}\right)_p}{29}\right| < 0.5 \times 10^{-d}$$

$$\therefore \left(y_{n+1}\right)_c - \left(y_{n+1}\right)_p < 29 \times 0.5 \times 10^{-d} \approx 15 \times 10^{-d}$$

Similarly for Adams – Bashfourth method we get,

$$\left(y_{n+1}\right)_c - \left(y_{n+1}\right)_p < \frac{270}{19} \times 0.5 \times 10^{-d} \approx 7 \times 10^{-d}$$

6.7 Stability :

Stability of a numerical method ensures that small changes in the initial conditions should not lead to large changes in the solution. This is particularly important as the initial conditions. May not be given exactly. The approximate solution computed with error in initial conditions is further used as the initial condition for computing solution at the next grid point. This accounts for large deviation in the solution started with small initial errors also round off error's in computations may also affect the accuracy of the solutions at a grid point.

Euler method is found to stable :

Stability is the necessary and sufficient condition for convergence.

6.7.1 Stability of Numerical Solutions :

Consider a differential equation

$$\frac{dy}{dx} = f\left(x, y\right)$$

with initial condition $y\left(x_0\right) = y_0$.

The solution with nearby initial values are close $+0^{y(x)}$ is called the stable. If the solution with near by initial values diverse from $y(x)$ then the solution is unstable.

For e.g. $\frac{dy}{dx} = y - 1$ with initial condition $y(0) = 1$

Exact solution of given DE is

$$\int \frac{dy}{y-1} = \int dx$$

$$\log(y-1) = x + \log c$$

$$y-1 = 1 + ce^{x}$$

By initial condition $y(0) = 1$

$$y(0) = 1 + ce^{0}$$

$$\boxed{c = 0}$$

$$\therefore y(x) = 1$$

Which is exact solution of given differential equation if $y^{1}(0) = 1.0001$ then find for the same $y(0) = 1.0001$.

$$y(0) = 1 + ce^{0}$$

$$1.0001 = 1 + e^{0}c$$

$$c = 0.0001 + 0$$

$$y(x) = 1.0001.$$

The exact solution of this differential equation is $y(x) = 1$ where $c = 0$.

However if we use other initial values $c \pm 0$ and the solution with diverge from the $y(x) = 1$.

It is difficult to obtain accurate numerical solution to an unstable initial value problem. If a small numerical error occur solution diverge from the two solution.

6.8 Model Differential Equation :

Consider the model first order differential equation

$$\frac{dy}{dx} = \lambda y \qquad \text{..........} \text{(I)}$$

with initial condition $y(0) = y_0$ where λ is constant and it may be oral as complex number

$$\frac{dy}{dx} = \lambda y$$

$$\therefore \frac{dy}{y} = \lambda dx$$

Integrating both sides

$$\int \frac{dy}{y} = \int \lambda dx$$

$$\log y = \lambda x + \log c$$

$$y(x) = ce^{\lambda x} \qquad \text{……….} \text{ (II)}$$

To find 'c' use $y(0) = 10$

$$\therefore y(0) = ce^{0}$$

$$y_0 = c\,.$$

Put $y_0 = c$ in equation (II) we get,

$$y(x) = y_0 e^{\lambda x}$$

Which is exact solution of differential equation.

$\therefore$ If $\lambda \leq 0$, a small change in the initial condition causes only small change in the solution and therefore the problem is a stable problem.

$\therefore$ If $\lambda > 0$ large changes in solutions will occurs and the problem is unstable.

6.9 Model Difference Problem :

Consider the model difference problem

$$y_{n+1} = k\, y_n, \qquad \text{n = 1, 2, 3, ...}$$

Where the initial value y_0 is given as 6 is complex number

$$\therefore Ey_n - k\, y_n = 0$$

$$(E - k) y_n = 0$$

Auxiliary equation is

$$E - k = 0$$

$$\therefore E = k$$

$$C \bullet F = Ak^n \text{ and } P \bullet F = 0$$

The complete solution is

$$y_n = C \bullet F + P \bullet F$$

$$y_n = Ak^n \qquad \text{……….} \text{ (I)}$$

To find A put $A = 0$

$$y_0 = Ak^0$$

$$y_0 = A$$

Put this value in equation I

$$y_n = \sigma^n y_0 .$$

The solution is bounded if $|6| \le 1$ which is the solution of difference equation.

The connection between the exact solution and the difference solution is clear if we evaluate the exact solution at the points $x_n = nh$ where $n = 1, 2, 3, ...$ and $h > 0$.

$$y = y_0 e^{\lambda x}$$

$$y(x_n) = e^{\lambda x_n} y_0$$

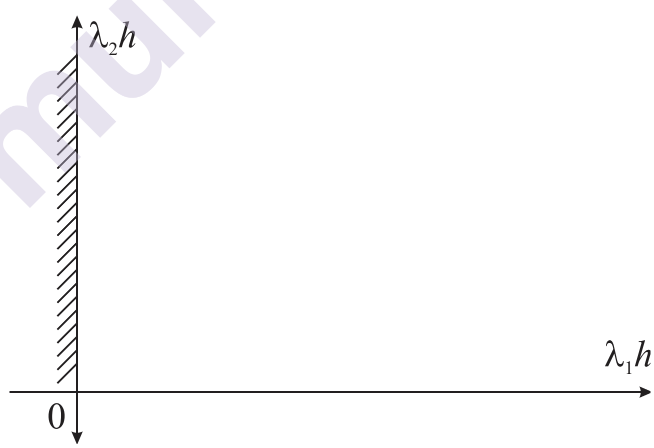
$$y_n = \left(e^{\lambda h}\right)^n y_0$$

$$y_n = k^n y_0$$

$$\boxed{\because k = e^{\lambda h}}$$

If exact solution is bounded then $|\sigma| = \left|e^{\lambda h}\right| \le 1$. this is possible if $\text{Re}(\lambda h) = \lambda_1 h \le 0$.

i.e. in the $\lambda_1 h - \lambda_2 h$ plane the region of stability of the exact solution is the left half plane as shown in fig.



Stability region of exact solution.

A single step method is called

i) Also olutely stable if $|k| \le 1$

ii) Relatively stable if $|k| \le e^{\lambda h}$

iii) Periodically stable if $|k| \le 1$ and λ is purely imaginary.

6.10 Stability of Euler Method :

Given differential equation is

$$\frac{dy}{dx} = \lambda y$$

with initial condition $y(0) = y_0$

$$\therefore f(x, y) = \lambda y$$

By Euler's method state that

$$y_{n+1} = y_n + hf(x_n, y_n)$$
$$= y_n + h\lambda y_n$$
$$= y_n (1 + hd)$$
$$y_{n+1} = ky_n$$

Where $k = 1 + \lambda h$.

$\therefore$ The solution of this differential equation is $y_n = k^n y_0$.

Since the exact problem has an exponentially decaying solution for $\lambda < 0$, a stable numerical method should exhibit the same behavior.

$\therefore$ In order to ensure stability of Euler's method we need that the so called growth factor $\|\lambda h\| < 1$

$\therefore y_n \to 0$ as $n \to \infty$ if $|k| = |1 + \lambda h| < 1$

Here we discuss the following cases.

Case I :

If λ is real and $\lambda \le 0$ then

$$|\sigma| = |1 + \lambda h| < 1$$
$$\Rightarrow -1 < 1 + \lambda h < 1$$
$$\Rightarrow -2 < \lambda h < 0$$
$$\therefore n < {}^{-2}\!/\!{}_{\lambda}.$$

Thus Euler's method is only conditionally stable. i.e. the step size has to be choosen sufficiently small to ensure stability. The set of λh function the growth factor is less than are is called the linear stability domain D.

Case II :

λ is purely imaginary

$$|k| = |1 + \lambda h| = |1 + i\lambda_2 h| = \sqrt{1 + (\lambda_2 h)^2} > 1$$

Euler method's unstable where λ is ure imaginary.

