Chapter 1

Topological Spaces

Chapter Structure

1.1 Introduction

Metric spaces are generalization of real line and the usual distance. We take the process of generalization one more step ahead when we eliminate the concept of distance and work with the collection of open sets. In the chapter on metric spaces we saw several concepts like closure, interior, boundary, continuity etc. which involve concept of distance. These concepts can be described equivalently using only open sets and closed sets. Thus a lot of analysis can be carried out only knowing the open sets (and the complements of open sets) without using the notion of distance.

1.2 Objectives

After going through this chapter you will know:

- Definition and examples of a topological space.
- Definition and examples of basis and subbasis of a topological space.
- Definition and properties of closure of a set.
- Definition and properties of interior of a set.
- Properties of continuous functions.

1.3 Topological spaces

Definition 1. A topology or a topological structure on X is a family τ of subsets of X satisfying the following axioms:

- $\emptyset \in \tau$ and $X \in \tau$.
- If $\{G_\lambda : \lambda \in \Lambda\}$ is a subfamily of τ, then $\cup_{\lambda\in\Lambda} G_\lambda$ belongs to τ. This means τ is closed under the union of arbitrary subfamilies of it.
- If $G_1, \dots, G_n$ belong to τ then $\cap_{1\le k\le n} G_k$ belongs to τ. This means the family τ is closed under intersection over finite subfamilies of it.

If τ is a topological structure on X then the ordered pair (X, τ) is called a topological space. Sets belonging to τ are called open sets of the topological space.

Remark 1.3.1. By definition, a topological space is closed under finite intersection and arbitrary union. Every metric space is a topological space. It is easy to check that the collection τ of usual open sets (i.e. open balls) forms a topology.

Remark 1.3.2. It is important to note that not every topological space is a metric space. Let X be a set such that $|X| > 1$. Then $\tau = \{\emptyset, X\}$ forms a topological space: the first axiom is obvious. The other two follow by observing that $\emptyset \cap X = \emptyset$ and $\emptyset \cup X = X$.

If X has more than one element then X cannot be a metric space for the following reason: recall first that every metric space is Hausdorff. We will prove here that the topological space (X, τ) is not Hausdorff which will imply that it is not a metric space. Let x, y be two distinct elements of X. If X were Hausdorff, we would get disjoint open sets U_1, U_2 in X such that $x \in U_1$ and $y \in U_2$. However, the only open sets in (X, τ) are the empty set and X itself. Hence, we cannot get sets U_1 and U_2 as claimed above.

Examples:

- We will consider here several examples of topologies on $X = \{a, b, c\}$. You should verify yourself that in each of these examples the above three axioms are satisfied.
  - $\{\emptyset, X\}$. This is called the **indiscrete topology** on X.
  - $\{\emptyset, \{a\}, X\}$.
  - $\{\emptyset, \{a\}, \{b\}, \{c\}, X\}$.
  - $\{\emptyset, \{a, b\}, X\}$.
  - $\{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$.
  - $\{\emptyset, \{a\}, \{a, b\}, X\}$.
  - $P(X)$. This is called the **discrete topology** on X. In this topology every singleton set is open and hence, every subset of X open too.
- Let X be a non-empty set. Let τ consist of $\emptyset, X$ and all subsets G of X such that $X \setminus G$ is finite. We call this as cofinite topology on X. Verify that it is a topology.
- Let X be any infinite set. Let τ consist of $\emptyset, X$ and all subsets G of X such that $X \setminus G$ is countable. We call this as cocountable topology on X. Verify that it is a topology.
- Let $X = \mathbb{N}$. Let $I_n = \{1, 2, 3, \ldots, n\}$ and $J_n = \{n, n+1, n+2, \ldots\}$. Let $\tau = \{\emptyset, I_1, I_2, I_3, \ldots, \mathbb{N}\}$. Let $\tau_1 = \{\emptyset, J_1 = \mathbb{N}, J_2, J_3, \ldots\}$. Then τ, τ_1 are both topologies on X.

Check Your Progress

- Prove that the cofinite topology and cocountable topology are actually topologies on X.
- Check that τ, τ_1 defined above are actually topologies on X.

All these examples are useful to understand the concept of a topological space and also to understand the concepts that will follow. We will now study families of subsets of X that generate a given topology. This is done via the notion of a base for a topological space.

1.3.1 Base

Definition 2. Let (X, τ) be a topological space. A base to the topology τ is a subfamily $\mathcal{B}$ of τ having the following property: given any $G \in \tau$ there exists a family $\{B_\lambda : \lambda \in \Lambda\} \subseteq \mathcal{B}$ such that $G = \cup_{\lambda\in\Lambda} B_\lambda$.

For example the family $\{\{x\} : x \in X\}$ is a base to the discrete topology on X. The family $\{\emptyset, \{a\}, \{b\}, X\}$ is a base to the topology $\{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ on $X = \{a, b, c\}$.

Let us prove a theorem that will help us to check if a given family is a base to a given topology.

Proposition 1.3.1. *A family $\mathcal{B}$ of subsets of X is a base to a topology on X if and only if*

(i) $X = \cup_{B\in\mathcal{B}} B$.

(ii) If B_1 and B_2 are in $\mathcal{B}$ and if $x \in B_1 \cap B_2$ then there exists $B \in \mathcal{B}$ such that $x \in B \subseteq B_1 \cap B_2$.

Proof. Suppose $\mathcal{B}$ is a base to a topology τ on X. Now, $X \in \tau$ and therefore by the defining property of the base $\mathcal{B}$ in τ, there exists a subfamily $\{B_\lambda : \lambda \in \Lambda\}$ such that $X = \cup_{\lambda\in\Lambda} B_\lambda = \cup_{B\in\mathcal{B}} B$. This proves (i). Next, let B_1, B_2 be any two elements of B then B_1, B_2 are open subsets of (X, τ) and therefore $B_1 \cap B_2 \in \tau$. Again by defining property of a base, there exists a family $\{B_\lambda : \lambda \in \Lambda\} \subseteq \mathcal{B}$ such that $B_1 \cap B_2 = \cup_{\lambda\in\Lambda} B_\lambda$. Consequently, if $x \in B_1 \cap B_2$ then there exists $\lambda \in \Lambda$ such that $x \in B_\lambda (:= B)$. This completes the proof of (ii).

Conversely, suppose the family $\mathcal{B}$ satisfies both the properties above. Let τ denote the family of subsets of X consisting of the empty set $\emptyset$ together with subsets of X of the type $G = \cup_{\lambda\in\Lambda} B_\lambda$ where $\{B_\lambda : \lambda \in \Lambda\}$ is a subfamily of $\mathcal{B}$. By the property (i) of $\mathcal{B}$, $X \in \tau$ and $\emptyset \in \tau$.

Clearly the family τ is closed under arbitrary unions. Now let G_1 and G_2 be any two non empty subsets in the family τ. If $x \in G_1 \cap G_2$ then $x \in G_1$ and therefore, there exists $B_1 \in \mathcal{B}$ such that $x \in B_1 \subseteq G_1$. Similarly if $x \in G_1 \cap G_2$ then $x \in G_2$ and therefore, there exists $B_2 \in \mathcal{B}$ such that $x \in B_2 \subseteq G_2$ which implies $x \in B_1 \cap B_2 \subseteq G_1 \cap G_2$. By property (ii) of $\mathcal{B}$, there exists $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq B_1 \cap B_2 \subseteq G_1 \cap G_2$. This implies $G_1 \cap G_2 = \cup\{B_x : x \in G_1 \cap G_2\}$ which means $G_1 \cap G_2 \in \tau$.

This proves that τ is a topology on X and the way in which τ is obtained from $\mathcal{B}$ implies that $\mathcal{B}$ is a base of τ. $\square$

Definition 3. A collection $\mathcal{C}$ of subsets of X is called a subbasis for the topological space (X, τ) if every set in τ is a union of finite intersections of sets in $\mathcal{C}$.

We do not use subbases extensively in our further discussion. We move on to our next important concept: the product topology. This is one of the ways to build new topological spaces out of old ones.

1.4 Product Topology and Subspace Topology

Let (X, τ) and (Y, τ') be topological spaces. The product topology on $X \times Y$ is the topology having as basis the collection $\mathfrak{B}$ of all sets of the form $U \times V$, where U is an open subset of X and V is an open subset of Y.

Theorem 1.4.1. *If $\mathfrak{B}$ is a basis for the topology of X, and $\mathfrak{C}$ is a basis for the topology Y, then the collection $D = \{B \times C \mid B \in \mathfrak{B}$ and $C \in \mathfrak{C}\}$ is a basis for the topology of $X \times Y$.*

Proof. Given any open set W of $X \times Y$ and a point $x \times y$ of W, there is a basis element $U \times V$ such that $x \times y \in U \times V \in W$. Because $\mathfrak{B}$ and $\mathfrak{C}$ are bases for X and Y respectively, we can choose $B \in \mathfrak{B}$ and $C \in \mathfrak{C}$ such that $x \in B \subseteq U$ and $y \in C \subseteq V$. Then $x \times y \in B \times C \subseteq W$. Thus D is a basis for $X \times Y$. □

We also study another important notion: that of **subspace topology.** A topological structure on a set X induces a topological structure on any subset Y of X.

Definition 4. Let (X, τ) be any topological space and let Y be a nonempty subset of X. We consider $\tau_Y = \{G \cap Y : G \in \tau\}$. Then τ_Y is a topological structure on Y. The topology τ_Y is said to be induced by topology τ on X. We also say that (Y, τ_Y) is a subspace of the topological space (X, τ).

Check Your Progress

- Show that the countable collection $\{(a, b) \times (c, d) \mid a < b$ and $c < d,$ and $a, b, c, d \in \mathbb{Q}\}$ is a basis for the usual topology on $\mathbb{R}^2$ i.e., the one induced by the usual metric on $\mathbb{R}^2$.

- Show that the set of all half open intervals $[a, b)$ with $a, b \in \mathbb{R}$ is a basis for the usual (metric) topology on $\mathbb{R}$.

1.5 Open and Closed sets

You have studied neighbourhoods in metric spaces. We extend this concept to topological spaces.

Definition 5. Let (X, τ) be a topological space. Let $x \in X$. A subset N of X is said to be a neighborhood of the point x if there exists $G \in \tau$ such that $x \in G \subseteq N$. Note that in general, N need not belong to τ. It is called open neighborhood if it belongs to τ. Collection of all neighborhoods of x is denoted by $N(x)$. It is easy to see that a nonempty subset G of X is open if and only if it is a neighborhood of each of its point. Let Nx be an open set containing x and contained in G. Then $G = \cup_{x \in G} Nx$ and G is open, being an union of open sets. On the other hand if G is open then G itself is the neighborhood of each of its point contained in G.

Examples:

- Let $X = \{a, b, c\}$ and let $\tau = \{\emptyset, \{a\}, X\}$ then we have $N(a) = \{\{a\}, \{a, b\}, \{a, c\}, X\}$ and $N(b) = N(c) = X$.
- Let $X = \mathbb{N}$ and let $\tau = \{\emptyset, J_1, J_2, J_3, \ldots, J_k, \ldots,\}$ where $J_k = \{k, k+1, k+2, \ldots\}$. Then $N(1) = \{J_1\} = \mathbb{N}$, $N(2) = \{J_1, J_2\}$, $N(3) = \{J_1, J_2, J_3 \cup \{1\}\}$, $N(4) = \{J_1, J_2, J_3 \cup \{1\}, J_4 \cup \{1\}, J_4 \cup \{2\}, J_4 \cup \{1, 2\}\}$.

Definition 6. Let A be a nonempty subset of (X, τ). A point $x \in X$ is said to be a limit point or an accumulation point of A if for every neighborhood N of x the set $A \cap (N \setminus \{x\})$ is non empty. The set of limit points of A is called the derived set of A and is denoted by A'. Note that a limit point of a set need not be an element of the set. For example consider the topological structure defined in example 2 above. Let $A = \{2\}$. The point 1 is a limit point of A but it is not in A.

Definition 7. For any set A in a topological space (X, τ), the closure of A in X denoted by $c(A)$ is defined as $c(A) = A \cup A'$.

Check Your Progress Let $X = \{a, b, c\}$ and let $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. Show that $\{a\}' = \{c\}$ and $\{c\}' = \emptyset$. Find the derived sets of other subsets of X.

Following results are similar to those in a metric space.

Lemma 1.5.1. *(i)* $(\emptyset)' = \emptyset$.

(ii) *If* $A \subseteq B$ *then* $A' \subseteq B'$.

(iii) $(A \cup B)' = A' \cup B'$.

Proof. (i) For any neighborhood N of any point x of X we have $\emptyset \cap N = \emptyset$ and hence $\emptyset \cap N \setminus \{x\} = \emptyset$. This proves (i).

(ii) Let $x \in A'$. Then for any neighborhood N of x we have $A \cap N \setminus \{x\} \neq \emptyset$. But A is a subset of B. Hence $A \cap N \setminus \{x\} \subseteq B \cap N \setminus \{x\}$ which gives $B \cap N \setminus \{x\} \neq \emptyset$ i.e. $x \in B'$. This proves $A' \subseteq B'$.

(iii) We have $A \subseteq A \cup B$ which give $A' \subseteq (A \cup B)'$ Similarly $B \subseteq A \cup B$; which gives $B' \subseteq (A \cup B)'$. These two together give us $A' \cup B' \subseteq (A \cup B)'$.

To get the reverse inclusion suppose $x \in (A \cup B)'$ but does not belong to A' and B'. Then there exist neighborhoods N_1 and N_2 of x with $A \cap N_1 \setminus \{x\} = \emptyset$ and $B \cap N_2 \setminus \{x\} = \emptyset$. Let $N = N_1 \cap N_2$. Then N is a neighborhood of x satisfying $A \cap N \setminus \{x\} = \emptyset$ and $B \cap N \setminus \{x\} = \emptyset$ which given $(A \cup B) \cap N \setminus \{x\} = (A \cap N \setminus \{x\}) \cup (B \cap N \setminus \{x\}) = \emptyset \cup \emptyset = \emptyset$. This shows that x does not belong to $(A \cup B)'$ This proves $(A \cup B)' \subseteq A' \cup B'$ which completes the proof of *(iii)*.

□

Definition 8. A set A in a topological space (X, τ) is said to be closed if $A' \subseteq A$.

There is a more simple characterization of closed sets.

Theorem 1.5.1. *A set A in (X, τ) is closed if and only if $X \setminus A$ is an open set in (X, τ).*

Proof. Let A be a closed set, which means $A \subseteq A$. To prove $G = X \setminus A$ is open in (X, τ). Let $y \in G$. Then y is not in A and hence not in A because $A \subseteq A$. Therefore there exists a neighborhood N of y such that $N \cap A \setminus \{y\} = \emptyset$. But y is not in A hence we get, $A \cap N = \emptyset$.

Now, by definition of neighbourhood there exists $G_y \in \tau$ with $y \in G_y \subseteq N$. Now $A \cap N = \emptyset$ implies, $A \cap G_y = \emptyset$; equivalently put, we have $y \in G_y \subseteq X \setminus A$.

This proves that for any $y \in X \setminus A$ there exists an open set G_y satisfying $y \in G_y \subseteq X \setminus A$. Hence $X \setminus A = \cup\{G_y : y \in X \setminus A\}$ which gives $X \setminus A$ is an open set in (X, τ).

Conversely, suppose $X \setminus A$ is open. Then, no point of $X \setminus A$ is a limit point of A. Because if y is a point of $X \setminus A$ then by openness of $X \setminus A$, we get that $X \setminus A$ is a neighborhood of y which is disjoint from A. Therefore y is not a limit point of A. This proves $A' \subseteq A$. That is A is closed. □

Next theorem also is familiar to you.

Theorem 1.5.2.

- *$\emptyset$ and X are closed sets in (X, τ).*
- *Any intersection of closed sets in (X, τ) is a closed set in (X, τ).*
- *Finite union of closed sets in (X, τ) is a closed set in (X, τ).*

Proof. We use the above theorem to prove this.

$\emptyset \in \tau$ implies that $X \setminus \emptyset$ is closed and $X \in \tau$ implies that $\emptyset = X \setminus X$ is closed.

- Let $\{C_\lambda : \lambda \in \Lambda\}$ be a family of closed sets. Then for each $\lambda \in \Lambda$ the set $G_\lambda = X \setminus C_\lambda$ is an open set in (X, τ). Hence $\cup\{G_\lambda : \lambda \in \Lambda\} \in T$. But $\cup\{G_\lambda : \lambda \in \Lambda\} = X \setminus \cap\{G_\lambda : \lambda \in \Lambda\}$. This shows that $\cap\{C_\lambda : \lambda \in \Lambda\}$ is closed.

- Let $\{C_k : k = 1, 2, 3, \ldots, n\}$ be a family of closed sets. Let $G_k = X \setminus C_k, 1 \leq k \leq n$. Then each G_k is open and therefore $\cap\{G_k : 1 \leq k \leq n\}$ is an open set in (X, τ). But $\cap_{1 \leq k \leq n} G_k = X \setminus \cup_{1 \leq k \leq n} G_k$ which shows that $\cap_{1 \leq k \leq n} G_k$ is closed. This completes the proof. □

Following is another familiar result. We have proved it earlier for metric spaces.

Theorem 1.5.3. *Closure of a set A in (X, τ) is the smallest closed subset of (X, τ) containing A.*

Proof. Let $\overline{A}$ be the smallest closed set containing A and let $c(A) = A \cup A'$. We will prove:

(i) $\overline{A} \subseteq c(A)$,

(ii) $c(A) \subseteq \overline{A}$.

These put together will prove $\overline{A} = c(A)$.

