1

FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS

Unit structure

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1.0 OBJECTIVES

After doing this unit, you will be able to:

- identify partial differential equation of order one.
- classify different types of partial differential equation
- solve problems of semi linear and quasilinear problems
- determine characteristic equations
- solve Cauchy problem
- find general solution
- find complete integral.

1.1 INTRODUCTION

A partial differential equation for a function $u(x_\alpha)$ of m independent variables $x_\alpha(\alpha = 1,\ 2,\ 3 \ldots\ldots m)$ is a relationship between the function and its partial derivatives $u_{x_\alpha}, u_{x_\alpha x_\beta}, \ldots..$ We represent this relationship in the form

$$F\left(x_1, \ldots\ldots, x_m; u;\ u_{x_1}, \ldots\ldots u_{x_m};\ u_{x_1x_1}; u_{x_1x_2} \ldots\ldots\right) = 0 \tag{1.1}$$

Or briefly

$$F\left(x_\alpha, u, \quad u_{x_\alpha}, \quad u_{x_\alpha x_\beta} \ldots\ldots\right) = 0$$

where only a finite number of derivatives occur on the left-hand side and the function F is defined over Domain D_3. The order of the partial derivative is the order of the highest derivatives appearing in the function F.

A genuine solution of the partial differential equation is a function$u = u(x_\alpha)$defined over a domain D ofx_α space such that all partial derivative of u appearing in the equation exist and are continuous in D,

$\left(x_\alpha, u(x_\alpha), u_{x_\beta}(x_\alpha), u_{x_\beta x_\gamma}(x_\alpha), \dots \dots\right) \in D_3$ when $x_\alpha \in D$ and

$F\left(x_\alpha, u(x_\alpha), u_{x_\beta}(x_\alpha), u_{x_\beta x_\gamma}(x_\alpha), \dots \dots\right) = 0$ for all $x_\alpha \in D$. We also say that the function satisfies equation (1.1). we shall refer the genuine solution simply as a solution.

While studying partial differential equations, we shall assume that all functions are real valued with real arguments unless otherwise stated.

The simplest partial differential equations to study are those of the first order for the determination of just one unknown function. Apart from the fact that they form the basis of the study of higher order equations called hyperbolic equations, they are the simplest kind of equations for which method of solutions are available and for which the existence, uniqueness and stability can be discussed in detail. In this chapter, we shall present some basic result concerning first order partial differential equation.

1.2 FIRST ORDER PARTIAL DIFFERENTIAL EQUATION AND CAUCHY PROBLEM

In this chapter while dealing with the partial differential equation in two independent variables, we shall denote the independent variable by x and y.

A first order partial differential equation in two unknowns in its most general form is given by

$$F(x, y, u, u_x, u_y) = 0 \tag{1.2}$$

where F is a known function of its arguments.

1.2.1 Classification of partial differential Equation

Linear equation: when the function F is linear in $u_x, u_y$and u. then the equation of the form

$$a(x, y)u_x + b(x, y)u_y = c_1(x, y)u + c_2(x, y) \tag{1.3}$$

is called linear equation. Where a, b, c_1, c_2 will depend on x and y.

For example,

$yx^2u_x + xy^2u_y = xyu + x^2y^2$and $xyu_x + x^2yu_y = 3xyu + xy^2$.

Semi linear equation: When the function F is linear in u_x, u_ythen the equation of the form

$$a(x,y)u_x + b(x,y)u_y = c(x,y,u) \tag{1.4}$$

is called semi-linear equation. Wherea and b depend on x and y whereas c depends on x, y and u.

For example,

$xy\,u_x + x^2y\,u_y = x^2yu^2$ and $2x^2y\,u_x + 3x^2y\,u_y = 5x^2yu^3$

Quasilinear equation: When the function F is linear in u_x, u_ythen the equation of the form

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u) \tag{1.5}$$

Is called quasi linear. Where a, b, c depend on x, y and u.

For example,

$xyu^2u_x + xu\,u_y = yu^2$and $(x^2 - yu)u_x + (y^2 - xu)\,u_y = u^2 - xy$

Non-linear equation: When the function F is not linear in u_x, u_ythen the equation (1.2) is called non-linear equation.

For example,

$u_x^2 + x^2u_y^3 = uy$ and $x^2u_x^2 + y^2u_y^2 = u^2$

The solution $u = u(x,y)$ represents a surface in (x,y,u) space. This surface is called integral surface of the partial differential equation.

While dealing with partial differential equations appearing in science and engineering, we rarely to find out or discuss properties of a solution in its most general form. Almost always we deal with those solution of differential equations which satisfy certain conditions. In the case of first order partial differential equations, the search for these specific solutions can be formulated as a Cauchy problem.

1.2.2. The Cauchy problem

Consider an interval I on the real line and three arbitrary functions $x_0(\eta), y_0(\eta) and\, u_0(\eta)$ of single variable $\eta \in I$ suchthat the derivatives $x_0'(\eta) and\ y_0'(\eta)$ are piecewise continuous and $(x_0')^2 + (y_0')^2 \neq 0$.

A Cauchy problem for a first order equation (1.2) is to find the domain D in (x,y) plane containing $\big(x_0(\eta), y_0(\eta)\big)$for all $\eta \in I$ and a solution $u = u(x,y)$ of the equation such that

$$u(x_0(\eta), y_0(\eta)) = u_0(\eta) \tag{1.6}$$

for all values of $\eta \in I$.

Geometrically, $x = x_0(\eta), y = x_0(\eta)$ represents a curve γ in (x, y) plane. We call the curve *datum curve.* The Cauchy problem is to determine the solution of $F(x, y, u, u_x, u_y) = 0$ in a neighbourhood of γ such that u takes prescribed values $u_0(\eta)$ on γ.

The solution of Cauchy problem also involves such questions as the conditions on the functions $F, x_0(\eta), y_0(\eta) and\, u_0(\eta)$ under which the solution exists and its unique.

1.3 SEMILINEAR AND QUASILINEAR EQUATIONS IN TWO INDEPENDENT VARIABLES

We start with a semi linear equation instead of linear equation as the theory of the former does not require any special treatment as compared to that of latter.

1.3.1 Semilinear Equation

Consider a single first order equation in two independent variables (x, y) for a single unknown quantity:

$$a(x, y)u_x + b(x, y)u_y = c(x, y, z) \tag{2.1}$$

We assume that a, b, c are continuously differential functions of their arguments and a and b are not simultaneously zero. $a, b \in C^1(D_1)$ and $c \in C^1(D_2)$, where D_1 and D_2 are domains in (x, y) plane and (x, y, u) space respectively, such that whenever $(x, y, u) \in D_2, (x, y) \in D_1$.

At a given point $(x, y) \in D_1$, $a(x, y)u_x + b(x, y)u_y$ represents a derivative of $u(x, y)$ in the direction of vector $(a(x, y), b(x, y))$. Therefore, if we consider a one parameter family of curves whose tangent at each point is in the above direction i.e. the family of curves defined by ordinary differential equation

$$\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)} \tag{2.2}$$

the variation of u along these curves is given by $\frac{du}{dx} = u_x + \frac{dy}{dx}u_y = \frac{au_x + bu_y}{a}$, which with the help of (2.1) gives

$$\frac{du}{dx} = \frac{c(x,y,u)}{a(x,y)} \tag{2.3}$$

Consider a curve represented by a solution of equation (2.2). we can choose a variable σ such that the curve has a parametric representation $x = x(\sigma), y = y(\sigma)$ and $x(\sigma)$ and $y(\sigma)$ satisfy a pair of ordinary differential equations

$\frac{dx}{d\sigma} = a(x, y), \frac{dy}{d\sigma} = b(x, y)$ (2.4)

The variation of u along the curve is given by

$\frac{du}{d\sigma} = c(x, y, u)$ (2.5)

The equations (2.2) or (2.4) are called *characteristic equations*. The solution of (2.2) can be written in the form

$f(x, y, C) = 0$ (2.6)

Where C is a constant of integration. This equation represents one parametric family of curves with C as a parameter. We call these curves the *characteristiccurves* of the partial differential equation. In the domain D_1consider another curve $x = x_0(\eta), y = y_0(\eta)$ such that it is nowhere tangential to characteristic curve.

Solving (2.4) with the condition $x = x_0(\eta), y = y_0(\eta)\ at\ \sigma = 0$, we get a solution of the form

$x = x(\sigma, \eta), y = y(\sigma, \eta)$ (2.7)

Because of the equivalence of (2.2) and (2.4), the equation (2.7) also represents the one-parameter family of characteristic curves of equation (2.1). in the parametric representation of (2.7), σ varies along a characteristic curve. η remains constant along characteristic curves. The equation (2.3) or (2.5) is called *compatibility condition* along a characteristic curve.

Suppose that $u(x, y)$ is assigned an initial value u_0 at point (x_0, y_0)in(x, y) −plane. Since $a(x, y), b(x, y)$ and $c(x, y, u)$ are D^1 function of their arguments, the initial value problem for the ordinary differential equations (2.4) and (2.5) with initial values x_0, y_0, u_0 has unique solution. Therefore, through the point (x_0, y_0) there passes a unique characteristic curve given by

$x = x(x_0, y_0, \sigma), y = y(x_0, y_0, \sigma)$ (2.8)

and along this curve

$u = u(x_0, y_0, u_0, \sigma)$ (2.9)

is uniquely determined by the equation (2.5). This shows that, if u is given at any point, it is uniquely determined everywhere along the characteristic curve denoted by C_c passing through the point, as long as it does not pass through a singular point and as long as (x, y, u) remains in D_2, where $c(x, y, u)$ is defined. This suggests the following method of solution of the Cauchy problem.

We take an arbitrary point $P_0(x_0(\eta), y_0(\eta), \sigma)$on the datum curve γ. The value of u at P_0 is $u_0(\eta)$. solving the characteristic equations and the compatibility condition with initial values $x = x_0(\eta), y = y_0(\eta), u = u_0(\eta)$ at $\sigma = 0$, we get

$$x = x(x_0(\eta), y_0(\eta), \sigma), y = y(x_0(\eta), y_0(\eta), \sigma) \quad (2.10)$$

and

$$u = u(x_0(\eta), y_0(\eta), u_0(\eta), \sigma) \quad (2.11)$$

Solving the pair of equations (2.10) for σ and η in terms of x, y and substituting in (2.11) we get a solution of the Cauchy Problem in neighbourhood of the curve y.

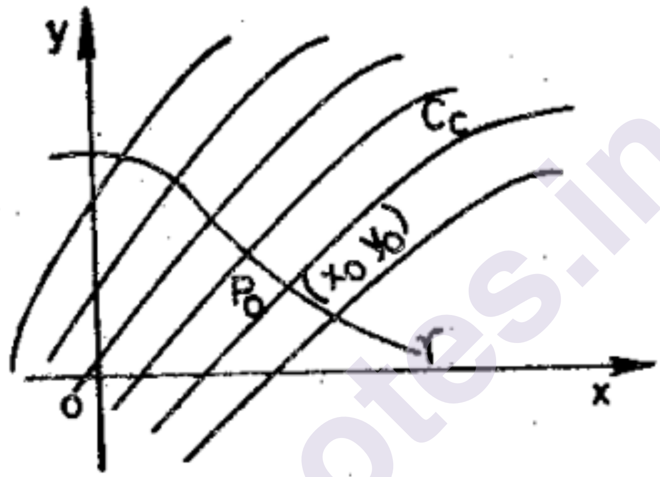

Fig 1.1 Solution of a cauchy problem with the help of characteristic Curves C_c.

the method fails if the curve γ coincide with the characteristic curve. From the compatibility condition (2.5) we also note that if γ is a characteristic curve, the variation of the Cauchy data $u_0(\eta)$ on γ is constrained by the relation (2.5) and so cannot be arbitrarily prescribed on it.

Example 2.1: Solve the Cauchy problem of partial differential equation

$2u_x + 3u_y = 1,$ (2.12)

with Cauchy data prescribed on the straight line $\gamma: x = \beta x - \alpha y = 0$, where α and β are constants. A parametric representation of Cauchy data is $x = \alpha\eta,\ y = \beta\eta, u(\alpha\eta, \beta\eta) = f(\eta)$

Solution: Initial values: $x_0 = \alpha\eta, y_0 = \beta\eta, \sigma = 0 a = 2, b = 3, c = 1$

Characteristic Equations

$$\frac{dx}{d\sigma} = 2, \frac{dy}{d\sigma} = 3 => dx = 2d\sigma, \qquad dy = 3d\sigma$$

$$=> x = 2\sigma + c_1,$$

Applying Initial Conditions, $\alpha\eta = 2 * 0 + c_1 => c_1 = \alpha\eta$

$=> x = 2\sigma + \alpha\eta$ (2.13)

And $y = 3\sigma + c_2$,

Applying Initial Conditions,$\beta\eta = 3 * 0 + c_2 => c_2 = \beta\eta$

$=> y = 3\sigma + \beta\eta$ (2.14)

$$\frac{du}{d\sigma} = c => \frac{du}{d\sigma} = 1 => du = d\sigma => u = \sigma + c_3$$

Applying Initial Conditions,$f(\eta) = 0 + c_3 => c_3 = f(\eta)$

$$=> u = \sigma + f(\eta)$$

Solving σ and η from (2.13) and (2.14)

$$=> \eta = \frac{3x - 2y}{(3\alpha - 2\beta)}, \qquad \sigma = \frac{\beta x - \alpha y}{2\beta - 3\alpha}$$

Substituting these values in $u = \sigma + f(\eta)$

We get, $u = \frac{\alpha y - \beta x}{3\alpha - 2\beta} + f\left(\frac{3x - 2y}{3\alpha - 2\beta}\right)$ (2.15)

Provided we assume that

$3\alpha - 2\beta \neq 0$ (2.16)

Equation (2.15) represents a genuine solution of the equation (2.12) if the given function $f(\eta)$ is continuously differential. Then u_x and u_y are C^1 function in the entire (x, y) −plane and satisfy the equation (2.12).

When the constants α and β are such that$3\alpha - 2\beta = 0$, the above method of finding the solution breaks down. In this case the straight line γ is itself a characteristic curve. Along a characteristic curve $\frac{dx}{d\sigma} = 2.$ The compatibility condition (2.5) shows that the function $f(\eta)$ in the above Cauchy problem cannot be arbitrarily prescribed but must satisfy the relation

$\frac{df(\eta)}{d\eta} = \frac{\alpha}{2}$ (2.17)

This condition completely determines the function $f(\eta)$expect for a constant of integration:

$$f(\eta) = \frac{\alpha}{2}\eta \qquad (2.18)$$

It is simple to check that the characteristic Cauchy problem with the Cauchy data

$$x = \alpha\eta, \qquad y = \frac{3}{2}\alpha\eta, \qquad u = \frac{\alpha}{2}\eta$$

has a solution of the form

$$u = \frac{x}{2} + g(3x - 2y) \quad (2.19)$$

Where $g(\xi)$ is an arbitrary C^1 function of ξ and satisfies $g(0) = 0$

This example verifies a general property namely, the solution of a characteristic Cauchy problem when it exists, is non unique in that it involves an arbitrary function.

Example 2.2: Solve the Cauchy problem of partial differential equation

$u_x + u_y = u,$ with initial conditions $u(x, 0) = 1.$

Solution: $a = 1, b = 1, c = u,$

With initial conditions, $x_0 = \eta, y_0 = 0, u_0 = 1, \sigma = 0$

Characteristic equation: $\frac{dx}{d\sigma} = 1 => dx = d\sigma$

$$=> x = \sigma + c_1$$

Applying Initial Conditions, $\eta = 0 + c_1 => c_1 = \eta$

$$=> x = \sigma + \eta => \eta = x - y$$

$$\frac{dy}{d\sigma} = 1 => y = \sigma + c_2,$$

Applying Initial Conditions, $0 = 0 + c_2 => c_2 = 0$

$$=> y = \sigma$$

$$\frac{du}{d\sigma} = u => \frac{du}{u} = d\sigma => log u = \sigma + c_3$$

Applying Initial Conditions, $log\, 1 = 0 + c_3 => c_3 = 0$

$$\log u = \sigma => u = e^{\sigma}$$

$$=> u = e^{y}$$

*Example 2.3:*Find the characteristic equation of the following PDE

$$yu_x - xu_y = 0$$

Solution: Characteristic equation

$$\frac{dy}{dx} = \frac{b}{a} = -\frac{x}{y} => ydy = -xdx$$

$$=> \int ydy = \int -xdx$$

$$=> \frac{y^2}{2} = -\frac{x^2}{2} + constt$$

$$=> x^2 + y^2 = c$$

This represents equation of circle with centre as origin.

*Example 2.4:*Find the characteristic equation of the following PDE

$$2xyu_x - (x^2 + y^2)u_y = 0$$

Solution: $a = 2xy, b = -(x^2 + y^2)$

$$\frac{dy}{dx} = \frac{b}{a} => \frac{dy}{dx} = -\frac{x^2 + y^2}{2xy} => 2xydy = -x^2dx - y^2dx$$

$$2xydy + y^2dx = -x^2dx$$

$$d(xy^2) = -x^2dx$$

Integrating both the sides,

$$\int d(xy^2) = \int -x^2dx => xy^2 = -\frac{x^3}{3} + c$$

$$=> xy^2 + \frac{x^3}{3} - c = 0$$

$$=> 3\mathrm{xy}^2 + x^3 - C = 0$$

EXERCISE 2.1

1. Find the characteristics of the equation $(x^2 - y^2 + 1)u_x + 2xyu_y = 0$

2. Show that characteristic of $u_x - u_y = 0$ touches the branch of the hyperbola $xy = 1$ in the first quadrant of the (x, y) −plane at the point $P(1, 1)$. Verify that the point P divides the hyperbola into two portions such that the Cauchy data prescribed on one portion determines the value of u on the other portion.

3. Find the solution of $yu_x - xu_y = 0$, given that $u(x,0) = x^2$ for $-\infty < x < \infty$

4. Show that if u is prescribed on the interval $0 \le y \le 1$ of the y −axis, the solution of $(x^2 - y^2 + 1)u_x + 2xyu_y = 0$ is completely determined in the first quadrant of the (x, y) −plane.

5. Find the solution of the partial differential equation $(x+1)^2 u_x + (y-1)^2 u_y = (x+y)u$ satisfying the condition $u(x,0) = -1 - x$ for $-1 < x < \infty$

6. Find the solution of the Cauchy problems and the domain in which they are determined in (x, y) −plane:
   (i) $yu_x + xu_y = 2u$ with $u(x,0) = f(x)$for $x > 0$,
   (ii) $yu_x + xu_y = 2u$ with $u(0,y) = g(y)$for $y > 0$,
   (iii) $u_x + u_y = u^2$ with $u(x,0) = 1\ for - \infty < x < \infty$

1.3.2 Quasilinear Equations

Now, we pass on to the general quasilinear equation of the first order

$$a(x,t,u)u_x + b(x,y,u)u_y = c(x,y,u) \qquad (2.20)$$

where the coefficients a and b depend on the dependent variable u also. We assume that a, b, c are C^1 functions in the domain D_2 of (x, y, u) −space. We recall here the geometrical interpretation of a solution $u = u(x, y)$ as a surface in (x, y, u) −space, called *integral surface*. The direction ratio of the normal to the surface are $(u_x, u_y, -1)$, so equation (2.20) can be written as $(a, b, c).(u_x, u_y, -1) = 0$ (2.21)

where the left-hand side is the scalar product of two vectors, we can interpret the equation as being equivalent to a condition that the integral surface at each point has the property that the vector (a, b, c) is tangential to the surface.

Monge direction: At any point (x, y, u)in D_2, the vector $(a(x,y,u), b(x,y,u), c(x,y,u))$ defines a direction, called *Monge direction.* Therefore, the coefficients in the equation (2.20) defines a direction field i.e. the field of Monge directions in the domain of D_2 of (x, y, z) −space.