Case III :

λ is a complex number

Let $z = \lambda \in c$

$\therefore |1 + x| < 1$

$\Rightarrow \left|1 + (\lambda_1 + i\lambda_2)\ h\right| < 1$

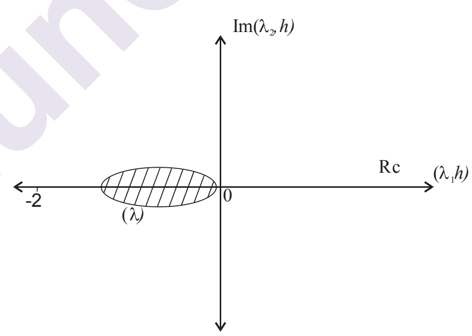
$\Rightarrow \left|1 + (\lambda_1 h + i\lambda_2 h)\right| < 1$

$\sqrt{(1 + \lambda_1 h)^2\ (\lambda_2 h)^2 < 1}$

$(\lambda_1 h + 1)^2 + (\lambda_2 h)^2 < 1$

A rather small circular subset of the left half of the complex plane.

Diagram



Stability region of Euler method.

where $\lambda = -1$

$|1 - \lambda| < 1$

$\therefore -1 < 1 - h < 1$

$-2 < h < 0$

$\therefore 0 < h < 2$

$\therefore$ Euler method is stable inside the circle.

6.11 Review

In this chapter we have learn

* Simultaneous first under differential equation solution by numerical method.
* Solution of second order differential equations by numerical method.
* Multi-step method : (Predication – correction method)

i) Milne – Simpson Method
ii) Adam – Bashfourth Maulton Method

* Accuracy of multi-step method.
* Stability.
*

6.12 Unit End Exercise :

1) Using Taylor's series method evaluate $x(0.2)$ and $y(0.2)$ given that $\frac{dx}{dt} - y = e^t \frac{dy}{dt} + x = \sin t$ with $x(0) = 1$ and $y(0) = 0$.

2) Using Taylor's series method compute $x(0.1)$ and $y(0.1)$ correct upto h-decimal places given that $\frac{dx}{dt} = y - t$ and $\frac{dy}{dt} = x + t$ with $x(0) = 1$ and $y(0) = 1$.

3) Using R-K method of 4th order find the approximate value of x and y at t = 0.1 the following system $\frac{dx}{dt} = 2x + y, \frac{dy}{dt} = x - 3y$ with $x_0 = 1, y_0 = 0$.

4) Using R-K method solve the differential equation $\frac{d^2x}{dt^2} = x + \frac{tdx}{dt}$ with $x(0) = 1, x^1(0) = 0$ taking $h = 0.1$ to find $x(0.2)$ and $x^1(0.2)$.

5) Use Taylor's series method to find $x(0.2)$ and $x^1(0.2)$ given that $\frac{d^2x}{dt^2} = t\left(\frac{dx}{dt}\right)^2 - x^2$ with $x(0) = 1$ and $x^1(0) = 0$ with $h = 0.2$.

6) Use Taylor series method to find the value of y at t = 0.1 correct upto decimal place if y satisfies the equations $\frac{d^2y}{dt^2} = ty$ given that $\frac{dy}{dt} = 1,\ y = 1$ when $t = 2$ with $h = 0.1$.

7) Given that $\frac{dy}{dx} = x^2 + y^2 - 2$ with $y(\infty) = 1.09,\ y(0.1) = 1,\ y(0.2) = 0.89$ and $y(0.3) = 0.7605$ use Milne's Simpson's method to determine $y(0.4)$ correct to 4 decimal places.

8) Given $\frac{dy}{dx} = x - y^2$ with $y(0) = 0$ evaluate $y(0.8)$ using Milne – Simpson's method obtain the starting values from Euler's method.

9) Using Adam – Bashforth method determine $y(4.4)$ given that $5xy^1 + y^2 = 2$ with $y(4) = 1$
$y(4.1) = 1.0049,\ y(4.2) = 1.0097,\ y(4.3) = 1.0143$.

10) Given $\frac{dy}{dx} = y - x^2$ with
$y(0) = 1,\ y(0.2) = 1.2186,\ y(0.4) = 1.4681,\ y(0.6) = 1.7378$ compute $y(0.8)$ and $y(1.0)$ by Adam – Bashforth method correction upto 4th decimal places.

11) Determine Milne Simpson's method to solve ordinary differential equation with initial condition $y(x_0) = y_0$.

12) Determine Adams – Moultan correction formula solve 1st order differential equation with initial condition $y(x_0) = y_0$.

13) Determine Adams – Bashforth predication formula to solve 1st order differential equation with initial condition $y(x_0) = y_0$.

14) Determine accuracy of multi-step method of Milne's Simpson's method.

7

Numerical Solutions of Partial Differential Equations

Unit Structure

7.0 Objective :

After studying this chapter you will be able to :

* Solve partial differential equation by numerical method.
* Solve Laplace equation using numerical method.
* Sole heat equation of one dimension by numerical method.
* Solve wave equation of are dimension by numerical method.

7.1 Introduction :

Partial differential equations are used in a number of physical problems such as fluid flow heat transfer, solid mechanics and biological process. There are three types of equations. Hyperbolic equations are most commonly associated with advection, parabolic equations are most commonly associated with diffusion and elliptic equation are most commonly associated with steady states of either parabolic or hyperbolic parabolic problems classification of general linear partial differential equations.

General partial differential equation is of the form.

$$A(x, y)\frac{\partial^2 y}{\partial x^2} + B(x, y)\frac{\partial^2 y}{\partial x\,\partial y} + C(x, y)\frac{\partial^2 y}{\partial y^2} + D(x, y)\frac{\partial y}{\partial x}$$
$$+ E(x, y)\frac{dy}{dx} + F(x, y)u + G(x, y) = 0 \quad \ldots.\ (I)$$

Where U is or known function of x and y and A, B, C, D, E, F and G are also functions.

This equations is called

1) Elliptic : If $B^2 - 4AC < 0$

For eg. $\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0$ Laplace equations

$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = f(x, y)$ Poisson's equations

2) Parabolic : If $B^2 - 4AC = 0$

For eg. $\frac{\partial U}{\partial y} = C^2 \frac{\partial^2 U}{\partial x^2}$ are dimensional heat conduction equation.

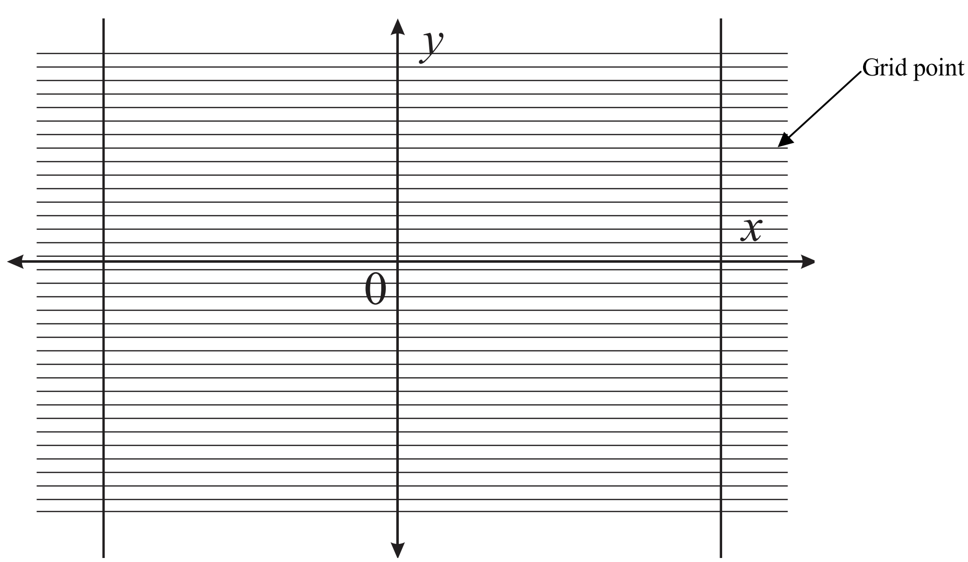
3) Hyperbolic : If $B^2 - 4AC > 0$

For eg. $\frac{\partial^2 U}{\partial y^2} = C^2 \frac{\partial^2 U}{\partial x^2}$ the wave equation.