Suppose $x \in X \setminus c(A)$ then x does not belong to A and x does not belong to A'. Therefore there exists an open set Gx such that $x \in Gx$ and $A \cap Gx \setminus \{x\} = \emptyset$. But x is not in A and therefore this set equation implies, $A \cap Gx = \emptyset$. Note that this also implies that no point of Gx is a limit point of A. In other words $Gx \subseteq X \setminus A'$. On the other hand $A \cap Gx = \emptyset$ implies $Gx \subseteq X \setminus A'$, we get $Gx \subseteq X \setminus A \cup A' = X \setminus c(A)$. Thus each point x of $X \setminus c(A)$ has a neighborhood Gx contained entirely in $X \setminus c(A)$. Therefore $X \setminus c(A)$ is an open subset of (X, τ). This means $c(A)$ is a closed subset of (X, τ) containing A. Now $\overline{A}$ being the smallest closed set containing A implies $\overline{A} \subseteq c(A)$. This proves (i). Also the set $\overline{A}$ is closed and hence $(\overline{A})' \subseteq \overline{A}$. But $A \subseteq \overline{A}$ implies $A' \subseteq \overline{A}'$. Thus we have $c(A) \subseteq A \cup A' \subseteq \overline{A}$. This proves (ii) completing the proof. □

1.6 Interior

This concept from metric spaces is extended to topological spaced as follows.

Let A be a non empty set in (X, τ). We define Interior of A as the union of all open subsets of A. We denote it by $i(A)$ or A°. $i(A)$ is an open set by definition. $i(A)$ is in fact the largest open set contained in A.

Theorem 1.6.1. *(1.)* $i(X) = X$

(2.) $i(A) \subseteq A$, *for all subsets* A *of* X.

(3.) $i(i(A)) = i(A)$, *for all subsets* A *of* X.

(4.) *If* $A \subseteq B$ *then* $i(A) \subseteq i(B)$.

(5.) $i(A \cap B) = i(A) \cap i(B)$, *for all subsets* A *and* B *of* X.

Proof. (1.) X is open. Hence X is the largest open set contained in X. This proves (1).

(2.) $i(A)$ is a union of subsets of A and hence is a subset of A. This proves (2).

(3.) $i(A)$ is open. Hence $i(i(A)) = i(A)$.

(4.) $i(A)$ is an open set contained in A. Hence $i(A)$ is an open set contained in B. This means $i(A)$ is contained in largest open set contained in B which is $i(B)$.

(5.) $A \cap B \subseteq A$ which gives $i(A \cap B) \subseteq i(A)$. Also $A \cap B \subseteq B$ which gives $i(A \cap B) \subseteq i(B)$. Together these two imply $i(A \cap B) \subseteq i(A) \cap i(B)$. Now let $y \in i(A) \cap i(B)$. By definition of interior there exist open sets G and H in (X, τ) satisfying $y \in G \subseteq A$ and $y \in H \subseteq B$. $G \cap H$ is an open subset of $A \cap B$ hence $G \cap H \subseteq i(A \cap B)$. This proves $y \in i(A \cap B)$ and hence $i(A) \cap i(B) \subseteq i(A \cap B)$. This completes the proof.

□

Definition 9. For any set A in the topological space the set $c(A) \setminus i(A)$ is defined as the boundary of A denoted by $b(A)$. It is also called frontier of A.

Check Your Progress Show that a point x of X is a boundary point of a subset A of X if and only if $N \cap A \neq \emptyset$ and $N \cap (X \setminus A) \neq \emptyset$ for all $N \in N(x)$.

1.7 Continuous functions

Now we generalize the concept of continuous functions to topological spaces.

Definition 10. Let $(X, \tau), (Y, \tau')$ be any two topological spaces and let $f : X \to Y$ be a map. We say f is continuous at a point a of X, if given any neighborhood $\widetilde{N}$ of $f(a)$ there exists a neighborhood N of a such that $f(N) \subseteq \widetilde{N}$. f is said to be continuous on a subset A of X if f is continuous at every point a of A.

Remark 1.7.1. $f(N) \subseteq \widetilde{N}$ implies $N \subseteq f^{-1}(\widetilde{N})$ and therefore $f^{-1}(\widetilde{N})$ becomes a neighborhood of a. Therefore the definition can be restated as: f is continuous at a if for every neighborhood $\widetilde{N}$ of $f(a)$ the set $f^{-1}(\widetilde{N})$ is a neighborhood of a. Since neighborhoods are supersets open sets containing that point continuity at a point can be rephrased as : f is continuous at a if given open set H of (Y, τ') with $f(a) \in H$ there exists open subset G of (X, τ) containing a such that $f(G) \subseteq H$.

We prove some theorems on continuous functions.

Theorem 1.7.1. *Let $(X_1, \tau_1), (X_2, \tau_2), (X_3, \tau_3)$ be topological spaces and let $f : (X_1, \tau_1) \to (X_2, \tau_2)$ and $g : (X_2, \tau_2) \to (X_3, \tau_3)$ be maps such that f is continuous on X_1 and g is continuous on X_2 then $h = gof : (X_1, \tau_1) \to (X_3, \tau_3)$ is continuous on X_1.*

Proof. Let $a \in X_1$. Let $f(a) = b$. Let $\widetilde{N}$ be a neighborhood of $g(b) = g(f(a))$. By continuity of g at $f(a) = b$ we have $g^{-1}(\widetilde{N})$ is a neighborhood of b. We have $g^{-1}(\widetilde{N})$ is a neighborhood of b. Also by continuity of f at a we have $f^{-1}(g^{-1}(\widetilde{N}))$ is a neighborhood of b. But $f^{-1}(g^{-1}(\widetilde{N})) = (g \circ f)^{-1}(\widetilde{N})$ which is a neighborhood of a. Thus we have verified that, for any neighborhood $\widetilde{N}$ of $(g \circ f)(a)$ the set $(gof)^{-1}(\widetilde{N})$ is a neighborhood of a. Therefore by the definition it follows that $g \circ f$ is continuous at a. □

Theorem 1.7.2. *Let $f : (X, \tau) \to (Y, \tau')$ be any map. The following conditions on f are equivalent:*

(a) f is continuous on X. (i.e. at every point of X).

(b) If $H \in \tau'$ then $f^{-1}(H) \in \tau$.

(c) If C is a closed subset of (Y, τ') then $f^{-1}(C)$ is a closed subset of (X, τ).

(d) For any subset A of X, $f(c(A))$ is a subset of $c(f(A))$.

Proof. We will prove the implications cyclically.

$(a) \implies (b)$. Let f be continuous on X. Let H be an open subset of (Y, τ'). Let $x \in f^{-1}(H)$. Then $f(x) \in H \in \tau'$. This means H is a neighborhood of $f(x)$. By continuity of f at x, $f^{-1}(H)$ is a neighborhood of x. This proves that $f^{-1}(H)$ is a neighborhood of each of its point. Hence $f^{-1}(H)$ is an open set of (X, τ).

$(b) \implies (c)$. Let C be a closed subset of (Y, τ'). Then $H = Y \setminus C$ is an open subset of (Y, τ'). By (b) this gives $f^{-1}(H)$ is an open subset of (X, τ). But $f-^{1}(H) = f^{-1}(Y \setminus C) = X \setminus f^{-1}(C)$. Hence $f^{-1}(C)$ is a closed subset of (X, τ).

$(c) \implies (d)$. Let A be any subset of X. Let $C = c(f(A))$. C is a closed subset of (Y, τ'). Therefore by (c), $f^{-1}(C)$ is a closed subset of (X, τ). But we have $A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(c(f(A))) = f^{-1}(C)$.

Now A is a subset of the closed set $f^{-1}(C)$. Hence $c(A) \subseteq f^{-1}(C)$ which gives $f(c(A)) \subseteq f(f^{-1}(C)) \subseteq C$. Thus $f(c(A)) \subseteq c(f(A))$ is proved.

$(d) \implies (c)$. Let C be a closed subset of (Y, τ'). Let $A = f^{-1}(C)$. We have $f(c(A)) \subseteq c(f(A)) \subseteq c(f(f-1(C)) \subseteq c(C) = C$. Now apply f^{-1} to $f(c(A)) \subseteq C$ we get $f^{-1}(f(c(A))) \subseteq f^{-1}(C) = A$. But $c(A) \subseteq f^{-1}(f(c(A)))$. Therefore we get $c(A) \subseteq A$. That is, A is a closed subset of (X, τ).

$(c) \implies (a)$. Let x be any point of X and let H be any open subset of (Y, τ') containing $f(x)$. Let $C = Y \setminus H$ then we have $f^{-1}(H) = X \setminus f^{-1}(Y \setminus H) = X \setminus f^{-1}(C)$. By the result (c), $f^{-1}(C)$ is closed and hence $f^{-1}(H) = X \setminus f^{-1}(C)$ is open. This proves that f is continuous at every point x in X. $\square$

Remark 1.7.2.

- If $(X, \tau) \to (Y, \tau')$ is continuous and x is a limit point of $x \in A \subseteq X$ then it is not necessary that $f(x)$ is a limit point of $f(A)$. For example take the constant map from the usual (or indiscrete) topology to the discrete topology on $\mathbb{R}$. Show that it is continuous. Now any point in the domain is a limit point of $\mathbb{R}$ but its image is not a limit point of $f(\mathbb{R})$.

- Any function defined on a discrete topological space is continuous.

- If $f : (X, \tau) \to (Y, \tau')$ is continuous then $A \in \tau$ does not mean $f(A) \in \tau'$.

1.8 Glossary

In this chapter, you have learnt the following:

- Definition of a topological space.
- Base of a topological space.
- Product and subspace topology.
- Open and Closed sets.
- Interior of a set.
- Continuity in topological spaces.

1.9 Bibliography

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1.10 Let Us Sum Up

A topological structure on X is a family τ of subsets of X which contains $\emptyset, X$ and is closed under the operations of unions over arbitrary subfamilies of τ and over intersections of finite subfamilies of τ. Any non empty family of subsets of X generates a unique topology on X. A subfamily $\mathfrak{B}$ of τ is called a base of τ if every non-empty set G in τ can be expressed as a union of a subfamily of $\mathfrak{B}$. A subset N of X is called a neighborhood of $x \in X$ if there exists G in τ satisfying $x \in G \subseteq N$.

Associated with any set A in a topological space (X, τ) there is another set A called derived set of A. The set A' consists of all limit points of A; where a limit point of A is a point x of X having the property that every neighborhood of it contains at least one point of A which is different from x. A subset A of a topological space X is said to be closed if A' is a subset of A. A set is closed if and only if its complement is open. The set $A \cup A'$ is called closure of A and is denoted by $c(A)$. Closure of a set satisfies following properties.

- $c(\emptyset) = \emptyset$.
- $A \subseteq c(A)$, for every $A \subseteq X$.
- $c(c(A)) = c(A)$, for every $A \subseteq X$.
- $c(A \cap B) = c(A) \cap c(B)$ for all $A, B \subseteq X$.
- $c(A)$ is the smallest closed subset of X containing A.

The definition of a continuous function from one topological space to another is a generalization of the classical $\varepsilon - \delta$ definition of continuity of a real valued function of a real variable. A map $f : (X, \tau) \to (Y, \tau')$

is continuous at a point a of X if for any neighborhood $\widetilde{N}$ of $f(a)$ in (Y,τ) there corresponds a neighborhood N of a in (X,τ) such that $f(N) \subseteq \widetilde{N}$. We say that f is continuous on a subset A of (X,τ) if it is continuous at every point a of A. Continuity of f on X has several equivalent forms. Some are listed below.

- $f^{-1}(H) \in \tau$ for every $H \in \tau'$.
- $f^{-1}(C)$ is a closed subset of (X,τ) whenever C is a closed subset of (Y,τ').
- $f(c(A)) \subseteq c(f(A))$ for every subset A of X.

1.11 References for further reading

1. W.J. Pervin: Foundations of General Topology, Academic press , New York, London.
2. Kelley J.L, General Topology, Van Nostrand Reinhold Co., New York, 1955.
3. Dugundji J.: Topology, Allyn and Bacon, Boston, 1966.

1.12 Chapter End Exercises

1. There are 26 topologies on $X = \{a,b,c\}$. List all of them.
2. Let Y be a non-empty subset of a topological space (X,τ) and let $y \in Y$. Prove that a subset M of Y is a neighborhood of y in the subspace (Y,τ') if and only if there exists a neighborhood N of y in (X,τ) such that $M = N \cap Y$.
3. Let I be the set of all bounded open intervals and let I^* be the subfamily of I consisting of all open intervals of I having rational end points. Prove that the topology on $\mathbb{R}$ generated by I^* is the same as that generated by I. (Note that the topology generated by I is the usual topology of $\mathbb{R}$.)
4. Prove that two bases $\mathfrak{B}$ and $\mathfrak{B}^*$ generate the same topology if and only if for each $x \in B \in \mathfrak{B}$ there exists $B^* \in \mathfrak{B}^*$ such that $x \in B^* \subseteq B$ and to each $y \in B' \in B^*$ there exists $B'' \in \mathfrak{B}$ such that $x \in B'' \subseteq B'$.

5. Let $\mathfrak{B}$ be a base to a topological structure τ on X and let Y be a nonempty subset of X. Let $\mathfrak{B}_y = \{B \cap Y : B \in \mathfrak{B}\}$. Is $\mathfrak{B}_y$ a base to the subspace topology τ_y on Y? Justify.

6. Give an example of a set X and two topologies τ and τ' on X such that $\tau \cup \tau'$ is not a topology on X.

7. If A is a subset of (X, τ) with the property that $B' \subseteq A \subseteq B$ for some subset B of X, prove that A is closed.

8. Let $X = \mathbb{N}$. Let $I_n = \{1, 2, 3, \ldots, n\}$ and $J_n = \{n, n+1, n+2, \ldots\}$. Let $\tau = \{\emptyset, I_1, I_2, I_3, \ldots, \mathbb{N}\}$ and $\tau' = \{\emptyset, J_1 = \mathbb{N}, J_2, J_3, \ldots\}$. Then check that τ and τ' both are topological structures on $\mathbb{N}$. Find $\{1\}'$ with respect to both τ and τ'.

9. Let $f : (X, \tau) \to (Y, \tau')$ be a constant map. Show that f is continuous.

10. Describe the closure of an infinite set in the co-finite topology.

Chapter 2

Countability and Separation Axioms

Chapter Structure

2.1 Introduction

The basic definition of a topological structure on a space X is in terms of a family τ of subsets of X satisfying only three conditions, namely the three topological axioms and yet it manages to capture in it the most basic geometric ingredient "neighbourhood" of the real number systems $\mathbb{R}, \mathbb{R}^2$ etc. But most of the other mathematically important properties of these number systems are lost in such a generalization. We get them back one by one by imposing additional conditions on a topological structure.

2.2 Objectives

After going through this chapter you will know:

- Countability axioms and their consequences
- Separation axioms and examples of metric spaces that satisfy them.
- Definition, examples and properties of Hausdorff spaces.
- Regular and normal topological spaces and their properties.

2.3 Homeomorphism and Heredity

The main aim of studying topological spaces is the study of properties that are invariant under homeomorphism. Homeomorphism is thus a concept of central importance in topology.

Definition 11. A map $f : (X, \tau) \to (Y, \tau')$ is said to be a homoemorphism if f is bijective, f is continuous and f^{-1} is also continuous.

For example, the identity map from $(X, \tau) \to (X, \tau)$ is clearly a homeomorphism and composite of two homeomorphisms is a homeomorphism. This makes homeomorphism an equivalence relation on the collection of topological spaces. Also inverse of a homeomorphism is a homeomorphism.

Definition 12. A property P of topological spaces is said to be a topological property if P is satisfied by a topological space implies that it is satisfied by another topological space homeomorphic to it. Also a property P is called hereditary if P is satisfied by a topological space implies that it is satisfied by every subspace of it.

Remark 2.3.1. Note that a bijective continuous map need not be a homeomorphism. Take for example a continuous map f from $[-\pi, \pi)$ to the unit circle S^1 in $\mathbb{R}^2$ defined by $f(x) = (cosx, sinx)$. f is bijective and continuous but the inverse of f is not continuous.

2.4 Cardinality

Though cardinality is not our main topic of discussion we need to revise some results in cardinality to understand the countability axioms. Cardinality of $\mathbb{N}$, the set of natural numbers and any set which is in bijection with $\mathbb{N}$ is denoted by $\aleph_0$. These sets are called denumerable.

It can be proved that the set of rational numbers is denumerable. A set which is either denumerable or finite is called as countable. The set of real numbers $\mathbb{R}$ and any interval in $\mathbb{R}$ are uncountable sets. It can be proved that cardinality of $\mathbb{R}$, $\mathbb{R}^2$ and any interval in $\mathbb{R}$ are same. We denote it by C. Continuum hypothesis states that there is no infinite set having cardinality between $\aleph_0$ and C. An important property of countable sets is that all its elements can be written in the form of a sequence.

2.4.1 Countability Axioms

Recall, if N is a neighborhood of x and if M is a subset of X with $N \subset M$ then M also becomes a neighborhood of x. This property of neighborhoods suggests that we need not know the entire family $N(x)$; it is enough to single out a subfamily of it which is such that any neighborhood of x is a superset of a set belonging to the subfamily. Such a subfamily, say $\mathfrak{B}(x)$ of $N(x)$, which is capable of describing all the neighborhoods of x as the supersets of sets belonging to $\mathfrak{B}(x)$ is called the base to the neighborhood systems of x. Thus we have two kinds of bases in our discussion:

- A base to the topological structure τ on X.
- A base to the complete neighborhood system $N(x)$ of a point x in (X, τ).

It turns out that the two concepts are not independent; the first kind of base, that is a base to a topological structure determines a base to the complete neighborhood system $N(x)$ of a point x of X which the topological structure assigns. Countability axioms are auxiliary axioms on the topological structure of a topological space which demand that the two bases be countable.

Definition 13. A base of the neighborhood system $N(x)$ of x is any subfamily $\mathfrak{B}(x)$ of $N(x)$ having the following property: If $N \in N(x)$ then, there exists $B \in \mathfrak{B}(x)$ such that $B \subseteq N$. If all the neighborhoods in $\mathfrak{B}(x)$ consist of open sets then we say that $\mathfrak{B}(x)$ is an open base of the neighborhood system on x. Note that, if $\mathfrak{B}(x)$ is a base of the neighborhood system $N(x)$ of x then so is $\{i(B) : B \in \mathfrak{B}(x)\}$ which is an open base.