Monge curve: A surface $u = u(x, y)$ is an integral surface if and only if, at each point of the surface the tangent plane contains the Monge direction at that point.Thus, at a given point (x, y, u) the tangent plane of the integral surface has one degree of freedom, i.e. it can rotate about Monge direction. A space curve whose tangent at every point coincide with Monge direction is called a *Monge curve* and it determined by the equations,

$$\frac{dx}{a(x,y,u)} = \frac{dy}{b(x,y,u)} = \frac{du}{c(x,y,u)} \qquad (2.22)$$

This equation also known as *Lagrange equation.*

In terms of parameter σ, such that $d\sigma$ is the common value of the three ratios in (2.22), we can write the characteristic equation and compatibility condition respectively as

$$\frac{dx}{d\sigma} = a(x,y,u), \frac{dy}{d\sigma} = b(x,y,u) \qquad (2.23)$$

and

$$\frac{du}{d\sigma} = c(x,y,u) \qquad (2.24)$$

As in §2.1, we consider a surface in D_2given by $x = x_0(\eta_1,\eta_2), y = y_0(\eta_1,\eta_2)$,

$u = u_0(\eta_1,\eta_2)$, such that it nowhere touches the Monge curve. Solving the system of equations (2.23) and (2.24), with the condition $x = x_0(\eta_1,\eta_2), y = y_0(\eta_1,\eta_2), u = u_0(\eta_1,\eta_2)$at $\sigma = 0$, we get a representation of the Monge curve in the form

$$x = x(\sigma,\eta_1,\eta_2), y = y(\sigma,\eta_1,\eta_2), u = u(\sigma,\eta_1,\eta_2) \qquad (2.25)$$

The totality of Monge curves form a two-parameter family of curves with parameter $\eta_1$andη_2. The projection of a Monge curve on (x,y) −plane is called *characteristic curve* of (2.20). Note that the characteristic equations (2.4) of the semilinear equation (2.1) are not coupled with the compatibility condition (2.5) and hence can be integrated independently. Thus, the one parameter family of characteristic curves of a semilinear equation can be drawn once for all without any reference to the compatibility condition. For the quasi-linear equation (2.20), the characteristic equation and compatibility condition are coupled. Therefore, to determine the characteristic in case of the quasi-linear equation, we have to draw them by solving the three equations (2.23) and (2.24) together. The totality of the characteristic curves in (x,y) −plane of quasilinear equation forms a two-parameter family of curves. For a given solution u is a known function of x and y, and the equation (2.23) for characteristics can be solve without any reference to the compatibility condition (2.24), as in the case of semilinear equations. In this case through any point (x,y), there is only one characteristic curve and the set of all characteristic curves from one characteristic curve and the set of all characteristic curves form one- parameter family of curves in the (x,y) −plane.

Example 2.5: consider the partial differential equation

$$uu_x + u_y = 0$$

The Monge curve through the point (x_0, y_0, u_0) is a straight line given by the equations

$$x - x_0 = u_0(y - y_0), u = u_0$$

The characteristic curves through an arbitrary point (x_0, y_0) in (x, y) −plane is the one parameter family of straight line passing through the point and depending on the parameter u_0.

Consider a surface generated by a one parameter sub-family of Monge curves. The tangent plane at the point of the surface contains the Monge direction at that point. Therefore, *every surface generated by a one parameter sub family of Monge curve is an integral surface of (2.20).* the converse of this statement is also true. Let $u = u(x, y)$ be an integral surface S. Let $x = x_0(\eta), y = y_0(\eta)$,

$u = u_0(\eta) \equiv u(x_0(\eta), y_0(\eta))$ be a space curve lying on S and suppose the function $x_0(\eta), y_0(\eta)$ are so prescribed that the curve is not Monge curve. Consider the solution of

$$\frac{dx}{d\sigma} = a\big(x, y, u(x, y)\big), \frac{dy}{dx} = b(x, y, u(x, y)) \qquad (2.26)$$

with $x = x_0(\eta), y = y_0(\eta)$ at $\sigma = 0$ in the form $x = x(\sigma, \eta), y = y(\sigma, \eta)$. In (2.26) u is known function of x, y from the equation of integral surface S. Then along the one parameter family of curves

$$x = x(\sigma, \eta), y = y(\sigma, \eta), u = u(x(\sigma, \eta), y(\sigma, \eta)) \qquad (2.27)$$

with η as parameter lying on S, we have

$$\frac{du}{d\sigma} = \frac{dx}{d\sigma}u_x + \frac{dy}{d\sigma}u_y = au_x + bu_y = c(x, y, u) \qquad (2.28)$$

In view of (2.26) and (2.28), we infer that the curves (2.27) are Monge curves. These Monge curves generate the integral surface S as ηvaries. We have shown that starting from a non-Monge curve on an integral surface, we can determine one parameter sub-family of Monge curve that generate the surface.Thus*any integral surface S is generated by a family of Monge curve depending on a single parameter* η.

Now we have also proved that through an arbitrary point of an integral surface there passes a Monge curve which lies entirely on the integral surface. This with the uniqueness theorem of the solution of an initial value problem of the ordinary differential equation (2.23) and (2.24) implies that if *Monge curve is tangential to an integral surface at any point, it lies entirely on the integral surface.*

We can now present a method for the solution of a Cauchy problem for the quasilinear equation (2.20). We first note that geometrically $x = x_0(\eta), y = y_0(\eta), u = u_0(\eta)$ represents a curve Γin (x, y, u) −space. We call this curve *initial curve*. The datum curve γ, on which the Cauchy data is prescribed, is the projection of Γ on the (x, y) −plane. A geometrical representation of a Cauchy problem for a first order partial differential equation is to find an integral surface of the equation passing through initial curveΓ. The result of the last two paragraphs shows that in the order to solve a Cauchy problem we just have to find the surface generated by the one parameter family of Monge curves, starting from the points $\left(x_0(\eta), y_0(\eta), u_0(\eta)\right)$, in the form

$$x = x(\sigma,\eta), y = y(\sigma,\eta), u = u(\sigma,\eta) \quad (2.29)$$

This is a parametric representation of required integral surface. We shall again have to exclude datum curve which are tangential to the characteristic curves. We present here a precise formation in the following theorem.

Theorem 2.1: Let $x_0(\eta), y_0(\eta),$ and$u_0(\eta)$ be continuously differential function of η in a closed interval say $[0, 1]$ and a, b, c be functions of x, y, u having continuous first order partial derivatives with respect to their arguments in some domain D_2 of (x, y, u) −space containing the initial curve

$$\Gamma : x = x_0(\eta), y = y_0(\eta), u = u_0(\eta); 0 \le \eta \le 1 \quad (2.30)$$

and satisfying the condition

$$\frac{dy_0(\eta)}{d\eta} a\left(x_0(\eta), y_0(\eta), u_0(\eta)\right) - \frac{dx_0(\eta)}{d\eta} b\left(x_0(\eta), y_0(\eta), u_0(\eta)\right) \neq 0.$$
(2.31)

Then there exists a solution $u = u(x, y)$ of the quasi-linear equation (2.20) in the neighbourhood of the datum curve $\gamma: x = x_0(\eta), y = y_0(\eta),$ and satisfying the condition $u_0(\eta) = u\left(x_0(\eta), y_0(\eta)\right), 0 \le \eta \le 1$ (2.32)

Proof: since a, b, c have continuous partial derivative with respect to x, y, u; the ordinary differential equation (2.23) and (2.24) have a unique continuously differential solution of the form (2.29) satisfying the initial condition

$$x(0,\eta) = x_0(\eta), y(0,\eta) = y_0(\eta), u(0,\eta) = u_0(\eta) \quad (2.33)$$

As $x_0(\eta), y_0(\eta),$ and$u_0(\eta)$ be continuously differential, the solution (2.29) is continuously differential with respect to η. In view of assumption (2.31) the Jacobian

$$\frac{\partial(x,\ y)}{\partial(\sigma,\ \eta)} \equiv \begin{vmatrix} x_\sigma & x_\eta \\ y_\sigma & y_\eta \end{vmatrix} = (a y_\eta - b x_\eta) \quad (2.34)$$

does not vanish at $\sigma = 0$ for $0 \le \eta \le 1$. Therefore, in the neighbourhood of $\sigma = 0$, we can uniquely solve for σ and η in terms of x and y from the first two relations in (2.29) and substitute in the third relation to get u as a function of x and y

i.e. $u(x, y) = u(\sigma(x, y), \eta(x, y))$ (2.35)

At any point of the datum curve, $u\left(x_0(\eta), y_0(\eta)\right) = u(0,\eta) = u_0(\eta)$, which shows that the initial condition (2.33) is satisfied.

From (2.24), i.e.$u_\sigma = c,$ we have $u_x x_\sigma + u_y y_\sigma = c$ or $au_x + bu_y = c$ showing that the function $u(x,y)$ given by (2.35) satisfies the equation (2.20).

To prove the uniqueness of the solution we first note that if a Monge curve is tangential to an integral surface at any point, it lies entirely on the surface. Let us assume now that there are two integral surfaces S and S' passing through the initial curve Γ, given by (2.30). Then for an arbitrary given value of η, the Monge curve (2.29) starting from the point $(x_0(\eta), y_0(\eta), u_0(\eta))$ lies entirely on both the surfaces S and S'.Hence S and S' are generated by same subfamily of Monge curves which implies that the two integral surfaces are same.

Example 2.6: Consider the equation

$$uu_x + u_y = 0 \tag{2.36}$$

with the Cauchy data $u(x,0) = x, 0 \leq x \leq 1.$

prescribed only on a portion of the x −axis. The Cauchy data can be put in the form of (2.30):

$x = \eta, y = 0, u = \eta, 0 \leq \eta \leq 1$(2.37)

Solving the characteristic equations and compatibility condition

$$\frac{dx}{d\sigma} = u, \frac{dy}{d\sigma} = 1, \frac{du}{d\sigma} = 0$$

With the initial data we get

$$x = \eta(\sigma + 1),\ y = \sigma,\ u = \eta \tag{2.38}$$

The characteristic curve passing through a point $x = \eta$ on the x-axis is a straight line $x = \eta(y + 1)$. These characteristic for all admissible but fixed value of ηi.e.$0 \leq \eta \leq 1$ pass through the same point $(0, -1)$ and cover the wedged shaped portion D of the (x,y)-plane bounded by two extreme characteristics $x = 0$ and $x = y + 1.$ $u = \eta$ in (2.38) shows that u is constant in those characteristics, being equal to the abscissa of the point where the characteristics intersects the x-axis. The solution is determined in the wedged shaped region D as shown in the Fig. 1.2

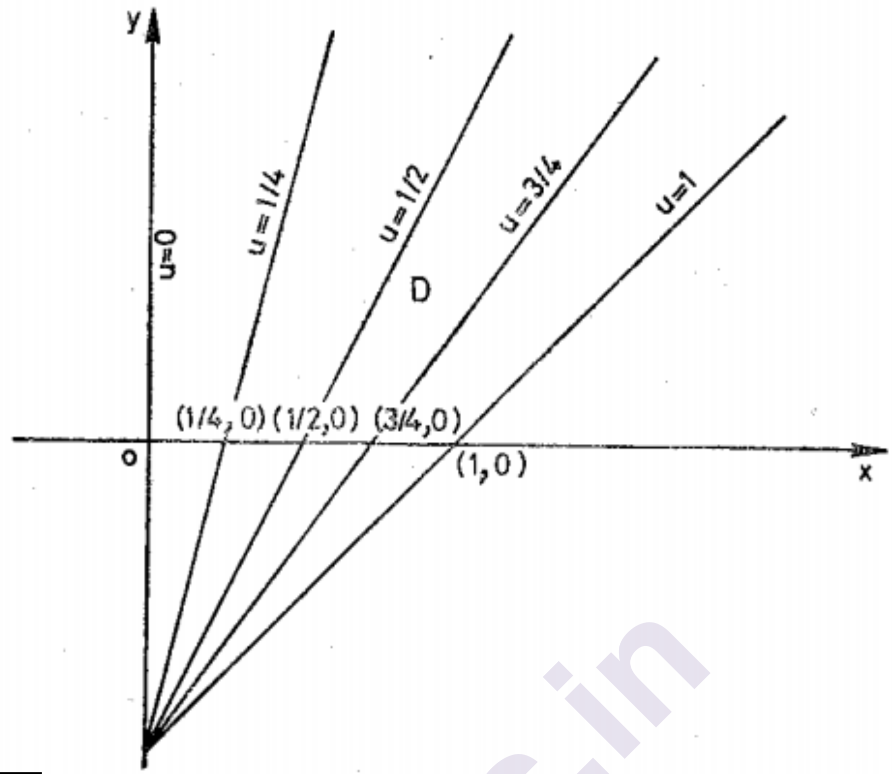

Fig 1.2 The solution is determined in the wedge shaped region D of the (x, y)-plane.

We note two very important aspects of quasi-linear equation from this example.

(i) The domain D in the (x, y)- plane in which the solution is determined depends on the data prescribed in the Cauchy problem. Had we prescribed $u(x, 0)$ =constant= $\frac{1}{2}$, say, for $0 \leq x \leq 1$, the characteristic would have been a family of parallel straight lines $y - 2x = -2\eta$ and the domain D would have been a family of parallel straight lines $y - 2x = -2\eta$ and domain D would have been the infinite strip bounded by extreme characteristics $y - 2x = 0$ and $y - 2x = -2$ as shown in the Fig. 1.3.

(ii) Even though the coefficient in the equation (2.36) and the Cauchy data (2.37) are regular, the solution develops a singularity at the point $(0, -1)$. Geometrically this is evident from the fact that the characteristic which carry different values of u all intersect at $(0, -1)$. Analytically, this is clear from the explicit form of solution obtained (2.38) after eliminating σ and η:

$$u = \frac{x}{y+1} \tag{2.39}$$

The appearance of the singularity in the solution of a Cauchy problem for certain Cauchy data is properly associated with non-linear differential equations.

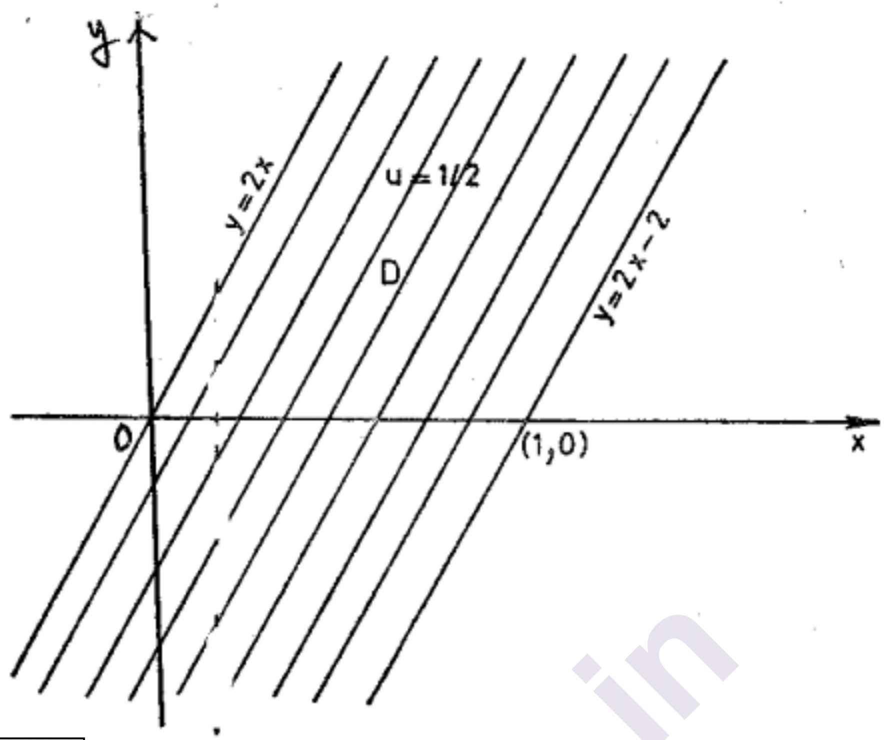

Fig 1.3 The domain D when the Cauchy data is $u(x, 0)=1/2$ for $0 \leqslant x \leqslant 1$.

1.3.3 the characteristic Cauchy problem

We have just seen that if the datum curve γ is such that Cauchy data satisfies (2.31), then the unique solution of the Cauchy problem exists in a neighbourhood of the curve. Now suppose that

$$\frac{dy_0(\eta)}{d\eta} a\big(x_0(\eta), y_0(\eta), u_0(\eta)\big) - \frac{dx_0(\eta)}{d\eta} b\big(x_0(\eta), y_0(\eta), u_0(\eta)\big) = 0$$
(2.40)

Everywhere along the curve γ, i.e.γ is a characteristic curve for a possible solution. Let us suppose further that a solution: $u = u(x, y)$, of Cauchy problem exists. Then from (2.40) and (2.20) it follows that

$$\frac{du_0(\eta)}{d\eta} = \frac{d}{d\eta} u\big(x_0(\eta), y_0(\eta)\big) = \frac{dx_0}{d\eta} u_x(x_0, y_0) + \frac{dy_0}{d\eta} u_y(x_0, y_0)$$

must be proportional to $c_0(x_0(\eta), y_0(\eta), u_0(\eta))$. Therefore the function $x_0(\eta), y_0(\eta), u_0(\eta)$ satisfy the equations

$$\frac{dx_0}{a(x_0(\eta), y_0(\eta), u_0(\eta))} = \frac{dy_0}{b(x_0(\eta), y_0(\eta), u_0(\eta))}$$
$$= \frac{du_0}{c(x_0(\eta), y_0(\eta), u_0(\eta))}$$

and the initial curve Γ is necessarily a Monge curve.

Consider now another curve Γ' in (x, y, u)- space which is not a Monge curve and which intersects Γ at some point. Then we can obtain an integral surface S' passing through Γ'. As on point of Γ lies on S' the entire original initial curve Γ will lie on S' and hence S' is an integral surface passing through Γ. Consider now another curve Γ'', which is not a Monge curve and which intersects Γ, but does not lie on S'. Then we get another integral surface S'' containing Γ and different from S'.

Therefore, *the solution of a characteristic initial value problem, if it exists, is nonunique.*

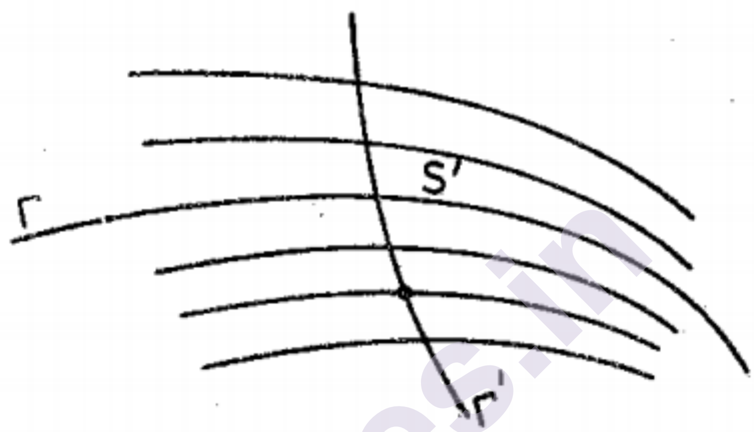

Fig 1.4 Integral surface containing an initial curve Γ which is a Monge curve

1.3.4 General Solution

Until now we have discussed only those solution of a first order differential equation which satisfy certain prescribed conditions (i.e. solution of a Cauchy problem). In general, these particular solutions are completely determined. For a single quasilinear equation of first order, it is possible to get an explicit form of *general solution* which is define to be a solution from which all particular solution can be obtained.

A relation of the form $f(x, y, u) = C$, where C is a constant is called a *first integral* of first order ordinary differential equations (2.22)(or (2.23) and (2.24)), if the function $f(x, y, u)$ has a constant value along an integral curve of (2.22) (i.e. along a Monge curve). It follows, therefore, that if $f(x, y, u) = C$ be a first integral of (2.22) and $x = x(\sigma), y = y(\sigma), u = u(\sigma)$ be a solution of these equations, then $f(x(\sigma), y(\sigma), u(\sigma))$ is independence of σ.

The general solution of the ordinary differential equation (2.22) consists of any two independent first integrals

$$\phi(x, y, u) = C_1 \quad and \quad \psi(x, y, u) = C_2 \qquad (2.41)$$

which together also constitute another representation of the two-parameter family of Monge curve of (2.20). The surface represented by a first integral, say $\phi(x, y, u) = C_1$, is generated by one parameter family of Monge curves by varying a parameter C_2 and hence represents an integral surface of (2.20). now it follows that each one of the two equations in (2.41) represents a one parameter family of integral surface of (2.20). Next,we prove a theorem which connect the two independent families of integral surface two the quasilinear equations.

Theorem 2.2: if $\phi(x, y, u) = C_1$ and $\psi(x, y, u) = C_2$ be two independent first integral of the ordinary differential equation (2.22), and $\phi_u^2 + \psi_u^2 \neq 0$ the general solution of the partial differential equation (2.20) is given by

$$h(\phi(x, y, u), \psi(x, y, u)) = 0 \tag{2.42}$$

where h is an arbitrary function.