7.2 Finite Difference Approximations to Derivatives :

Let the Parabolic : If (x, y) plane be divided into a network of rectangles of sizes $\Delta x = h$ and $\Delta y = k$ by drawing the sets of lines $x = i\,h,\ i = 0, 1, 2, \ldots..$

$y = jk \qquad j = 0, 1, 2, \ldots..$



The point of intersection of these families of lines are called mesh points, lattice points on grid points.

Let $\mathrm{U}(x, \mathrm{y})$ Bethe function of the variable x and y

$$\mathrm{U}_x = \frac{\mathrm{U}(x+h, \mathrm{y}) - \mathrm{U}(x, \mathrm{y}) + \mathrm{O}(h)}{h}$$

Expand $\mathrm{U}(x + h, y)$ in Taylor's series expansion

$$\mathrm{U}(x + h, y) = \mathrm{U}(x, y) + h\mathrm{U}_x + \frac{h^2}{2!}\mathrm{U}xx + ...$$

$$\therefore \frac{\mathrm{U}(x + h, y) - \mathrm{U}(x, y)}{h} = \mathrm{U}_x + \frac{h^2}{2!}\mathrm{U}xx + ...$$

$$\therefore \mathrm{U}_x = \frac{\mathrm{U}(x + h, y) - \mathrm{U}(x, y)}{h} + \mathrm{O}(h)$$

This is forward difference approximation for Ux. Similarly we have the approximation.

$$\mathrm{U}_x = \frac{\mathrm{U}(x, y) - \mathrm{U}(x-h, y) + \mathrm{O}(h)}{h}$$

This is backward difference approximation for Ux.

$$\mathrm{U}_x = \frac{\mathrm{U}(x+h, y) - \mathrm{U}(x-h, y) + \mathrm{O}(h^2)}{2h}$$

This is central difference approximation for Ux.

Let $\mathrm{U}(x, y) = \mathrm{U}(ih, ij) = \mathrm{U}(i, j)$

$$\mathrm{U}x = \frac{\mathrm{U}(i+1, j) - \mathrm{U}(i, j) + \mathrm{O}(h)}{h} \text{ for forward}$$

$$\mathrm{U}x = \frac{\mathrm{U}(i, j) - \mathrm{U}(i-1, j) + \mathrm{O}(h)}{h} \text{ for backward}$$

$$\mathrm{U}x = \frac{\mathrm{U}(i+1, j) - \mathrm{U}(i-1, j) + \mathrm{O}(h^2)}{2h} \text{ for central}$$

$$\mathrm{U}xx = \frac{\mathrm{U}(i-1, j) - 2\mathrm{U}(i, j) + \mathrm{U}(i+1, j)}{h^2} + \mathrm{O}(h^2)$$

Similarly we have the approximation

$$\mathrm{U}y = \frac{\mathrm{U}(i, j+1) - \mathrm{U}(i, j)}{h} + \mathrm{O}(k) \text{ for forward}$$

$$\mathrm{U}y = \frac{\mathrm{U}(i, j) - \mathrm{U}(i, j-1)}{h} + \mathrm{O}(k) \text{ for backward}$$

$$\mathrm{U}y = \frac{\mathrm{U}(i, j+1) - \mathrm{U}(i, j-1)}{2k} + \mathrm{O}(k) \text{ for central}$$

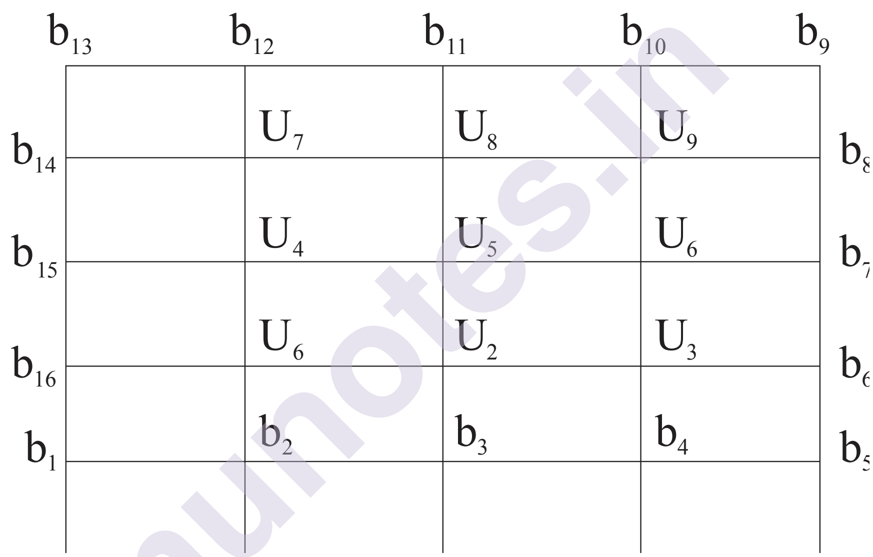
$$Uyy = \frac{U(i,\ j-1) - 2U(i,j) + U(i,\ j+1)}{k^2} + O(k^2)$$

7.3 Laplace Equation of Two Dimension :

Consider the Laplace equation of two diameter are given by,

$$Uxx + Uyy = 0 \qquad \text{..... (I)}$$

Consider a square region R for which $U(x,\ y)$ is known at the boundary and divide R into small squares of side h as shown in fig. where $b_1, b_2, \ldots, b_{16}$ are boundary values.



We know that

$$Uxx = U_{i-1,j} - 2U_{i,j} + U_{i+1,j}/h^2 \qquad \text{...... (II)}$$

$$Uyy = \frac{U_{i,j-1} - 2U_{i,j} + U_{i,j+1}}{k^2} \qquad \text{...... (III)}$$

Replace (II) and (III) in equation (I) we get,

$$\frac{U_{(i-1,j)} - 2U_{i,j} + U_{i+1,j}}{h^2} + \frac{U_{(i,j-1)} - 2U_{i,j} + U_{i,j+1}}{k} = 0 \qquad \text{..... (IV)}$$

For value of $h = k$ i.e. for square grid of the mesh size h equation can be written as

$$U_{i-1,j} - 2U_{i,j} + U_{i+1,j} + U_{i,j-1} - 2U_{i,j} + U_{i,j+1} = 0$$

$$\therefore U_{i,j} = \frac{1}{4}\left[U_{i-1,j} + U_{i+1,j} + U_{i,j-1} + U_{i,j+1}\right] \qquad \text{...... (V)}$$

These shows that the values of $U(x, y)$ is the average of its four neighbors to the East, West, North, South is called standard five points formula [S, F, P, F].

This formula is also known as Liebman's averaging procedure.

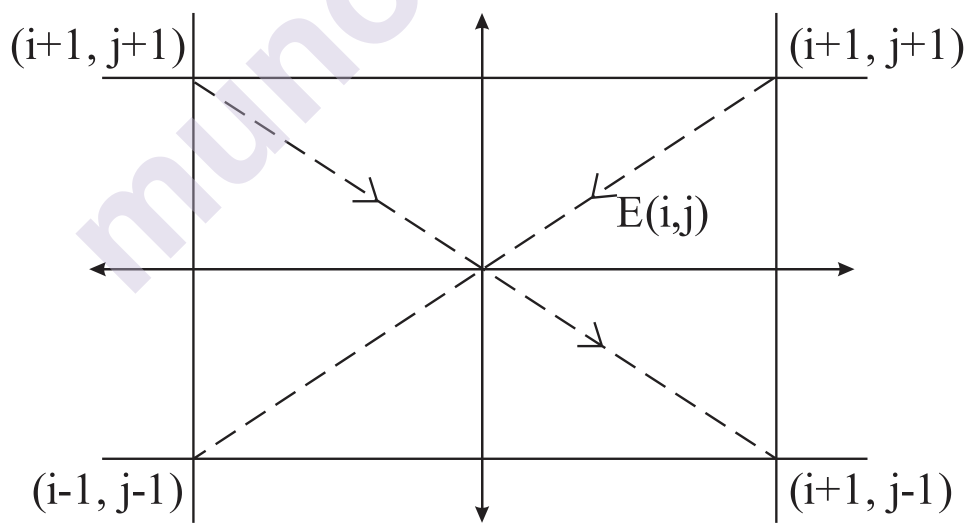
Note :

The Laplace equation remains unchanged when coordinate are rotated through 45°.