Here are some illustrative examples of neighborhood bases.

- In case of discrete topology $P(X)$ on a set X, $\mathfrak{B}(x) = \{\{x\}\}$ is a base to $N(x)$.
- In τ the usual topology on $\mathbb{R}$, $\mathfrak{B}(x) = \{(x-1/n, x+1/n) : n \in \mathbb{N}\}$ is a countable base to $N(x)$.

- Let (X, d) be any metric space. Then $\mathfrak{B}(x) = \{B(x, 1/n) : n \in \mathbb{N}\}$ is a countable base to the complete neighborhood system of each x in X.

Definition 14. We say that the topological space (X, τ) is first countable space if for each x in X the complete neighborhood system $N(x)$ has a countable base. This countable base is called the fundamental system of neighborhoods of x. We also say that such a space is a C_1-space.

Topological spaces discussed in examples 2 and 3 discussed above are C_1- spaces. In addition the discrete space $(X, P(X))$ is C_1 because $\{\{x\}\}$ is a fundamental system of neighborhoods for any x in X and the indiscrete space $(X, \{\emptyset, X\})$ is C_1- spaces because X is the only neighborhood of every point x in X. If $\{N_n : n \in N\}$ is a fundamental system of neighborhoods of a point x then putting $\widetilde{N_1} = N_1 \cap N_2 \cap N_3 \cdots$ we get that $\{\widetilde{N_n} : n \in N\}$ is another countable neighborhood base of x in which the neighborhoods decrease with increasing n : $\widetilde{N_1} \supset \widetilde{N_2} \supset \widetilde{N_3} \supset \widetilde{N_n} \supset \cdots$ Thus in a C_1- space we can always choose a countable, monotonically decreasing fundamental neighborhood system of neighborhoods of each of its points. Now we verify that first countability is a topological property.

Let $f : (X, \tau) \to (Y, \tau')$ be a homeomorphism. Let (X, τ) be a C_1-space. We verify that each $y \in Y$ has a countable fundamental neighborhood system. Let $x \in X$ satisfy $f(x) = y$. If $M_n = f(N_n)$ then for each n, M_n are neighborhoods of y. In fact $\{M_n : n \in \mathbb{N}\}$ is a fundamental neighborhood system of y. This proves that (Y, τ') is C_1-space. Thus homeomorphic image of a C_1-space (X, τ) is a C_1- space.

We will verify that first countability is also a hereditary property. Let (X, τ) be a C_1-space and let Y be a non-empty subset of X. Let $y \in Y$. Then $y \in X$. By the C_1- property of (X, τ), y has a fundamental neighborhood system $\{N_n : n \in \mathbb{N}\}$ in (X, τ). Then $\{N_n^* := N_n \cap Y : n \in \mathbb{N}\}$ is a fundamental neighborhood system of y in (Y, τ_y). This proves that the subspace (Y, τ_y) is also C_1- space. Hence it is a hereditary property.

Definition 15. A topological space (X, τ) is second countable (or satisfies the second countability axiom) if the topology has a countable base.

A second countable space is first countable, for if $\mathfrak{B}$ is a countable basis for (X, τ), then $\mathfrak{B}(x) = \{B \in \mathfrak{B} \text{ for which } x \in B\} \subseteq \mathfrak{B}$ and hence is countable. Thus, $\mathfrak{B}(x)$ is a countable base to the complete neighborhood system $N(x)$ and hence C_1- axiom is satisfied.

The converse is not true. A space may be first countable without being second countable. Consider the discrete topological structure on $\mathbb{R}$. This topological space $(\mathbb{R}, P(\mathbb{R}))$ has no countable base. But it is C_1- space because for every point $x \in X$ the singleton family $\{\{x\}\}$ is a fundamental neighborhood system of x. This shows that second countability is stronger than first countability. Some examples of second countable spaces are given below.

- Let $X = \mathbb{N}$. Let $I_n = \{1, 2, 3, \ldots, n\}$ and $J_n = \{n, n+1, n+2, \ldots\}$. Let $\tau = \{\emptyset, I_1, I_2, I_3, \ldots, X\}$ and $\tau' = \{\emptyset, J_1 = \mathbb{N}, J_2, J_3, \ldots\}$. Then τ and τ' are both topological structures on $\mathbb{N}$. Both are countable families and hence are second countable structures.

- Let $(\mathbb{R}, d)$ be the usual metric space. $\mathfrak{B} = \{B(x, 1/n) : n \in \mathbb{N}, x \in \mathbb{Q}\}$ is a countable base to the metric space. Hence the usual metric space is second countable.

2.5 Separation Axioms

Definition 16. A topological space (X, τ) is a T_0-space if for any two distinct points x and y in X there exists an open subset of (X, τ) which contains only one of x and y but not the other.

Here are some examples:

- If X contains at least two points then the indiscrete topological space $\{X, \{\emptyset, X\}]\}$ is not T_0 because an open set will either contain both the points or contain no points.

- Let $X = \{a, b, c\}$. Then $\tau = \{\emptyset, \{a, b\}, X\}$ is not T_0 because the condition fails for the pair a, b.

- Any metric space satisfies the Hausdorff property and hence is T_0.

- If τ is the co-finite topology on any set having more than element then it is T_0.

- Cocountable topology on $\mathbb{R}$ is also a T_0 space.

Theorem 2.5.1. • *Being T_0 is a topological property*

- *Being T_0 is a hereditary property.*

Proof. • Let $f : (X, \tau) \to (Y, \tau')$ be a homeomorphism. Suppose (X, τ) is a T_0 space. To prove (Y, τ') is also T_0, consider two distinct points y_1 and y_2 in Y. There exist two distinct points x_1 and x_2 in X satisfying $f(x_1) = y_1$ and $f(x_2) = y_2$.

By the T_0 property of (X, τ) there exists an open set G of (X, τ) which contains one of the points x_1, x_2 avoiding the other. Without loss of generality suppose $x_1 \in G$ and $x_2 \notin G$. Let $f(G) = H$. Then by defining property of the homeomorphism $H \in \tau'$. $y_1 \in H$ but $y_2 \notin H$. This proves (Y, τ') is a T_0 space.

- Let (X, τ) be a T_0 space let Y be a non empty subset of X. We want to prove that (Y, τ_y) is a T_0 space. Let y_1, y_2 be two distinct points of Y. They are also distinct points of the T_0 space (X, τ). Therefore by the T_0 property of (X, τ) there is an open subset G of it, which contains only one of these points. Suppose $y_1 \in G$ and $y_2 \notin G$. Let $H = G \cap Y$. Then H is an open subset of (Y, τ_y) which contains y_1 but not y_2. This proves that (Y, τ_y) is a T_0 space.

$\square$

Theorem 2.5.2. *(X, τ) is a T_0 space if and only if distinct one point subsets of it have distinct closures.*

Proof. First suppose that (X, τ) is a T_0 space. Let x and y be distinct points of X. We prove that $c(\{x\}) \neq c(\{y\})$. Since $x \neq y$, there exists an open subset G of (X, τ) with say $x \in G$ but $y \notin G$. Now $y \notin G$ implies that the closed set $X \setminus G$ contains y but does not contain x. Clearly $c(\{y\}) \subseteq X \setminus G$ and as such $x \notin c(\{y\})$. But $c(\{x\})$ must contain point x. This proves $c(\{y\}) \neq c(\{x\})$.

Conversely, suppose for every pair of points $x, y \in X$ with $x \neq y$ we have the inequality of sets $c(\{x\}) \neq c(\{y\})$. Then either $x \notin c(\{y\})$ or $y \notin c(\{x\})$. For if $x \in c(\{y\})$ then $\{x\} \subseteq c(\{y\})$ which gives $c(\{x\}) \subseteq c(\{y\})$.

Similarly $y \in c(\{x\})$ implies $c(\{y\}) \subseteq c(\{x\})$. These two together give $c(\{x\}) = c(\{y\})$, which contradict the assumption. Suppose $x \notin c(\{y\})$. Let $G = X \setminus c(\{y\})$. Then G is an open subset of (X, τ). $X \in G$ and $y \notin G$. This proves the T_0 axiom for (X, τ).

$\square$

We now move on to the next separation axiom.

Definition 17. A topological space (X, τ) is T_1 space if for any two points $x, y \in X$ there exit two open sets G, H of (X, τ) such that $x \in G$, $y \notin G$ and $y \in H$, $x \notin H$.

Clearly every T_1 space is a T_0 space. But there are spaces which are T_0 but not T_1.

Here are some examples

- Let $X = \mathbb{N}$. Let $I_n = \{1, 2, 3, \ldots, n\}$. Let $\tau = \{\emptyset, I_1, I_2, \ldots, \mathbb{N}\}$. (X, τ) is T_0 but not T_1 since there exists an open set that contains 2 but not 1, but there is no open set containing 1 and not containing 2. This proves that the above space is not T_1.

- Discrete topological space $(X, P(X))$ and metric spaces are examples of T_1 spaces.

- On $X = \{a, b, c\}$ we define $\tau = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}$. (X, τ) is T_0 but not T_1. The pair a, b does not satisfy the requirements of a T_1 space.

Theorem 2.5.3.

- *A topological space is T_1 space if and only if every one point subset of it is a closed subset.*

- *In a T_1 space (X, τ) if x is a limit point of a subset A of X then every neighborhood of x contains infinitely many points of A.*

Proof.

- First we assume that (X, τ) is a T_1 space and prove that for any $x \in X$, $\{x\}$ is a closed set. Equivalently we prove that $G = X \setminus \{x\}$ is an open set of (X, τ).

  Let $y \in G$ be arbitrary. Then $y \neq x$. Therefore by the T_1 property there exists $G_y \in \tau$ such that $y \in G_y$ and $x \notin G_y$. Thus $y \in G_y \subseteq X \setminus \{x\}$. Consequently we have $X \setminus \{x\} = \cup\{G_y : y \in X \setminus \{x\}\}$. This shows that $X \setminus \{x\}$ is an open set and hence $\{x\}$ is closed. Conversely suppose that $\{x\}$ is closed subset of (X, τ) for every $x \in X$. Now let x and y be two distinct points of X, then $x \in X \setminus \{y\} = G_x \in \tau$ and similarly we have $y \in X \setminus \{x\} = G_y \in \tau$ Thus for the distinct points x and y in X we have the open sets G_x, G_y with $x \in G_x$, $y \notin G_x$ and $y \in G_y$, $x \notin G_y$. This proves that (X, τ) is a T_1 space.

- Suppose $x \in X$ is such that a neighborhood N of x contains only finitely many points of A. Say, $A \cap N \setminus \{x\} = \{x_1, x_2, x_3, \ldots, x_n\}$. Now being one point subsets of the T_1 space each of the sets $\{x_1\}, \{x_2\}, \{x_3\}, \ldots, \{x_n\}$ are closed subsets of (X, τ). Consequently their union $\{x_1, x_2, x_3, \ldots, x_n\}$ is also a closed subset of (X, τ). This further implies, the set $X \setminus \{x_1, x_2, x_3, \ldots, x_n\}$ is an open neighborhood of X. The intersection of this neighborhood with N which is $N \setminus \{x_1, x_2, x_3, \ldots, x_n\}$ is also a neighborhood of x. $A \cap (N \setminus \{x_1, x_2, x_3, \ldots, x_n\}) \setminus \{x\} = \emptyset$.

  This is a contradiction to the assumption that x is a limit point of A. Therefore the assumption that $A \cap N$ is finite is wrong. This proves that every neighborhood of x contains infinitely many points of A.

$\square$

Like T_0, being T_1 also is topological and hereditary property. You can prove this as an exercise. We continue our discussion of separation axioms in the next section where T_2-spaces are discussed with the name Hausdorff spaces.

2.6 Hausdorff Topological Spaces

Definition 18. A topological space (X, τ) is said to be Hausdorff space or a T_2 space if for any two distinct points x, y of X there exist open subsets G, H of (X, τ) such that $x \in G, y \in H$ and $G \cap H = \emptyset$.

Remark 2.6.1.

- Axiom T_2 is a topological property as well as hereditary property.
- Axiom T_2 is stronger than T_1. If a topological space satisfies axiom T_2 then it satisfies axiom T_1.
- The converse is not true. On an infinite set the cofinite topology satisfies T_1 axiom but does not satisfy T_2 axiom.
- On $X = \{a, b, c\}$ we can not define a topology which is T_1 but not T_2. Because if the topology is T_1 then singletons are closed sets. This means that all the sets having 2 elements are open sets. This forces the topology to be $P(X)$ which is T_2.
- We have proved earlier that every metric space is a Hausdorff space i.e. satisfies T_2 axiom.

Definition 19. A sequence in a metric space (X, τ) is a map $a : \mathbb{N} \to X$. We denote it by the notation (a_n). The sequence (a_n) is said to be convergent to an element l in X if for any neighborhood N of l there exists $n_0 \in \mathbb{N}$ such that $a_n \in N$ for all $n > n_0$. We use the notation $\lim a_n = l$.

Following theorem proves a connection between limit of a sequence and limit point of a set.

Theorem 2.6.1. *If (a_n) is a sequence of distinct points of A which converges to l then l is a limit point of A.*

Proof. Let N be any neighborhood of l. Since (a_n) converges to l, there exists $n_0 \in \mathbb{N}$ such that $a_n \in N$ for all $n > n_0$. But since the sequence is in A we get $a_n \in A \cap N$, for all $n > n_0$. Moreover all a_n's are distinct implies $a_n \neq l$ for at least one $n > n_0$. In other words there exists $n > n_0$ such that $a_n \in A \cap N \setminus \{l\}$. This proves $A \cap N \setminus \{l\} \neq \emptyset$ for every neighborhood N of l i.e. l is a limit point of A. $\square$

In general in a topological space limit of a sequence is not unique and hence limit is not a well defined concept. For example in the indiscrete topology $(X, P(X))$ any sequence converges to each point in the metric space. Thus if the set X has more than one point then each sequence has more than one limit. Another such example is $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{c\}, \{a, b\}, X\}$. Now define the sequence $\{a_n : n \in \mathbb{N}\}$ by $a_{2k} = a$ and $a_{2k+1} = b$ for all $k \in \mathbb{N}$. This sequence converges to both b and c.

Following is an example of a sequence that converges to infinitely many points.

Let $X = \mathbb{N}$. Let $J_n = \{n, n+1, n+2, \ldots\}$ and $\tau = \{\emptyset, J_1 = \mathbb{N}, J_2, J_3, \ldots\}$. Define the sequence (a_n) by $a_n = n$ for every n. The sequence converges to every $m \in \mathbb{N}$.

We cannot find such examples in Hausdorff spaces because of the following

Theorem 2.6.2. *In a Hausdorff topological space a convergent sequence has a unique limit.*

Proof. Let (X, τ) be a Hausdorff space. If possible suppose the sequence $\{a_n : n \in \mathbb{N}\}$ converge to two different points l and l^* of X. By the Hausdorff property there exist open sets G and H of (X, τ) such that $l \in G$, $l^* \in H$ and $G \cap H = \emptyset$. Now since (a_n) converges to l there exists $n_1 \in \mathbb{N}$ such that $a_n \in G$, for all $n > n_1$. Also since (a_n) converges to l^* there exists $n_2 \in \mathbb{N}$ such that $a_n \in H$, for all $n > n_2$. Let $m = \max\{n_1, n_2\}$. Now for all $n > m, n > n_1$ and $n > n_2$ and so $a_n \in G \cap H = \emptyset$. This is a contradiction which proves uniqueness of limits. $\square$

We now define an important property of a topological space.

Definition 20. We call a topological space (X, τ) separable if there exists a countable dense subset of it. That is, there exists $A = \{a_n : n \in \mathbb{N}\}$ satisfying $c(A) = X$.

The real line is a separable metric space because the set of rational numbers is a countable dense subset. But $\mathbb{R}$ with the co-countable topology is not a separable topological space. Because if you take a countable set A then $\mathbb{R} \setminus A$ is a co-countable set. This means $\mathbb{R} \setminus A$ is open and A is a closed set which gives $c(A) = A$ is a proper subset of $\mathbb{R}$ as $\mathbb{R}$ is uncountable.

Lemma 2.6.1. *Being separable is a topological property.*

Proof. Suppose $f : (X, \tau) \to (Y, \tau')$ is a homeomorphism. Suppose $A = (a_n : n \in \mathbb{N})$ is a countable dense set in X. We want to show that the countable set $B = f(A)$ is dense in Y. If $c(B)$ is a proper subset of Y then $H = Y \setminus c(B)$ is a nonempty subset of (X, τ). Let $G = f^{-1}(H)$. Then G is a non empty subset of (X, τ). Now $f^{-1}(c(B)) = c(f^{-1}(B)) = c(A) = X$. This contradicts that $G = X \setminus f^{-1}(c(B)) \neq \emptyset$. This proves $c(B) = Y$ which means that the countable subset B of Y is dense in Y. $\square$

Remark 2.6.2. Being separable is not hereditary.

In fact we can prove that any non-separable topological space is a subspace of a separable space. Let (Y, τ') be a non separable topological space. Take any object w which is not a member of Y. Let $X = Y \cup \{w\}$. Introduce a topology τ on X as follows: $\tau = \{\emptyset, G \cup \{w\} \mid G \in \tau'\}$. (X, τ) is a topological space and (Y, τ') is a subspace of it. $\{w\}$ is dense in (X, τ). This proves that (X, τ) is separable but subspace (Y, τ') is not separable.