Proof: since the first integral $\phi(x, y, u) = C_1$ represents an integral surface, the equation (2.20) is satisfied by $u_x = -\frac{\phi_x}{\phi_u}$, $u_y = -\frac{\phi_y}{\phi_u}$. This gives

$$a\phi_x + b\phi_y + c\phi_u = 0 \tag{2.43}$$

Similarly

$$a\psi_x + b\psi_y + c\psi_u = 0 \tag{2.44}$$

If $f(x, y, u) = 0$ be the equation of an integral surface of (2.20), we also have

$$af_x + bf_y + cf_u = 0 \tag{2.45}$$

Since $a^2 + b^2 + c^2 \neq 0$, it follows from (2.43)−(2.45) that the Jacobian $\frac{\partial(f,\phi,\psi)}{\partial(x,y,u)} \equiv 0$. This implies that $f = h(\phi, \psi)$ where h is an arbitrary function of its arguments, showing that the equation of any integral surface is given by (2.42).

The two-parameter family of Monge curve in $(x, y, u)-$space is represented by the equation (2.41). The integral surface (2.42) is generated by one parameter sub-family of the Monge curves, obtained by restricting the values of C_1 and C_2 by the relation

$$h(C_1, C_2) = 0 \tag{2.46}$$

For a given Cauchy problem, it is simple to determine the one parameter of subfamily of Monge curves which generate the integral surface passing through the initial curve Γ represented by (2.30). The parameter C_1 and C_2 for which the Monge curve intersect the curve Γ, satisfy

$$\phi(x_0(\eta), y_0(\eta), u_0(\eta)) = C_1$$

and

$$\psi(x_0(\eta), y_0(\eta), u_0(\eta)) = C_2$$

Eliminating η from these two, we get a relation of the form (2.46) between C_1 and C_2. This determines the function $h.$ the solution of the Cauchy problem is obtained by solving u in terms of x and y from (2.42).

Example2.7: Find the general solution of differential equation

$$(y + 2ux)u_x - (x + 2uy)u_y = \frac{1}{2}(x^2 - y^2) \qquad (2.47)$$

Solution: The characteristic equations and the compatibility conditions are $\frac{dx}{y+2ux} = \frac{dy}{-(x+2uy)} = \frac{du}{\frac{1}{2}(x^2-y^2)}$

To get one first integral we derive from these,

$$\frac{xdx + ydy}{2u(x^2 - y^2)} = \frac{2du}{x^2 - y^2} => \frac{xdx + ydy}{2u} = 2du$$

$$=> xdx + ydy = 4udu$$

Integrating both the sides we get,

$$\phi(x, y, u) \equiv x^2 + y^2 - 4u^2 = C_1 \qquad (2.48)$$

For another independent first integral we derive a second combination

$$\frac{ydx + xdy}{y^2 - x^2} = \frac{2du}{x^2 - y^2} => ydx + xdy = -du$$

$$\psi(x, y, u) \equiv xy + 2u = C_2 \qquad (2.49)$$

The general integral of the equation (2.47) is given by

$$h(x^2 + y^2 - 4u^2, xy + 2u) = 0 \text{ or } f(xy + 2u) = x^2 + y^2 - 4u^2 \quad (2.50)$$

where $h \text{ or } f$ are arbitrary functions of their arguments.

Consider a Cauchy problem in which u is prescribed to be zero on the straight line $x - y = 0$. Parametrically, we can write it in the form

$$x = \eta, y = \eta, u = 0$$

From (2.48) and (2.49) we get, $2\eta^2 = C_1$and $\eta^2 = C_2$ which gives $C_1 = 2C_2$.

Therefore, the solution of Cauchy problem is obtained, when we take

$h(\phi, \psi) = \phi - 2\psi$. This gives $u = \frac{1}{2}\{\sqrt{x - y^2 + 1} - 1\}$ (2.51)

We know that the solution of the Cauchy problem is determined uniquely at all points in the (x, y)-plane.

Example 2.8: Find the general solution of the following quasi linear equations.

a) $\frac{y^2u}{x}u_x + xu\,u_y = y^2$

Sol: $\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c} \Rightarrow \frac{dx}{\frac{y^2u}{x}} = \frac{dy}{xu} = \frac{du}{y^2}$

Taking 1st two terms together

$$\frac{dx}{\frac{y^2u}{x}} = \frac{dy}{xu} \Rightarrow \frac{xdx}{y^2u} = \frac{dy}{xu}$$

$$\Rightarrow \frac{xdx}{y^2} = \frac{dy}{x}$$

$$x^2dx = y^2dy \Rightarrow \frac{x^3}{3} = \frac{y^3}{3} + c$$

$$\Rightarrow \phi(x, y, u) \equiv x^3 - y^3 = C_1$$

Taking 1st and 3rd term together,

$$\frac{dx}{\frac{y^2u}{x}} = \frac{du}{y^2} \Rightarrow xdx = udu \Rightarrow \frac{x^2}{2} = \frac{u^2}{2} + c'$$

$$\psi(x, y, u) \equiv x^2 - u^2 = C_2$$

The general solution is $h(x^3 - y^3, x^2 - u^2) = 0$

b) $x^2p + y^2q = u$

Solution: $\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c} \Rightarrow \frac{dx}{x^2} = \frac{dy}{y^2} = \frac{du}{u}$

Taking first two terms together, $\frac{dx}{x^2} = \frac{dy}{y^2}$

$$\Rightarrow \phi(x, y, u) \equiv \frac{1}{x} - \frac{1}{y} = C_1$$

Taking last two terms together, $\frac{dy}{y^2} = \frac{du}{u} \Rightarrow -\frac{1}{y} = log\,u + c$

$$\Rightarrow \psi(x, y, u) \equiv \frac{1}{y} + log\,u = C_2$$

The general solution is $h\left(\frac{1}{x} - \frac{1}{y}, \frac{1}{y} + \log u\right) = 0$

c) $\tan x\, u_x + \tan y\, u_y = \tan u$

Solution: $\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{du}{\tan u}$

Solving, $\frac{dx}{\tan x} = \frac{dy}{\tan y}$

$$=> \cot x\, dx = \cot y\ dy$$

$$=> \log sinx = \log siny + \log c$$

$$=> \log sinx\ = \log \sin y * C_1$$

$$=> sinx = siny\ C_1 => \phi(x,y,u) \equiv \frac{\sin x}{siny} = C_1$$

Solving $\frac{dy}{\tan y} = \frac{du}{\tan u}$

$$=> \psi(x,y,u) \equiv \frac{\sin y}{\sin u} = C_2$$

The general solution is $h\left(\frac{\sin x}{siny}, \frac{\sin y}{\sin u}\right) = 0$

d) $u_x + 3u_y = 5u + \tan\,(y - 3x)$

Solution: $\frac{dx}{1} = \frac{dy}{3} = \frac{du}{5u+\tan\,(y-3x)}$

$$\frac{dx}{1} = \frac{dy}{3} => 3dx = dy$$

$$=> 3x = y + c => \phi(x,y,u) \equiv y - 3x = C_1$$

$$\frac{dy}{3} = \frac{du}{5u + \tan(y - 3x)}$$

$$=> \frac{dy}{3} = \frac{du}{5u + \tan C_1}$$

$$\frac{y}{3} = \log(5u + \tan C_1) * \frac{1}{5} + c'$$

$$\psi(x,y,u) \equiv 5y - 3\log(5u + \tan(y - 3x)) = C_2$$

The general solution is

$h(y - 3x, 5y - 3\log(5u + \tan(y - 3x)) = 0$or

$$f(y - 3x) = 5y - 3\log(5u + \tan(y - 3x)$$

EXERCISE 2.2

1. Show that all the characteristic curves of the partial differential equation $(2x+u)u_x+(2y+u)u_y=u$ through the point (1, 1) are given by the straight line $x-y=0$.

2. Discuss the solution of the differential equation $uu_x+u_y=0$, $y>0, -\infty<x<\infty$ with Cauchy data
   $u(x,0)=\alpha^2-x^2$ for $|x|\leq\alpha$
   $=0$ for $|x|>0$

3. Find the general solution of the equation
   $$(2x-y)y^2u_x+8(y-2x)x^2u_y=2(4x^2+y^2)u$$
   and deduce the solution of the Cauchy problem when $u(x,0)=\frac{1}{2x}$ on a portion of the x- axis.

4. Show that the result of elimination of an arbitrary function $h(\phi,\psi)$ of two arguments from the relation
   $h\big(\phi(x,y,u),\psi(x,y,u)\big)=0$

1.4 FIRST ORDER NON-LINEAR EQUATIONSIN TWO INDEPENDENT VARIABLES

The most general first order equation, i.e. an equation of the form $F(x,y,u,p\ q)=0$ (3.1) where F is a given function of its arguments and

$$p=u_x,\ q=u_y \tag{3.2}$$

In this section we shall consider a non-linear partial differential equation, i.e. equation (3.1) where F is not linear in p and q. we assume here that the function F possess continuous second order partial derivatives over a domain D_3 of (x,y,u,p,q)-space with $F_p^2+F_q^2\neq 0$. Let the projection of D_3 on (x,y,u)-space be denoted by D_2.

1.4.1 Monge strip and Charpit's Equation

Let $u=u(x,y)$ represent an integral surface S of (3.1) in (x,y,u)- space, then $(p,q,-1)$ are direction ratios of the normal to S.

The differential equation (3.1) states that at any point $P(x_0,y_0,u_0)$ on S, there is a relation between $p_0$andq_0. This relation $f(x_0,y_0,u_0,p_0,q_0)=0$ between $p_0,\&q_0$ is not linear. Hence all the tangent to integral surface do not pass through the fixed line but form a family of planes enveloping a conical surface, called the *Monge cone*with P as its vertex. The differential equation thus assigns a Monge cone at every point, i.e. a field of Monge cones in the domain D_2 of (x,y,u)-space. The problem of solving the differential equation (3.1) is to find the surface which fit in the field, i.e.

surfaces which touch the Monge cone at each point along a generator. Also note that Monge cone need not to be closed.

Example3.1: Consider the partial differential

$$p^2 - q^2 = 1 \tag{3.3}$$

At every point of the (x, y, u)-space the relation (3.3) can be expressed parametrically as

$$p_0 = \cosh\lambda,\ q_0 = \sinh\lambda - \infty < \lambda < \infty \tag{3.4}$$

The equation of tangent planes at (x_0, y_0, u_0) are

$$(x - x_0)\cosh\lambda + (y - y_0)\sinh\lambda - (u - u_0) = 0 \tag{3.5}$$

The envelope of these planes is λ- eliminant of (3.5) and

$$(x - x_0)\sinh\lambda + (y - y_0)\cosh\lambda = 0 \tag{3.6}$$

Which is obtained by differentiating (3.5) partially with respect to λ. Therefore, the Monge cone of (3.3) is

$$(x - x_0)^2 - (y - y_0)^2 - (u - u_0)^2 = 0 \tag{3.7}$$

This is the right circular cone with semi vertical angle $\frac{\pi}{4}$ and whose axis is the straight line passing through (x_0, y_0, u_0) and parallel to x-axis.

Since an integral surface is touched by a Monge curve along a generator, we proceed to determine the equations to a generator of the Monge cone of (3.1). At a given point (x_0, y_0, u_0), the relation between p_0 and q_0 can be expressed parametrically in the form

$$p_0 = p_0(x_0, y_0, u_0, \lambda),\quad q_0 = q_0(x_0, y_0, u_0, \lambda) \tag{3.8}$$

which satisfy

$$F\big(x_0, y_0, u_0, p_0(x_0, y_0, u_0, \lambda), q_0(x_0, y_0, u_0, \lambda)\big) = 0 \tag{3.9}$$

For all values of the parameter λ for which p_0 and q_0 in (3.8) are defined.

The equations of the tangent planes for λ and $\lambda + \delta\lambda$ are

$$p_0(x_0, y_0, u_0, \lambda)(x - x_0) + q_0(x_0, y_0, u_0, \lambda)(y - y_0) = u - u_0 \tag{3.10}$$

and

$$p_0(x_0, y_0, u_0, \lambda + \delta\lambda)(x - x_0) + q_0(x_0, y_0, u_0, \lambda + \delta\lambda)(y - y_0) = u - u_0 \tag{3.11}$$

The limiting position of the line of intersection of these planes as $\delta\lambda \to 0$ is a generator of the Monge cone at (x_0, y_0, u_0). Expanding p_0 and q_0 in (3.11) in powers of $\delta\lambda$, using (3.10) and retaining only the first-degree terms, we get

$$\frac{dp_0}{d\lambda}(x - x_0) + \frac{dq_0}{d\lambda}(y - y_0) = 0 \qquad (3.12)$$

(3.10) and (3.12) are the equations to the generators in the terms of the parameter λ. We can eliminate the derivatives $\frac{dp_0}{d\lambda}$ and $\frac{dq_0}{d\lambda}$with the help of (3.9) which gives

$$F_p \frac{dp_0}{d\lambda} + F_q \frac{dq_0}{d\lambda} = 0 \qquad (3.13)$$

From (3.10), (3.12) and (3.13) we get the following equations of the generator of the Monge cone at (x_0, y_0, u_0)

$$\frac{x-x_0}{F_p} = \frac{y-y_0}{F_q} = \frac{u-u_0}{pF_p+qF_q} \qquad (3.14)$$

If we replace $x - x_0, y - y_0, u - u_0$by dx, dy, du, respectively, corresponding finite infinitesimal moment, $x - x_0 = dx, y - y_0 = dy, u - u_0 = du$, from (x_0, y_0, u_0) along the generator, then (3.14)tends to

$$\frac{dx}{F_p} = \frac{dy}{F_q} = \frac{du}{pF_p+qF_q} \qquad (3.15)$$

We note that, for quasilinear equation (2.20), equations (3.15) reduce to (2.22) showing that the Monge cone degenerates into the Monge line element.

Suppose we are given an integral surface $S: u = u(x, y)$, where $u(x, y)$ has continuous second order partial derivatives with respect to x and y. At the point of S we know $u, p\ and\ q$ as function of x and y.Also at each point of the surface S, there exist Monge cone which touches the surface along a generator of the cone. The line of contact between the tangent plane of S and the corresponding cones, that is the generators along with the surface is touched, define a direction field on the surface, which is called *Monge direction* on S (Fig. 3.1). Monge direction for a quasilinear equation and Monge direction on an integral surface for a non-linear equation has the common property that they are special direction tangential to the integral surface. However, in the non-linear case, they have no exitance of their own but are defined only when an integral surface is prescribed.

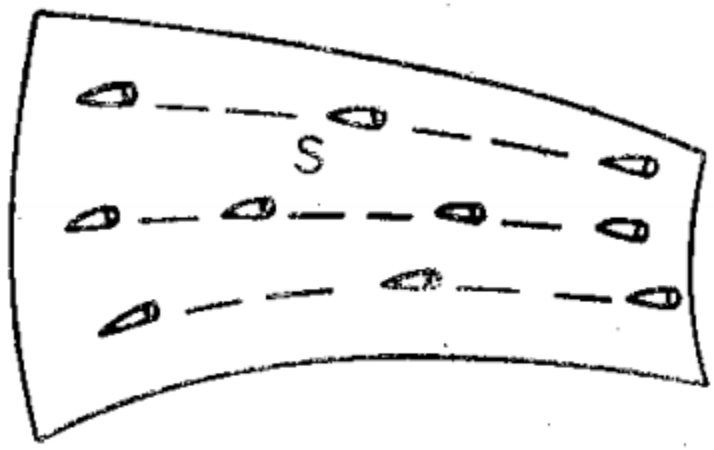

Fig 1.5 Monge directions on an integral surface S.

The above direction field also defines a one parameter family of curves on S, we call these curves *Monge curves* on S, and these curves generates S. Denoting the ratios in (3.15) by $d\sigma$, we notice that the Monge curves on S can be determined solving the ordinary differential equations

$\frac{dx}{d\sigma} = F_p(x, y, u(x, y), u_x(x, y), u_y(x, y)\ )$ (3.16)

and

$\frac{dy}{d\sigma} = F_q(x, y, u(x, y), u_x(x, y), u_y(x, y)\ )$ (3.17)

In the form of

$x = x(\sigma, x_0, y_0),\ \ y = y(\sigma, x_0, y_0)$ (3.18)

and then determining u from

$u = u(\sigma, x_0, y_0) \equiv u(x(\sigma, x_0, y_0), y(\sigma, x_0, y_0))$ (3.19)

Here $(x_0, y_0, u(x_0, y_0))$is a point on the surface S and the Monge curve on S given by (3.18) and (3.19) passes through the point. Since

$$\frac{du}{d\sigma} = u_x \frac{dx}{d\sigma} + u_y \frac{dy}{d\sigma}$$

It follows from (3.18) and (3.19) that along these curves u varies according to

$\frac{du}{d\sigma} = pF_p + qF_q$(3.20)

Where $u = u(x, y)$ has been substituted in the expression on the right-hand side.

Example 3.2: consider the function

$u = x\cos\phi + y\sin\phi\,, \phi =$constant. (3.21)

Which represents an integral surface of the equation

$F \equiv p^2 + q^2 - 1 = 0$ (3.22)

Then (3.16) and (3.17) give

$$\frac{dx}{d\sigma} = 2p = 2\cos\phi$$

$$\frac{dy}{d\sigma} = 2q = 2\sin\phi$$

Therefore, the Monge curves of (3.22) on integral surface (3.21) are given by

$$x = x_0 + 2\sigma\cos\phi\,,\ \ y = y_0 + 2\sigma\sin\phi$$

and

$$u = x_0\cos\phi + y_0\sin\phi + 2\sigma$$

Along the Monge curves on S the variation of p and q are known from the expressions $p = u_x(x(\sigma, x_0, y_0), y(\sigma, x_0, y_0))$ and $q = u_y(x(\sigma, x_0, y_0), y(\sigma, x_0, y_0))$ respectively. Now we shall determine the rates of change of p and q along a Monge curve on S. Since (3.1) is identically satisfied by $u = u(x, y)$, differentiating with respect to x we get the identity

$F_x + F_u u_x + F_p u_{xx} + F_q u_{yx} = 0$ on S. (3.23)

Along Monge curve on S

$$\frac{du_x}{d_\sigma} = u_{xx}\frac{dx}{d\sigma} + u_{xy}\frac{dy}{d\sigma} = u_{xx}F_p + u_{xy}F_q$$

For sufficiently smooth solution, $u_{xy} = u_{yx}$ so that from (3.23), we get

$\frac{dp}{d\sigma} = -(F_x + pF_u)$ (3.24)

Similarly, the variation of q along a Monge curve on S is

$\frac{dq}{d\sigma} = -(F_y + qF_u)$ (3.25)

Given an integral surface, we have shown that there exist a family of Monge curves, which generate the surface and along which x, y, u, p, q vary according to

$\frac{dx}{d\sigma} = F_p$ (3.26)

$\frac{dy}{d\sigma} = F_q$ (3.27)

$\frac{du}{d\sigma} = pF_p + qF_q$ (3.28)

$\frac{dp}{d\sigma} = -F_x - pF_u$ (3.29)

and

$\frac{dq}{d\sigma} = -f_y - qf_u$ (3.30)

We have discussed Monge curves exist only on a given integral surface. We now reverse the process by disregarding the fact that the system of ordinary differential equations (3.26) to (3.30) was derived with the help of integral surface. we call the first two equations (3.26) and (3.27) *characteristic equations,* the last three equations (3.28) - (3.30) *compatibility conditions* and the system formed with all the five equations (3.26)-(3.30), *Charpit's equations.*

A set $(x(\sigma), y(\sigma), u(\sigma), p(\sigma), q(\sigma))$ of five differential function is said to be a strip, if when we consider the curve $x = x(\sigma), y = y(\sigma), u = u(\sigma)$, the planes with the normals given by $(p(\sigma), q(\sigma), -1)$ are tangential to it.

A solution $x = x(\sigma), y = y(\sigma), u = u(\sigma), p = p(\sigma)$and $q = q(\sigma)$ of the Charpit's equations satisfied the strip condition

$\frac{du}{d\sigma} = p(\sigma)\frac{dx}{d\sigma} + q(\sigma)\frac{dy}{d\sigma}$ (3.31)

Note that not every set of five functions can be interpreted as a strip (Fig. 3.2). A strip requires that the plane with normal$(p, q, -q)$ be tangent to curve, i.e. they must satisfy the strip condition (3.31) and the normal should vary continuously along the curve. For a solution of Charpit's equation (3.26)-(3.30), the strip condition is guaranteed by the first three equations.