A formula similar to the (VI) is sometimes used with convenience it is given as

$$U_{ij} = \frac{1}{4}\left(U_{i+1,j+1} + U_{i+1,j-1} + U_{i-1,j+1} + U_{i-1,j-1}\right) \qquad \text{...... (VI)}$$

This is known as diagonal five point formula as these points Lies on the diagonals (DFPE). But it is less accurate than standard five point's formula.



We use the following five point formula to set the initial value of U at the centre.

$$U_5 = \frac{1}{4}\left(b_1 + b_5 + b_9 + b_{13}\right)$$

Then the approximate values of U_1, U_3, U_7, U_9 are calculated by the diagonal five point formula.

$$U_1 = \frac{1}{4}\left(b_1 + b_3 + b_5 + b_{15}\right)$$

$$U_3 = \frac{1}{4}\left(b_3 + b_5 + b_7 + U_5\right)$$

$$U_7 = \frac{1}{4}\left(b_{15} + U_5 + b_{11} + b_{13}\right)$$

$$U_9 = \frac{1}{4}\left(U_5 + b_7 + b_9 + b_{11}\right)$$

The values of the remaining interion point i.e. U_2, U_{41}, U_6 and U_8 are obtained by the standard five point formula.

$$U_2 = \frac{1}{4}\left(b_3 + U_3 + U_5 + U_1\right)$$

$$U_4 = \frac{1}{4}\left(U_1 + U_5 + U_7 + b_{15}\right)$$

$$U_6 = \frac{1}{4}\left(U_3 + b_7 + U_9 + U_5\right)$$

$$U_8 = \frac{1}{4}\left(U_5 + U_9 + b_{11} + U_7\right)$$

Thus we obtain all initial values $U_1, U_2, U_3.....U_g$ once their accuracy can be improved by the repeated application of either Jacobi iteration formula as Gauss – Seidel iteration formula.

7.4 Jacobi's Iteration Formula :

Let $U_{i,j}^{n}$ be the nth iterative value of $U_{i,j}$ then Jacobi's iterative procedure is given below.

$$U_{i,j}^{(n+1)} = \frac{1}{4}\left[U_{i-1,j}^{n} + U_{i+1,j}^{n} + U_{i,j-1}^{n} + U_{i,j+1}^{n}\right]$$

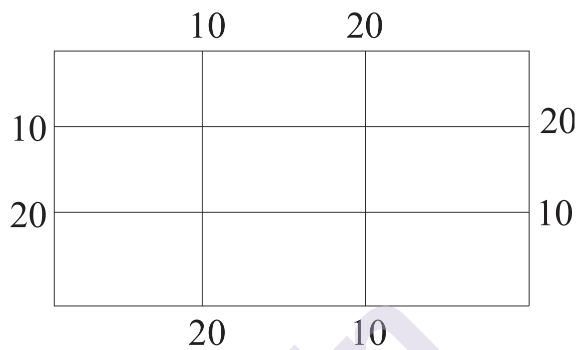
* **Gauss – Seidel Method :**

This method utilizes the latest iterative value available and scans the mesh points symmetrically from left to right along successive rows. The formula is given below.

$$U_{(i,j)}^{n+1} + \frac{1}{4}\left[U_{i-1,j}^{n+1} + U_{i+1,j}^{n} + U_{i,j-1}^{n-1} + U_{i,j+1}^{n}\right]$$

Ex. :

Solve Laplace equation $Uxx + Uyy = 0$ in the domain of the figure given below by Gauss – Seidel method.



Sol^n. :

$$U_1^{(n+1)} = \frac{1}{4}\left[10 + 10 + U_2^{n} + U_1^{(n)}\right]$$

$$U_2^{(n+1)} = \frac{1}{4}\left[20 + 20 + U_1^{(n+1)} + U_3^{(n)}\right]$$

$$U_3 = \frac{1}{4}\left[10 + 10 + U_1^{(n+1)} + U_3^{(n+1)}\right]$$

$$U_4^{(n+1)} = \frac{1}{4}\left[20 + 20 + U_1^{(n+1)} + U_3^{(n+1)}\right]$$

We use SFPF

$$U_{ij} = \frac{1}{4}\left[U_{i-1,j} + U_{i+1,j} + U_{i,j-1} + U_{i,j+1}\right]$$

and DFPF as

$$U_{ij} = \frac{1}{4}\left[U_{i-1,j-1} + U_{i-1,j+1} + U_{i-1,j-1} + U_{i+1,j+1}\right]$$

Now initially $U_1 = 0, U_2 = 0, U_3 = 0, U_4 = 0,$

First Iteration :

$$U_1^{(1)} = \frac{1}{4}\left[10 + 0 + 10 + 0\right] = 5$$

$$U_2^{(2)} = \frac{1}{4}\left[20 + 0 + 20 + 5\right] = 11.25$$

$$U_3^{(1)} = \frac{1}{4}\left[10 + 11.25 + 10 + 0\right] = 7.8125$$

$$U_4^{(1)} = \frac{1}{4}\left[20 + 7.8125 + 20 + 5\right] = 13.20$$

Second Iteration :

$$U_1^{(2)} = \frac{1}{4}\left[10 + 10 + 11.25 + 13.20\right] = 11.1125$$

$$U_2^{(2)} = \frac{1}{4}\left[20 + 20 + 11.1125 + 7.8125\right] = 14.73$$

$$U_3^{(2)} = \frac{1}{4}\left[10 + 10 + 14.73 + 13.20\right] = 11.98$$

$$U_4^{(2)} = \frac{1}{4}\left[20 + 20 + 11.1125 + 11.98\right] = 15.77$$

Third Iteration :

$$U_1^{(3)} = \frac{1}{4}\left[10 + 10 + 14.73 + 15.77\right] = 12.63$$

$$U_2^{(3)} = \frac{1}{4}\left[20 + 20 + 12.63 + 11.98\right] = 16.15$$

$$U_3^{(3)} = \frac{1}{4}\left[10 + 10 + 16.15 + 15.77\right] = 12.98$$

$$U_4^{(3)} = \frac{1}{4}\left[20 + 20 + 12.63 + 12.98\right] = 16.40$$

Fourth Iteration :

$$U_1^{(4)} = \frac{1}{4}\left[10 + 10 + 16.15 + 16.40\right] = 13.14$$

$$U_2^{(4)} = \frac{1}{4}\left[20 + 20 + 13.14 + 12.98\right] = 16.53$$

$$U_3^{(4)} = \frac{1}{4}\left[10 + 10 + 16.53 + 16.40\right] = 13.23$$

$$U_4^{(4)} = \frac{1}{4}\left[20 + 20 + 13.14 + 13.23\right] = 16.59$$

Fifth Iteration :

$$U_1^{(5)} = \frac{1}{4}\left[10 + 10 + 16.53 + 16.59\right] = 13.28$$

$$U_2^{(5)} = \frac{1}{4}\left[20 + 20 + 13.28 + 13.23\right] = 16.63$$

$$U_3^{(5)} = \frac{1}{4}\left[10 + 10 + 16.63 + 16.59\right] = 13.31$$

$$U_4^{(5)} = \frac{1}{4}\left[20 + 20 + 13.28 + 13.31\right] = 16.65$$

Sixth Iteration :