Theorem 2.6.3. *Every second countable topological space is separable.*

Proof. Let (X, τ) be a topological space. Let $\{B_n : n \in \mathbb{N}\}$ be a base to the topology τ. We choose a point b_n from each of the non-empty sets B_n and form a countable set $B = \{b_n : n \in \mathbb{N}\}$. This set must be a dense subset of (X, τ). Because if it is not dense then $G = X \setminus c(B)$ is a non-empty open set of (X, τ). But $\{B_n : n \in \mathbb{N}\}$ is a base. Hence a subfamily $\{B_{n_k} : k \in \mathbb{N}\}$ generates G. Hence $G = \cup_{k\in\mathbb{N}} B_{n_k}$. Hence $\{b_{n_k} : k \in \mathbb{N}\} \subseteq G$. This contradicts the construction of G. Therefore $\{b_n : n \in \mathbb{N}\}$ is a countable dense subset of X. Hence (X, τ) is separable. $\square$

2.7 Regular and normal topological spaces

Definition 21. Suppose one point sets are closed in X. Then X is said to be regular if each pair consisting of one point x and a closed set B disjoint from x, there exist disjoint open sets containing x and B respectively. The space is called normal if for each pair A, B of disjoint closed sets of X there exist disjoint open sets containing A and B respectively.

Following theorem gives us a characterization of normal and regular spaces.

Theorem 2.7.1. *Let (X, τ) be a topological space in which one point sets are closed. Then*

- *X is regular if and only if given a point x of X and a neighborhood U of x, there is a neighborhood V of x such that $c(V) \subseteq U$.*

- *X is normal if and only if given a closed set A and an open set U containing A, there is an open set V containing A such that $c(V) \subseteq U$.*

Proof. • Suppose that X is regular and suppose that the point x and the neighborhood U of x are given. Let $B = X \setminus U$; then B is a closed set. By hypothesis, there exist disjoint open sets V and W containing x and B respectively. The set $c(V)$ is disjoint from B since if $y \in B$ the set W is a neighborhood of y disjoint from V. Therefore $c(V) \subseteq U$ as desired.

To prove the converse, suppose the point x and the closed set B not containing x are given. Let $U = X \setminus B$. By hypothesis there is a neighborhood V of x such that $c(V) \subseteq U$. The open sets V and $X \setminus c(V)$ are disjoint open sets containing x and B respectively. Thus X is regular.

- This proof is similar. Just replace the point x by the set A throughout.

□

2.8 Glossary

In this chapter, you have learnt the following:

- Homeomorphism between topological spaces.
- Hereditary property of a topological space.
- First and second countable spaces.
- Hausdorff topological space.
- Regular and normal topological spaces.
- Characterization of regular and normal topological spaces.

2.9 Bibliography

- James Munkres: Topology, Pearson
- George Simmons: Topology and Modern Analysis, TataMcgraw-Hill.
- M. A. Armstrong: Basic Topology, Springer UTM
- Wolfgang Thorn : Topological Structures; Holt, Rinehart and Winston, New York, Chicago.

2.10 Let Us Sum Up

A topological system is a generalization of the classical number system $\mathbb{R}$. But in such a wild generalization many important properties of the number system are lost. We regain these properties by introducing a number of additional axioms. There are many types of such axioms. One such type of axioms are separation axioms. They are denoted by $T_0, T_1, T_2, \ldots$. They are about enclosing a pair of points in disjoint open subsets, disjoint closed subsets etc. Of these we study T_0, T_1 and T_2. These properties are both topological and hereditary.

A T_0 space separates a pair of points from the space by a single open set. A space (X, τ) is T_0 if and only if distinct singleton sets have distinct closures. A space is T_1 if given any pair of distinct points there are open sets each one of them containing one point but not the other. A topological space is T_1 if and only if singleton sets are closed. In a T_1 space if p is a limit point of a set A then every neighborhood of p contains infinitely many points of A.

A T_2 space is also called a Hausdorff space. It separates distinct points by disjoint open sets. In a Hausdorff space a convergent sequence has unique limit. Regular and normal spaces satisfy stronger separation axioms. Countability axioms are other type of auxiliary axioms. We study first and second countability axioms. Both these axioms are topological and hereditary. A metric space is called separable if it has a countable dense subset. A second countable topological space is separable.

2.11 References for further reading

1. W.J. Pervin: Foundations of General Topology, Academic press , New York, London.

2. Kelley J.L, General Topology, Van Nostrand Reinhold Co., New York, 1955.

3. Dugundji J.: Topology, Allyn and Bacon, Boston, 1966.

2.12 Chapter End Exercises

1. Show that being T_1 is a topological and hereditary property.
2. Show that being T_2 is a topological and hereditary property.
3. Let τ be a topology on $\mathbb{R}$ generated by the collection $\{[a, b), a, b \in \mathbb{R}, a < b\}$. Prove that (R, τ) is a separable, Hausdorff, first countable space.
4. Show that in a separable topological space every collection of non-empty pairwise disjoint open sets is countable.
5. Prove that (X, τ) is a T_1 space if and only if for each $x \in X$, $\{x\} = \cap\{G : G \in \tau, x \in G\}$.
6. Show that subspace of a regular space is a regular space.
7. Show that if X is regular, every pair of points of X have neighborhoods whose closures are disjoint.
8. Show that if X is normal, every pair of disjoint closed sets have neighborhoods whose closures are disjoint.
9. Show that a closed subspace of a normal space is normal.
10. Let X be the set of all irrational numbers with usual metric d. Is (X, d) separable?

Chapter 3

Compactness

Chapter Structure

3.1 Introduction

Compactness is a word we use daily in our life, to indicate that objects occupy less space. Even in mathematics the word has the same sense in $\mathbb{R}^n$ namely that compact objects are those which are closed and bounded. However, in an arbitrary topological space such a nice formulation may fail, as we shall see from examples.

In an arbitrary topological space the correct notion of compactness is defined in terms of open covers. Using this defintion, we prove that compactness is preserved under continuous functions and under finite products. We then move on to the notion of local compactness and also study spaces which are not too far away from being compact: namely spaces which admit a one-point compactification.

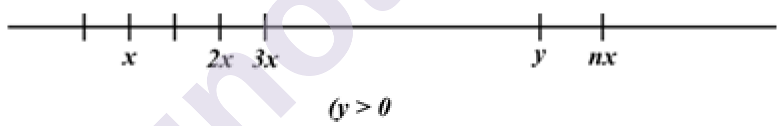
We also study a notion similar to compactness: namely the notion of a Lindelöf space.

3.2 Objectives

After going through this chapter you will know:
- General definition of compactness.
- Continuity preserves compactness.
- A finite product of compact spaces is compact.
- Definition of local compactness.
- Construction of one point compactification.
- Lindelöf Spaces.

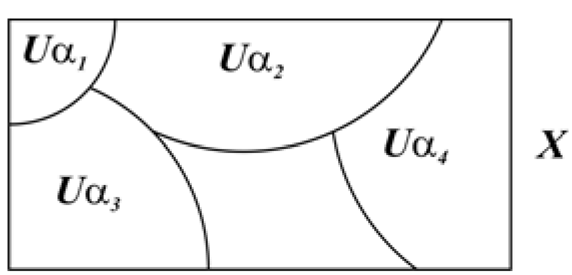
3.3 General definition of compactness

Before going to the notion of compactness in topological spaces, we begin by recalling an important property of real numbers: the Archimedean property. Let $x, y \in \mathbb{R}$ with $x > 0$. Then there exists a natural number n such that $nx > y$. Stated in words, this just means that by taking enough number of units of x, we can get past any given real number y.



In fact we will often use the following version of this property: if $y \in \mathbb{R}$, there exists $n_y \in \mathbb{N}$ such that $n_y > y$. (Use the above statement with $x = 1$.) Before going into the definition of compactness, let us understand what an open cover of a topological space is.

Definition 22. (Open cover) Let X be a topological space and $\{U_\alpha\}_{\alpha \in \Lambda}$ be a family of open subsets of X. Then $\{U_\alpha\}_{\alpha \in \Lambda}$ is said to be an open cover of X if $X \subset \cup_{\alpha \in \Lambda} U_\alpha$. (Note that this condition is same as saying $X = \cup_{\alpha \in \Lambda} U_\alpha$, as $U_\alpha \subset X$, for all $\alpha \in \Lambda$.)



From the above definition it is easy to check that $\{U_\alpha\}_{\alpha\in\Lambda}$ is an open cover of X if and only if $X = \cup_{\alpha\in\Lambda} U_\alpha$, for open subsets U_α of X.

Let us see some examples of open covers:

1. Let $X = \{1, 2, 3\}$ with the discrete topology. Then $\{1\}, \{2, 3\}$ is an open cover of X.

2. Let $X = (0, 1)$. Let $n \geq 1$ and $U_n = (0, \frac{1}{n})$. Then $\{U_n\}$ is an open cover of X.

3. Let $X = \mathbb{R}$. Let $n \geq 1$ and $U_n = (-n, n)$. Then, $\{U_n\}$ is an open cover of X.

4. Let $X = \mathbb{R}^2$. For $n \geq 1$, let $U_n = \{(x, y) \mid x^2 + y^2 = n^2\}$. Then $\{U_n\}$ is an open cover of X.

Check Your Progress Find other open covers of X in the examples above, distinct from the given ones.

We now study the definition of compactness for arbitrary topological spaces: this abstract defintion was first formulated in 1906 by Maurice Fréchet. (See the website: $www : //http - history.mcs.st - andrews.ac.uk/Biographies/Frechet.html$ and $http : //en.wikipedia.org/wiki/Compactness$ for further details.)

Definition 23. Let X be a topological space. X is said to compact if every open cover of X admits a finite subcover i.e. if there exists a family $\{U_\alpha\}_{\alpha\in\Lambda}$ of open subsets of X such that $X \subset \cup_{\alpha\in\Lambda} U_\alpha$, then there exist finitely many $\alpha_1, \ldots, \alpha_n \in \Lambda$ such that $X \subset U_{\alpha_1} \cup U_{\alpha_2} \cup \cdots \cup U_{\alpha_n}$. (In this case we say that the family $\{U_\alpha\}_{\alpha\in\Lambda}$ admits a finite subcover.)

Remark 3.3.1. The main stress in this definition is on **every open cover**. If you are able to prove that a specific open cover of X admits a finite subcover that does not mean the space is compact. For example, take $X = (0, 1)$ and for $n \geq 1$, $U_n = (0, \frac{1}{n})$. Then, clearly U_1 is the finite subcover of X.

On the other hand, if one takes another subcover of X, defined for $n \geq 1$, by $V_n = (0, 1 - \frac{1}{n})$, then this does not admit a finite subcover: for if there is a finite subcover say $V_{n_1}, V_{n_2}, \ldots, V_{n_k}$, let $n_j = \max\{n_1, \ldots, n_k\}$. Then, one can check that $(0, 1) \subset \cup_{n_i} V_{n_i} \subset V_{n_j} = (0, 1 - \frac{1}{n_j})$, which is impossible by the Archimedean property.

In other words, in order to say that a topological space X is not compact, it is enough to produce an open cover of X which admits no finite subcover. For example, the real numbers with the usual metric topology is not compact. Take the open cover $\{U_n := (-n, n)\}_{n\geq 1}$ of $\mathbb{R}$. This cannot admit a finite subcover, for if it is does, then we would get $\mathbb{R} \subset (-k, k)$, for some $k \in \mathbb{N}$, which is a contradiction.

Here are easy examples of compact sets:

Example 1. Every finite set is a compact set. (Since any open cover of this finite set will require atmost finitely many open sets to cover it.)

Example 2. Let X be a non-empty topological space with indiscrete topology. Then, X is compact.

Check Your Progress

1. Show that a finite union of compact sets is compact.

2. Show that every subset of the real line $\mathbb{R}$ with the finite complement topology is compact.

3.4 Continuity and compactness

One of the main themes in mathematics is to study objects with structures and maps between these objects. In topology, the objects under study are topological spaces and one is more interested in maps between such spaces that are continuous. This is because continuous maps preserve many topological properties. For example, we will prove here that compactness is a property that behaves well with respect to continuity i.e., continuous image of a compact set is always compact. This property helps us to get more examples of compact spaces.

Before proving that continuity preserves compactness, let us prove the following useful result for compactness for subspaces:

Lemma 3.4.1. *Let Y be a subspace of X (i.e., Y is a subset of X with the induced topology.) Then Y is compact if and only if every open covering of Y by sets open in X contains a finite subcollection covering of Y.*

Proof. We first recall that a subset $V_{\alpha,Y}$ of Y is open in Y if and only there exists an open set $V_{\alpha,X}$ of X such that $V_{\alpha,Y} = V_{\alpha,X} \cap Y$. (This is precisely the subspace topology on Y.)

(a) Suppose Y is compact and let $\{V_{\alpha,X}\}_{\alpha\in\Lambda}$ be an open cover of Y by sets open in X. Then, $\{V_{\alpha,X} \cap Y\}_{\alpha\in\Lambda}$ be an open cover of Y by sets open in Y. As Y is compact, we get finitely many α's that cover Y. This proves the existence of a finite subcollection covering of Y.

(b) For the other way, start with an open covering of Y, say $\{V_{\alpha,Y}\}_{\alpha\in\Lambda}$. Hence there exists a family of open sets $\{V_{\alpha,X}\}_{\alpha\in\Lambda}$ of X such that $V_{\alpha,Y} = V_{\alpha,X} \cap Y$, i.e., $V_{\alpha,Y} \subset V_{\alpha,X}$. Then clearly we get:

$$Y \subset \cup_{\alpha\in\Lambda} V_{\alpha,Y} = \cup_{\alpha\in\Lambda}(V_{\alpha,X} \cap Y) \subset \cup_{\alpha\in\Lambda} V_{\alpha,X}.$$

Thus, we get a covering of Y by open subsets of X. The hypothesis now gives a finite subcollection covering Y, i.e. there exist $\alpha_1, \ldots, \alpha_n$ such that $Y \subset V_{\alpha_1,X} \cup V_{\alpha_2,X} \cdots \cup V_{\alpha_n,X}$. Taking intersection of both sides with Y, we get that

$$Y \subset (V_{\alpha_1,X} \cup V_{\alpha_2,X} \cdots \cup V_{\alpha_n,X}) \cap Y = \cup_{i=1}^n (V_{\alpha_i,X} \cap Y) = \cup_{i=1}^n V_{\alpha_i,Y}.$$

Thus we get a finite subcover of Y, proving that Y is compact.

□

We will now prove the following:

Lemma 3.4.2. *Let X, Y be topological spaces, with X compact. If $f : X \to Y$ is continuous, then $f(X)$ is compact in Y.*

Proof. Let $\{V_{\alpha,Y}\}_{\alpha\in\Lambda}$ be an open cover of $f(X)$. Then, it is easy to check that $\{f^{-1}(V_{\alpha,Y})\}_{\alpha\in\Lambda}$ is an open cover of X. As X is compact, we get $\alpha_1, \ldots, \alpha_n$ such that $X \subset f^{-1}(V_{\alpha_1,Y}) \cup f^{-1}(V_{\alpha_2,Y}) \cup \cdots \cup f^{-1}(V_{\alpha_n,Y})$. By noting that $(f \circ f^{-1})V_{\alpha_i,Y} \subset V_{\alpha_i,Y}$, we get that $f(X) \subset V_{\alpha_1,Y} \cup \cdots \cup V_{\alpha_n,Y}$. This proves that the continuous image of a compact set is compact. □

Example 3. Using Lemma 3.4.2 above, we will prove that the unit circle S^1 is compact. (Recall: $S^1 := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$.) Let $f : \mathbb{R} \to \mathbb{R}^2$ be given by $f(\theta) = (\cos(\theta), \sin(\theta))$. Then, clearly f is continuous and image of f is S^1. In fact by observing that $f([0, 2\pi]) = S^1$ and noting that $[0, 2\pi]$ is a compact subset of $\mathbb{R}$, we have that S^1 is a compact subset of $\mathbb{R}^2$.

Lemma 3.4.3. *Every closed subset of a compact space is compact.*

Proof. Let X be a compact space and let F be a closed subset of X. Let $\{V_\alpha\}_{\alpha\in\Lambda}$ be an open cover of F. Note that by the definition of subspace topology on F, there exist open subsets U_α of X for each $\alpha \in \Lambda$ with the property that $V_\alpha = U_\alpha \cap F$.

Since F is closed, F^c is open and $\{U_\alpha\}_{\alpha\in\Lambda} \cup F^c$ is an open cover, say $\mathcal{R}$ of X. As X is compact, $\mathcal{R}$ admits a finite subcover $\mathcal{R}'$.

If $F^c \in \mathcal{R}'$, then it is easy to check that $\{U_\alpha \cap F := V_\alpha \mid U_\alpha \in \mathcal{R}' \setminus F^c\}$ gives an open cover of F. If $F^c \notin \mathcal{R}'$, then $\{U_\alpha \cap F := V_\alpha \mid U_\alpha \in \mathcal{R}\}$ is an open cover of F. □

Lemma 3.4.4. *Every compact subset of a Hausdorff space is closed.*

Proof. Let C be a compact subset of a Hausdorff space X. We will prove that $X \setminus C$ is open, which will prove C is closed in X.

Let x be a point of $X \setminus C$. For each point $y \in C$, using the Hausdorff property, choose disjoint neighbourhoods U_x, V_y in X of x and y respectively. Then $\mathcal{R} = \{V_y \mid y \in C\}$ is an open covering of C and as C is compact, this open cover admits a finite subcover of C, say

$V_{y_1}, \ldots, V_{y_n}$. Hence the open set $V := V_{y_1} \cup \cdots \cup V_{y_n}$ contains C. Let $U := U_{y_1} \cap \cdots \cap U_{y_n}$. Clearly U being a finite intersection of open sets is open in X. Also, $U \cap V = \emptyset$. (For if there exists $r \in U \cap V$, then $r \in V_{y_i}$ for some $1 \leq i \leq n$ and $r \in U_{y_i}$ for all $1 \leq i \leq n$ implies that $r \in U_{y_i} \cap V_{y_i}$, which is not possible as $U_{y_i} \cap V_{y_i} = \emptyset$.) $\square$

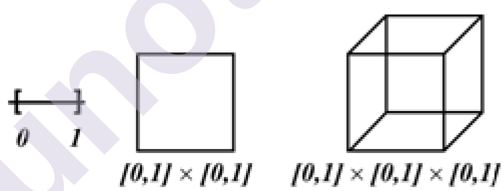
Check Your Progress

1. Show from first principles (i.e., using the definition of compactness) that $\mathbb{R}$ with the usual topology is not compact. (Hint: It is enough to produce an open cover which does not admit a finite subcover.)