Along a solution of the Charpit's equations, we have

$\frac{dF}{d\sigma} = F_x\frac{dx}{d\sigma} + F_y\frac{dy}{d\sigma} + F_u\frac{du}{d\sigma} + F_p\frac{dp}{d\sigma} + F_q\frac{d_q}{d\sigma}$ (3.32)

which becomes identically equal to zero when we use (3.26) - (3.30). Therefore, Fremains constant along an integral curve of Charpits Equations in $(x, y, u, p, q) -$ space.If $F = 0$ is satisfied at an integral point $\sigma = 0, F = 0$ everywhere along the solution of Charpit's equations.

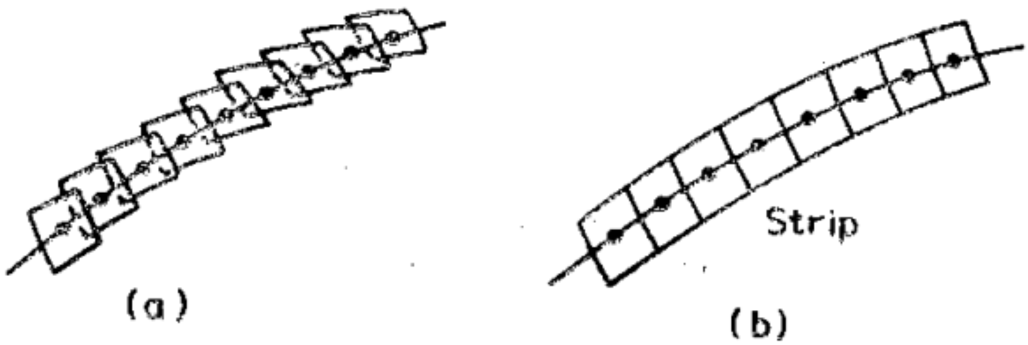

Fig 1.6 . Any set of five functions does not form a strip as in (a). The planes must be tangent to the curve and their normal should vary continuously (b).

The initial value for a solution of Charpit's equations can be prescribed by specifying x, y, u, p, q on the four-dimensional surface in (x, y, u, p, q) −space. Therefore, the system of Charpit's equations define a four-parameter family of strips. From this four-parameter family we choose a three parameter sub-family of strips by imposing the condition that $F = 0$ at $\sigma = 0$. Which implies $F = 0$along these strips. We call this three parameter sub-family of strips *Monge strips* and the projection on (x, y)-plane of the corresponding space curves in (x, y, u)-space, *characteristic curve*.

We shall show that *if a Monge strip, say M has one element (i.e. the values of* $x(\sigma), y(\sigma), u(\sigma), p(\sigma), q(\sigma), for\ some\ \sigma, say\ \sigma = 0$ *common with an integral surface* $S: u(x, y)$, *then the strip belong entirely to the integral surface.* let us suppose that at the point P, the integral surface S and the strip M has common values of (x, y, u, p, q). Since S is an integral surface, we can find a unique Monge curve on S through P. This together with p and q at points on this curve, gives a Monge strip M' on S.Since both strip M and M' satisfy Charpit's equations (3.26) – (3.30) with the same initial condition at P, it follows from the uniqueness theorem of solution of ordinary differential equation that M and M' are the same. As M' belongs entirely to the integral surface, the result follows.

EXERCISE 3.1

1. Show that the Monge cone of equation $p = q^2$ is an open cone which is generated by a one parameter family of straight lines whose one end is fixed but the other and moves on a parabola.

2. Consider the partial differential equation $F \equiv u(p^2 + q^2) - 1 = 0$

(i) Show that the general solution of the Charpit equation is a four parameter family of strips represented by

$$x = x_0 + \frac{2}{3}u_0(2\sigma)^{\frac{3}{2}}\cos\theta, y = y_0 + \frac{2}{3}u_0(2\sigma)^{\frac{3}{2}}\sin\theta$$

$$u = 2u_0\sigma, p = \frac{\cos\theta}{\sqrt{2\sigma}}, q = \frac{\sin\theta}{\sqrt{2\sigma}}$$

Where x_0, y_0, u_0 *and* θ are the parameters.

(ii) Find the three-parameter subfamily representing the totality of all Monge strips.

(iii)Show that all characteristic curves consist of all straight line in the (x, y)-plane.

1.4.2 Solution of a Cauchy Problem

If there exits an integral surface passing through a space curve Γ:

$$x = x_0(\eta), y = y_0(\eta), z = z_0(\eta); \qquad (3.33)$$

The first order partial derivatives $p = p_0(\eta)$ and $q = q_0(\eta)$, evaluated from the equation of integral surface at the point of Γ, satisfy the equation (3.1), i.e. $F\big(x_0(\eta), y_0(\eta), u_0(\eta), p_0(\eta), q_0(\eta)\big) = 0$ (3.34)

Moreover since $u_0(\eta) = u(x_0(\eta), y_0(\eta))$, differentiating with respect to η, we find the strip condition with respect to η:

$$u_0'(\eta) = p_0(\eta)x_0'(\eta) + q_0(\eta){y_0}'(\eta) = 0 \quad (3.35)$$

is satisfied at every point of Γ. Therefore, irrespective of choice of S, we can now solve for $p_0(\eta)$ and $q_0(\eta)$ from (3.34) and (3.35) to get an initial strip

$$x = x_0(\eta), y = y_0(\eta), u = u_0(\eta), p = p_0(\eta), q = q_0(\eta) \qquad (3.36)$$

We solve the Charpit's equations (3.26)-(3.30) with initial values of x, y, u, p and q at $\sigma = 0$ given by (3.36) and get the Monge strips starting from the various points of Γ. Since p_0, q_0 satisfy the strip condition (3.35) with respect to η, these Monge strips smoothly join to form a surface. Due to (3.34), F is identically zero along each Monge strip, hence the surface thus generated is integral surface of (3.1) passing through Γ. We note that there can be more than one integral surface passing through Γ, since there can more than one pair of function $p_0(\eta), q_0(\eta)$ satisfying the equations (3.34) and (3.35). However, once a set of values p_0 and q_0 ae selected, we expect to get a unique solution of Cauchy problem. In order that the solution exists and unique, it will be necessary to impose some restriction on the initial curve Γ.

Theorem 3.1: suppose the function $F(x,y,u,p,q) \in C^2(D_3)$ where D_3 is a domain in (x, y, u, p, q)-space. Further suppose that along a datum curve $x = x_0(\eta), y = y_0(\eta)$ on $I = \{\eta: 0 \leq \eta \leq 1\}$ the initial value $u = u_0(\eta)$ are assigned. Let the function $x_0(\eta), y_0(\eta), u_0(\eta)$ belong to $C^2(I)$; the functions $p_0(\eta), q_0(\eta)$, satisfying two equations (3.34) and (3.35), belongs to $C^1(I)$ and the set $\big(x_0(\eta), y_0(\eta), u_0(\eta), p_0(\eta), q_0(\eta)\big) \in D_3$ for $\eta \in I$ and satisfies

$$\frac{dx_0}{d\eta} F_q(x_0, y_0, u_0, p_0, q_0) - \frac{dy_0}{d\eta} F_p(x_0, y_0, u_0, p_0, q_0) \neq 0 \qquad (3.37)$$

Then we can find a domain D in (x, y)-plane containing the datum curve and a unique solution in D:

$$u = u(x, y) \qquad (3.38)$$

such that for $\eta \in I$

$$u\big(x_0(\eta), y_0(\eta)\big) = u_0(\eta) \qquad (3.39)$$

$$u_x\big(x_0(\eta), y_0(\eta)\big) = p_0(\eta) \text{and} u_y\big(x_0(\eta), y_0(\eta)\big) = q_0(\eta) \quad (3.40)$$

Proof: since the function appearing on a right hand side of the Charpit's equation (3.26)-(3.30) belong to $C^1(D_3)$ and
$x_0(\eta), y_0(\eta), u_0(\eta), p_0(\eta), q_0(\eta)$

are $C^1(I)$, there exists a unique solution of the Charpit's equation with initial condition $(x, y, u, p, q) = (x_0(\eta), y_0(\eta), u_0(\eta), p_0(\eta), q_0(\eta))$ at $\sigma = 0$:

$x = X(\sigma, \eta), y = Y(\sigma, \eta), u = U(\sigma, \eta), p = P(\sigma, \eta), q = Q(\sigma, \eta)$
(3.41) whose partial derivative with respect to σ and η exists and continuous.

From (3.26), (3.27) and (3.37) it follows that

$$\frac{\partial(X,Y)}{\partial(\eta,\sigma)}(at\ \sigma = 0) = \frac{dx_0}{d\eta} F_q(x_0, y_0, u_0, p_0, q_0) - \frac{dy_0}{d\eta} F_p(x_0, y_0, u_0, p_0, q_0) \neq 0 \quad (3.42)$$

Therefore, there exists a neighbourhood $N(x_0, y_0)$ of a point $(x_0(\eta), y_0(\eta))$ on the datum curve in (x, y)-plane (corresponding to $\sigma = 0$, such that in $N(x_0, y_0)$ we can solve the first two equations of (3.41) uniquely in the form

$$\sigma = \sigma(x, y), \eta = \eta(x, y) \qquad (3.43)$$

Substituting (3.43) in the expressions of u, p and q in (3.41) we get

$$u = U\big(\sigma(x, y), \eta(x, y)\big) \equiv u(x, y) \qquad (3.44)$$

$$p = P\big(\sigma(x, y), \eta(x, y)\big) \equiv p(x, y) \qquad (3.45)$$

$q = Q\big(\sigma(x,y),\eta(x,y)\big) \equiv q(x,y)$ (3.46) which are continuously differential differentiable function of x and y. we shall now show that (3.44) is the solution the Cauchy problem. It is obvious that on the datum curve $\sigma = 0$, the function (3.44) takes the prescribed value $u_0(\eta)$. Further, on the family of Monge strip (3.41), $F(x,y,u,p,q)$ has a constant value $F(x_0,y_0,u_0,p_0,q_0)$which is zero i.e.

$$F\big(x,y,u(x,y),p(x,y),q(x,y)\big) = 0 \; for \in N(x_0,y_0) \qquad (3.47)$$

Therefore, the function $u(x,y)$ in (3.44) is a solution of the differential equation (3.1) provided, we can show that

$$u_x(x,y) = p(x,y), u_y(x,y) = q(x,y) \qquad (3.48)$$

Consider the function

$$W(\sigma,\eta) = U_\eta - PX_\eta - QY_\eta \qquad (3.49)$$

whose value, $W(0,\eta)$, on the datum curve is zero. Differential (3.49) with respect to

$$\frac{\partial W}{\partial \sigma} = U_{\eta\sigma} - PX_{\eta\sigma} - QY_{\eta\sigma} - P_\sigma X_\eta - Q_\sigma Y_\eta$$

$$= \frac{\partial}{\partial \sigma}(U_\sigma - PX_\sigma - QY_\sigma) + P_\eta X_\sigma + Q_\eta Y_\sigma - P_\sigma X_\eta - Q_\sigma Y_\eta$$

$$= 0 + P_\eta X_\sigma + Q_\eta Y_\sigma + X_\eta(F_x + PF_u) + Y_\eta(F_y + QF_u)$$

where we have used the Charpit's equation in the result. Adding and subtracting $F_u U_\eta$ we get,

$$\frac{\partial W}{\partial \sigma} = \big(F_x X_\eta + F_y Y_\eta + F_u U_\eta + F_p P_\eta + F_q Q_\eta\big) - F_u\big(-PX_\eta - QY_\eta + U_\eta\big)$$

$$= F_\eta - F_u W$$

Since F identically zero along each of the Monge strips (3.14), $F_\eta \equiv 0$. The function Wnow satisfies the following linear homogeneous ordinary differential equation

$$\frac{\partial W}{\partial \sigma} = -F_u(\sigma,\eta)W \qquad (3.50) \text{ with solution}$$

$$W = w(0,\eta)\exp\{-\textstyle\int_0^\sigma F_u(\sigma,\eta)\; d\sigma\} \qquad (3.51)$$

Since $W(0,\eta) = 0, W(\sigma,\eta) = 0$ for all values of (σ,η) such that $(x,y) \in N(x_0,y_0)$

Therefore,

$$U_\eta = PX_\eta + QY_\eta \qquad (3.52)$$

From the Charpit's Equation, we also have

$U_\sigma = PX_\sigma + QY_\sigma$ (3.53)

From (3.44) we get,

$$u_x = U_\sigma \sigma_x + U_\eta \eta_x = \sigma_x (PX_\sigma + QY_\sigma) + \eta_x (PX_\eta + QY_\eta)$$

$$= P(X_\sigma \sigma_x + X_\eta \eta_x) + Q(Y_\sigma \sigma_x + Y_\eta \eta_x)$$

$= P\frac{\partial x}{\partial x} + Q\frac{\partial y}{\partial x} = P.1 + Q.0 = P(\sigma,\eta) = p(x,y)$ (3.54) where we have used the expressions of x and y from the first two equations (3.41). similarly, we can show that

$u_y = q(x,y)$ (3.55)

Therefore from (3.47) it follows that $u(x,y)$ given by (3.44) is a solution of the differential equation (3.1), in the domain $N(x_0, y_0)$.

To prove the uniqueness of the solution, let us assume that S' is another integral surface represented by the solution $u = u'(x,y)$ of the Cauchy's problem. The surface S'can be covered by the family of Monge strips after solving (3.16) and (3.17) with u replaced by u'. These Monge strips satisfy the same initial condition at their point of intersection with the initial curve Γ, as the strips (3.14). from the uniqueness theorem for a solution of the Charpit's ordinary differential equations, it follows that this family of Monge strips on the integral surface S' must be the same as the strips (3.41). Therefore, the integral surface S coincide with S', i.e.$u = u'$ in $N(x_0, y_0)$

*Example 3.1:*Consider the equation

$p^2 + q^2 = 1$ (3.56)

And straight line in (x,y)- plane.

$x = x_0 \equiv \eta \sin\beta \cos\alpha \,, y = y_0 \equiv \eta \sin\beta \sin\alpha$ (3.57)

On which u prescribed by

$u = u_0 \equiv \eta \cos\beta$ (3.58)

where α and β are constants.

The Monge cone at (x_0, y_0, u_0) is the envelope of the planes

$$(x - x_0)\cos\lambda + (y - y_0)\sin\lambda - (u - u_0) = 0$$

The Monge cone is therefore represented by the equation

$$(x - x_0)^2 + (y - y_0)^2 = (u - u_0)^2$$

which gives a right circular cone with vertex at (x_0, y_0, u_0), axis parallel to u-axis and semi vertical angle $\frac{\pi}{4}$.

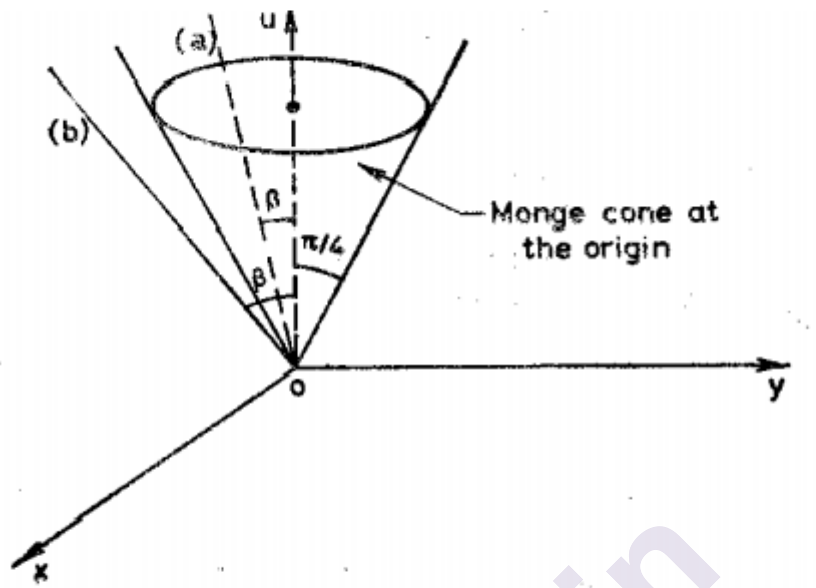

Fig 1.7

Case (a): When $\beta < \pi/4$ the initial curve Γ is in the interior of the Monge cone.
Case (b): When $\beta > \pi/4$, Γ is outside the Monge cone.

For the initial strip we have to solve the equations

$$p_0^2 + q_0^2 = 1 \qquad (3.59)$$

and

$$p_0 \sin\beta \cos\alpha + q_0 \sin\beta \sin\alpha = \cos\beta \qquad (3.60)$$

If $\beta < \frac{\pi}{4}$, the equations (3.59) and (3.60) do not possess a real solution for p_0 and $q_0$showing that the solution of Cauchy problem does not exist. This can be explained from the fact that the space curve given by (3.57) and (3.58) through which the integral surface should pass, lies in the interior of the Monge cone at the origin. Naturally it is not possible for an integral surface to touch the Monge cone along a generator of the cone and also to pass through a line within it.

For $\pi \backslash 4 < \beta < \pi/2$, we get two sets of values of p_0 and q_0

$$p_0 = \cot\beta \cos\alpha \pm \sin\alpha\,(1 - \cot^2\beta)^{1/2} \qquad (3.61)$$

$$q_0 = \cot\beta \sin\alpha \mp \cos\alpha\,(1 - \cot^2\beta)^{1/2} \qquad (3.62)$$ which is independent of η.

The Charpit's equations are

$$\frac{dx}{d\sigma} = 2p\,, \quad \frac{dy}{d\sigma} = 2q$$

$\frac{du}{d\sigma} = 2(p^2 + q^2) = 2(1) = 2$, using (3.56)

$\frac{dp}{d\sigma} = 0$and $\frac{dq}{d\sigma} = 0$

Solving thesewith the initial values (3.57), (3.58), (3.61) and (3.62), we get

$x = 2p_0\sigma + \eta \sin\beta \cos\alpha\,, y = 2q_0\sigma + \eta \sin\beta \sin\alpha\,, u = 2\sigma + \eta\, cos\beta,$

$p = p_0, q = q_0$ (3.63)

Eliminating σ and η from (3.63) we get the two solution of Cauchy problem corresponding to the two sets of values of p_0 and q_0.

$u = \cot\beta\,(x\cos\alpha + y\sin\alpha) \pm \sqrt{1 - \cot^2\beta}\ (x\sin\alpha - y\cos\alpha)$
(3.64)

They represent two planes which pass through the initial line Γ and touch the Monge cones along two generators.

1.4.3 Solution of a Characteristic Cauchy Problem

We have seen that when the condition (3.37) is satisfied, i.e. when the data is such that datum curve γ in (x, y)-plane is nowhere tangential to the characteristic curve for a possible solution of the Cauchy problem exists and unique. However, when $F_q x_0'(\eta) - F_q y_0'(\eta) = 0$ hold everywhere along γ and the initial manifold M: $(x_0(\eta), y_0(\eta), u_0(\eta), p_0(\eta), q_0(\eta))$ belongs to the integral surface S, then following the arguments of §3.1 for the derivation of Charpit's equation (3.26) – (3.30) we can show that the strip M must be a Monge strip on S with the parameter σ replaced by η. Hence in exception case, $F_q x_0' - F_q y_0' = 0$, a necessary condition for the existence of a solution of the Cauchy problem is that the initial strip M is a Monge strip. This condition is also sufficient. In fact, if this condition is satisfied, there exist not only one but an infinite number of solutions of the characteristic Cauchy problem.

If $F_q x_0' - F_q y_0' = 0$ and the initial strip is not a Monge strip, then it follows from above that there exists no solution of the Cauchy problem having continuous derivatives up to second order in the neighbourhood of the datum curve.

EXERCISE 3.2

1. Solve the Cauchy problems:
   - (i) $\frac{1}{2}(p^2 + q^2) = u$ with Cauchy data prescribed on the circle $x^2 + y^2 = 1$ by $u(\cos\theta\,, \sin\theta) = 1, 0 \le \theta \le 2\pi$
   - (ii) $p^2 + q^2 + \left(p - \frac{x}{2}\right)\left(q - \frac{y}{2}\right) - u = 0$, with Cauchy data prescribed on x-axis by $u(x, 0) = 0$

(iii) $2pq - u = 0$, with Cauchy data prescribed on y-axis by $u(0, y) = \frac{y^2}{2}$

(iv) $2p^2x + qy - u = 0$, with Cauchy data $u(x, 1) = -\frac{x}{2}$

2. Consider two parameter family of functions $u = \phi(x, y, a, b)$ where ϕ is a known functions of its arguments and a, b are parameters. If the rank of the matrix $\begin{bmatrix} \phi_a & \phi_{xa} & \phi_{ya} \\ \phi_b & \phi_{xb} & \phi_{yb} \end{bmatrix}$ is 2, show that the result of elimination of a and b from the relation $\phi(x, y, a, b) = u, \phi_x(x, y, a, b) = u_x,$ $\phi_y(x, y, a, b) = u_y$ leads to a first non-linear equation
$$F(x, y, u, u_x, u_y) = 0$$

3. Two first order partial differential equations are said to be compatible, if they have a common solution. Show that the necessary and sufficient condition for two equations $F(x, y, u, p , q) = 0$ and $G(x, y, u, p , q) = 0$ to be compatible is that $\frac{\partial(F,G)}{\partial(x,p)} + p\frac{\partial(F,G)}{\partial(u,p)} + \frac{\partial(F,G)}{\partial(y,q)} + q\ \frac{\partial(F,G)}{\partial(u,q)} = 0$ is satisfied either identically or as a consequence of relations $F = 0$ and $G = 0$.