$$U_1^{(6)} = \frac{1}{4}\left[10 + 10 + 16.53 + 16.55\right] = 13.33$$

$$U_2^{(6)} = \frac{1}{4}\left[20 + 20 + 13.32 + 13.31\right] = 16.66$$

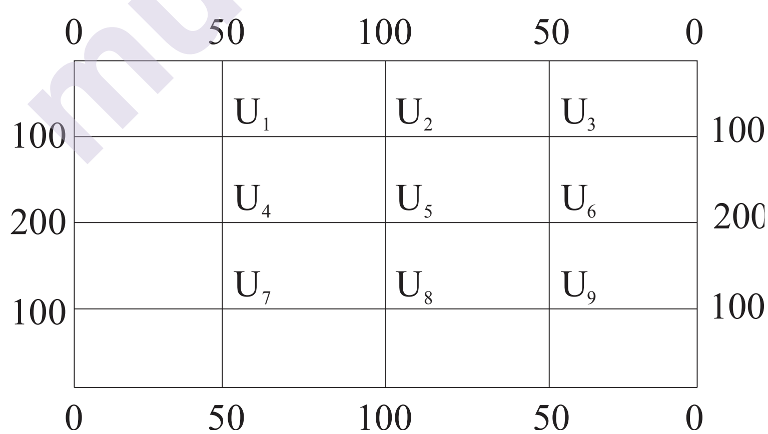
$$U_3^{(6)} = \frac{1}{4}\left[10 + 10 + 16.67 + 16.65\right] = 13.33$$

$$U_4^{(6)} = \frac{1}{4}\left[20 + 20 + 13.32 + 13.33\right] = 16.66$$

$\therefore U_1 = 13.33,\ U_2 = 16.66,\ U_3 = 13.33,\ U_4 = 16.66$.

Ex. 2 :

Solve $Uxx + Uyy = 0$ by h i.e. bman iteration process for the domain of the figure given below :



Sol^n. :

We use standard five point formula

$$U_{ij} = \frac{1}{4}\left[U_{i-1,j} + U_{i+1,j} + U_{i,j-1} + U_{i,j+1}\right]$$

and diagonal five point formula

$$U_{ij} = \frac{1}{4}\left[U_{i-1,j} + U_{i-1,j+1} + U_{i-1,j+1} + U_{i+1,j-1} + U_{i+1,j+1}\right]$$

Values given on the figure are symmetrical about middle line

$\therefore U_1 = U_2 = U_3 = U_7$ and $U_2 = U_8$ & $U_4 = U_6$.

$$U_5 = \frac{1}{4}\left[200 + 100 + 200 + 100\right] = 150 \qquad \text{[Standard formula]}$$

$$U_4 = \frac{1}{4}\left[0 + 200 + 150 + 100\right] = 112.5 \qquad \text{[Diagonal formula]}$$

$\therefore$ Similarly $U_1 = U_3 = U_5 = U_7 = 112.5$

$$U_2 = \frac{1}{4}\left[100 + 112.5 + 150 + 122.5\right]$$
$$= 118.75.$$

$$U_8 = U_2 = 118.75$$

$$U_4 = \frac{1}{4}\left[112.5 + 200 + 112.5 + 150\right]$$
$$= 143.75.$$

$$U_4 = U_6 = 143.75$$

$\therefore U_1 = 112.5,\ U_2 = 118.75,\ U_3 = 112.5,\ U_4 = 143.75,\ U_5 = 150,$
$U_6 = 143.75,\ U_7 = 112.5,\ U_8 = 118.75,\ U_9 = 112.5.$

Now by Gauss – Seidel Method :

$$U_{ij}^{n+1} = \frac{1}{4}\left[U_{i-1,j}^{(n+1)} + U_{i+1,j}^{n} + U_{i,j-1}^{(n+1)} + U_{i,j+1}^{n}\right]$$

$$U_1^{(n+1)} = \frac{1}{4}\left[100 + U_2^{(n)} + 50 + U_4^{n}\right]$$
$$= U_3^{(n+1)} = U_5^{(n+1)} = U_7^{(n+1)}$$

$$U_2^{(n+1)} = \frac{1}{4}\left[U_1^{n+1} + U_3^{(n+1)} + 100 + U_5^{(n+1)}\right]$$
$$= U_8^{(n+1)}$$

$$U_4^{(n+1)} = \frac{1}{4}\left[200 + U_5^{n} + U_1^{(n+1)} + U_7^{(n+1)}\right]$$

$$= U_6^{(n+1)}$$

$$U_5^{(n+1)} = \frac{1}{4}\left[U_4^{(n+1)} + U_6^{(n+1)} + U_2^{(n+1)} + U_8^{(n+1)}\right]$$

First Iteration we get,

$$U_1^{(1)} = \frac{1}{4}[100 + 118.75 + 50 + 143.75]$$

$$= 103.125.$$

$$U_2^{(1)} = \frac{1}{4}[103.125 + 103.125 + 100 + 150]$$

$$= 114.06.$$

$$\therefore U_2^{(1)} = U_8^{(1)} = 114.06$$

$$U_4^{(1)} = \frac{1}{4}[200 + 150 + 103.125 + 103.125]$$

$$= 139.06.$$

$$\therefore U_4^{(1)} = U_6^{(1)} = 139.06.$$

$$U_5^{(1)} = \frac{1}{4}[139.06 + 139.06 + 114.06 + 114.06]$$

$$= 126.56.$$

Similarly the Liebman's iteration are given by,

Iteration	$U_1 = U_3 = U_9 = U_7$	$U_2 = U_8$	$U_4 = U_6$	U_5
2nd	100.8	106.9	132.1	119.5
3rd	97.3	103.5	128.8	116.2
4th	95.6	101.9	126.9	114.4
5th	94.7	101.0	726	135.5
6th	94.2	100.5	125.5	113
7th	94	100.3	125.3	112.8
8th	93.9	100.2	125.2	112.7
9th	93.9	100.1	125.1	112.6

$$\therefore U_1 = U_3 = U_7 = U_9 = 93.9$$

$$U_2 = U_8 = 100$$

$$U_4 = U_6 = 125.1$$

$$U_5 = 112.6.$$

7.5 One Dimensional Heat Equation (Parabolic Equation) :

Consider a one dimensional heat equation

$$\frac{\partial^2 y}{\partial x^2} = \alpha \frac{\partial U}{\partial t} \qquad \text{...... (I)}$$

with initial condition $\mathrm{U}(x, 0) = f(x)$ and boundary conditions $\mathrm{U}(0, t) = T_0$ and $\mathrm{U}(1, t) = T_1$.