2. Show that closed subsets of compact sets are compact. (Hint: Imitate the proof in Lemma 3.4.3.)

3.5 Finite products and compactness

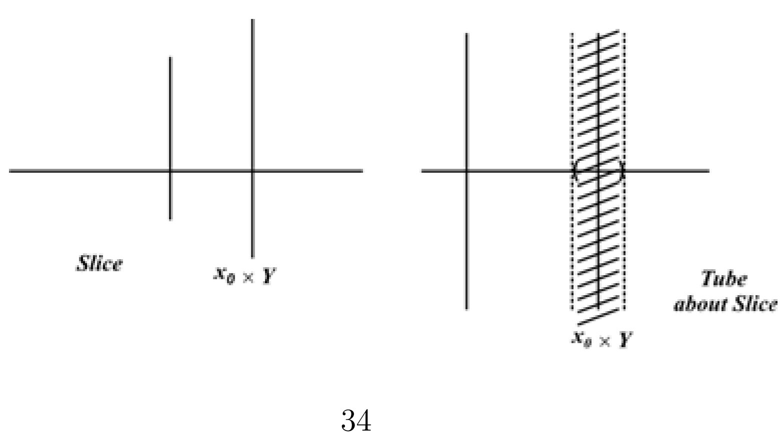
Compactness is preserved under taking finite products i.e., product of finitely many compact topological spaces is compact. This immediately implies for example that the torus i.e., $S^1 \times S^1$ is compact in $\mathbb{R}^2$ and so is the unit cube in $\mathbb{R}^3$, being the finite product of the unit interval $[0, 1]$ with itself.



In order to prove this result, we require a technical lemma, called the Tube Lemma. Before proving this lemma, we define a few terms.

Definition 24. Let X, Y be topological spaces. If $x_0 \in X$, the set $x_0 \times Y$ is called a slice of $X \times Y$. If W is an open subset of X containing x_0, the set $W \times Y$ is called a tube about $x_0 \times Y$.

Here is a picture to illustrate the above concepts:



Lemma 3.5.1. *Let X, Y be topological spaces. Then the projection maps $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ are continuous in the product topology. Let $x_0 \in X$. Then the slice $x_0 \times Y$ is homeomorphic to Y. In particular, if Y is compact, then the slice $x_0 \times Y$ is a compact subset of $X \times Y$, where $X \times Y$ is given the product topology.*

Proof. Consider the projection map $p_2 : x_0 \times Y \to Y$ by $p_2(x_0, y) = y$. We will check that p_2 is continuous by checking that the inverse image of an open set V of Y is open in $x_0 \times Y$. For this note that $p_2^{-1}(V) = x_0 \times V$, which is open in $x_0 \times Y$. Clearly p_2 is injective as well as surjective and hence it is a bijection. We now need to prove that the direct image of a basic open subset in $x_0 \times Y$ is open in Y. This will then prove that $x_0 \times Y$ is homeomorphic to Y and Y compact implies that $x_0 \times Y$ is compact as well, since compactness is preserved under continuous maps.

Note that a set is open in $x_0 \times Y$ if and only if it is the intersection of an open set in $X \times Y$ with $x_0 \times Y$. Hence, it can be checked (recall basic open sets) that $W \subset x_0 \times Y$ is open if and only if there exist open subsets V_i of Y such that $W = \cup_i (x_0 \times V_i)$. Then, clearly $p_2(W) = \cup_i V_i$, is an open subset of Y, proving that p_2 is a homeomorphism. $\square$

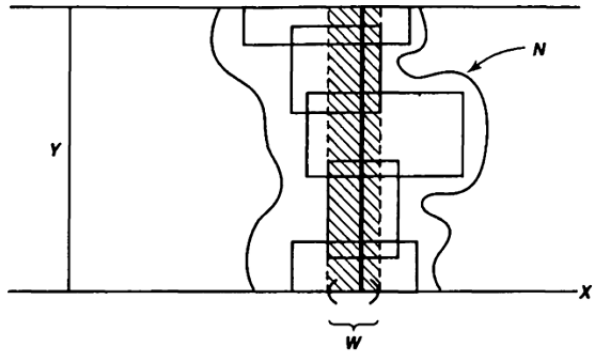
Lemma 3.5.2. *(Tube Lemma) Let X, Y be topological spaces with Y compact. If N is an open subset of $X \times Y$ containing the slice $x_0 \times Y$, then N contains a tube $W \times Y$ about $x_0 \times Y$, where W is a neighbourhood of x_0 in X.*

Proof. Cover the slice $x_0 \times Y$ by basis elements $U \times V$, with $U \times V$ lying in N. (Here U, V are open subsets of X and Y respectively and $X \times Y$ is given the product topology. This can be achieved, as $x_0 \times Y \subset N$ and N is an open set in the product topology.) By Lemma 3.5.1, $x_0 \times Y$ being compact admits a finite subcover by these basic elements, say $U_1 \times V_1, \ldots, U_n \times V_n$. As U_i are open in X, so is their finite intersection $U_1 \cap \cdots \cap U_n$, say W. As $x_0 \in U_i$, for all $1 \leq i \leq n$, we get that $x_0 \in \cap_{i=1}^n U_{:} = W$.

We now claim that the tube $W \times Y \subset N$, by proving that $U_i \times V_i$ cover $W \times Y$, i.e., we prove that $W \times Y \subset \cup_{i=1}^n (U_i \times V_i)$.

For this, let $(w, y) \in W \times Y$. Consider the point (x_0, y), i.e. the point with the same y co-ordinate as the point we started with. Then, $(x_0, y) \in U_j \times V_j$ for some j, as $U_i \times V_i$ is a cover of $x_0 \times Y$. Then, $(w, y) \in U_j \times V_j$, as $w \in W$ implies that $w \in U_j$ too. This proves that $W \times Y \subset \cup_{i=1}^n (U_i \times V_i) \subset N$.

Here is a picture of what tube lemma achieves:



$\square$

Remark 3.5.1. One has a more general theorem which says that an arbitrary product of compact spaces is compact too. This theorem is known as Tychonoff's theorem. The proof of this is beyond the scope of these notes, but it is a very useful theorem.

We use the Tube lemma now to prove that the product of two compact spaces is again compact. The result for finitely many compact spaces then follows by induction.

Theorem 3.5.1. *Let X, Y be compact topological spaces. Then their Cartesian product $X \times Y$ (with the product topology) is also compact.*

Proof. Let $\mathcal{R}$ be an open cover of $X \times Y$. Given $x_0 \in X$, the slice $x_0 \times Y$ is compact by Lemma 3.5.1. Hence, there exist finitely many open subsets, say $U_1, \ldots, U_n$ of $\mathcal{R}$ which cover $x_0 \times Y$. Then, $N := U_1 \cup \cdots \cup U_n$ is an open set containing the slice $x_0 \times Y$. Hence, by Tube Lemma, there exists a tube say $W_{x_0} \times Y$ around $x_0 \times Y$, contained in N. Clearly, $W_{x_0} \times Y$ has a finite subcover, as $W_{x_0} \times Y \subset N = U_1 \cup \cdots \cup U_n$.

Now for each $x \in X$, repeat the procedure above to get a tube $W_x \times Y$ containing the slice $x \times Y$. Now observe that $\{W_x | x \in X\}$ is an open cover of X, as W_x are open in X by construction. As X is compact, there exist $x_1, \ldots, x_n \in X$ such that $X \subset W_{x_1} \cup \cdots \cup W_{x_n}$. Hence,

$$X \times Y \subset (W_{x_1} \times Y) \cup \cdots \cup (W_{x_n} \times Y).$$

As each of $W_{x_i} \times Y$ has a finite cover, so does $X \times Y$. Hence, $X \times Y$ is compact. $\square$

Check Your Progress

1. Let f be a continuous real-valued function on $[a, b]$. Prove that the graph of f, i.e., the set $\{(x, f(x)) \mid x \in [a, b]\}$ is a compact subset of $\mathbb{R}^2$.

2. State true or false with correct justification (if false, give a counter-example): An arbitrary union of compact sets is compact.

3.6 Local compactness and one-point compactification

There is another more useful notion of compactness called local compactness. This is a weaker notion than compactness. We shall see examples of spaces which are locally compact but not compact.

Definition 25. (Local Compactness) A topological space X is said to be locally compact at $x \in X$ if there is some compact subset C of X that contains a neighbourhood of x. If X is locally compact at each of its points, then X is said to be locally compact.

Every compact space is locally compact, because the compact subset C containing a neighbourhood of x in X can be taken to be the whole space X itself. The converse need not hold: for example, $\mathbb{R}$ with the usual topology is locally compact but not compact.

Check Your Progress

1. Verify that $\mathbb{R}$ is locally compact but not compact.
2. Prove that the space $\mathbb{Q}$ of rationals is not locally compact.
3. The infinite dimensional product space $\mathbb{R}^{\mathbb{N}}$ is not locally compact.

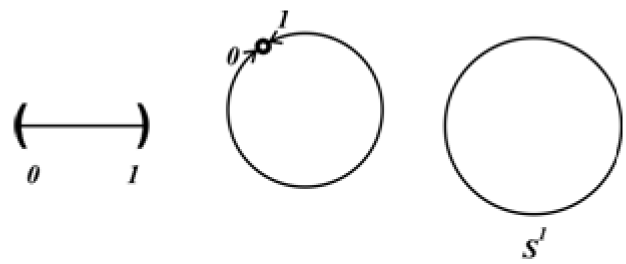
Definition 26. Let X be a topological space and $x \in X$. We say x has a local base of compact neighbourhoods, if x has arbitrarily small compact neighbourhoods in X.

Lemma 3.6.1. *Let X be a regular space and suppose X is locally compact at x. Then, x has a local base of compact neighbourhoods in X.*

Proof. Let X be regular and locally compact at x. Then x has a compact neighbourhood C in X. Let $U \subset C$ be a neighbourhood of x in X. By regularity, there exists a closed neighbourhood F of x such that $F \subset U$. Note that $F \cap C \subset U$ is a closed neighbourhood of x in F, since C is closed in X.

Since F is compact, so is $F \cap C$. Thus, we get a chain of decreasing compact neighbourhoods $x \in F \cap C \subset C$. Repeat the argument above with $F = F \cap C$, to get a local base of compact neighbourhoods for $x \in X$. $\square$

Sometimes your space need not be compact, but adding just one point to it makes it compact. For example, consider the open interval $(0, 1)$ as a string of thread. If you are able to fuse the two ends of it, by joining 0 and 1 together, then you would get a circle, which is now a compact space! Thus by adding one point suitably you are sometimes able to get a compact space out of your old space. Such a process is called as one point compactification.



More precisely, the mathematical definition goes as follows:

Definition 27. (One point compactification) Let X be a locally compact Hausdorff space. Choose y not in X. (It is a convention to denote such a y by the symbol ∞.) Consider the set $Y = X \cup \{\infty\}$. Declare the following subsets of Y to be open:

(a) U, with U open in X. (We call these open sets of Type I.)

(b) $Y \setminus C$, with C compact subset of X. (We call these open sets of Type II.)

We will check below that Y becomes a topological space when the above sets are declared as open sets. Such a space Y is called a one-point compactification of X.

Proposition 3.6.1. *Let Y be the set defined above with the open sets of the two types. Then, Y is a topological space.*

Proof. We need to check the following axioms for open sets:

1 Finite intersection of open sets in Y is again an open set in Y.

2 Arbitrary union of open sets in Y is again an open set in Y.

Before we check the first axiom, we make the following observations:

1(a) Let U_1, U_2 be both open sets of Type I. Then clearly, $U_1 \cap U_2$ is again an open set of Type I.

1(b) Let U_1 be an open set of Type I and U_2 be an open set of Type II. We will prove that $U_1 \cap U_2$ is again an open set in Y. Since U_2 is of Type II, there exists a compact subset C_2 of X such that $U_2 = Y \setminus C_2$. Hence, $U_1 \cup U_2 = U_1 \cup (Y \setminus C_2) = U_1 \cup (X \setminus C_2)$. Note that as C_2 is compact subset of a Hausdorff space, hence closed in X. Thus, $X \setminus C_2$ is open in X and hence $U_1 \cup (X \setminus C_2)$ is open of Type I.

1(c) Let U_1, U_2 be both open sets of Type II i.e, there exist compact sets $C_1, C_2 \in X$ such that $U_1 = Y \setminus C_1$ and $U_2 = Y \setminus C_2$. Hence, $U_1 \cup U_2 = Y \setminus (C_1 \cap C_2)$. As a finite intersection of compact sets is again compact, we observe that $U_1 \cup U_2$ is an open set of Type II.

Now let $U_1, U_2, \ldots, U_n$ be open subsets of Y. By rearranging the indices that occur namely, $1, 2, \ldots, n$, we may assume that $U_1, \ldots, U_{n_1}$ are of Type I; $U_{n_1+1}, U_{n_1+2}, \ldots, U_n$ are of Type II, for some $1 \leq n_1 \leq n$. Using the observation above, we see that $U_1 \cap \cdots \cap U_{n_1}$ is again a subset of Type I and $U_{n_1+1} \cap \cdots \cap U_n$ is again a subset of Type II.

Hence, $U_1 \cap \cdots \cap U_n$ is again an open subset of Y, proving that the first axiom for open sets holds in Y.

We now check that an arbitrary union of open subsets of Y is open in Y by making the following observations:

1(a) Let $\{U_\alpha\}_{\alpha \in \Lambda}$ be a family of open sets of Type I. Then clearly, $\cup_{\alpha \in \Lambda} U_\alpha$ is again an open set of Type I.

1(b) Let $\{U_\alpha\}_{\alpha \in \Lambda}$ be a family of open sets of Type I and $\{U_\beta\}_{\beta \in \Lambda'}$ be a family of open subsets of Type II.

We will prove that $(\cup_{\alpha \in \Lambda} U_\alpha) \cup (\cup_{\beta \in \Lambda'} U_\beta)$ is again an open set in Y. For this observes that $(\cup_{\alpha \in \Lambda} U_\alpha) \cup (\cup_{\beta \in \Lambda'} U_\beta)$ equals $U \cup (Y \setminus C) = Y \setminus (C \setminus (U \cap C))$. Now note that $U \cap C$ is open in C and hence, $C \setminus (U \cap C)$ is closed in C and hence compact, as C is compact. Thus, $(\cup_{\alpha \in \Lambda} U_\alpha) \cup (\cup_{\beta \in \Lambda'} U_\beta)$ is a set of Type II and hence open.

1(c) Let $\{U_\beta\}_{\beta \in \Lambda'}$ be a family of open sets of Type II. Then $\cup_{\beta \in \Lambda'} U_\beta$ is again an open set of Type II as this equals $\cup_{\beta \in \Lambda'} Y \setminus C_\beta$, for compact subsets C_β of X. This union now equals $Y \setminus \cap_{\beta \in \Lambda'} C_\beta$. Now note each C_β is a compact subset of the Hausdorff space X and hence is closed in X. Hence, $\cap_{\beta \in \Lambda'} C_\beta$ is a closed subset of X. Moreover, for a fixed $\beta_0 \in \Lambda$, $\cap_{\beta \in \Lambda'} C_\beta \subset C_{\beta_0}$ and hence we have a closed subset of a compact set and hence $\cap_{\beta \in \Lambda'} C_\beta$ is a compact subset of X. This implies that we get that $\cup_{\beta \in \Lambda'} U_\beta$ is an open set of Type II.

Using these observations, it easily follows that an arbitrary union of open sets is again open.

□

Remark 3.6.1. Every space admits a unique one point compactification upto homeomorphism.

Here is an example to keep in mind:

Example 4. One-point compactification of $(0, 2\pi)$ is homeomorphic with the circle S^1.

Define a map $f : (0, 2\pi) \cup \{\infty\} \to S^1$ by: $f(\theta) = (\cos(\theta), \sin(\theta))$, for all $x \in (0, 2\pi)$ and $f(\infty) = (1, 0)$. Clearly, f is a bijection.

We now prove that f is continuous. Let $V \subset S^1$ be open. Then, either $(1, 0) \notin V$ or $(1, 0) \in V$. If $(1, 0) \notin V$, then V is an arc of S^1 not containing $(1, 0)$ and hence $f^{-1}(V)$ is of the form (θ_1, θ_2) for some $\theta_1 < \theta_2 \in (0, 2\pi)$. In this $f^{-1}(V)$ is an open subset of $(0, 1)$.

If $(1, 0) \in V$, then there exists an arc around $(1, 0)$ contained in V. Thus, this arc consists of some points in the first quadrant and some points in the fourth quadrant, which are determined by angles $\theta_1 < \theta_2$ with $\theta_1 > 0$ and $\theta_2 < 2\pi$. Thus, in this case $f^{-1}(V) = \{\infty\} \cup (0, \theta_1) \cup (\theta_2, 2\pi)$. Clearly, then $f^{-1}(V) = Y \setminus C$, where C is the compact subset $[\theta_1, \theta_2]$ of $(0, 1)$. This finishes the proof that f is continuous.

Let $Y = (0, 2\pi) \cup \{\infty\}$. We will now prove that the direct image of an open subset of Y under f is again open. Similar to the case above, it is easy to check that if U is an open subset of $(0, 2\pi)$, then $f(U)$ is open in S^1. Let U be an open subset of $(0, 2\pi) \cup \{\infty\}$ containing ∞. Then, U is of the form $Y \setminus C$, for some compact subset C of X, then $f(U) = f(Y) \setminus f(C)$. As C is compact and we have proved that f is continuous, we get that $f(C)$ is compact and hence closed in S^1. Hence, $f(U) = S^1 \setminus f(C)$ is an open subset of S^1, proving that f is an open map. This completes the proof that f is a homeomorphism.