1.5 COMPLETE INTEGRAL

In problem 2 in Exercise 3.2 we saw that the result of elimination of two arbitrary constants a and b from a relation

$u = \phi(x, y, a, b)$ (4.1)

leads to a non-linear equation

$F(x, y, u, u_x, u_y) = 0$ (4.2)

We note that (4.1) satisfied (4.2) for all values of a and b.

We shall show that a solution of the form (4.1) and (4.2) is sufficiently general in the sense of all other solution of this equation can be obtained from it merely by simple operation of differentiation and elimination of the constants.

Definition: A two parameter family of solution (4.1) of the equation (4.2) is called *complete integral* of the equation if the rank the matrix $\begin{bmatrix} \phi_a & \phi_{xa} & \phi_{ya} \\ \phi_b & \phi_{xb} & \phi_{yb} \end{bmatrix}$ is 2 in an appropriate domain of the variables x, y, a, b.

The condition that the above matrix has rank 2 assures that the function ϕ depends on two independent parameters and elimination of a and b from (4.1) and

$u_x = \phi_x(x, y, a, b), u_y = \phi_y(x, y, a, b)$ (4.3)

leads to equation (4.2).

Note 1: If a and b be combined into one parameter $c = c(a, b)$, then two rows of the matrix become linearly dependent and its rank becomes one.

2: If the rank is two, a and b can solved from (4.3) and these can be substituted in (4.1).

1.5.1 Determination of complete integral

It is simple to determine a complete integral for a given partial differential equation (4.2). the problem of Exercise 3.2 gives the condition for the existence of a common solution of two equations $F(x, y, u, u_x, u_y) = 0$ and $G(x, y, u, u_x, u_y) = 0$. Once these two equations have a common solution, we first solve them simultaneously for u_x and u_y in terms of x, y and u.

$u_x = h(x, y, u)$ and $u_y = k(x, y, u)$

and then the differential relation

$$h(x, y, u)dx + k(x, y, u)dy = du \quad (4.4)$$

will possess an integrating factor and can be integrated giving a relation between x, y and u and an arbitrary constant b. Therefore, a complete integral of (4.2) can be determined if we can determine a compatible equation $G(x, y, u, p, q) = 0$ containing an arbitrary constant a. But this is simple since the result of problem 3.2 shows that any G satisfying the equation:

$$F_p \frac{\partial G}{\partial x} + F_q \frac{\partial G}{\partial y} + (pF_p + qF_q) \frac{\partial G}{\partial u} - (F_x + pF_u) \frac{\partial G}{\partial p} - (F_y + qF_u) \frac{\partial G}{\partial q} = 0 \quad (4.5)$$

would be a compatible equation.

This is the first order linear homogeneous partial differential equation for G in five independent variables x, y, u, p and q. For the equation (4.5), the characteristic equations and compatibility conditions are

$$\frac{dx}{F_p} = \frac{dy}{F_q} = \frac{du}{pF_p + qf_q} = \frac{dp}{-(F_x + pF_u)} = \frac{dq}{-(F_y + qF_u)} \quad (4.6)$$

Since the compatibility condition implies that $G =$constant on the characteristic curve in (x, y, u, p, q)-space, it follows that if we can get any first integral, say $s(x, y, u, p, q) = a$ of the characteristic equations, then $G \equiv s(x, y, u, p, q) - a = 0$ is the required equation containing an arbitrary constant a and compatible with $F(x, y, u, u_x, u_y) = 0$

The characteristic equations of (4.5) are nothing but the Charpit's equations (3.26) - (3.30) of the equation (4.2)

Example 4.2: Find the complete integral of the partial differential equation

$$x^2p^2 + y^2q^2 - 4 = 0. \quad (4.7)$$

Solution: Charpit's equation for the given PDE,

$$\frac{dx}{2px^2} = \frac{dy}{2qy^2} = \frac{du}{2(x^2p^2 + y^2q^2)} = \frac{dp}{-2xp^2} = \frac{dq}{-2yq^2}$$

We take the relation $\frac{dx}{2px^2} = \frac{dy}{2qy^2}$ which gives

$$G \equiv xp = \text{constant} = a, \text{ say.} \quad (4.8)$$

Taking one of the value of p and q from (4.7) and (4.8) and substituting in (4.4), we get

$$du = \frac{a}{x}dx + \frac{\sqrt{4-a^2}}{y}dy$$

Integrating this we get a complete integral

$$u = a\log x + \sqrt{4-a^2}\log y + b$$

Containing two arbitrary constants a and b.

Example 4.3 Find the complete integral of the PDE $u = px + qy + p^2 + q^2$.

Solution: $F(x, y, u, p, q) = 0$

$$F = px + qy + p^2 + q^2 - u = 0$$

$$F_x = p, F_y = q, F_u = -1, F_p = x + 2p, F_q = y + 2q$$

Charpit's equation for the given PDE,

$$\frac{dp}{-p+p} = \frac{dq}{-q+q} = \frac{du}{p(x+2p)+q(y+2q)} = \frac{dx}{x+2p} = \frac{dy}{y+2q}$$

Taking $\frac{dp}{0} = \frac{dq}{0}$,

$$dp = 0, dq = 0 => p = a, q = b$$

Complete integral $u = ax + by + a^2 + b^2$.

EXERCISE 1.1

1. Show that the compete integral of

a) $F(p, q) = 0$, where F involves only p and q and $F(p, Q(p)) = 0$ is $u = ax + Q(a)y + b$.

b) $F \equiv f(x, p) - g(y, q) = 0$ is obtained by solving p and q from $f(x, p) = a, g(y, q) = a$ and integrating $du = pdx + qdy$.

c) $F \equiv u - px - qy - f(p, q) = 0$ is $u = ax + by + f(a, b)$

2. If independent variable x and y do not appear in the equation $F(u, p, ap) = 0$, then show that the complete integral can be obtained by solving p form $F(u, p, ap) = 0$, taking $q = ap$ and integrating $du = pdx + qdy$.

Note: These all are standard results and can be used to find complete integral of any PDE satisfying the given condition.

1.5.2 Solution of a Cauchy Problem

Once we know a complete integral, we can find solution of the Cauchy problem.

We are required to construct an integral surface S of (4.2) passing through an initial curve

$\Gamma: x = x_0(\eta), y = y_0(\eta), u = u_0(\eta)$ (4.9)

At the point of intersection of Γ and any member of (4.1) the parameter η satisfies

$\phi(x_0(\eta), y_0(\eta), a, b) = u_0(\eta)$ (4.10)

Differentiating both the sides with respect to η

$$\frac{\partial}{\partial\eta}\phi(x_0(\eta), y_0(\eta), a, b) = u_0{}'(\eta) \qquad (4.11)$$

Eliminating η from these two equations we get a relation between a and b. which is required integral surface.

Example 4.4: Solve the Cauchy problem

$$2p^2x + qy = u \qquad (4.12)$$

with Cauchy data $u(x,1) = -\frac{x}{2}$

Solution: Cauchy data can be put in the form

$x = x_0(\eta) \equiv \eta, y = y_0(\eta) = 1, u = u_0(\eta) \equiv -\frac{1}{2}\eta$
(4.13)

To derive a complete integral, the Charpit's equations:

$$\frac{dx}{4px} = \frac{dy}{y} = \frac{du}{4p^2x + qy} = \frac{dp}{-2p^2 + p} = \frac{dq}{0}$$

Which gives a compatible equation

$$q = a \qquad (4.14)$$

Containing an arbitrary constant. From (4.12) and (4.26), we get

$$p = \sqrt{\frac{u - ay}{2x}} \qquad (4.15)$$

The complete integral is given by

$$du = pdx + qdy$$

$$du = \sqrt{\frac{u - ay}{2x}}dx + ady \;=>\; \frac{du - a\,dy}{\sqrt{u - ay}} = \frac{dx}{\sqrt{2x}} \;=>\; \sqrt{u - ay}$$

$$= \sqrt{\frac{x}{2}} + b$$

$$=> \left(u - ay - \frac{x}{2} - b\right)^2 = 2bx \qquad (4.16)$$

Substituting (4.13) in (4.16),

$$(\eta + a + b)^2 = 2b\eta \qquad (4.17)$$

Which after differentiating with respect to η gives

$2(\eta + a + b) = 2b \quad (4.18)$

Eliminating η from (4.17) and (4.18), we get

$$b = -2a$$

Substituting this value of b in (4.16), we get the solution of Cauchy problem

$u = \frac{xy}{2(y-2)}$.

EXERCISE 1.2

1. Use the method of complete integrals to solve the following Cauchy problems:

i) $2pq - u = 0, u(\eta, 1) = \frac{1}{2}\eta$

ii) $p - q = \frac{1}{2}(x^2 + y^2), u(\eta, \eta) = \frac{\eta^2}{2}\ for - \infty < \eta < \infty$

iii) $p^2 + q^2 = u, u(\cos\eta, \sin\eta) = 1\ for\ 0 \le \eta \le 2\pi$

iv) $u = px + qy + p + q - 2pq,\ u(\eta, \eta) = 2\eta\ for\ -\infty < \eta < \infty$

2. Given any two complete integrals $u = \phi(x, y, a, b), u = \psi(x, y, c, d)$ of a first order partial differential equation, show that one complete integral can be derived from the other.
3. Find the complete integral of $4(p + q)(u - xp - yq) = 1$.

1.6 LET US SUM UP:

In this unit two main types of partial differential equations, semilinear and quasi linear, out of fourare discussed. Cauchy problem and its characteristics is discussed. General solution can be determined. We have also discussed Monge curve and Charpit's equation. Solution of characteristic Cauchy problem and complete integral is discussed.

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2

SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS

Unit structure

2.0 Objectives

2.1 Introduction

2.2 Classification of second order partial differential equation in two independent variables.

2.3 Classification of partial differential equation in more than two independent variables

2.4.1 The Cauchy problem

2.4.2 The Solution of Cauchy's Problem

2.5 Method of reduction to normal form

2.6 Potential Theory and Elliptical differential Equation.

2.7 Harmonic function

2.8 Poisson's formula

2.9 Let's sum up

2.10 List of references

2.11 Bibliography

2.1 OBJECTIVE

After doing this unit, you will be able to:

- Classify the 2nd order PDE in two variables into hyperbola, parabola and ellipse
- Classify the 2nd order PDE in more than two variables
- Find the characteristics equations of all three types of PDE.
- To solve the Cauchy's problem.
- To reduce the 2nd order PDE in its normal form.
- To find the potential equations.
- To study about harmonic function.
- To derive Poisson's integral formula.
- To learn about Maximum - minimum properties

2.1 INTRODUCTION

We have studies in previous chapter about first order partial differential equation and its types, Cauchy equation and how to find the general solution of this. Now we will learn about second order partial differential equation in two variables and its classification also study the partial differential equation more than two variable.

2.2 CLASSIFICATION OF SECOND ORDER PARTIAL DIFFERENTIAL EQUATION IN TWO INDEPENDENT VARIABLES.

Consider a general partial differential equation of second order for a function of two independent variables x and y in the form:

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0 \qquad (1.1)$$

Where R, S, T are continuous functions of x and y only possessing partial derivatives defined in some domain D on xy −plane.

And $r = \frac{\partial^2 z}{\partial x^2}$, $s = \frac{\partial^2 z}{\partial x \partial y}$, $t = \frac{\partial^2 z}{\partial y^2}$, Then (1.1) is said to be

(i) Hyperbolic at a point (x, y) in domain D if
$$S^2 - 4RT > 0$$
(ii) Parabolic at a point (x, y) in domain D if
$$S^2 - 4RT = 0$$
(iii) Elliptic at a point (x, y) in domain D if
$$S^2 - 4RT < 0$$

Note that the type of (1.1) is determine solely by its principal part ($Rr + Ss + Tt$, which involves the higher order derivative of z) and that the type will generally change with the position in the xy −plane unless R, S, T are constants.

Remark: some authors use u in place of z. Then we will have

$$r = \frac{\partial^2 u}{\partial x^2}, s = \frac{\partial^2 u}{\partial x \partial y}, \qquad t = \frac{\partial^2 u}{\partial y^2}$$

Examples:

i) Consider the one-dimensional wave equation $\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial y^2}$ i.e. $r - t = 0$

Sol. Comparing it with (1.1), here , $R = 1, S = 0, T = -1$

$S^2 - 4RT = 0 - 4(1)(-1) = 4 > 0$, so the given equation is hyperbolic.

ii) Consider the one dimensional diffusion equation

iii) Consider the two-dimensional Laplace equation $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0\ i.e. r + t = 0$

Sol. Comparing it with (1.1), here , $R = 1, S = 0, T = 1$

$S^2 - 4RT = 0 - 4(1)(1) = -4 < 0$,so the given equation is elliptic.

Ex.2. classify the following partial differential equations.

i) $2\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial y} - 3\frac{\partial^2 u}{\partial y^2} = 2$

Sol: $R = 2, S = 4, T = -3$

$S^2 - 4RT = 16 - 4 * 2 * -3 = 16 + 24 = 40 > 0$ =>hyperbolic

ii) $\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial y} + 4\frac{\partial^2 u}{\partial y^2} = 0$

$R = 1, S = 4, T = 4,\ S^2 - 4RT = 16 - 4 * 1 * 4 = 0$=> parabolic

iii) $xyr - (x^2 - y^2)s - xyt + py - qx = 2(x^2 - y^2)$

Sol.: $R = xy, S = -(x^2 - y^2), T = -xy$

$$S^2 - 4RT = \{-(x^2 - y^2)\}^2 - 4 * xy * -xy,$$

$$= \{(x^2 - y^2)\}^2 + 4x^2y^2 = x^4 + y^4 - 2x^2y^2 + 4x^2y^2$$

$= x^4 + y^4 + 2x^2y^2 = (x^2 + y^2)^2 > 0$ =>hyperbolic

iv) $x(xy - 1)r - (x^2y^2 - 1)s + y(xy - 1)t + xp + yq = 0$

Sol.: $(x^2y^2 - 1)^2 - 4x(xy - 1) * y(xy - 1)$

$$= (x^2y^2 - 1)^2 - 4xy(xy - 1)^2$$

$$= (xy - 1)^2(xy + 1)^2 - 4xy(xy - 1)^2$$

$$= (xy - 1)^2\{(xy + 1)^2 - 4xy\}$$

$$= (xy - 1)^2\{x^2y^2 + 1 + 2xy - 4xy\}$$

$$= (xy - 1)^2\{x^2y^2 + 1 - 2xy\}$$

$$= (xy - 1)^2(xy - 1)^2 = (xy - 1)^4 > 0 => hyperbolic$$

2.3 CLASSIFICATION OF SECOND ORDER PARTIAL DIFFERENTIAL EQUATION IN THREE INDEPENDENT VARIABLES.

A linear partial differential equation of the second order in three independent variables $x_1, x_2\ and\ x_3$, is given by,

$\sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{3} b_i \frac{\partial u}{\partial x_i} + cu = 0$ (1.2)

Where $a_{ij}(= a_{j\,i}), b_i$ and c are constant or some functions of independent variables $x_1, x_2, x_3$and u is thedependent variable.

Since $a_{ij} = aj\ i$ $A = [a_{ij}]_{3\times 3}$ is real and symmetric of order 3×3. The eigen values of matrix A are roots of the characteristic equation of A, namely $|A - \lambda I| = 0$.

With the help of matrix A, (1.2) is classified as follows

i) If the eigen values od A are non-zero and have same sign, except precisely one of them then (1.2) is known *as hyperbolic type of equation.*

ii) If $|A| = 0$,i.e. anyone of the eigen value of A is zero, then (1.2) is known *as parabolic type of equation.*

iii) If all the eigen values of A are non-zero and of the same sign, then (1.2) is known *as elliptic type of equation.*

The matrix A can be remembered as

$$A = \begin{bmatrix} coeff.\ of\ u_{xx} & coeff.\ of\ u_{xy} & coeff.\ of\ u_{xz} \\ coeff.\ of\ u_{yx} & coeff.\ of\ u_{yy} & coeff.\ of\ u_{yz} \\ coeff.\ of\ u_{zx} & coeff.\ of\ u_{zy} & coeff.\ of\ u_{zz} \end{bmatrix}$$

Ex.1.$classify\ the\ PDE\ \ u_{xx} + u_{yy} = u_{zz}$

Sol.:$u_{xx} + u_{yy} - u_{zz} = 0$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Characteristic equation of A,

$$\lambda^3 - tr(A)\lambda^2 + (A_{11} + A_{22} + A_{33})\lambda - |A| = 0$$

$$\lambda^3 - \lambda^2 + (-1 - 1 + 1)\lambda + 1 = 0$$

$$=> \lambda^3 - \lambda^2 - \lambda + 1 = 0$$

$$=> \lambda^2(\lambda - 1) - 1(\lambda - 1) = 0$$

$$(\lambda - 1)(\lambda^2 - 1) = 0$$

$$=> (\lambda - 1)(\lambda - 1)(\lambda + 1) = 0$$

$$=> \lambda = 1, 1, -1$$

It is showing that all eigen values are non-zero and have the same sign except one. Hence the given equation is <u>hyperbolic type.</u>

Ex.2.$u_{xx} + u_{yy} + u_{zz} + u_{zy} + u_{yz} = 0$

Sol.: The given equation can be rewritten as $u_{xx} + 0.u_{xy} + 0.u_{xz} + 0.u_{yx} + u_{yy} + u_{yz} + 0.u_{zx} + u_{zy} + u_{zz} = 0$

The matrix A of the given equation is as follows

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Determinant of A, $|A| = 0$

=>given equation is parabolic type.

Ex.3. Classify $u_{xx} + 2u_{yy} + u_{zz} = 2u_{xy} + 2u_{yz}$

Sol.:the given can be rewritten as

$$u_{xx} + 2u_{yy} + u_{zz} - u_{xy} - u_{yx} - u_{yz} - u_{zy} = 0$$

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix},$$

$$|A| = 1(2-1) + 1(-1-0) + 0 = 1 - 1 = 0$$

=>given equation is parabolic type

Ex4.: classify the following equations.

i) $u_{xx} + u_{yy} + u_{zz} = 0$
ii) $u_{xx} + u_{yy} = u_z$
iii) $3u_{xx} + 3u_{zz} + 4u_{xy} + 8u_{zz} + 4u_{yz} = 0$

Sol.: Try yourself.

Ans.: i) elliptic ii) parabolic iii) hyperbolic

2.4.1 THE CAUCHY'S PROBLEM

We start with the general quasilinear second order equation for a function $u(x, y)$ of two independent variables:

$u_{x\,x} + 2\,b\,u_{x\,y} + c\,u_{y\,y} = d(1)$

where a, b, c, d depend on x, y, u, u_x, u_y. The Cauchy problem consists in finding a solution of (2.4.1) with given values of u and its normal derivative on a curve C in the (x, y) plane.

Let the parametric representation of C be: $x = x_0(s), y = y_0(s), s \in I$, where I is aninterval on the real line. We are given two functions $u_0(s)$

and $u_1(s), s \in I$ The Cauchy problem consists in finding a solution $u(x, y)$ of (2.4.1) which satisfies the following conditions:

$u\big(x_0(s), y_0(s)\big) = u_0(s), s \in I$and $\frac{\partial u}{\partial v}\big(x_0(s), y_0(s)\big) = u_t(s), s \in I$(2)

where $\frac{\partial}{\partial v}$ denotes a normal derivative to C.