Divide xt – plane into small rectangles of size h and k in x and t directions respectively

Let $\mathrm{U}(x, t) = \mathrm{U}(ih, jk)$ where $i, j = 0, t1, t2, ...$

We write partial derivative in equation (I) we get,

$$\frac{\partial^2 y}{\partial x^2} = \frac{U_{i-1,j} - 2U_{ij} + U_{i+1,j}}{h^2}$$

and $$\frac{\partial U}{\partial t} = \frac{U_{i,j+1} - U_{ij}}{k}$$

$\therefore$ From equation (I)

$$\alpha \frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}$$

$$\alpha \left(\frac{\mathrm{U}_{ij+1} - \mathrm{U}_{ij}}{k} \right) = \frac{\mathrm{U}_{i-1,j} - 2\mathrm{U}_{i,j} + \mathrm{U}_{i+1,j}}{h^2}$$

$$\left(\mathrm{U}_{i,j+1} - \mathrm{U}_{ij} \right) = \frac{k}{\alpha h^2} \left(\mathrm{U}_{i-1,j} - 2\mathrm{U}_{i,j} + \mathrm{U}_{i+1,j} \right)$$

Put $\lambda = k / \alpha h^2$ ..... (II)

We get,

$$\mathrm{U}_{i,j+1} - \mathrm{U}_{i,\mathrm{j}} = \lambda \left(\mathrm{U}_{i-1,\mathrm{j}} - 2\mathrm{U}_{i,j} + \mathrm{U}_{i+1,j} \right)$$

$$\mathrm{U}_{i,j+1} = \lambda \mathrm{U}_{i+1,j} + (1 - 2\lambda) \mathrm{U}_{i,\mathrm{j}} + \lambda \mathrm{U}_{i+1,j} \qquad \text{..... (III)}$$

It gives formula for unknown temperature $\mathrm{U}_{i,j+1}$ at $(i, j+1)$ when reaming values are known. Hence the method is called explicit method. These method is valid $\mathrm{U} < \lambda \leq 1/2$ choose k in such a way that co-efficient of $\mathrm{U}_{i,j}$ in equation (III) will become zero.

i.e. $1 - 2\lambda = 0 \Rightarrow \lambda = 1/2$

put $\lambda = 1/2$ in equation (III) we get,

$$\frac{1}{2} = \frac{k}{\alpha h^2}$$

$$k = \frac{\alpha h^2}{2}$$

$\therefore$ Equation (III) reduce to the form

$$U_{i,j+1} = \frac{1}{2} U_{i-1,j} + U_{i+1,j} \qquad \ldots\ldots\ldots\ldots$$

(IV)

This is called the Bendre Schmidt or Schmidt recurrence relation.

$$U_{i,j+1} = \frac{1}{2} U_{i-1,j} + U_{i+1,j}$$

Given the values of U at the interval points when the boundary conditions are known.

7.6 Crank – Nicholson Difference Method :

Crank – Nicholson proposed a method in which $\frac{\partial^2 y}{\partial x^2}$ is replaced by the average of its finite difference approximation on the j^{th} and $(j+1)^{th}$ rows thus.

$$\frac{\partial^2 y}{\partial x^2} + \frac{1}{2}\left[\frac{U_{i-1,j} + 2U_{i,j} + U_{i+1,j}}{h^2} + \frac{U_{i-1}, U_{j-1} - 2U_{i,j+1} + U_{i+1,j+1}}{h^2}\right]$$

$\therefore$ heat equation

$$\alpha \frac{\partial U}{\partial t} + \frac{\partial^2 y}{\partial x^2} \text{ can be written as}$$

$$\alpha\left[\frac{U_{i,j+1} + U_{i,j}}{k}\right] = \frac{1}{2}\left[\frac{U_{i-1,j} - 2U_{i,j} + U_{i+1,j}}{h^2} = \frac{U_{i-1,j+1}, 2U_{i,j+1} + U_{i+1,j+1}}{h^2}\right]$$

$$U_{i,j+1} + U_{i,j} = \frac{k}{2\alpha h^2}\left[\frac{U_{i-1,j} - 2U_{i,j} + U_{i+1,j}}{h^2} - \frac{U_{i-1,j-1}, 2U_{i,j+1} + U_{i+1,j+1}}{h^2}\right]$$

Put $\lambda = \frac{k}{\alpha h^2}$

$$2\left[U_{i,j+1} + U_{i,j}\right] = \lambda\left[U_{i-1,j} - 2U_{i,j} + U_{i+1,j} - U_{i-1,j-1}, 2U_{i,j+1} + U_{i+1,j+1}\right]$$

$$\therefore 2(1+\lambda) U_{i,j+1} - \lambda\left[U_{i-1,j+1} + U_{i+1,j+1}\right] = 2(1-\lambda) U_{i,j} + \lambda\left[U_{i+1,j} + U_{i+1,j}\right] \qquad \ldots\ldots \text{(I)}$$

This is known as "Crank – Nicolson" Difference formula.
This formula is convergent for all values of λ.

To choose the value of k in such away that the co-efficient of $U_{i,j}$ in equation (I) will become zero i.e. $\lambda = 1$.

$$\therefore 1 = \frac{k}{\alpha h^2}$$

$$\therefore k = \alpha h^2$$

For $k = \alpha h^2$ equation (I) become reduce to

$$U_{i,j+1} = \frac{1}{4}\left[U_{i-1,j} + U_{i+1,j} + U_{i-1,j+1} + U_{i+1,j+1}\right].$$

Ex. :

Solve one dimensional heat equation $\frac{2\partial^2 U}{\partial x^2} = \frac{\partial U}{\partial t}$

When $0 < t < 1.5$ and $0 < x < h$ with initial condition $U(x, 0) = 50(h - x)$ $0 \le x \le h$ and the boundary conditions.

$$U(0, t) = 0 \qquad 0 \le t \le 1.5$$

$$U(h, t) = 0 \qquad 0 \le t \le 1.5$$

Using Schmidt method.

Sol^n. :

Given equation

$$\frac{2\partial^2 U}{\partial x^2} = \frac{\partial U}{\partial t}$$

with heat equation we get,

$$\alpha = 2, \qquad h = 1, \qquad k = 1/2$$

$$t = \frac{h^2}{2k} = \frac{(1)^2}{2 \times 2} = 0.25$$

The bender Schmidt equation

$$U_{i,j+1} = \frac{1}{2}\left[U_{i-1,j} + U_{i+1,j}\right]$$

$$U(x, 0) = 50(h - x) \qquad t = 0.25, \qquad \Delta x = t = 1$$

Where $i = 0, 1, 2, 3, 4$ and $j = 0.25, 0.50, 0.75, 1.00, 1.25, 1.5$.

$$U(x, 0) = 50(h - x) \Rightarrow U_{i,0} = 50(h - i)$$

$$U(0, 0) = 0 \quad U(1, 0) = 50(4 - 1) = 150$$
$$U(2, 0) = 50(4 - 1) = 100$$

$$U(3, 0) = 50(4 - 3) \quad U(4, 0) = 50(4 - 4) = 0$$

Thus we can generate successfully $U(x, t)$

j	0	1	2	3	4
0.00	0	150	100	50	0
0.25	0	50	100	50	0
0.50	0	50	50	50	0
0.75	0	25	25	25	0
0.10	0	12.5	25	12.5	0
1.25	0	12.5	12.5	12.5	0
1.3	0	6.25	12.5	6.25	0

$\therefore U_{1,1.5} = 6.25, U_{2,1.5} = 12.5, U_{3,1.5} = 6.25$.

7.7 Given One Dimensional Heat Equation $\frac{\partial^2 U}{\partial x^2} = \frac{\partial U}{\partial t}$ under the conditions.

$U(x, 0) = 20$ for $0 \le x \le 3$ and $U(0, t) = 0$

$U(3, t) = 30$ for $t > 0$ taking $h = 1, k = 1$ and using Crank – Nicolson method to compute 'U' for one formed step on.