We now study the basic properties of one-point compactification.

Theorem 3.6.1. *Let X be a locally compact Hausdorff space which is not compact. Let Y be the one-point compactification of X. Then, Y is a compact, Hausdorff space; X is a subspace of Y, the set $Y \setminus X$ consists of a single point and $\overline{X} = Y$.*

Proof. We will prove first that X is a (topological) subspace of Y i.e., we will prove a set is open in X if and only if it is the intersection with X of some open set in Y. Let $U \subset X$ be open. Then, clearly $U = Y \cap U$, and U is open in Y by the topology on Y. Conversely, let $U = V \cap X$, for an open set V in Y. Since open sets in Y are of two types, suppose V is of the first type: i.e., let $V = U_1$, for some open subset U_1 of X. Then clearly V is open in X.

Suppose V is of the second type i.e., $V = (Y \setminus C) \cap X$, for some C compact in X. Then, $V = (Y \cap X) \setminus (C \cap X) = X \setminus C = C^c$, as $C \subset X$. Since C is a compact subset of a Hausdorff space, it is closed in X and hence C^c is open in X, proving that V is open in X.

We will now prove that ∞ is a limit point of X and this will imply $\overline{X} = Y$. For this we need to show that every open subset around ∞ intersects X in a point different from ∞. For this observe that every open subset around ∞ is the complement in Y of a compact subset C

of X. As X is not compact, C is a proper subset of X and hence the complement $X \setminus C$ contains at least one point. This then implies that every open subset around ∞ intersects X in a point different from ∞, proving that ∞ is a limit point of X.

Clearly, $Y \setminus X$ is the single point infinity and it now remains to prove that Y is compact and Hausdorff. First we prove compactness of Y. Let $\mathcal{R}$ be an open covering of Y. As ∞ should belong to this collection, $\mathcal{R}$ contains an open subset of Type II, say of the form $Y \setminus C$, with C compact in X. Now take all members of $\mathcal{R}$ other than $Y \setminus C$ and intersect them with X. They form a collection of open sets in X covering C. Compactness of C implies that there exist finitely many of them covering C. Take the corresponding finitely many elements of $\mathcal{R}$ alongwith $Y \setminus C$ to get an open cover of Y.

Let us now check that Y is Hausdorff: let $x, y \in Y$. If both x, y lie in X, then X Hausdorff implies that there exist disjoint open sets U, V in X such that $x \in U$ and $y \in V$. If $x \in X$ and $y = \infty$, then local compactness of X allows us to choose a compact set C in X containing a neighbourhood U of x. Then, U and $Y \setminus C$ are disjoint neighbourhoods of x and ∞ respectively in Y.

□

Check Your Progress

1. Prove that the composition of two homeomorphisms is again a homeomorphism.

2. Show that the one-point compactification of $(0, 1)$ is also homeomorphic to S^1. (Hint: Use that $(0, 1)$ and $(0, 2\pi)$ are homeomorphic and use the exercise above.)

3. Check that the open sets in X described in Theorem 3.6.1 above do give a topology on X.

3.7 Lindelöf Topological Spaces

Definition 28. A topological space X is said to be a Lindelöf space, if every open cover of X admits a countable subcover.

Lemma 3.7.1. *The Lindelöff property is preserved under continuous functions i.e., if X is a Lindelöf topological space and $f : X \to Y$ is continuous, then $f(X)$ is Lindelöf.*

Proof. Let X be a Lindelöf space and $f : X \to Y$ be continuous. We will prove that $f(X)$ is Lindelöf. Let $\{V_\alpha\}_{\alpha \in \Lambda}$ be an open cover of $f(X)$. Then, it is easy to check that $\{f^{-1}(V_\alpha)\}_{\alpha \in \Lambda}$ is an open cover of X. As X is Lindelöf, $\{f^{-1}(V_\alpha)\}_{\alpha\in\Lambda}$ admits a countable subcover, say $\{f^{-1}(V_{\alpha_i})\}_{\alpha_i\in\Lambda, i\in\mathbb{N}}$. Then it can be checked that $(V_{\alpha_i})_{\alpha_i\in\Lambda, i\in\mathbb{N}}$ is a countable subcover of $\{V_\alpha\}_{\alpha\in\Lambda}$ for $f(X)$. This proves that $f(X)$ is Lindelöf. $\square$

Check Your Progress

- Show that if A is a closed subspace of a Lindelöf topological space X, then A is Lindelöf.
- Show that if X is compact and Y is Lindelöf, then $X \times Y$ is Lindelöf.

Theorem 3.7.1. *For a metric space (X, d) the following are equivalent:*

(a) (X, d) is Lindelöf.

(b) (X, d) is separable.

(c) (X, d) is second countable.

Proof. $(a) \implies (b)$ Let (X, d) be a Lindelöf metric space. We need to show that X has a countable dense subset. For each $n \in \mathbb{N}$, let $\mathcal{C}_n = \{B(x, \frac{1}{n}) \mid x \in X\}$. Then, clearly $\mathcal{C}_n$ is an open cover of X. As X is Lindelöf, there exists a countable subcover $\mathcal{D}_n = \{B(x_{ni}, \frac{1}{n}) \mid i \geq 1, i \in \mathbb{N}\}$ of $\mathcal{C}_n$, for a fixed n. Let $\mathcal{D} = \{x_{n,i} \mid n \in \mathbb{N},\ i \in \mathbb{N}\}$. Clearly $\mathcal{D}$ is a countable subset of X, as $\mathcal{D}$ is a countable union of countable sets. We will now show that $\mathcal{D}$ is dense in X, which will prove that (X, d) is separable.

Let $y \in X$. We will prove that for every $r > 0$, $B(y, r) \cap \mathcal{D} \neq \emptyset$, which will prove that $\mathcal{D}$ is dense in X. To see this, choose $m \in \mathbb{N}$ such that $\frac{1}{m} < r$. Then, $B(y, \frac{1}{m}) \subseteq B(y, \frac{1}{r})$. Since for this m, $\mathcal{D}_m$ is a cover of X there exists $k \in \mathbb{N}$, such that $y \in B(x_{m,k}, \frac{1}{m}) \in \mathcal{D}_m$ which means that $d(y, x_{m,k}) < \frac{1}{m}$ i.e., $d(x_{m,k}, y) < \frac{1}{m}$ i.e., $x_{m,k} \in B(y, \frac{1}{m}) \subseteq B(y, r)$. Thus $x_{m,k} \in B(y, r) \cap \mathcal{D}$, proving that $B(y, r) \cap \mathcal{D} \neq \emptyset$, as required.

$(b) \implies (c)$ We will prove that if (X, d) is a separable metric space, then it is second countable. Let (X, d) be a separable metric space i.e., there exists a subset A of X such that A is countable and $c(A) := \overline{A} = X$. Consider $\mathcal{B} := \{B(x, r) \mid x \in A; r \in \mathbb{Q}; r > 0\}$. Clearly, $\mathcal{B}$ is a countable collection of open sets in X. We will show that $\mathcal{B}$ is a basis for (X, d), which will prove that (X, d) is second countable.

For this, let $x \in X$ and let U be an open set in X such that $x \in U$. We will prove that there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$. Since $x \in U$ and U is open, there exists $r > 0, r \in \mathbb{Q}$ such that $x \in B(x, r) \subseteq U$. As $\overline{A} = X$, there exists $a \in A$ such that $a \in A \cap B(x, r) \setminus \{x\}$. Let $B = B(a, \frac{r}{2})$. Then, B belongs to $\mathcal{B}$ and using the triangle inequality, it is easy to check that $x \in B \subseteq U$. This prove that $\mathcal{B}$ is a countable basis for (X, d), proving that (X, d) is second countable.

$(c) \implies (a)$ Let (X,d) be a second countable space. Let $\{U_i\}_{i\in I}$ be an open cover of X. Since X has a countable basis, denote this by $\{V_n\}_{n\in\mathbb{N}}$, where V_n are open subsets of X. Let $S = \{n \in \mathbb{N} \mid V_n \subseteq U_i \text{ for some i } \in \text{ I}\}$. Clearly, S is countable as a subset of a countable set is countable. Using the fact that $\{V_n\}_{n\in S}$ is a basis of X, it is easy to check that $\{V_n\}_{n\in S}$ is an open cover of X. For each $n \in S$, choose $i(n) \in I$ such that $V_n \subseteq U_{i(n)}$, for some $i(n) \in I$. Then, $\{U_{i(n)}\}_{n\in S}$ gives the required countable subcover of X, proving that X is Lindelöf. (Note here that we did not use the fact that X was a metric space. Thus this result is more general, but here we will restrict ourselves to metric spaces.) $\square$

In general, Lindelöff property is not hereditary. Here is a counter-example. Let X be an uncountable set and let $x_0 \in X$. Let $\tau = \{A \subseteq X \mid x_0 \notin A\}$. It can be checked that τ is indeed a topology on X. It is also easy to check that X is Lindelöf, in fact X is compact. It can now be checked that (Y, τ_Y) with $Y = X \setminus \{x_0\}$ with the subspace topology τ_Y is not Lindelöf.

In general, even a finite product of Lindelöff spaces is not Lindelöff. Other results regarding these spaces are very technical. We shall not go into further details about this property, but the interested reader can look up the references for further reading.

3.8 Glossary

In this chapter, you have learnt the following:

- Open covers and finite subcover of a given cover.
- Definition of Compactness: Every open cover has a finite subcover.
- Continuous image of a compact set is compact.
- Closed subset of a compact topological space in compact.
- Tube lemma.
- Products of finitely many compact spaces is compact.
- Local compactness: every point x is contained in a compact set containing a neighbourhood of x.
- A regular space, locally compact at x, admits a local base of compact neighbourhoods of x.
- One-point compactification.

- Basic properties of one-point compactification.

3.9 Bibliography

- Bartle and Sherbert, Introduction to Real Analysis, Wiley.
- James Munkres: Topology, Pearson.
- Rudin, W.: Principles of Mathematical Analysis. Third Edition, Mc-Graw Hill International Editions.
- Singer, I. M., Thorpe, J. A.: Lecture Notes on Elementary Topology and Geometry.

3.10 Let Us Sum Up

In this chapter, we first learn the abstract definition of compactness. A topological space is said to be compact, if every open cover of it admits a finite subcover. Compactness is a nice property of a topological space, as it is preserved under continuity and finite products. These properties help us to get more examples of compact spaces, like the unit circle S^1 and the torus, $S^1 \times S^1$. One important ingredient in the proof of the second property (namely, finite product of compact spaces is compact) is the Tube Lemma.

Sometimes, a topological space may fail to be compact, but it may be locally compact i.e., every point has a compact neighbourhood. For example, the real line with the usual topology is locally compact, but not compact. Out of such locally compact spaces, some spaces are not very far away from being compact. These spaces can then be made compact by adding one point and defining a suitable topology on this new space, such that it becomes compact. This process is called the Alexandroff one-point compactification. We have studied the properties of one-point compactification and seen one explicit example of such a compactification.

3.11 References for further reading

1. George Simmons: Topology and Modern Analysis, TataMcgraw-Hill.

2. M. A. Armstrong: Basic Topology, Springer UTM

3. W.J. Pervin: Foundations of General Topology, Academic press , New York, London.

4. Kelley J.L, General Topology, Van Nostrand Reinhold Co., New York, 1955.

5. Wolfgang Thorn: Topological Structures; Holt, Rinehart and Winston, New York, Chicago.

3.12 Chapter End Exercises

1. Show that an arbitrary intersection of compact sets is again a compact set.

2. This is a generalization of the above exercise: Let $f : X \to Y$, with Y compact. Then f is continuous if and only if the **graph** of f, $G_f := \{(x, f(x)) \mid x \in X\}$ is closed in $X \times Y$.

3. Show that the one-point compactification of $\mathbb{R}$ is homeomorphic to S^1.

4. A relation C on a set X is said to be a simple order if it has the following properties:

   (a) (Comparibility) For every $x, y \in X$ for which $x \neq y$, either xCy or yCx holds.

   (b) (Non-reflexivity) For no $x \in A$ does the relation xCx hold.

   (c) (Transitivity) If xCy and yCz, then xCz.

   A set X with a simple order is said to have the least upper bound property if every non-empty subset X_0 of X that is bounded above has a least upper bound.

   Show that every simply ordered set with the least upper bound property is locally compact.

5. Consider the set $\mathbb{Q}$ of all rational numbers as a metric space with the usual metric given by $d(p, q) = |p - q|$. Let E be the set of all $p \in \mathbb{Q}$ such that $2 < p^2 < 3$. Show that E is closed and bounded in $\mathbb{Q}$, but not compact.

6. Suppose that M is compact and $f : M \to N$ is continuous, one-one and onto. Prove that f is a homeomorphism.

7. Show that if Y is compact, then the projection $\pi_1 : X \times Y \to X$ is a closed map i.e., π_1 carries closed sets to closed sets.

8. Prove that local compactness is preserved under continuous, open functions.

9. Let X be an infinite set with a distinguished point x_0. Let $\mathcal{T}$ consist of the empty set and all subsets of X containing x_0. Prove that $(X, \mathcal{T})$ is a locally compact space.

10. Let $K \subset \mathbb{R}$ consist of 0 and the numbers $\frac{1}{n}$, for all $n \geq 1$. Prove that K is a compact subset of $\mathbb{R}$.

Chapter 4

Compact and Complete Metric Spaces

Chapter Structure

4.1 Introduction

The aim of this unit is to give for metric spaces other equivalent formulations of compactness: sequential compactness and limit point compactness. The main theorem here is that a subset E of $\mathbb{R}^n$ is compact if and only if E is closed and bounded in $\mathbb{R}^n$. We then study complete metric spaces and characterize them in terms of compactness and total boundedness. We briefly study the notion of completion of a metric space. We then try to understand compactness better using the notion of Lebesgue covering. This helps us to prove a very important theorem: every continuous map from a compact metric space to any other metric space is uniformly continuous.

4.2 Objectives

After going through this chapter you will be able to:
• Give various equivalent definitions of compactness for metric spaces.
• Show $E \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded.
• Define complete metric spaces and completion of a metric space.
• State Lebesgue covering lemma and find Lebesgue number of a covering.
• Prove uniform continuity theorem.

4.3 Equivalent formulations of compactness for metric spaces

In the previous chapter, we have studied the notion of compactness using open covers. However, checking whether a space is compact via this definition may not be always be easy and so, one tries to see if there are other equivalent definitions which might work. In this section we study these other notions of compactness: sequential and limit point compactness and prove that all three definitions of compactness are equivalent for metric spaces.

Definition 29. (Sequentially compact) Let Y be a topological space. Y is said to be sequentially compact, if every sequence in Y has a convergent subsequence.

Definition 30. (Limit point compact) Let Y be a topological space. Y is said to be limit point compact if every infinite subset of Y has a limit point.

We state a helpful lemma which is a consequence of sequential compactness.

Lemma 4.3.1. *Let X be sequentially compact. Then for every $\epsilon > 0$, there exists a finite covering of X by ε-balls.*

Proof. We prove the contrapositive of this statement: if there exists $\epsilon > 0$ such that X cannot be covered by finitely many ϵ-balls, then X is not sequentially compact.

Suppose X cannot be covered by finitely many ϵ-balls. We shall construct an infinite sequence of points (x_n) such that (x_n) has no convergent subsequence. Start with any $x_1 \in X$. Then, there exists $x_2 \in$

$X \setminus B(x_1, \epsilon)$. (This is so as by assumption, X cannot be covered by a single ball $B(x_1, \epsilon)$.) Having chosen $x_1, x_2, \ldots, x_n$ continue by induction to get $x_{n+1} \notin B(x_1, \epsilon) \cup B(x_2, \epsilon) \cdots \cup B(x_n, \epsilon)$.

The choice of x_n's implies that $d(x_{n+1}, x_i) \geq \epsilon$, for all $1 \leq i \leq n$. Hence the sequence (x_n) can have no convergent subsequence. □

The above three notions of compactness are in general not equivalent. The class of metric spaces is a good class of topological spaces where the three notions of compactness agree. In fact, the three notions of compactness are equivalent for a slightly bigger class, namely for the class of metrizable spaces. (See the book by James Munkres for the definition and further details.) We prove some implications of this equivalence below:

Theorem 4.3.1. *Let (X, d) be a metric space. Then one has the following implications:*

(1) X is compact implies X is limit point compact.

(2) X is limit point compact implies X is sequentially compact.

Proof.

- We will first prove X compact implies X limit point compact.

  Let X be compact and let A be an infinite subset of X. We have to prove that A has a limit point in X. We prove the contrapositive of this statement: if A has no limit point in X, then A is finite.

  Assume A has no limit point. Then A contains all its limit points and is hence closed. Thus A is a closed subset of a compact space and hence itself compact. Since no $a \in A$ is a limit point of A, there exists a neighbourhood U_a of a disjoint from $A \setminus \{a\}$. In fact, $U_a = \{a\}$, for if it contains any other $b \neq a$, then it would not be disjoint from $A \setminus \{a\}$. Clearly, $\mathcal{R} := \{U_a \mid a \in A\}$ is an open covering of A. Since A is compact, $\mathcal{R}$ has a finite subcover, say $U_{a_1}, \ldots, U_{a_n}$. Hence, $A \subset U_{a_1} \cup \cdots \cup U_{a_n} = \{a_1, \ldots, a_n\}$. Since every subset of a finite set is again finite, we get that A is finite, as required.

- We now prove X limit point compact implies X is sequentially compact.

  We will prove that every sequence in Y has a convergent subsequence. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in Y. Consider the set $A = \{x_n \mid n \in \mathbb{N}\}$. We make two cases: A is finite or A is infinite.