For discussion of the Cauchy problem here, we assume that a, b, c and d are analyticfunctions, regular in some domain D. Our aim is to examine whether there exists a unique analytic solution of (2.4.1), which takes given values on C. To do so, we formally construct a solution using a Taylor's series expansion about any point of C. The first step in such a solution is to show that the partial derivatives of u of all orders are uniquely determined at every point of C Let suffix 0 denote the values of partial derivatives of u at point of Ci.e.$u\big(x_0(s), y_0(s)\big) = u_0(s)$, and so on. Then $u_{x_0}(s)$ and $u_{y_0}(s)$satisfy the following linear equations:

$$x_0' u_{x_0}(s) + y_0' u_{y_0}(s) = {u_0}'(s)$$

and $-y_0' u_{x_0}(s) + x_o' u_{y_0}(s) = \sqrt{x_0^2 + y_0^2} * u_1(s)$ (3)

where a prime ($'$)denotes differentiation with respect to s. Except at points where x_0' and y_0'vanish simultaneously u_{x_0} and u_{y_0} can be determined uniquely.

Regarding second order derivatives, namely, $u_{xx_0}(s)$ and $u_{xy_0}(s)$ and $u_{yy_0}(s)$ they can be determined as solutions of the linear equations:

$$a u_{xx_0}(s) + 2b u_{xy_0}(s) + c u_{yy_0}(s) = d$$

$$x_0'(s) u_{xx_0}(s) + y_0'(s) u_{yy_0}(s) = \{u_{x_0}(s)\}'$$

$x_0'(s) u_{xy_0}(s) + y_0'(s) u_{yy_0}(s) = \{u_{y_0}(s)\}'$ (4)

These equations determine $u_{xx_0}(s), u_{xy_0}(s)\ and\ u_{yy_0}(s)$ uniquely provided the determinant of the coefficient matrix is nonzero. This requires that

$$a y_0'^2 - 2b x_0' y_0' + c x_0'^2 \neq 0$$

Or$Q(-y_0', x_0') \neq 0$ (5)

where Q is the characteristic quadratic form.Further we can show that the derivatives of u of all orders can be uniquely determined at points of C provided

$$Q(-y_0', x_0') \neq 0$$

In this way we can formally develop a unique Taylor's series expansion solution in the neighbourhood of any point of C satisfying the given conditions on C. The difficulty is to show that such an expansion is convergent in some region around C. The Cauchy-Kowalewski method (see Garabedian, 1964) provides a majorant series ensuring convergence.

On the other hand, if $Q(-y_0', x_0') = 0$, then the partial derivatives of u on the curve C cannot be determined uniquely. The exceptional curves C, on which if u and its normal derivative are prescribed, no unique solution of (1) can be found satisfying these conditions, are called characteristic curves. These curves satisfy the homogencous equation

$Q(-y_0', x_0') = 0.$

If the curve $C: x = x_0(s), y = y_0(s)$ in the (x, y) plane is given by the equation $\phi(x, y)$ =constant.

By eliminating s, then ϕ satisfied the PDE $Q(\phi_x, \phi_y) = 0$ on$\phi(x, y)$ =constant (6)

Since $-\frac{y_0'}{x_0'} = \frac{\phi_x}{\phi_y} = -\frac{dy}{dx}$

From the result it follows that there are two distinct families of characteristic curves satisfying equation (6), if the equation is hyperbolic. There are precisely $\xi(x, y)$ =constant and $\psi(x, y)$ =constant, ξ and ψ are referred as characteristic variable or coordinates.

For the hyperbolic equation in its normal form, namely,

$$u_{\xi\eta} + D(\xi, \eta, u, u_\xi, u_\eta) = 0 \qquad (7)$$

$\xi(x, y)$ =constant and $\psi(x, y)$ =constanare the characteristic curves. If, for example in the Cauchy problem u and$u_\xi$are proscribed on a characteristic carve. C: ξ =constant, then we cannot determine $u_{\xi\xi}$ uniquely on ξ = constant from the given equation (since the coefficient of $u_{\xi\xi}$is zero in the linear second order equation (7)). Since u and $u_\xi$are prescribed on ξ =constant as $u_0(\eta)$ and $u_1(\eta)$, say, respectively, u and $u_\xi$can be computed on ξ =constant and the equation (7) will reduce to the compatibility condition

$$u_1(\eta) + D(\xi, \eta, u_0, u_1, u_0') = 0$$

on ξ =constant. Compatibility conditions to be satisfied on charactoristic curves are typical, as tho equation gives no additional information in this case (like the value of use in equation (7)), but merely insists on a relation between already known quantities. If the compatibility condition is satisfied there will be an infinity of solutions of the Cauchy problem (choosing $u_{\xi\xi}$ arbitrarily in (7)), or else there will be no solution. The above discussion holds for data prescribed on η = constant as well. For a hyperbolic equation, we have two compatibility conditions, one each on the characteristic curves ξ =constant and η =constant. For a parabolic equation, we have one compatibility condition on the single family of characteristic curves.

In the canonical elliptic form, $u_{\xi\xi}$ and $u_{\eta\eta}$can always be determined Whenever u and its normal derivative are prescribed on any curve in the (x, y) −plane, since $Q(\phi_x, \phi_y) \neq 0$ on any real curve $\phi(x, y) =$ constant. We can always find a unique solution for the. Cauchy problem in this case.

In the case of m independent variables, those surfaces $\phi(x_1, x_2, \ldots \ldots . x_m) = 0,$ on which, when the function and its normal derivative are prescribed, no unique solution, exists satisfying the prescribed conditions, are called characteristic surfaces. Following a similar process, as in the case of two independent variables, it follows that ϕ satisfies the equation ,namely

$$Q_1(\phi) = a_{\alpha\beta}\phi_{x_\alpha}\phi_{x_\beta} = 0 \; on \; \phi = 0$$

The characteristic condition $Q_1(\phi) = 0$ is required to be satisfied on ϕ=0 but this does not require that ϕ satisfies the equation $Q_1(\phi) =$ 0identically.

EXERCISE 2.1

1. Let $u(x, y)$ satisfy the equation
$$u_{xx} - 2u_{xy} + u_{yy} + 3u_x - u + 1 = 0$$
in a region of the (x, y) plane. Classify the equation and find its characteristics. Construct a solution, if it is exists, for each of the Cauchy data:
(i) $u = 2, u_y = 0$ on the line $y = 0$
(ii) $u = 2, u_x = 0$ on the line $x + y = 0$

2.4.2 THE SOLUTION OFCAUCHY'S PROBLEM

Consider the second order partial differential equation

$Rr + Ss + Tt + f(x, y, z, p, q) = 0(1)$

In which R, S, T are functions of $x \, and \, y$ only. The Cauchy problems consists of the problem of determining the solution of (1) such that on a given space curve C it takes on prescribed value of $z \, and \, \frac{\partial z}{\partial n}$, where n is the distance measured along the normal to curve.

As an example of Cauchy's problem for second order partial differential equation, consider the following problem.

To determine the solution of $\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial y^2}$ which of the following data prescribed on the x-axis. $z(x, 0) = f(x), z_y(x, 0) = g(x)$. Observe that y-axis is the normal to the given curve(x-axis here).

Characteristic equation and characteristic curves:

Corresponding to equation (1), consider the λ −quadratic

$R\lambda^2 + S\lambda + T = 0$ (2)

Where $S^2 - 4RT \geq 0$, (2) has real roots. Then the ordinary differential equations

$\frac{dy}{dx} + \lambda(x,y) = 0$ (3)

are called the *characteristic equations.*

The solution of (3) are known as *characteristic curves* or simply the *characteristics* of the second order partial differential equation (1).

Now consider the following cases:

Case 1: if $S^2 - 4RT > 0$ (i.e. if (1) is hyperbolic), then equation (2) has two distinct real roots λ_1 *and* λ_2 say so that we have two characteristic equations

$$\frac{dy}{dx} + \lambda_1(x,y) = 0, \frac{dy}{dx} + \lambda_2(x,y) = 0$$

Solving these we get two distinct families of characteristic curves.

Case2. if $S^2 - 4RT = 0$ (i.e. if (1) parabolic),then equation (2) has two equal real roots each λ, so that we have only one characteristic equation $\frac{dy}{dx} + \lambda(x,y) = 0$

Solving these we get only one family of characteristic curve.

Case 3. if $S^2 - 4RT < 0$ (i.e. if (1) is elliptic), then equation (2) has no real roots i.e., two complex roots. Hence there are no real characteristics. Thus, we get two distinct families of complex characteristic curves when (1) is elliptic.

Ex.1. Find the characteristics of $y^2r - x^2t = 0$

Sol: $R = y^2, S = 0, T = -x^2$

$S^2 - 4RT = 0 - 4(y^2)(-x^2) = 4x^2y^2 > 0$, hence the given equation is hyperbolic everywhere except on the co-ordinate axes $x = 0$ and $y = 0$.

The λ −quadratic is $R\lambda^2 + S\lambda + T = 0$

i.e., $y^2\lambda^2 - x^2 = 0$

$=> \lambda^2 = \frac{x^2}{y^2} => \lambda = \pm\frac{x}{y}$ are two distinct roots. Corresponding characteristics equation are

$$\frac{dy}{dx} + \frac{x}{y} = 0 \ and \ \frac{dy}{dx} - \frac{x}{y} = 0$$

$=> \frac{dy}{dx} = -\frac{x}{y}$ and $\frac{dy}{dx} = \frac{x}{y}$

$$=> ydy = -xdx \ and \ ydy = xdx$$

Integrating, $x^2 + y^2 = c_1$and $x^2 + y^2 = c_2$, which are required family of Characteristic Curves.

Ex.2. Find the characteristics of $x^2r + 2xys + y^2t = 0$

Sol: $R = x^2, S = 2xy, T = y^2$

$S^2 - 4RT = 4x^2y^2 - 4(x^2)(y^2) = 0$, hence the given equation is hyperbolic.

The λ −quadratic is $R\lambda^2 + S\lambda + T = 0$

$$=> x^2\lambda^2 + 2xy\lambda + y^2 = 0$$

$$=> (x\lambda + y)^2 = 0 => \lambda = -\frac{y}{x}, -\frac{y}{x}$$

$\frac{dy}{dx} - \frac{y}{x} = 0 => \frac{dy}{y} = \frac{dx}{x} => \log y = \log x + \log c \quad => y = cx$ this is required family of Characteristic Curves. Here it represents a family of straight lines passing through the lines.

Ex.3. Find the characteristics of $4r + 5s + t + p + q - 2 = 0$

Sol.: Try yourself. **Ans.:** $y - x = c_1, y - \frac{x}{y} = c_2$

Ex.4. Find the characteristics of $(\sin^2 x)r + (2\cos x)s - t = 0$

Sol.: Try yourself. Ans: $y + cosec\, x - \cot x \ = c_1 , y + cosec\, x + \cot x \ = c_2$

2.5 METHOD OF REDUCTION TO NORMAL FORM

Consider the second order partial differential equation of the type

$Rr + Ss + Tt + f(x, y, z, p, q) = 0$, where R, S, T are continuous function of x and y possessing continuous partial derivative of as high an order as necessary. There is a certain method to solve different types of PDE's which we are going to discuss in detail as follows.

2.5.1 working rule for reducing a hyperbolic equation to its normal form

Step1. let the given equation $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, (1)

be hyperbolic so that $S^2 - 4RT > 0$

Step 2. Write λ −quadratic equation $\quad R\lambda^2 + S\lambda + T = 0$ (2)

Let λ_1 *and* λ_2 be its two distinct roots.

Step 3. Then corresponding characteristic equations are

$\frac{dy}{dx} + \lambda_1 = 0$ and $\frac{dy}{dx} + \lambda_2 = 0$.

Solving these, we get $f_1(x, y) = c_1$ and $f_2(x, y) = c_2$ (3)

Step4. We select u, v such that $u = f_1(x, y)$ and $v = f_2(x, y)$ (4)

Step5. Using relation (4), find p, q, r, s, t in terms of u *and* v .

Step6. Substituting the value ofp, q, r, s, t obtained in step 4 in equation (1) and simplifying we shell get the following canonical form of (1):

$$\frac{\partial^2 z}{\partial u \partial v} = \phi\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right)$$

Ex.1. Write a canonical form of $\frac{\partial^2 z}{\partial^2 x} - \frac{\partial^2 z}{\partial^2 y} = 0$

Sol: Re writing the given equation $r - t = 0$ ------(1)

$S^2 - 4RT = 0 - 4(1)(-1) = 4 > 0$ =>hyperbolic

$$\lambda^2 + 0 * \lambda - 1 = 0 => \lambda^2 = 1 => \lambda = \pm 1$$

Characteristic equation $\frac{dy}{dx} + 1 = 0 => dy + dx = 0 => y + x = C_1$

And$\frac{dy}{dx} - 1 = 0 => dy - dx = 0 => y - x = C_2$

Let $u = y + x, v = y - x$

Jacobian form of u and v =$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 1 + 1 = 2 \neq 0$

=>u and v are independent function.

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} * \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} * \frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} * 1 + \frac{\partial z}{\partial v} * -1 = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}$$

$$=> \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} => \frac{\partial}{\partial x} = \frac{\partial}{\partial u} - \frac{\partial}{\partial v}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} * \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} * \frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$$

$$=> \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} => \frac{\partial}{\partial y} = \frac{\partial}{\partial u} + \frac{\partial}{\partial v}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$$
$$= \frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial u \partial v} - \frac{\partial^2 z}{\partial v \partial u} + \frac{\partial^2 z}{\partial v^2}$$

$$r = \frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}$$

Using these values in eq. (1) the required canonical form is

$$\frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} - \left(\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right) = 0$$

$$\frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} - \frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} - \frac{\partial^2 z}{\partial v^2} = 0$$

$=> -4\frac{\partial^2 z}{\partial u \partial v} = 0 => \frac{\partial^2 z}{\partial u \partial v} = 0$which is required equation.

Ex.2.$t - s + p - q\left(1 + \frac{1}{x}\right) + \left(\frac{z}{x}\right) = 0$ ---------(1)

Sol:$S^2 - 4RT = (-1)^2 - 0 = 1 > 0 =>$ this is hyperbolic.

$0 * \lambda^2 - \lambda + 1 = 0 => \lambda = 1$ corresponding ch eq. $y + x = c_1$

$let\ u = y + x, v = x$ ----(2)

Jacobian form of u and v =$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0$

v and u are independent functions.

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} * \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} * \frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} * 1 + \frac{\partial z}{\partial v} * 1$$

$=> \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} => \frac{\partial}{\partial x} = \frac{\partial}{\partial u} + \frac{\partial}{\partial v}$ ---------(3)

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} * \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} * \frac{\partial v}{\partial y} = \frac{\partial z}{\partial u}$$

$=> \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} => \frac{\partial}{\partial y} = \frac{\partial}{\partial u}$ ---------(4)

$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u}\right) = \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v}$ ---------(5)

$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right) = \frac{\partial^2 z}{\partial u^2}$ ------(5)

Using the values p, q, s, t in eq. (1)

$$\frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\left(1 + \frac{1}{v}\right) + \left(\frac{z}{v}\right) = 0$$

$$\frac{\partial^2 z}{\partial u \partial v} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} - \frac{\partial z}{\partial u} - \frac{1}{v}\frac{\partial z}{\partial u} + \left(\frac{z}{v}\right)$$

$\frac{\partial^2 z}{\partial u \partial v} = \frac{\partial z}{\partial v} - \frac{1}{v}\frac{\partial z}{\partial u} + \left(\frac{z}{v}\right)$ which is required equation.

2.5.2 working rule for reducing a parabolic equation to its normal form

Step1. let the given equation $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, (1)

be hyperbolic so that $S^2 - 4RT = 0$

Step 2. Write $\lambda -$quadratic equation $R\lambda^2 + S\lambda + T = 0$ (2)

Let λ be its root.

Step 3. Then corresponding characteristic equation is

$\frac{dy}{dx} + \lambda = 0.$

Solving this, we get $f(x, y) = c$ (3)

Step4. We select u, v such that $u = f_1(x, y)$ and $v = f_2(x, y)$ (4)

Where $f_2(x, y)$ is an arbitrary function of x and y and is independent of $f_1(x, y)$. for this verify that Jacobian J of u and v given by (4) is non – zero.

$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial u}{\partial x} * \frac{\partial v}{\partial y} - \frac{\partial v}{\partial x} * \frac{\partial u}{\partial y} \neq 0$ (5)

Step5. Using relation (4), find p, q, r, s, t in terms of u and v .

Step6. Substituting the value ofp, q, r, s, t obtained in step 4 in equation (1) and simplifying we get the following canonical form of (1).

$$\frac{\partial^2 z}{\partial u^2} = \phi\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right) \quad or \quad \frac{\partial^2 z}{\partial v^2} = \phi\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right)$$

Ex.1. Reduce $r + 2s + t = 0$ to its canonical form.

Sol: $r + 2s + t = 0 ---------(1)$

$S^2 - 4RT = 4 - 4 = 0$, the given equation is parabolic.

$$1 * \lambda^2 + 2\lambda + 1 = 0 => (\lambda + 1)^2 = 0 => \lambda = -1, -1$$

$$\frac{dy}{dx} - 1 = 0 => y - x = c$$

Let $u = y - x, v = y$ -----------(2)

Jacobian form of u and v $= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} = -1 \neq 0$ u and v are independent.

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} * \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} * \frac{\partial v}{\partial x} = -\frac{\partial z}{\partial u}$$

$\frac{\partial z}{\partial x} = -\frac{\partial z}{\partial u} => \frac{\partial}{\partial x} = -\frac{\partial}{\partial u}$ --------(3)

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} * \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} * \frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} * 1 + \frac{\partial z}{\partial v} * 1$$

$=> \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} => \frac{\partial}{\partial y} = \frac{\partial}{\partial u} + \frac{\partial}{\partial v}$--------(4)

$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \left(-\frac{\partial}{\partial u}\right)\left(-\frac{\partial z}{\partial u}\right) = \frac{\partial^2 z}{\partial u^2}$---------(5)

$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = -\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = -\frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial u \partial v}$-----(6)

$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}$------(7)

Using the values r, s, t in equation (1), we get the required equation,

$$\frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} = 0$$

$$=> \frac{\partial^2 z}{\partial v^2} = 0$$

Ex.2.$y^2 r - 2xys + x^2 t - \left(\frac{y^2}{x}\right)p - \frac{x^2}{y}q = 0$

Sol:$y^2 r - 2xys + x^2 t - \left(\frac{y^2}{x}\right)p - \frac{x^2}{y}q = 0$ (1)

$S^2 - 4RT = 4x^2y^2 - 4y^2x^2 = 0$ => this equation is parabolic.

The λ −quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to

$$y^2\lambda^2 - 2xy\lambda + x^2 = 0 => (y\lambda - x)^2 = 0 => \lambda = \frac{x}{y}, \frac{x}{y}$$

The corresponding characteristic equation is $\frac{dy}{dx} + \frac{x}{y} = 0 => \frac{y^2}{2} + \frac{x^2}{2} = c_1$

Let $u = \frac{y^2}{2} + \frac{x^2}{2}, v = \frac{x^2}{2} - \frac{y^2}{2}$ (2)

Jacobian form of u and v $= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} x & y \\ x & -y \end{vmatrix} = -xy - xy = -2xy \neq 0$

$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} * \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} * \frac{\partial v}{\partial x} = x\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$
(3)

$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} * \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} * \frac{\partial v}{\partial y} = y\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$
(4)

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(x\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\right)$$

$$= \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} + x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right]$$

$= \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} + x^2\left(\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right)$ using (2) (5)

$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left[y\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\right] = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} + y\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$, by (4)

$$= \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} + y\left\{\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right\}$$

$= \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} + y^2\left(\frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right)$
(6)

And $s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(y\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\right) = y\left\{\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right\}$

$s = xy\left(\frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial v^2}\right)$
(7)

Using (3), (4),(5), (6) and (7) in (1) and simplifying, we get

$4x^2y^2\left(\frac{\partial^2 z}{\partial v^2}\right) = 0$ so that $\frac{\partial^2 z}{\partial v^2} = 0$, which is the required canonical form.

2.5.3 working rule for reducing elliptic equation to its normal form

Step 1. $Rr + Ss + Tt + f(x, y, z, p, q) = 0$-------(1)

be elliptic so that $S^2 - 4RT < 0$

Step.2. write λ $-$quadratic $R\lambda^2 + S\lambda + t = 0$ --------(2)

Then λ_1 and λ_2 be two distinct roots.

Step 3. Then the corresponding characteristic equations are $\frac{dy}{dx} + \lambda_1 = 0$ and $\frac{dy}{dx} + \lambda_2 = 0$

Solving these we get $f_1(x, y) + if_2(x, y) = c_1$ and $f_1(x, y) - if_2(x, y) = c_2$--------(3)

Step 4. We select u and v such that

$u = f_1(x, y) + if_2(x, y)$ and $v = f_1(x, y) - if_2(x, y)$--------(4)

Step 5. Let α and β are new real independent variables such as $u = \alpha + i\beta$ and $v = \alpha - i\beta$

Where $\alpha = f_1(x, y), \beta = f_2(x, y)$

Step 6. Find values $p,\ q,\ r,\ s,\ t$ in terms of α and β.