Sol^n.:

Crank – Nicolson difference formula is given by

$$2(1 + \lambda) U_{i,j+1} - \lambda\left[U_{i-1,j+1} + U_{i+1,j+1}\right] = 2(1 + \lambda) U_{i,j} + \lambda\left[U_{i-1,j} + U_{i+1,j}\right]$$
………. (I)

Given that $\alpha = 1, h = 1, k = 1$

$$\lambda = \frac{k}{\alpha h^2} = \frac{1}{1\lambda(1)^2} = 1$$

Put $\lambda = 1$ in equation (I) we get,

$$4\,U_{i,j+1} - \left[U_{i-1,j+1} + U_{i+,j+1}\right] = \left[U_{i-1,j} + U_{i+1,j}\right]$$

$$\therefore U_{i,j+1} = \frac{1}{4}\left[U_{i-1,j} + U_{i+1,j} + U_{i-1,j+1} + U_{i+1,j+1}\right]$$

………. (II)

$U(x, 0) = 20 \Rightarrow U(ih, 0) = 20 \Rightarrow U_{i,0} = 20 \Rightarrow i = 0, 1, 2, 3$

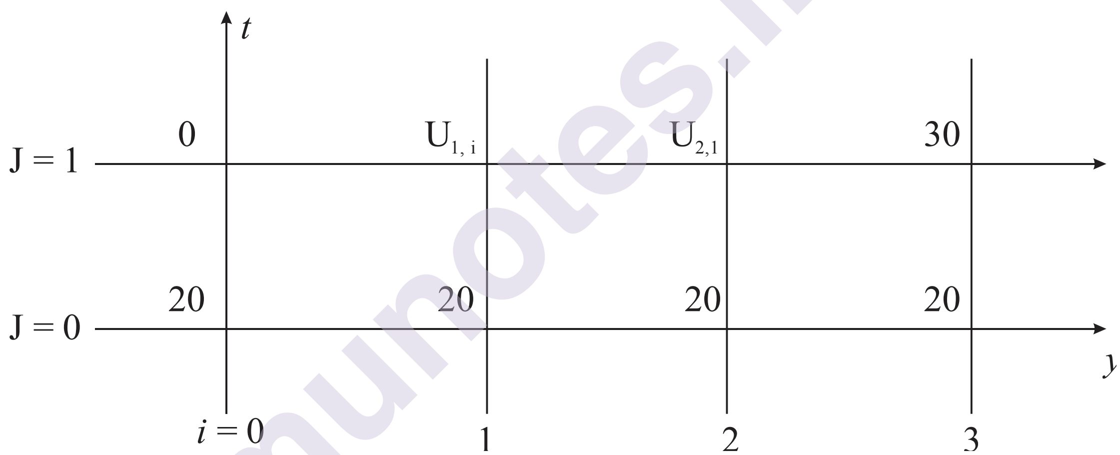
$U_{0,0} = 20, U_{1,0} = 20, U_{2,0} = 20, U_{3,0} = 20$

$U(0, t) = 0 \Rightarrow U(0, jk) = 0 \Rightarrow U_{0,j} = 0$

$\therefore U_{0,1} = 0$.

$U(3, t) = 30 \Rightarrow U(3h, 3k) = 30$

$U_{(3,1)} = 30 \Rightarrow U_{3,1} = 30$.



The mesh point U_{11} and U_{21} we need to find using equation (II)

$$\therefore U_{1,1} = \frac{1}{4}\left[U_{0,0} + U_{2,0.5} + U_{0,1} + U_{2,1}\right]$$

$$U_{1,1} = \frac{1}{4}\left[20 + 20 + 0 + U_{2,1}\right]$$

$$4\left(U_{1,1}\right) = 40 + U_{2,1}$$

$$4\left(U_{1,1}\right) - U_{2,1} = 40 \qquad \text{.......... (III)}$$

For $U_{2,1}$

$$U_{2,1} = \frac{1}{4}\left[U_{1,0} + U_{3,0} + U_{1,1} + U_{3,1}\right]$$

$$U_{2,1} = \frac{1}{4}\left[20 + 20 + U_{1,1} + 30\right]$$

$$U_{2,1} = 70 + U_{1,1}$$

$$U_{2,1} - U_{1,1} = 70 \qquad \text{.......... (IV)}$$

Solving equation (III) and (IV) we get,

$$U_{1,1} = 15.33$$

$$U_{2,1} = 21.33 .$$

7.8 One Dimensional Wave Equation (Hyperbolic Equation) :

Consider a one dimensional wave equation $\frac{\partial^2 \partial^2 U}{\partial x^2} = \frac{\partial^2 U}{\partial t^2}$ under the boundary condition $U(0, t) = U, U(1, t) = 0$ and initial conditions $U(x, 0) = f(x), U_t(x, 0) = g(x)$.

Divide *xt* – plane with small rectangles of sides *h* and *k* in *x* and *t* directions resp.

Let $U(x, t) = U(ih, ik)$ where $i, j = 0 + 1 + 2$. First we unite initial and boundary conditions in difference notations.

$$U(0, t) = 0 \Rightarrow U(0, jk) = 0 \Rightarrow U_{0,j} = 0$$

$$U(\lambda, t) = 0 \Rightarrow U(nk, jk) = 0 \text{ where } \ell = nh$$

$$\therefore U_{n,i,j} = 0$$

$\therefore \mathrm{U}(x, 0) = f(x) \Rightarrow \mathrm{U}(ih, 0) = f(ih)$

$\mathrm{U}_{i,0} = f(ih)$

$\mathrm{U}_t(x, 0) = g(x) \Rightarrow \dfrac{\mathrm{U}_{i-1} - \mathrm{U}_{(i-1)}}{2k} = g(ih)$

$$\therefore \left[\frac{\partial \mathrm{U}}{\partial t}(x, t), \frac{\mathrm{U}_{i, j-1} - \mathrm{U}_{i, j+1}}{2k}\right]$$

Partial derivative of second order is given by

$$\frac{\partial^2 \mathrm{U}}{\partial t^2} = \mathrm{U}_{i-1, j} - 2\mathrm{U}_{ij} + \mathrm{U}_{i+1, j} / h^2$$

$$\frac{\partial^2 \mathrm{U}}{\partial t^2} = \mathrm{U}_{i, j-1} - 2\mathrm{U}_{i, j} + \mathrm{U}_{\mathrm{i}, j+1} / k^2$$

Put $\dfrac{\partial^2 \mathrm{U}}{\partial x^2}$ and $\dfrac{\partial^2 \mathrm{U}}{\partial t^2}$ in equation (I) we get,

$$\alpha^2 \left[\frac{\mathrm{U}_{i-1, \mathrm{j}} - 2\mathrm{U}_{ij} + \mathrm{U}_{i+1, j}}{h^2}\right] = \left[\frac{\mathrm{U}_{i, j-1} - 2\mathrm{U}_{ij} + \mathrm{U}_{i, j+1}}{k^2}\right]$$

$$\therefore \left[\mathrm{U}_{i, \mathrm{j}-1} - 2\mathrm{U}_{ij} + \mathrm{U}_{i, j+1}\right] = \frac{\alpha^2 k^2}{h^2}\left[\mathrm{U}_{i-1, \mathrm{j}} - 2\mathrm{U}_{ij} + \mathrm{U}_{i+1, j}\right]$$

Put $\lambda = \dfrac{\alpha k}{h}$

$\therefore \mathrm{U}_{i, \mathrm{j}+1} = 2(1 - \lambda^2)\mathrm{U}_{i, j} + \lambda^2\left[\mathrm{U}_{i-1, j} + \mathrm{U}_{i+1, j} - \mathrm{U}_{i, j-1}\right]$ ……….
(III)

This scheme is called the explicit scheme for one dimensional wave equation.

Equation (III) is valid where $0 < \lambda \le 1$ and invalid where $\lambda > 1$.

In case of $\lambda = 1$ we get equation (III)

$$\mathrm{U}_{i, j+1} = \mathrm{U}_{i-1, j} + \mathrm{U}_{i+1, j} - \mathrm{U}_{i, j-1}$$

Ex. :

Solve $\mathrm{U}_{xx} = \mathrm{U}_{tt}$ with conditions

$\mathrm{U}(x, 0) = 0, \mathrm{U}(1, t) = 0, \mathrm{U}(x, 0) = \dfrac{x(1-x)}{2}$ and $\mathrm{U}(x, 0) = 0$ taking $h = 0.1$ and $k = 0.1$ for $0 \le t \le 0.2$.