  - Suppose A be finite. We claim that there exists x such that $x = x_n$, for infinitely many $n \in \mathbb{N}$. To see this, define $f : \mathbb{N} \to$

A by $f(n) = x_n$. Then, $\mathbb{N} = \cup_{x \in A} f^{-1}(x)$. If the number of elements in $f^{-1}(x)$ had been finite for every $x \in A$, then we would get that the set of natural numbers is a finite set, which is a contradiction. Hence, there exists $x \in A$ such that $f^{-1}(x)$ is an infinite subset of $\mathbb{N}$ i.e., there exist infinitely many n such that $f(n) := x_n = x$. Now it is obvious that $(x_n)_{n \in f^{-1}(x)}$, being the constant sequence is a convergent subsequence of $(x_n)_{n \in \mathbb{N}}$.

– Suppose A is infinite. As X is limit point compact by assumption, we have that A has a limit point, say x. We will now define a subsequence of (x_n) converging to x as follows: since x is a limit point of A, $B(x, 1)$, the ball around x of radius 1 contains a point of A other than x. Thus, one can choose n_1 such that $x_{n_1} \in B(x, 1)$. Applying inductively the same argument again, given a positive integer n_{i-1} we can choose an index $n_i > n_{i-1}$ such that $x_{n_i} \in B(x, 1/i)$. (The guarantee that $n_i > n_{i-1}$ is due to the fact that A is infinite.)

The choice of x_{n_i} now implies that the subsequence $x_{n_1}, x_{n_2}, \ldots$ converges to x in A.

□

Having come this far, it is natural to ask if a metric space (X, d) is sequentially compact then is it compact too? The answer is yes, but requires the notion of Lebesgue number of a covering. This will be developed in the last section of this chapter and we will then prove that sequential compactness implies compactness for metric spaces.

In general, the above implications need not hold. We will now see some examples of these.

Example 5. Let X be a two-point space in the indiscrete topology. Then, $X \times \mathbb{N}$ is limit point compact but not compact.

Lemma 4.3.2. *Let X be a sequentially compact topological space. Then, X is limit point compact.*

Proof. Let A be an infinite subset of X. Then, A contains a sequence in X. Since X is sequentailly compact, this sequence in A has a convergent subsequence. The limit of this convergent subsequence is a limit point of A. □

4.4 Compact in $\mathbb{R}^n$ if and only if closed and bounded

In this section, we are going to characterize compact sets in $\mathbb{R}^n$. Before proving that, let us recall some basic notions related to metrics on $\mathbb{R}^n$. First recall the definition of a metric:

Definition 31. A metric on a set X is a function $d : X \times X \to \mathbb{R}$ having the following properties:

- For all $x, y \in X$, $d(x, y) \geq 0$ where equality holds if and only if $x = y$.
- For all $x, y \in X$, $d(x, y) = d(y, x)$.
- For all $x, y, z \in X$, $d(x, y) \leq d(x, z) + d(z, y)$. (This inequality is called the triangle inequality.)

Check Your Progress

1. For $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and $y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$, check that $d(x, y) = (\sum_i (x_i - y_i)^2)^{\frac{1}{2}}$ is a metric. (This is called the Euclidean metric on $\mathbb{R}^n$.)
2. For $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and $y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$, check that $\rho(x, y) = \max\{|x_i - y_i|\}$ is also a metric on $\mathbb{R}^n$.
3. With notations as above, show that $\rho(x, y) \leq d(x, y) \leq \sqrt{n}\rho(x, y)$.
4. Use the fact above to prove that the topologies induced by the Euclidean metric and the square metric are the same.
5. Also show that the topologies induced by the Euclidean metric and square metric are same as the product topology on $\mathbb{R}^n$.

We will prove an important theorem in this section. The ideas in here are also used once again in the later sections. This theorem also has an important application: it helps us to characterize completely compact subsets of $\mathbb{R}^n$.

We fix up some notation:

Definition 32. If $a < b$, the set of all points in $\mathbb{R}$ satisfying $a \leq x \leq b$ is called a closed interval.

Lemma 4.4.1. *Let $\{I_n\}$ be a sequence of closed intervals in $\mathbb{R}$ such that for all $n \in \mathbb{N}$, $I_n \supseteq I_{n+1}$. Then, $\cap_{i=1}^{\infty} I_n \neq \emptyset$.*

Proof. The proof is based on the least upper bound property (lub property) of real numbers. If $I_n = [a_n, b_n]$, let E be the set of all a_n. Then E is non-empty and bounded above by b_1, for example. Let x be the least upper bound of E, which exists as E is a non-empty bounded subset of $\mathbb{R}$. Now observe that for all $m, n \in \mathbb{N}$ $a_n \leq a_{n+m} \leq b_{n+m} \leq b_m$ which then implies that $x \leq b_m$ for each m, x being the least among upper bounds of E. By definition, $a_m \leq x$, for all $m \in \mathbb{N}$. Thus, for all $m \in \mathbb{N}$, $x \in [a_m, b_m] := I_m$, proving that $x \in \cap_{i=1}^{\infty} I_n$. This proves that $\cap_{i=1}^{\infty} I_n \neq \emptyset$. $\square$

Theorem 4.4.1. *Every closed and bounded interval in $\mathbb{R}$ is compact.*

Proof. Let $I = [a, b]$ be a closed and bounded interval in $\mathbb{R}$. Then, for all $x, y \in I$, $|x - y| \leq b - a$. Let $b - a = \delta$. We will prove I is compact. Suppose not, i.e., suppose that there exists an open cover $\{G_\lambda\}_{\lambda \in \Lambda}$ of I which contains no finite subcover of I. Let c be the mid-point of $[a, b]$ i.e., $c = \frac{a+b}{2}$. Then, clearly $I = [a, c] \cup [c, b]$. Then atleast one of the intervals $[a, c]$ or $[c, b]$ cannot be covered by any finite collection of $\{G_\lambda\}_{\lambda \in \Lambda}$, (otherwise so would be I.) Denote the interval which cannot be covered by any finite collection by $[a_1, b_1]$ and apply the above argument again to this interval $I_1 := [a_1, b_1]$. Next subdivide I_1 as above to get an $I_2 \subset I_1$ which cannot be covered by any finite collection of $\{G_\lambda\}_{\lambda \in \Lambda}$. Thus continuing further, we get a sequence of closed intervals I_n satisfying the requirements of Lemma 4.4.1 and also having for all $x, y \in I_n$, $|x - y| \leq 2^{-n}\delta$. By Lemma 4.4.1, there exists a point x lying in I_n for all $n \in \mathbb{N}$. Since $\{G_\lambda\}_{\lambda \in \Lambda}$ is an open cover of I, there exists an α such that $x \in G_\alpha$ with $G_\alpha \in \{G_\lambda\}_{\lambda \in \Lambda}$. Since G_α is open, there exists $r > 0$ such that $|y - x| < r$ implies that $y \in G_\alpha$. By Archimedean property, there exists $n \in \mathbb{N}$ large such that $2^{-n}\delta < r$. This then implies that $I_n \subset G_\alpha$, which contradicts the choice of I_n.

Hence, I is compact. $\square$

Theorem 4.4.2. *A subset A of $\mathbb{R}^n$ is compact if and only if it is closed and bounded (in the euclidean metric d or the square metric ρ.)*

Proof. It is enough to prove that A is bounded under d. This follows from the inequalities $\rho(x, y) \leq d(x, y) \leq \sqrt{n}\rho(x, y)$ as A is bounded under d if and only if it is bounded under ρ.

Let A be a compact subset of $\mathbb{R}^n$. By Lemma3.4.4, it is closed in $\mathbb{R}^n$. Consider the collection of open sets $\{B_\rho(0, m) \mid m \in \mathbb{N}\}$. Clearly this is a nested open cover of $\mathbb{R}^n$ and its intersection with A gives a nested open cover of A. Compactness of A implies that there exists a finite subcover of A indexed by $m_1, m_2, \ldots, m_n$. Let $s = \max\{m_1, \ldots, m_n\}$. Then, it is clear that $A \subset B_\rho(0, s)$, proving that A is bounded under ρ.

Conversely, suppose that A is closed and bounded in ρ i.e., suppose $\rho(x, y) \leq N$ for every pair of points $x, y \in A$. Choose $x_0 \in A$ and let $\rho(x_0, 0) = b$. Then, $\rho(x, 0) \leq \rho(x, x_0) + \rho(x_0, 0) \leq N + b$, for every $x \in A$. Let $P = N + b$. Then $A \subset [-P, P]^n$. Since $[-P, P]^n$ is a cube in $\mathbb{R}^n$, it is compact, being a finite product compact subsets of $\mathbb{R}$. (See Theorem 3.5.1.) As A is a closed subset of this compact set, A is also closed in $\mathbb{R}^n$. (See Lemma 3.4.3.) $\square$

4.5 Completeness and completion in metric spaces

We will study another very helpful notion in this section: completeness. The word "completeness" is used in mathematics in the same sense as in English: absence of gaps. We will then study the relation between compactness and completeness by introducing a geometric concept of total boundedness.

In the earlier chapters, we have studied the notion of one-point compactification, which helps us to get a compact space out of a non-compact one. Here too, we will "complete" spaces that are not complete: for example the set of rational numbers is not a complete metric space and its completion gives us the real line. Though the formal definition may look very intimidating, the underlying idea is simple: one is just trying to fill up the gaps in a space which is not complete.

4.5.1 Complete metric spaces

Before we go to complete metric spaces, we look at the notion of total boundedness.

Definition 33. Let (X, d) be a metric space. A subset A of X is said to be totally bounded if given $\epsilon > 0$ there exist a finite number of subsets $A_1, \ldots, A_n$ of X such that $\operatorname{diam} A_k < \epsilon$ (for all $1 \leq k \leq n$) and such that $A \subset \cup_{k=1}^{n} A_k$.

Theorem 4.5.1. *If a subset A of a metric space (X, d) is totally bounded, then A is bounded.*

Proof. If A is totally bounded, then there exist nonempty subsets $A_1, A_2, \ldots, A_n$ of X such that $\operatorname{diam} A_k < 1$, for all $1 \leq k \leq n$ and $A \subset \cup_{k=1}^{n} A_k$. For each k between 1 and n let $a_k \in A_k$ be any point. Let $D = \sum_{i=1}^{n-1} d(a_i, a_{i+1})$.

Now let $x, y \in A$. Then, without loss of generality there exist $1 \leq l \leq m \leq n$ such that $x \in A_l$ and $y \in A_m$. Then, $d(x, y) \leq d(x, a_l) + \sum_{t=l}^{m-1} d(a_t, a_{t+1}) + d(a_m, y)$. Since $\operatorname{diam} A_t < 1$, we have $d(x, a_l), d(a_m, y) < 1$. Hence, $d(x, y) < 1 + D + 1 = D + 2$, for all $x, y \in A$. This proves that A is bounded. □

We state without proof an important equivalent condition for total boundedness in metric spaces. (For details see the book by Goldberg, Methods of Real Analysis.)

Theorem 4.5.2. *Let (X, d) be a metric space. A subset A of X is totally bounded if and only if every sequence of points of A contains a Cauchy subsequence.*

Using Theorem 4.5.2, we show that bounded need not imply totally bounded.

Example 6. Consider the metric space l^2, consisting of all sequences (x_n) such that $\sum x_n^2 < \infty$. Then, l^2 is a metric with $d(x,y) := ||x-y||_2$, where for a sequence $s = (s_n) \in l^2$, $||s||_2 = (\sum_{i=1}^{\infty} s_n^2)^{\frac{1}{2}}$. For each $i \in \mathbb{N}$, let e_i be the sequence all whose terms are zero, except the i-th term which is one. Let E be the set of all e_i as above. Then one can easily check that for $j \neq k$, $d(e_j, e_k) = ||e_j - e_k||_2 = \sqrt{2}$. Thus, E is bounded with diameter $\sqrt{2}$. However E is not totally bounded as $e_1, e_2, \ldots$ cannot have any Cauchy subsequence, since $d(e_j, e_k) = \sqrt{2}$, which remains quite large.

4.5.2 Completion of a metric space

Definition 34. (Completion of a metric space) Let (X, d_1) and (Y, d_2) be two metric spaces.

- An isometric imbedding of X into Y is a map $f : X \to Y$ such that $d_2(f(x), f(x')) = d_1(x, x')$, for all $x, x' \in X$.

- If there exists an isometric imbedding f of (X, d_1) into a complete metric space (Y, d_2) such that $f(X)$ is dense in Y, then (Y, d_2) is called a completion of (X, d_1).

A trivial examples of an isometric imbeddings is that of the identity map from $\mathbb{R}$ to itself.

The main theorem of this section is to prove the existence of a completion of a metric space (X, d_1). Before that we outline the constructions and state certain results required in the construction. The completion of a metric space is a certain quotient space of the space of all Cauchy sequences in X. (The material in this section depends heavily on Section 4 of the book by K. D. Joshi, Introduction to General Topology. Readers are requested to look up the section there for further details.)

Let us recall the definition of a Cauchy sequence.

Definition 35. Let (X, d) be a metric space. A sequence (x_n) in X is said to be Cauchy, if for every $\epsilon > 0$ there exists an integer $N \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ for all $n, m \geq N$.

We now introduce a relation on the set of Cauchy sequences. The equivalence classes under this relation will help us to get a complete metric space out of our given space.

Definition 36. Let (X, d) be a metric space. A sequence (x_n) in X is said to be Cauchy, if for every $\epsilon > 0$ there exists an integer $N \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ for all $n, m \geq N$.

Definition 37. Two Cauchy sequences $(x_n), (y_n)$ in (X, d) are said to be equivalent, if $d(x_n, y_n) \to 0$ as $n \to \infty$.

It is easy to check that the above relation is an equivalence relation and let S denote the set of equivalence classes of Cauchy sequences in (X, d). Given a Cauchy sequence $x = (x_n)$, let $\hat{x}$ denote the image of x in S. We are now in a position to state the main theorem:

Theorem 4.5.3. *Let S be the set of Cauchy sequences in a metric space (X, d). Let $\hat{X}$ denote the set of all equivalence classes of S under the relation above. We can make $\hat{X}$ into a metric space by defining $e([\hat{x}], [\hat{y}]) = \lim_{n\to\infty} d(x_n, y_n)$. Let $h : X \to \hat{X}$ be given by $h(x) = [\hat{x}]$. Then, $\hat{X}$ is the completion of the metric space (X, d).*

Proof. To prove the theorem, we need to prove several things. We will only list them here first and ask to readers to supply details themselves one by one.

- Proving that the relation on S is an equivalence relation.
- Proving the function e is well-defined.
- Proving that the function e is a metric.
- Proving that h is an isometric imbedding.
- Proving that $h(X)$ is dense in $\hat{X}$.
- Proving that $\hat{X}$ is a complete metric space.

□

Remark 4.5.1. The property of completion is slightly different from its sister property of compactness: compactness is a topological property, but completion is **not** a topological concept i.e., it is not invariant under homeomorphism. For example, there is a homeomorphism $f : (\frac{-\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ with $\mathbb{R}$ is complete but $(\frac{-\pi}{2}, \frac{\pi}{2})$ not complete.

4.6 Lebesgue Covering Lemma

In this section, we seek a measure of how big sets can be. This quantification is made more precise using the notions of diameter of a set and associating a number to every covering of a compact topological space. (See *http : //en.wikipedia.org/wiki/Henri_Lebesgue* for more details on the mathematician Lebesgue, after whom this lemma is named.)

We begin with the definition of the diameter of a bounded set:

Definition 38. Let A be a bounded subset of a metric space (X, d). Then, diameter of A, denoted by $\text{diam}(A)$ is defined to be $\text{lub}\{d(a_1, a_2) \mid a_1, a_2 \in A\}$.

Examples:

(1) Let $X = \mathbb{R}$ with the absolute metric. Let $A = [0, 1]$. Then, $\text{diam}(A) := \text{lub}\{|x - y| \mid x, y \in A\}$. Note that for all $x, y \in A$, $|x - y| \leq 1$. It is also easy to check that one is the least upper bound of A. Hence, $\text{diam}(A) = 1$.

(2) Let $X = \mathbb{R}^2$ with the usual distance metric. Let $A = S^1$. Then, $\text{diam}(S^1) := \text{lub}\{d(x, y) \mid x, y \in S^1\}$. By definition of S^1, If $x = (x_1, x_2)$ and $y = (y_1, y_2)$ then we have $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \leq 2$. Also, note that the pair of point $(-1, 0)$ and $(1, 0)$ are at a distance 2, proving that 2 is the diameter. (This matches with our usual notion of diameter from school days.)

Definition 39. Let $\mathfrak{R}$ be an open covering of the metric space (X, d). If there exists $\delta > 0$ such that for each subset of X having diameter less than δ, there exists an element of $\mathfrak{R}$ containing it, then δ is called a Lebesgue number for the covering $\mathfrak{R}$.

The main aim of this section is to prove the existence of a Lebesgue number for compact metric spaces. Before we do this, we prove that sequential compactness guarantees the existence of a Lebesgue number for every open covering of X.

Lemma 4.6.1. *Let X be a sequentially compact space and let $\mathcal{R}$ be any open cover of X. Then $\mathcal{R}$ has a Lebesgue number δ.*

Proof. We shall prove the contrapositive of this statement: suppose there is no $\delta > 0$ such that every set of diameter less than δ lies in at least one element of $\mathcal{R}$ implies X is *not* sequentially compact.

Now suppose there is no such δ, i.e., for each $\delta > 0$, there exists a subset of X having diameter less than δ which does not lie inside any element of $\mathcal{R}$. In particular, for each $n \in \mathbb{N}$ we can choose a set C_n having diameter less than $\frac{1}{n}$ which is not contained in any element of $\mathcal{R}$. Choose for each n a point $x_n \in C_n$.