Step 7. Substitution the values of $p,\ q,\ r,\ s,\ t$ in equation (1) and simplifying we shall get the following canonical form,

$$\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = \phi\left(\alpha, \beta, z, \frac{\partial z}{\partial \alpha}, \frac{\partial z}{\partial \beta}\right)$$

Ex.1. Reduce the PDE in canonical form $r + x^2 t = 0$

Sol: $r + x^2 t = 0$ ---------(1)

$S^2 - 4RT = 0 - 4x^2 = -4x^2 < 0$ => which is elliptic equation.

$$R\lambda^2 + S\lambda + t = 0 => \lambda^2 + x^2 = 0 => \lambda = \pm ix$$

Characteristic equation $\frac{dy}{dx} + ix = 0 => y + \frac{ix^2}{2} = c_1$

$$\text{and} \frac{dy}{dx} - ix = 0 => y - \frac{ix^2}{2} = c_2$$

Let $u = y + i\frac{x^2}{2}$ and $v = y - i\frac{x^2}{2}$

Choose $\alpha = y,\ \beta = \frac{x^2}{2}$

Jacobian of $\alpha,\ \beta$=$\begin{vmatrix} \frac{\partial\alpha}{\partial x} & \frac{\partial\alpha}{\partial y} \\ \frac{\partial\beta}{\partial x} & \frac{\partial\beta}{\partial y} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ x & 0 \end{vmatrix} = -x \neq 0$ =>α and β are independent.

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \alpha} * \frac{\partial \alpha}{\partial x} + \frac{\partial z}{\partial \beta} * \frac{\partial \beta}{\partial x} = \frac{\partial z}{\partial \alpha} * 0 + \frac{\partial z}{\partial \beta} * x => \frac{\partial z}{\partial x} = x\frac{\partial z}{\partial \beta}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial \alpha} * \frac{\partial \alpha}{\partial y} + \frac{\partial z}{\partial \beta} * \frac{\partial \beta}{\partial y} = \frac{\partial z}{\partial \alpha} * 1 + \frac{\partial z}{\partial \beta} * 0 => \frac{\partial z}{\partial y} = \frac{\partial z}{\partial \alpha} => \frac{\partial}{\partial y} = \frac{\partial}{\partial \alpha}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(x\frac{\partial z}{\partial \beta}\right) = \frac{\partial x}{\partial x} * \frac{\partial z}{\partial \beta} + \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial \beta}\right)$$
$$= \frac{\partial z}{\partial \beta} + x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial \beta}\right)$$

$$= \frac{\partial z}{\partial \beta} + x\left[\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \beta}\right) * \frac{\partial \alpha}{\partial x} + \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \beta}\right) * \frac{\partial \beta}{\partial x}\right]$$
$$= \frac{\partial z}{\partial \beta} + x\left[\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \beta}\right) * 0 + \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \beta}\right) * x\right]$$

$$r = \frac{\partial z}{\partial \beta} + x^2\left[\frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \beta}\right)\right] = \frac{\partial z}{\partial \beta} + x^2\frac{\partial^2 z}{\partial \beta^2}$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha}\right) = \frac{\partial^2 z}{\partial \alpha^2}$$

Using these values of r and t in equation (1)

$$r + x^2 t = 0 => \frac{\partial z}{\partial \beta} + x^2\frac{\partial^2 z}{\partial \beta^2} + x^2\frac{\partial^2 z}{\partial \alpha^2} = 0 => \frac{\partial z}{\partial \beta} + x^2\left(\frac{\partial^2 z}{\partial \beta^2} + \frac{\partial^2 z}{\partial \alpha^2}\right) = 0$$

$$\frac{\partial^2 z}{\partial \beta^2} + \frac{\partial^2 z}{\partial \alpha^2} = -\frac{1}{x^2}\frac{\partial z}{\partial \beta}$$

$=> \frac{\partial^2 z}{\partial \beta^2} + \frac{\partial^2 z}{\partial \alpha^2} = -\frac{1}{2\beta}\frac{\partial z}{\partial \beta}$, which is required canonical form of given equation.

Ex.2. find the canonical for of$r + y^2 t - y = 0$

Sol: $r + y^2 t - y = 0$ ----------(1)

$S^2 - 4RT = 0 - 4y^2 = -4y^2 < 0$ => given equation is elliptic.

$$R\lambda^2 + S\lambda + t = 0 => \lambda^2 + y^2 = 0 => \lambda = \pm iy$$

$$\frac{dy}{dx}+iy=0=>\frac{dy}{y}+idx=0=>logy+ix=c_1$$

and $\frac{dy}{dx}-iy=0=>logy-ix=c_2$

Let $u=logy+ix, v=logy-ix$

Let $\alpha=\log y\,,\beta=x$

$$p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial \alpha}*\frac{\partial \alpha}{\partial x}+\frac{\partial z}{\partial \beta}*\frac{\partial \beta}{\partial x}=\frac{\partial z}{\partial \alpha}*0+\frac{\partial z}{\partial \beta}*1=\frac{\partial z}{\partial \beta}=>\frac{\partial z}{\partial x}=\frac{\partial z}{\partial \beta}=$$
$$>\frac{\partial}{\partial x}=\frac{\partial}{\partial \beta}$$

$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial \alpha}*\frac{\partial \alpha}{\partial y}+\frac{\partial z}{\partial \beta}*\frac{\partial \beta}{\partial y}=\frac{\partial z}{\partial \alpha}*\frac{1}{y}+\frac{\partial z}{\partial \beta}*0=\frac{1}{y}\frac{\partial z}{\partial \alpha}=>\frac{\partial z}{\partial y}=\frac{1}{y}\frac{\partial z}{\partial \alpha}$$

$$r=\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \beta}\right)=\frac{\partial^2 z}{\partial \beta^2}$$

$$t=\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial y}\left(\frac{1}{y}\frac{\partial z}{\partial \alpha}\right)=\frac{\partial}{\partial y}\left(\frac{1}{y}\right)\frac{\partial z}{\partial \alpha}+\frac{1}{y}\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial \alpha}\right)$$

$$=-\frac{1}{y^2}\frac{\partial z}{\partial \alpha}+\frac{1}{y}-\frac{1}{y^2}\frac{\partial z}{\partial \alpha}$$
$$+\frac{1}{y}\left[\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha}\right)\left(\frac{1}{y}\right)+\frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \alpha}\right)*0\right]\left[\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha}\right)\frac{\partial \alpha}{\partial y}\right.$$
$$\left.+\frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \alpha}\right)\frac{\partial \beta}{\partial y}\right]$$

$$t=-\frac{1}{y^2}\frac{\partial z}{\partial \alpha}+\frac{1}{y}\left[\frac{1}{y}\frac{\partial^2 z}{\partial \alpha^2}\right]=-\frac{1}{y^2}\frac{\partial z}{\partial \alpha}+\frac{1}{y^2}\frac{\partial^2 z}{\partial \alpha^2}$$

Using these values in equation(1),

$$r+y^2t-y=0=>\frac{\partial^2 z}{\partial \beta^2}+y^2\left(-\frac{1}{y^2}\frac{\partial z}{\partial \alpha}+\frac{1}{y^2}\frac{\partial^2 z}{\partial \alpha^2}\right)-y=0$$

$$\frac{\partial^2 z}{\partial \beta^2}-\frac{\partial z}{\partial \alpha}+\frac{\partial^2 z}{\partial \alpha^2}-e^{\alpha}=0$$

$$=>\frac{\partial^2 z}{\partial \beta^2}+\frac{\partial^2 z}{\partial \alpha^2}=\frac{\partial z}{\partial \alpha}+e^{\alpha}$$

Ex.3. If the reduced canonical form is $\frac{\partial^2 z}{\partial u\partial v}=0$ find its solution.

Sol.: Integrating w. r. t. u ,$\frac{\partial z}{\partial v}=\phi(v),\ \phi$ =arbitrary function.

Integrating w. r. t. v, $z = \int \phi(v)dv + f(u)$,where f is arbitrary function.

$$z = g(v) + f(u)$$

$\Rightarrow z = g(y - x) + f(y + x)$ it is the required general solution.

EXERCISE 2.2

Reduce the following equation into their normal forms.

Q.1. $x^2r - y^2t + px - qy = x^2$

Q.2. $r + 2xs + x^2t = 0$

Q.3. $r - 4s + 4t = 0$

Q.4. $xr + t = x^2$

2.6 POTENTIAL THEORY AND ELLIPTICAL DIFFERENTIAL EQUATION:

Boundary data rather than initial data serve to fix properly the solution of an elliptic differential equation. It is usually necessary to find an answer "in the large," namely in the domain bounded by a closed boundary, and this need for "global" constructions, rather than "local" treatment makes it especially difficult to study nonlinear elliptic equations. We shall restrict ourselves mainly to the linear potential equation or Laplace's equation in \$m\$-space variables. The boundary value problems of potential theory are suggested by physical phenomena from such varied field as electrostatics, steady heat conduction and incompressible fluid flow.

Boundary Value Problems and Cauchy Problem

The general linear homogeneous second order partial differential equation in m −space variables $x_1, x_2, x_3, \dots \dots \dots . . x_m$ is

$$Lu \equiv a_{\alpha\beta}u_{x_\alpha x_\beta} + b_\alpha u_{x_\alpha} + cu = 0,\ \alpha, \beta = 1,2, \dots \dots m \qquad (1)$$

where the coefficients $a_{\alpha\beta}, b_\alpha$ and c are continuous functions of the independent variables $x_1, x_2, x_3, \dots \dots \dots . . x_m$ and $a_{\alpha\beta} = a_{\beta\alpha}$. Equation (1) is said to be elliptic in a domain D of m −dimensional space, when the quadratic form

$$Q(\lambda) = a_{\alpha\beta}\lambda_\alpha\lambda_\beta \qquad (2)$$

can be expressed as the sum of squares with coefficients of the same sign, or equivalently, $Q(\lambda)$ is either positive or negative definite in D. The simplest case is that of the Laplace equation or potential equation:

$$\Delta_m u = u_{x_\alpha x_\alpha} = 0 \qquad (3)$$

i.e. $u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3} + \cdots \dots \dots u_{x_mx_m} = 0$

We shall first state three boundary value problems associated with Laplace equation and then consider the Cauchy problem. Let D be a domain in $(x_1, x_2, x_3, \dots \dots \dots \dots x_m)$ $-$space bounded by a piecewise smooth boundary ∂D.Let continuous boundary values be prescribed on ∂D, by means of a function f.

1st order BVP (Dirichlet problem): The first boundary value problem, also called the Dirichlet problem, requires a solution u of the Laplace equation (3) in the domain D, which is continuous in $D + \partial D$ and coincides with f on ∂Di.e.

$$u = f \; on \; \partial D \qquad (4)$$

2nd order BVP (Neumann problem): It requires the determination of solution u in the domain D, which is continuous with first order partial derivatives in $D + \partial D$, such that the normal derivative $\frac{\partial u}{\partial v}$of u on ∂Dtakes prescribed values f, i.e. harmonic function $u(x, y)$satisfies

$$\frac{\partial u}{\partial v} = f \text{on } \partial D \qquad (5)$$

where $\frac{\partial}{\partial v}$ is the directional derivative along outward normal and ∂D must have a continuously normal.

3rd order BVP (Robin Problem): It is a modification of the first two BVP where the solution $u(x, y)$ is a linear combination of $u \; and \; \frac{\partial u}{\partial v}$, takes prescribed value of ∂Di.e.

$$\frac{\partial u}{\partial v} + \alpha u = f on \; \partial D \qquad (6)$$

where α is a constant.

Before we discuss the Cauchy problem, we shall examine, in general, the requirements to be satisfied by' a reasonable mathematical problem. There are two requirements:

1. Existence requirement:-There is at least one u satisfying the equation and the given boundary/Cauchy data.

2. Uniqueness requirement:-There is utmost one such u.

If the mathematical problem is to be also physically realistic an extra requirement has to be satisfied:

3. Stability requirement:-Small changes in the boundary or Cauchy data result in small changes in the solution u.

The first two requirements ensure the existence and uniqueness of the solution of a mathematical problem, while all three requirements ensure, further, stability or continuous dependence on given data for a physical problem. If the three requirements are satisfied by a problem, it is said to*well posed.*

The Cauchy-Kowalewski theorem shows that the solution of an analytic Cauchy problem for an elliptic equation exists and is unique. However, a Cauchy problem for Laplace's equation is not always well posed.

Hadamard gave an example of a Cauchy problem, which violates the stability requirement. Consider the Laplace equation in two independent variables x, y with the following initial conditions:

(a) $u(x,0) = 0, u_y(x,0) = 0$
(b) $u(x,0) = 0, u_y(x,0) = \frac{\sin kx}{k}$
(7)

A solution satisfying condition (a) is

$$u(x,y) = 0 \qquad (8)$$

A solution satisfying condition (b) is

$$u(x,y) = \frac{1}{k^2}\sin kx \sinh ky \qquad (9)$$

For sufficiently large k, the Cauchy or initial values (a) and (b) are arbitrarily cluse, but the sulutions are not, since sinh ky behaves like $e^{ky}$for large k.

Having noted that a Cauchy problem could be illposed for an elliptic equation, we shall concentrate our attention hereafter only on the three boundary value problems mentioned earlier and show that they are really wellposed.

2.7 HARMONIC FUNCTION

A function $u(x)$ is called *harmonic* function in D, if $u(x) \in C^o$in $D + \partial D \in C^2$ and $\Delta_m u = 0$ in D.

In case of two or three variable, the general solution of potential equation can easily be obtained.For $m = 2, (x_1 = x, x_2 = y), i.e.\ u_{xx} + u_{yy} = 0$, this is the real and imaginary part of any analytic function of the complex variable $x + iy$. For $m = 3, (x_1 = x, x_2 = y, x_3 = z)$, consider an arbitrary function $p(w,t)$ analytic in the complex variable w for fixed real t. Then, for arbitrary values of t, both the real and imaginary parts of the function:

$$u = p(z + ix\cos t + iy\sin t, t) \qquad (10)$$

of the real variable x, y, z are solution of the equation $\Delta u = 0$. Further solutions may now be obtained by superposition.

$$u = \int_a^b p(z + ix\cos t + iy\sin t\,, t\,)\,dt \qquad (11)$$

If $u(x, y\,)$ is a solution of Laplace's equation in the domain D of (x, y)plane, the function.

$$v(x, y) = u\left(\frac{x}{r^2}, \frac{y}{r^2}\right),\ r^2 = x^2 + y^2, \qquad (12)$$

Also satisfies the potential equation and is in the domain D' obtain from D by inversion with respect to unit circle.

In general, m-dimension, if $u(x_1, x_2 \dots\dots x_m)$ satisfies potential equation in a bounded domain D then

$$v = u\left(\frac{x_1}{r^2}, \frac{x_2}{r^2} \dots\dots\dots\dots \frac{x_m}{r^2}\right),\ x_1^2 + x_2^2 + \cdots \dots\dots\dots + x_m^2 = r^2 \qquad (13)$$

also satisfies the potential equation and is regular in the region D' obtained from D by inversion with respect to m-dimensional unit sphere. Therefore, except for the r^{2-m}, the harmonic character of the function is invariant with respect to sphere. Besides, the harmonic property is retained completely under rotations, translations and simple reflections across planes.

2.8 POISSON'S FORMULA

Ex. Dirichlet problem for a circle in the x, y −plane.

Sol: let a circle C is given by $|z| = R, z = x + iy$

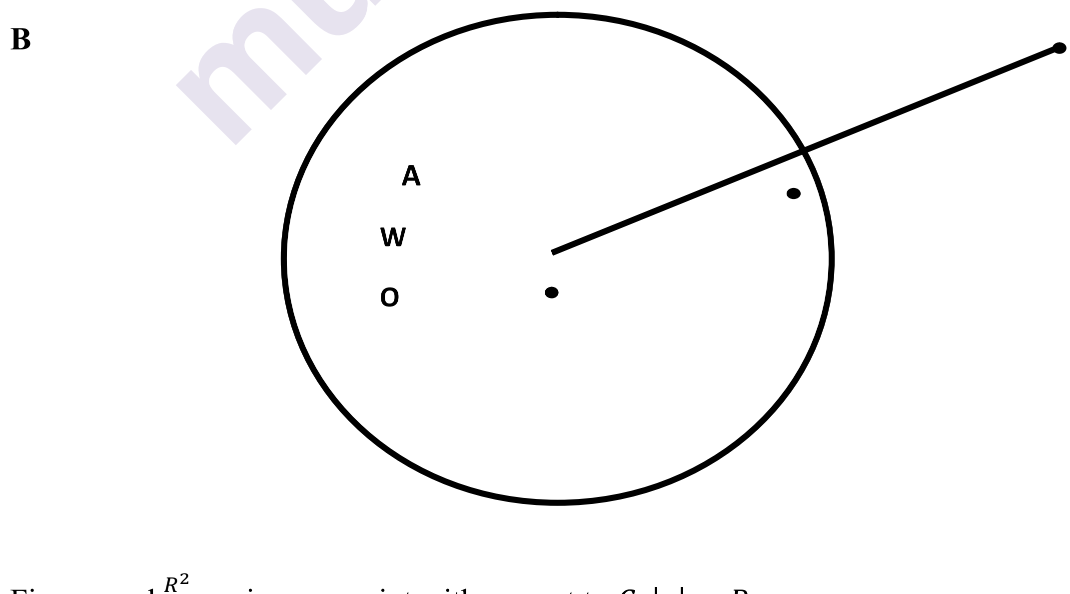

Fig. w and $\frac{R^2}{\bar{w}}$ are inverse point with respect to $C{:}\,|z| = R$

The problem is to find $u(x,y)$ $s.t. \Delta_2 u = 0 \Rightarrow u_{xx} + u_{yy} = 0$,

$$where\ u = f(\theta) on\ C.$$

$$z = Re^{i\theta}\ on\ C.$$

Let $f(z)$ be analytical function in the region enclosed by C s.t. the real part of $f(z)\ on\ |z| = R\ is\ f(\theta)$

Let$z_0$be a complex number in that region. The inverse point of z_0w.r.t. to C is $\frac{R^2}{\overline{z_0}}$ which lies outside C. According to Cauchy integral formula

$f(z_o) = \frac{1}{2\pi i}\int_c \frac{f(z)}{z-z_0} dz$-----(1)

$0 = \frac{1}{2\pi i}\int_c \frac{f(z)}{\left(z-\frac{R^2}{\overline{z_0}}\right)} d\,z$ -------(2)

Equation (1) – equation (2)

$$f(z_o) = \frac{1}{2\pi i}\int_c \left[\frac{f(z)}{z - z_0} - \frac{f(z)}{\left(z - \frac{R^2}{\overline{z_0}}\right)}\right] dz$$

$$= \frac{1}{2\pi i}\int_c \left[\frac{1}{(z - z_0)} - \frac{\overline{z}_0}{(z\overline{z}_0 - R^2)}\right] f(z)dz$$

$$= \frac{1}{2\pi i}\int_c \left[\frac{(z\overline{z}_0 - R^2) - \overline{z}_0(z - z_0)}{(z - z_0)(z\overline{z}_0 - R^2)}\right] f(z)dz$$

$$= \frac{1}{2\pi i}\int_c \left[\frac{z\overline{z}_0 - R^2 - \overline{z}_0 z + \overline{z}_0 z_0}{-z(R^2 + z_0\overline{z}_0) + \overline{z}_0 z^2 + z_0 R^2}\right] f(z)dz$$

$$= \frac{1}{2\pi i}\int_c \left[\frac{-(R^2 - \overline{z}_0 z_0)}{-\{z(R^2 + z_0\overline{z}_0) - \overline{z}_0 z^2 - z_0 R^2\}}\right] f(z)dz$$

$f(z_o) = \frac{1}{2\pi i}\int_c \left[\frac{(R^2 - \overline{z}_0 z_0)}{\{z(R^2 + z_0\overline{z_0}) - \overline{z_0} z^2 - z_0 R^2\}}\right] f(z)dz$ --------(3)

We know that $z\ lies\ on\ C\ and\ z_0\ lies\ inside\ C$

Let $z = Re^{i\theta},\ z_o = re^{i\phi} \Rightarrow |z_0| = r \Rightarrow z_0^2 = r^2 \Rightarrow \overline{z}_0 z_0 = r^2\ , r < R$

From eq. (3) $f\left(re^{i\phi}\right) = \frac{1}{2\pi i}\int_0^{2\pi} \left[\frac{(R^2 - r^2)}{\{Re^{i\theta}(R^2 + r^2) - \overline{z_0} R^2 e^{2i\theta} - re^{i\phi} R^2\}}\right] f\left(Re^{i\theta}\right) Re^{i\theta} id\theta$

Taking real part

$u(x,y) = \frac{(R^2 - r^2)}{2\pi}\int_0^{2\pi} \left[\frac{f(\theta)d\theta}{\{R^2 + r^2 - 2rR\ cos(\theta - \phi)\}}\right]$ ----------(4)

Where $r^2 = x^2 + y^2, \tan\phi = \frac{y}{x}$

Eq. (4) is called *Poisson's integral* formula in 2D.