Sol^n. :

Given equation $U_{xx} = U_{tt}$

i.e. $\frac{\partial^2 U}{\partial x^2} = \frac{\partial^2 U}{\partial t^2}$

$\therefore \alpha^2 = 1, \lambda = 2\frac{k}{h} = \frac{1 \times 0.1}{0.1} = 1$

Where $\lambda = 1$ solution of value equation is

$\therefore U_{i,j+1} = U_{i-1,j} + U_{i+1,j} - U_{i,j-1}$ .... (II)

$U(0, t) = U(1, t) = 0, U(o, j) = 0$ and $U(o, j) = 0$

$U(x, 0) = \frac{x(1-x)}{2}$

$U(i, 0) = \frac{i(1-i)}{2}$

$U(0.1, 0) = \frac{0.1(1-0.1)}{2} = 0.045$

$U(0.2, 0) = \frac{0.2(1-0.2)}{2} = 0.08$

Similarly we get

$U(0.3, 0) = 0.105, \quad 4(0.4, 0) = 0.12$

$U(0.5, 0) = 0.125, \quad U(0.6, 0) = 0.12$

$U(0.7, 0) = 0.105$

Now, $U_x(x, 0) = 0$

$\frac{U_{i,j+1} - U_{ij} = 0}{k}$ for $j = 0\ (t = 0)\ U_{i,1} = U_{i,j}$

Putting $j = 0$ in equation (I) we get,

$U_{i,1} = U_{i-1,0} + U_{i+1,0} - U_{i-1}$

$2U_{i,1} = U_{i-1,0} + U_{i+1,0} \left[\because U_{i,1} = U_{i-1}\right]$

$\therefore U_{i,1} = \frac{1}{2}\left[U_{i-1,0} + U_{i+1,0}\right]$

Now, for $i = 1$ $U_{1,1} = \frac{1}{2}\left[U_{0,0} + U_{2,0}\right]$

$= \frac{1}{2}[0 + 0.80]$

$= 0.040$.

For $i = 2$ $\quad U_{2,1} = \frac{1}{2}\left[U_{1,0} + U_{3,0}\right]$

$$= \frac{1}{2}\left[0.045 + 0.105\right]$$

$$= 0.075.$$

For $i = 3$ $\quad U_{3,1} = \frac{1}{2}\left[U_{2,0} + U_{4,0}\right]$

$$= \frac{1}{2}\left[0.08 + 0.120\right]$$

$$= 0.1.$$

For $i = 4$ $\quad U_{4,1} = \frac{1}{2}\left[U_{3,0} + U_{5,0}\right]$

$$= \frac{1}{2}\left[0.105 + 0.125\right]$$

$$= 0.115.$$

For $i = 5$ $\quad U_{5,1} = \frac{1}{2}\left[U_{4,0} + U_{6,0}\right]$

$$= \frac{1}{2}\left[0.12 + 0.12\right]$$

$$= 0.12.$$

For $i = 6$ $\quad U_{6,1} = \frac{1}{2}\left[U_{5,0} + U_{7,0}\right]$

$$= \frac{1}{2}\left[0.125 + 0.105\right]$$

$$= 0.115.$$

Putting $j = 2$ in equation (II) we get,

$$U_{i,2} = U_{i-1,1} + U_{i+1} - U_{i,0}$$

For $i = 1$ $\quad U_{1,2} = U_{0,1} + U_{2,1} - U_{1,0}$

$$= 0 + 0.075 - 0.045$$

$$= 0.03.$$

For $i = 2$ $\quad U_{2,2} = U_{1,1} + U_{3,1} - U_{2,0}$

$= 0.040 + 0.100 - 0.08$

$= 0.060$.

For $i = 3$ $\quad U_{3,2} = U_{2,1} + U_{4,1} - U_{3,0}$

$= 0.075 + 0.115 - 0.105$

$= 0.085$.

For $i = 4$ $\quad U_{4,2} = U_{3,1} + U_{5,1} - U_{4,0}$

$= 0.1 + 0.12 - 0.12$

$= 0.1$.

For $i = 5$ $\quad U_{5,2} = U_{4,1} + U_{6,1} - U_{5,0}$

$= 0.115 + 0.115 - 0.125$

$= 0.105$.

$\therefore U_{1,2} = 0.03, U_{2,2} = 0.06, U_{3,2} = 0.985, U_{4,2} = 0.1, U_{5,2} = 0.105$.

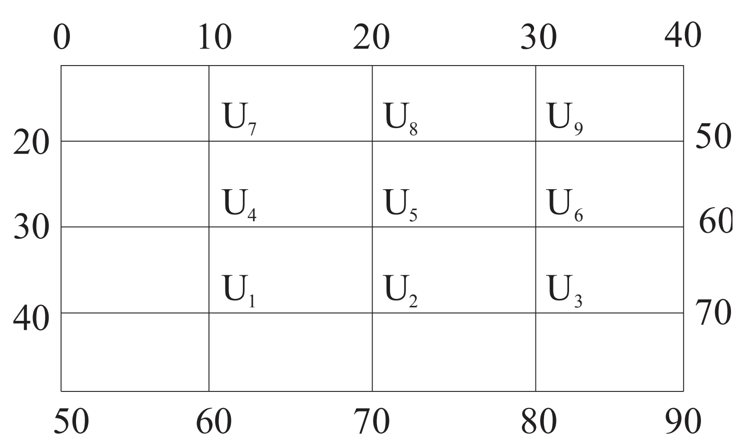
7.9 Review :

In this chapter we have learn,

* Classification of partial differential equation.
* Numerical methods of solving elliptic partial differential equation.
* Numerical methods by solving parabolic partial differential equations.
* Numerical methods of solving hyperbolic partial differential equations.

7.10 Unit End Exercise :

1) Solve the Laplace equations $\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0$ at the interior mesh p + s of the square region with boundary p + s shown in fig.



2) The temperature U in the steady heat below in a square plate bounded by $x = 0,\ y = 0,\ y = x = 4$ satisfies Laplace equation $\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0$.

3) Given one – dimensional heat $\frac{-\partial^2 U}{\partial x^2} = \frac{\partial U}{\partial t}$ under the condition $U(0, t) = 0 = U(1, t)$ for $t > 0$ take $h = \frac{1}{3},\ k = \frac{1}{36}$ and use explicit method to compute U for one time step only.

4) Using Schmidt method find the values of $U(x, t)$ satisfying the parabolic equation $\frac{4\partial^2 U}{\partial x^2} = \frac{\partial U}{\partial t}$ subject to the conditions.

5) Given one – dimensional heat equation $\frac{\partial^2 U}{\partial x^2} = \frac{\partial U}{\partial t}$ under the conditions $U(x, 0) = 40$ for $0 \le x \le 3$ and $U(0, t) = 0,\ U(3, t) = 60$ for $t > 0$ take $h = 1,\ k = 1$ and use Crank Nicolson method to compute x for one time step only.

6) Solve by Crank Nicolson method $\frac{\partial^2 U}{\partial x^2} = \frac{\partial U}{\partial t}$ $(0 < x < 1 \text{ and } t < 0)$ given that $U(x, 0) = 100x$ for $0 \le x \le 1$ and $U(0, t) = 0 = U(1, t)$ for $t > 0$ take $h = \frac{1}{2},\ k = \frac{1}{4}$ and compute U for one time step only.

7) Given one dimensional wave equation $\frac{6\partial^2 U}{\partial x^2} = \frac{\partial^2 U}{\partial t^2}$ under the conditions $U(0, t) = 0 = U(5, t)$ and $U(x, 0) = x^3 - 5x^2$ for $0 \le x \le 5$ for $t > 0$ and $U_t(x, 0) = 0$ for $0 < x < 5$.

8) Given hyperbolic equation $\frac{\partial^2 U}{\partial x^2} = \frac{\partial^2 U}{\partial t^2}$ subject to the condition $U(0, t) = 0 = U(1, t)$ for $t > 0$ and $U_t(x, 0) = \sin^3 \pi x$ for $0 \le x \le 1$ take $h = 0.25$ for $k = 0.2$ and use explicit method to compute U for two time steps.

9) Derive the five point formula for Laplace's equation.

10) What is Crank – Nicholson Method? Why is it known as implicit method?

11) What is Bender – Schmidt recurrence equation? Derive the formula.

12) Discuss the impact of size of the incremental width Δt for the time variable on the solution of a heat flow equation.

♦ ♦ ♦ ♦ ♦