We claim that such a chosen sequence (x_n) has no convergent subsequence. For suppose (x_n) had a subsequence (x_{n_i}) converging to x. Now x lies in some element $A \subset \mathcal{R}$. As A is open, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subset A$. Choose i large enough so that $d(x_{n_i}, x) < \frac{\epsilon}{2}$ and $\frac{1}{n_i} < \frac{\epsilon}{2}$. Now C_{n_i} lies in the $\frac{1}{n_i}$ neighbourhood of x_{n_i} and hence $C_{n_i} \subset B(x, \epsilon)$. Then, $C_{n_i} \subset A$, contradicting the choice of the sets C_n. $\square$

Check Your Progress The proof above is a nice illustration of one of the ways of attacking a problem in mathematics: the existential approach. The above proof asserts the existence of a δ without actually telling how one can find or construct it. Such a method of proof was first developed by one of the influential and universal mathematicians of the nineteenth and twentieth centuries, David Hilbert. (See *https : //en.wikipedia.org/wiki/David_Hilbert*). During those days, Hilbert was criticized for such a method of proof and in fact he was told by Gordan (another great mathematician) that this was not mathematics, but it was theology. Later, however Hilbert's method of thought became a very important way for going about a proof, specially in pure mathematics. Many of the proof in mathematics today are of this kind and one then often needs to write down algorithms to make the computations explicit.

Can you find at least two more examples of such existential proofs? (Hint: Linear Algebra).

Let us now prove that for a metric space (X, d) sequential compactness implies compactness. (See Theorem 4.3.1.)

Theorem 4.6.1. *Let (X, d) be a metric space. If (X, d) is sequentially compact, then (X, d) is compact.*

Proof. Let $\mathcal{R}$ be an open covering of X. Since X is sequentially compact, by Lemma 4.6.1 $\mathcal{R}$ has a Lebesgue number δ. Apply Lemma 4.3.1 with $\epsilon = \delta/3$, to get a finite covering of X by balls of radius $\delta/3$. Each of these balls has diameter atmost $2\delta/3 < \delta$ so we can choose for each of these balls an element of $\mathcal{R}$ containing it. Thus, we get a finite subcollection of $\mathcal{R}$ that covers X, proving that X is compact. $\square$

Remark 4.6.1. The property of completion is slightly different from its sister property of compactness: compactness is a topological property, but completion is **not** a topological concept i.e., it is not invariant under homeomorphism. For example, there is a homeomorphism $f : (\frac{-\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ with $\mathbb{R}$ is complete but $(\frac{-\pi}{2}, \frac{\pi}{2})$ not complete.

We can now record the Lebesgue covering lemma for compact spaces. The proof of it follows immediately from the facts proved above.

Lemma 4.6.2. *Let $\mathfrak{R}$ be an open covering of the metric space (X, d). If X is compact, there is a $\delta > 0$ such that for each subset of X having diameter less than δ, there exists an element of $\mathfrak{R}$ containing it. Such a number δ is called a Lebesgue number for the covering $\mathfrak{R}$.*

Here is an important consequence of completeness:

Theorem 4.6.2. *Let (X, d) be a complete metric space. For each $n \in \mathbb{N}$ let F_n be a closed and bounded subset of X such that $F_1 \supset F_2 \supset \cdots \supset F_n \supset F_{n+1} \supset \cdots$ and $\operatorname{diam}(F_n) \to 0$ as $n \to \infty$. Then $\cap_{n=1}^{\infty} F_n$ contains precisely one point.*

Proof. For each $n \in \mathbb{N}$ let a_n be any arbitrary point of F_n. Then as the F_n's form a nested sequence, we get $a_n, a_{n+1}, a_{n+2}, \ldots \in F_n$, i.e., $a_{n+k} \in F_n$ for all $k \geq 0$. (Call this property P.)

Since the diameter of F_n's tends to zero as n tends to infinity, given $\epsilon > 0$ there exists an integer $N \in \mathbb{N}$ such that $\operatorname{diam}(F_n) < \epsilon$ and $a_N, a_{N+1}, a_{N+2}, \ldots$ all lie in F_N. Thus for $m, n \geq N$ we have $d(a_n, a_m) \leq \operatorname{diam}(F_N) < \epsilon$. This proves that $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Since X is complete this Cauchy sequence converges to a point, say a in X. By property P and the fact that $a \in X$, we get that a is a limit point of F_n, for every $n \geq 1$ and for every closed subset F_n. Since for all $n \geq 1$, F_n is closed, we get that $a \in F_n$ for all $n \geq 1$. This shows that $\cap_{n=1}^{\infty} F_n$ is non-empty.

We now prove uniqueness: if there exist $a, b \in X$ such that both $a, b \in \cap_{n=1}^{\infty} F_n$, then there exists a K such that $d(a, b) > \operatorname{diam}(F_K)$ for K sufficiently large. Thus, b cannot lie in $\cap_{n=1}^{\infty} F_n$, a contradiction. □

Here is an important theorem which relates the various concepts that you have studied till now:

Theorem 4.6.3. *A metric space (X, d) is compact if and only if (X, d) is complete and totally bounded.*

Proof.

- Let (X, d) be compact. By Theorem 4.6.1 it is enough to prove that every sequence (x_n) has a subsequence which converges to a point of X.

  Suppose that for each point $x \in X$, there exists an open ball B_x which contains x for only finitely many values of n. The family of all such B_x would then be an open cover of X By hypothesis, X would then be covered by a finite number of B_x, which is impossible as the union of finitely many B_x would contain only finitely many x_n.

  Hence there exists $x \in X$ such that every open ball around x contains x_n for infinitely many n. Hence there exists n_1 such that $x_{n_1} \in B(x, 1)$, there exists $n_2 > n_1$ such that $x_{n_2} \in B(x, \frac{1}{2})$, and continuiung for any k, there exists $n_k > n_{k-1}$ such that $x_{n_k} \in B(x, \frac{1}{k})$. This subsequence (x_{n_k}) of (x_n) converges to the point x in X, thus proving that X is complete.

- Let (X, d) be complete and totally bounded. Suppose X is not compact, i.e. there exists an open cover $\mathcal{R}$ of X such that no finite number of sets of $\mathcal{R}$ form a cover of X.

  As X is totally bounded, it can be written as a union of a finite number of bounded subsets each of whose diameter is less than one. Then, one of these subsets, say A_1 cannot be covered by finitely many subsets of $\mathcal{R}$. (Else X would be covered by finitely many subsets of $\mathcal{R}$.) As $\operatorname{diam}\overline{A}_1 = \operatorname{diam}A_1$, $\overline{A_1}$ is a closed subset of X with diameter less than one and which cannot be covered by a finite number of sets from $\mathcal{R}$.

Since $\overline{A}_1$ is itself totally bounded, the same reasoning shows that there exists $A_2 \subset \overline{A_1}$ such that $\operatorname{diam}A_2 < \frac{1}{2}$ and A_2 cannot be covered by finitely many elements of $\mathcal{R}$. Proceed inductively to get a nested family of closed subsets of X such that $\overline{A}_1 \supset \overline{A}_2 \supset \cdots \overline{A_n} \supset \overline{A_{n+1}} \cdots$ with $\operatorname{diam}\overline{A}_n < \frac{1}{n}$ and such that no finite number of sets in $\mathcal{R}$ form a covering of any $\overline{A}_n$. By Theorem 4.6.2, $\cap_{n=1}^{\infty}\overline{A_n}$ contains precisely one point, say x. Since $\mathcal{R}$ is a covering of X, there is a set $G \subset \mathcal{R}$ such that $x \in G$. Since G is open, there exists $r > 0$ such that $B(x,r) \subset G$. Now if $N \in \mathbb{N}$ is such that $\frac{1}{N} < r$, then $\operatorname{diam}\overline{A_N} < \frac{1}{N} < r$. Since $x \in \overline{A_N}$, we have $\overline{A_N} \subset B(x,r) \subset G$. Thus, G alone covers $\overline{A_N}$. This is a contradiction to the fact that finitely many subsets of $\mathcal{R}$ cannot cover $\overline{A_N}$. This contradiction completes the proof of the theorem. □

4.7 Uniform continuity theorem

Using the notion of Lebesgue covering that we have developed, we prove that every continuous function from a compact metric space to another metric space is uniformly continuous. We begin by recalling the definitions of continuity and uniform continuity.

Definition 40. (Continuity) Let (X, d_1) and (Y, d_2) be metric spaces. We say $f : X \to Y$ is continuous at $x_0 \in X$, if for every $\epsilon > 0$ there exists $\delta > 0$ such that $d_1(x, x_0) < \delta$ implies $d_2(f(x), f(x_0)) < \epsilon$. The function f is said to be continuous if it is continuous at every point of X.

Note that in this definition, δ depends in general on ϵ as well as x.

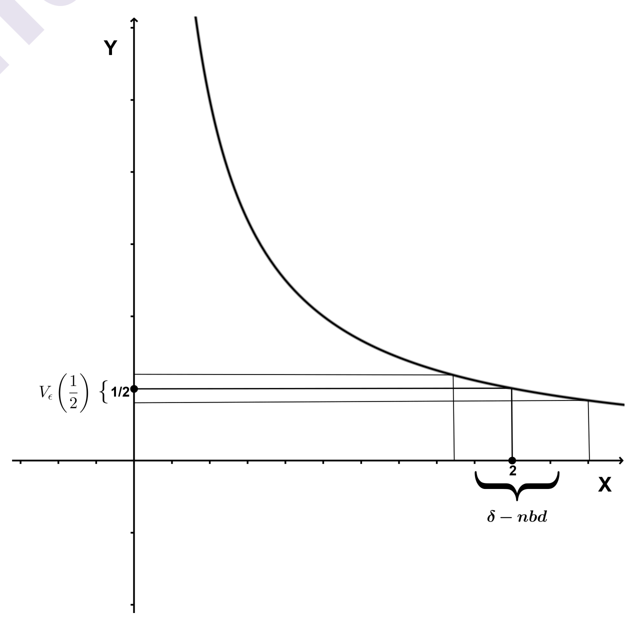
Definition 41. (Uniform continuity) Let (X, d_1) and (Y, d_2) be metric spaces. We say $f : X \to Y$ is uniformly continuous on X, if for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in X$, $d_1(x, y) < \delta$ implies $d_2(f(x), f(y)) < \epsilon$.

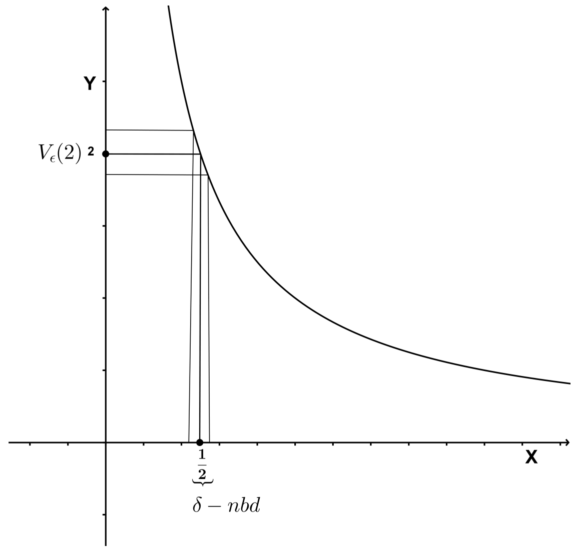
Note that in the definition of uniform continuity the δ chosen depends only on ϵ: ie., it is independent of the choice of $x \in X$. Thus in this case, the same δ works impartially for every $x \in X$, hence the name uniform continuity.

Example 7. We give two examples here: one of a uniformly continuous function and another of a function which is not uniformly continuous.

- (Uniformly continuous) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 2x$. Then, for every $\epsilon > 0$, the choice $\delta = \epsilon/2$ is such that $|x-y| < \epsilon/2$ implies that $|f(x) - f(y)| = 2|x - y| < (2\epsilon)/2 < \epsilon$. Since δ is independent of the choice of the point x chosen, we conclude that f is continuous.

- (Not uniformly continuous) Let $S = \{x \in \mathbb{R} \mid x > 0\}$. Define $g : S \to \mathbb{R}$ by $g(x) = \frac{1}{x}$. Then, we want $|g(x) - g(y)| = |\frac{y-x}{xy}| = \frac{|y-x|}{|xy|} < \epsilon$, whenever $|x - y| < \delta$. Since this should work for all x, y, we may choose $\delta < x^2\epsilon$. This equation tells that δ depends on both x, ϵ and cannot be made independent of x. For if δ were independent of x, then $\delta < x^2$, for every x, however small. Hence, $\delta = 0$, a contradiction. Hence the function g is not uniformly continuous. A simple pictorial illustration below explains how the choice of δ depends on $x \in \mathbb{R}$.





Check Your Progress

1. Prove that every uniformly continuous function is continuous. The converse need not hold.

2. Show that the function $f(x) = \frac{1}{1+x^2}$ for $x \in \mathbb{R}$ is uniformly continuous on $\mathbb{R}$.

3. Show that the function $h(x) = \frac{1}{x^2}$ is uniformly continuous on the set $K = [1, \infty)$ but not uniformly continuous on $(0, \infty)$.

Theorem 4.7.1. *(Uniform continuity theorem) Let $(X, d_X), (Y, d_Y)$ be metric spaces with X compact. Let $f : X \to Y$ be a continuous map. Then f is uniformly continuous.*

Proof. To prove f is uniformly continuous, we will prove that given $\epsilon > 0$, there exists $\delta > 0$ such that for any $x_1, x_2 \in X$, $d_X(x_1, x_2) < \delta$ implies $d_Y(f(x_1), f(x_2)) < \epsilon$.

Given $\epsilon > 0$, consider the open covering of Y by $B(y, \epsilon/2)$, balls of radius $\epsilon/2$ around y. Hence, $f(X) \subset Y = \cup_{y \in Y} B(y, \epsilon/2)$. Hence,

$$X \subset f^{-1}(\cup_{y \in Y} B(y, \epsilon/2)) \subset \cup_{y \in Y} f^{-1}(B(y, \epsilon/2)).$$

As f is continuous, we have $f^{-1}(B(y, \epsilon/2))$ is an open subset X. This proves that $\mathcal{R} = \{f^{-1}(B(y, \epsilon/2)) \mid y \in Y\}$ is an open cover of X.

Let δ be the Lebesgue covering of this open covering $\mathcal{R}$. Then if $x_1, x_2 \in X$ are such that $d_X(x_1, x_2) < \delta$, then the set $\{x_1, x_2\}$ has diameter less than δ. Hence its image $\{f(x_1), f(x_2)\}$ lies in some ball $B(y, \epsilon/2)$. Then, $d_Y(f(x_1), f(x_2)) < \epsilon$. This proves that f is uniformly continuous. $\square$

4.8 Glossary

In this chapter, you have learnt the following:

- Sequential compactness and limit point compactness.
- Equivalence of compactness, sequential compactness and limit point compactness in metric spaces.
- A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.
- Complete metric space and completion of a metric space.
- Lebesgue number and Lebesgue covering lemma.
- Uniform continuity and the fact that every continuous function from a compact metric space to another metric space is uniformly continuous.

4.9 Bibliography

- Goldberg, R. R, Methods of Real Analysis, Oxford's Indian Edition.
- James Munkres, Topology, Pearson.
- Joshi, K. D., Introduction to General Topology, New Age International (P) Limited.

4.10 Let Us Sum Up

In this chapter we first learnt various equivalent notions of compactness for metric spaces. Compactness was also characterized in terms of completeness and total boundedness. We then defined the notion of a Lebesgue covering and this gave rise to the notion of Lebesgue number of covering. The main theorem was that every sequentially compact space has a Lebesgue number. Since the notions of sequential compactness and compactness are equivalent in metric spaces, this also shows that compact metric spaces have a Lebesgue number. Existence of this Lebesgue number then helps us to prove an important theorem about continuous functions from a compact metric space to another metric space: these functions are also uniformly continuous.

4.11 References for further reading

1. Armstrong, M. A., Basic Topology, Springer UTM
2. Rudin, W., Principles of Mathematical Analysis. Third Edition, Mc-Graw Hill International Editions.
3. Sutherland, W. A., Introduction to metric and topological spaces. Oxford University Press.
4. Willard, S., General Topology, Dover Books on Mathematics.
5. Kumaresan, S., Topology of Metric Spaces, Morgan and Claypool.

4.12 Chapter End Exercises

1. Show that if f, g are uniformly continuous on $\mathbb{R}$ then so is $f+g$.
2. If $f(x) = x$ and $g(x) = \sin(x)$ then show that both f, g are uniformly continuous on $\mathbb{R}$ but their product fg is not uniformly continuous on $\mathbb{R}$.
3. Show that if f and g are uniformly continuous on $\mathbb{R}$ and if they are *both* bounded on $\mathbb{R}$, then their product fg is uniformly continuous on $\mathbb{R}$.
4. Prove that if f, g are uniformly continuous on $\mathbb{R}$, then so is their composition $f \circ g$.
5. Prove that the completion of a totally bounded metric space is compact.
6. Let (X, d) be a metric space and let $C(X)$ be the space of all bounded real-valued functions on X with a metric e defined by $e(f, g) = \sup\{|f(x) - g(x)| \,|x \in X\}$. Fix $a \in X$. For $x \in X$, define $h_x : X \to \mathbb{R}$ by $h_x(u) = d(x, u) - d(a, u)$. Prove that $h_x \in C(X)$.
7. With notations as in the above exercise, define $h' : X \to C(X)$ by $h'(x) = h_x$. Prove that h' is an isometric imbedding of X into $C(X)$.

8. Prove that a metric space X is complete, if it contains a dense subset D such that every Cauchy sequence in D has a limit point in X.

9. Prove the converse of the above statement.

10. Let $X = (0, 1)$ with metric d given by $d(x, y) = |\frac{1}{x} - \frac{1}{y}|$. Show that the sequence $\{\frac{1}{n}, n \in \mathbb{N}\}$ is not a Cauchy sequence in (X, d). (Recall that the sequence is a Cauchy sequence in the usual metric.)