Maximum principle:suppose that $u(x, y)$ be harmonic in a bounded domain D and continuous in $\overline{D} = D \cup \partial D$ then $u(x, y)$ attains its maximum on the boundary $\partial D\ of\ D$.

Minimum principle: suppose that $u(x, y)$ be harmonic in a bounded domain D and continuous in $\overline{D} = D \cup \partial D$ then $u(x, y)$ attains its minimum on the boundary $\partial D\ of\ D$.

2.9 LET US SUM UP:

In this unit we have learnt to identify different 2nd order PDE, to find the characteristic curve and to solve Cauchy's problem. We also discussed to how to reduce the PDE's in its normal form.

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3

GREEN FUNCTIONS I

Unit Structure:

3.1 Objectives

3.2 Introduction

3.3 Singularity functions and the fundamental solution,

3.4 Green functions

3.5 Greens identities

3.6 Lets sum up

3.7 Unit End exercise

3.8 Reference

3.1 OBJECTIVES

After going through this chapter students will be able to:

- Singularity functions.
- The fundamental solution of Laplace equation.
- Definition of Green functions using fundamental solution.
- Green's first identity.
- Green's second identity

3.2 INTRODUCTION

Singularity functions are used in the solution of differential equations in which the known terms are non-smooth in the independent variable. In particular, these functions are particularly useful in the study of bars, shafts, and beams subjected to non-smooth loading, such as point loading and distributed loading, that exhibits finite jumps.

The method of Green's functions is an important technique for solving boundary value and, initial and boundary value problems for partial differential equations.We shall learn Green's function method for finding the solutions of partial differential equations. This is accomplished by constructing Green's theorem that is appropriate for the second order differential equations. These integral theorems will then be used to show how BVP and IBVP can be solved in terms of appropriately defined Green's functions for these problems. More precisely, we shall study the construction and use of Green's functions for the Laplace, the Heat and the Wave equations.

3.3 SINGULARITY FUNCTIONS AND THE FUNDAMENTAL SOLUTION:

Singularity functions are discontinuous functions or their derivatives are discontinuous. A singularity is a point at which a function does not possess a derivative. In other words, a singularity function is discontinuous at its singular points. Hence a function that is described by polynomial in *t* is thus a singularity function. The commonly used singularity functions are:

Step Function, Ramp Function, and Impulse Function.

Step Function: One of the most common singularity functions is the Heaviside* step function *H(x),* defined as

$$H(x) = \begin{cases} 0, x < 0 \\ 1, x > 0 \end{cases} \quad (1)$$

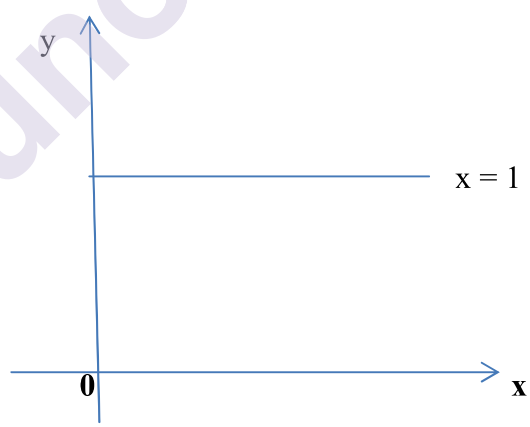

Figure 1 The Heaviside step function *H(x)*

Note that the Heaviside function *H(x)* is undefined at $x = 0$, although it is sometimes taken to be equal to $\frac{1}{2}$. Clearly, the Heaviside function $H(x- a)$ is analogous to the function plotted in Fig. 1,only shifted so as undergo the step at $x = a$.

Ramp Function: The integral of the Heaviside step function is the ramp function written as $< x >$. With Eq. (1) taken into account, the ramp function is given by

$$< x > = \begin{cases} 0, x \leq 0 \\ x, x > 0 \end{cases} \qquad (2)$$

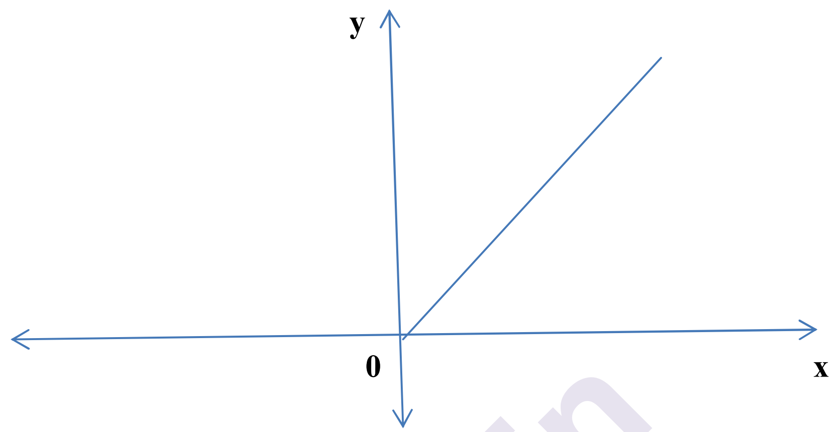

Figure 2 The ramp function

It is easy to see that the ramp function can be raised to any positive power, with

$$< x >^n = \begin{cases} 0, x \leq 0 \\ x^n, x > 0 \end{cases} \; For\; n > 1 \qquad (3)$$

While *H(x)* does not have a derivative in the usual sense of a smooth function, such a derivative can be defined as what in mathematics is termed a distribution from the limit of a sequence of continuous approximations to the discontinuous step function,as shown in Fig.3 below

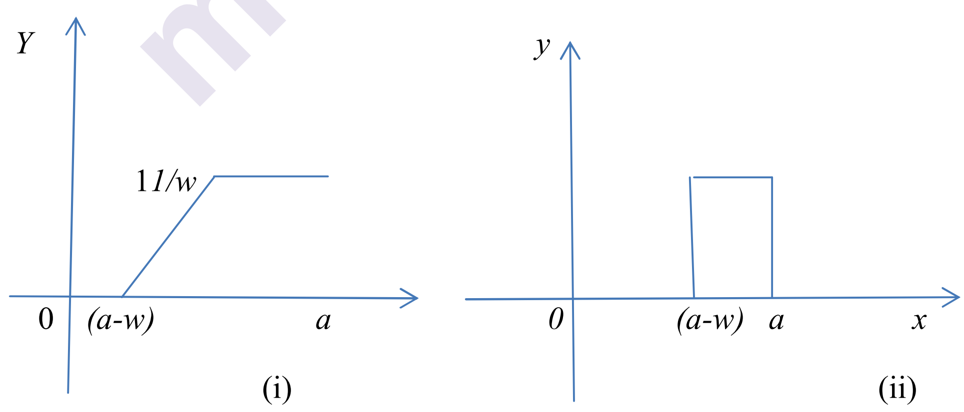

Figure 3 Ramp approximation to the step function and its derivative

The function depicted in Fig. 3(i) may be expressed as

$$H_w(x-a) = \frac{1}{w}[< x - a + w > - < x - a >]$$

and it becomes $H(x-a)$ in the limit as $\omega \to 0$, that is,

$$H(x-a) = \frac{d < x-a >}{dx}$$

Impulse Function (Dirac delta function):The derivative of $H_w(x-a)$ (depicted in Fig. 3(ii) is, in accordance with

$$\frac{d}{dx}H_w(x-a) = \frac{1}{w}[H(x-a+w) - H(x-a)]$$

and in the limit as $\omega \to 0$ it formally becomes (by the standard definition of the derivative) the derivative of $H(x-a)$, that is,

$$\frac{d}{dx}H(x-a) = \delta(x-a)$$

This limit is known as the Dirac delta function and is usually denoted $\delta(x-a)$.

We now turn to studying Laplace's equation

$\Delta u = 0$

and its inhomogeneous version, Poisson's equation,

$-\Delta u = f.$

We say a function u satisfying Laplace's equation is a harmonic function.

3.3.1 The Fundamental Solution of Laplace's equation:

Consider Laplace's equation in$\mathbb{R}^n$,

$\Delta u = 0 \; x \in \mathbb{R}^n$.

Clearly, there is a lot of functions u which satisfy this equation. In particular, any constant function is harmonic. In addition, any function of the form $u(x) = a_1x_1 + a_2x_2 + \cdots .. + a_nx_n$ for constants a_i is also a solution. Of course, we can list a number of others. Here, however, we are interested in finding a particular solution of Laplace's equation which will allow us to solve Poisson's equation.

Given the symmetric nature of Laplace's equation, we look for a radial solution. That is, we look for a harmonic function u on $\mathbb{R}^n$ such that $u(x) = v(|x|)$. In addition, to being a natural choice due to the symmetry of Laplace's equation, radial solutions are natural to look for because they reduce a PDE to an ODE, which is generally easier to solve. Therefore, we look for a radial solution.

If $u(x) = v(|x|)$, then

$$u_{x_i} = \frac{x_i}{|x|} v'(|x|) \qquad where\ |x| \neq 0$$

$$\Rightarrow u_{x_i x_i} = \frac{1}{|x|} v'(|x|) - \frac{{x_i}^2}{|x|^3} v'(|x|) + \frac{{x_i}^2}{|x|^2} v''(|x|) \qquad where\ |x| \neq 0$$

Therefore,

In $\mathbb{R}^n$ the solutions $v(|x|)$ of the potential equation $\Delta u = 0$, which depend only on the distance $r = |x| \neq 0$ of a fixed point x from a fixed point a, given by the equation

$$\Delta u = \frac{n-1}{|x|} v'(|x|) + v''(|x|)$$

Letting $r = |x| \neq 0$, we see that $u(x) = v(|x|)$, is a radial solution of Laplace's equation implies v satisfies

$$\frac{n-1}{r} v'(r) + v''(r) = 0$$

Therefore,

$$v''(r) = \frac{n-1}{r} v'(r)$$

$$\Rightarrow \frac{v''(r)}{v'(r)} = \frac{n-1}{r}$$

$$\Rightarrow \log v'(r) = (n-1) log\, r + C$$

$$\Rightarrow v'(r) = \frac{C}{r^{n-1}}$$

Therefore,

$$v(r) = \begin{cases} c_1 log r + c_2 & n = 2 \\ \dfrac{c_1}{(n-2) r^{n-2}} + c_2 & n \geq 3 \end{cases}$$

From these calculations, we see that for any constants c_1, c_2, the function

$$u(|x|) = \begin{cases} c_1 log|x| + c_2 & n = 2 \\ \frac{c_1}{(n-2)|x|^{n-2}} + c_2 & n \geq 3 \end{cases} \quad \text{(I)}$$

for x $\in\mathbb{R}^n, |x| \neq 0$ is a solution of Laplace's equation in $\mathbb{R}^n - \{0\}$. We notice that the function *u* defined in (I) satisfies $\Delta u(x) = 0$for $|x| \neq 0$, but at $x = 0$, Δu(0) is undefined.

Therefore, these solution exhibits so called characteristic singularity at $r = 0$. We defined as

$$\phi(x) = \begin{cases} \dfrac{-1}{2\pi} log|a-x| & n = 2 \\ \dfrac{1}{n(n-2)\alpha_n} |a-x|^{2-n} & n \geq 3 \end{cases}$$

Where w_n is the surface area of the unit sphere in *n*-dimensions given by

$\alpha_n = {2(\pi)^{\frac{n}{2}}}\Big/{\tau\left(\frac{n}{2}\right)}$ for singularity function $\Delta u = 0$.

$\phi(x)$has the property that $S \in C^\infty$ and $\Delta S = 0$ for $x \neq a$, with the singularity $x = a$.

For $n = 3$, $\phi(x)$ correspond physically to the gravitational potential at the point *x* of a unit mass concentrated at the point *a*. Every solution of a potential equation $\Delta u = 0$ in *D* of the form

$$\gamma(a,x) = \phi(x) + \varphi(x), \quad a \in D$$

Define the function$\phi(x)$as follows. For$x \neq 0$, let

$$\phi(x) = \begin{cases} \dfrac{-1}{2\pi} log|x| & n = 2 \\ \dfrac{1}{n(n-2)\alpha_n} |x|^{2-n} & n \geq 3 \end{cases}$$

As we will show in the following claim, $\phi(x)$satisfies $-\Delta_x\phi(x) = \delta_0$. For this reason, we call $\phi(x)$the fundamental solution of Laplace's equation.

Theorem: For $\phi(x)$ satisfies $-\Delta_x \phi(x) = \delta_0$ in the sense of distributions. That is, for all $g \in D$, $-\int_{\mathbb{R}^n} \phi(x)\Delta_x g(x)dx = g(0)$.

Proof: Let $F_\emptyset$be the distribution associated withthe fundamental solution $\emptyset$. That is let $F_\emptyset : D \to \mathbb{R}$ be define such that

$$(F_\emptyset, g) = \int_{\mathbb{R}^n} \phi(x)g(x)dx$$

for all $g \in D$. Recall that the derivative of a distribution F is defined as the distribution G such that

$$(G, g) = -(F, g')$$

for all $g \in D$. Therefore, the distributional Laplacian of ϕ is defined as the distribution $F_{\Delta\emptyset}$, such that

$$(F_{\Delta\emptyset}, g) = (F_{\emptyset}, \Delta g)$$

for all $g \in D$. We will show that

$$(F_{\emptyset}, \Delta g) = -(\delta_0, g) = -g(0)$$

and, therefore,$(F_{\Delta\emptyset}, g) = -g(0)$.

which means $-\Delta_x\, \phi(x) = \delta_0$in the sense of distributions. By definition,

$$(F_{\emptyset}, \Delta g) = \int\limits_{\mathbb{R}^n} \phi(x) g(x) dx$$

Now, we would like to apply the divergence theorem, but ϕ has a singularity at $x = 0$. We get around this, by breaking up the integral into two pieces: one piece consisting of the ball of radius δ about the origin, $B(0, \delta)$ and the other piece consisting of the complement of this ball in $\mathbb{R}^n$. Therefore, we have

$$(F_{\emptyset}, \Delta g) = \int\limits_{\mathbb{R}^n} \phi(x) g(x) dx$$

$$= \int\limits_{B(0,\delta)} \phi(x) \Delta g(x) dx + \int\limits_{\mathbb{R}^n - B(0,\delta)} \phi(x) \Delta g(x) dx = I + J$$

We look first at term I. For $n = 2$, term I is bounded as follows

$$\left| -\int\limits_{B(0,\delta)} \frac{1}{2\pi} log|x| \Delta g(x) dx \right| \leq C |\Delta G|_{L^\infty} \left| \int\limits_{B(0,\delta)} log|x| dx \right|$$

$$\leq C \left| \int_0^{2\pi} \int\limits_{B(0,\delta)} log|x| dx \right| \leq C \log|\delta| \delta^2.$$

For $n \geq 3$, term I is bounded as follows,

$$\left| - \int_{B(0,\delta)} \frac{1}{n(n-2)\alpha_n} |x|^{2-n} \Delta g(x) dx \right| \leq C|\Delta G|_{L^\infty} \left| \int_{B(0,\delta)} |x|^{2-n} dx \right|$$

$$\leq n\alpha(n) \int_0^\delta r\, dr = \frac{n\alpha(n)}{2} \delta^2$$

Therefore, as $\delta \to 0^+$, $|I| \to 0$.Next, we look at term J. Applying the divergence theorem, we have

$$\int_{\mathbb{R}^n - B(0,\delta)} \phi(x) \Delta_x g(x) dx$$

$$= \int_{\mathbb{R}^n - B(0,\delta)} \Delta_x \phi(x) g(x) dx$$

$$- \int_{\partial(\mathbb{R}^n - B(0,\delta))} \frac{\partial \phi(x)}{\partial v} g(x) dS(x)$$

$$+ \int_{\partial(\mathbb{R}^n - B(0,\delta))} \phi(x) \frac{\partial g(x)}{\partial v} dS(x)$$

$$= - \int_{\partial(\mathbb{R}^n - B(0,\delta))} \frac{\partial \phi(x)}{\partial v} g(x) dS(x) + \int_{\partial(\mathbb{R}^n - B(0,\delta))} \phi(x) \frac{\partial g(x)}{\partial v} dS(x)$$

$$= J_1 + J_2$$

Using the fact that$\Delta_x \emptyset(x) = 0$ for $x \in \mathbb{R}^n - B(0, \delta)$

We first look at term J_1. Now, by assumption, $g \in D$, and, therefore, g vanishes at ∞. Consequently, we only need to calculate the integral over ∂B(0, $\in$) where the normal derivative v is the outer normal to $\mathbb{R}^n - B(0, \delta)$. By a straightforward calculation, we see that

$$\nabla_x \emptyset(x) = \frac{-x}{n\alpha(n)|x|^n}.$$

The outer unit normal to $\mathbb{R}^n - B(0, \delta)$.on$B(0, \delta)$is given by

$$v = \frac{-x}{|x|}$$

Therefore, the normal derivative of $\emptyset$ on $B(0, \delta)$ is given by

$$\frac{\partial \phi(x)}{\partial v} = \left(\frac{-x}{n\alpha(n)|x|^n} \right) \cdot \left(\frac{-x}{|x|} \right) = \frac{1}{n\alpha(n)|x|^{n-1}}$$

Therefore, J_1 can be written as

$$-\int_{B(0,\delta)} \frac{\partial\phi(x)}{\partial v} g(x)dS(x) = -\int_{B(0,\delta)} \frac{1}{n\alpha(n)|x|^{n-1}} g(x)dS(x)$$

$$= -\int_{B(0,\delta)} g(x)dS(x)$$

Now if g is a continuous function, then

$-\int_{B(0,\delta)} g(x)dS(x) \to -g(0)$as $\delta \to 0$.

Lastly, we look at termJ_2 . Now using the fact that g vanishes as $|x| \to +\infty$, we only need to integrate over $\partial B(0, \delta)$. Using the fact that $g \in D$, and, therefore, infinitely differentiable, we have

$$\int_{B(0,\delta)} \phi(x)\frac{\partial g(x)}{\partial v} dS(x) = \left|\frac{\partial g(x)}{\partial v}\right|_{L^\infty \partial B(0,\delta)} \int_{B(0,\delta)} \phi(x)dS(x)$$

$$\leq C \int_{B(0,\delta)} \phi(x)dS(x)$$

Now first, for n = 2,

$\int_{B(0,\delta)}|\phi(x)|dS(x) = C\int_{B(0,\delta)}\big|log|x|\big|dS(x) \ \leq C\delta\ |\log|\delta||$.

Next, for n ≥ 3,

$$\int_{B(0,\delta)} |\phi(x)|dS(x)$$

$$= C\int_{B(0,\delta)} \frac{1}{|x|^{n-2}} dS(x) \leq \frac{C}{\delta^{n-2}} \int_{B(0,\delta)} dS(x)$$

$$= \frac{C}{\delta^{n-2}}\ n\alpha(n)\delta^{n-1} \leq C\delta.$$

Therefore, we conclude that term J_2 is bounded in absolute value by

$C\delta\ |\log|\delta||$for$n = 2$

$C\delta$for$n \geq 3$

Therefore, $|J_2| \to 0$ as $\delta \to 0^+$

Combining these estimates, we see that

$$\int_{\mathbb{R}^n} \phi(x)\Delta_x g(x)dx = \lim_{\delta\to 0^+} I + J_1 + J_2 = -g(0).$$

Hence prove.

Theorem : Assume $f \in C^2(\mathbb{R}^n)$and has compact support. Let

$$u(x) = \int_{\mathbb{R}^n} \emptyset(x-y)f(y)dy$$

where$\emptyset$ is the fundamental solution of Laplace's equation. Then

a) $u \in C^2(\mathbb{R}^n)$
b) $-\Delta u = f$ in $\mathbb{R}^n$.

Proof: a) By a change of variables, we write

$$u(x) = \int_{\mathbb{R}^n} \emptyset(x-y)f(y)dy = \int_{\mathbb{R}^n} \emptyset(y)f(x-y)dy$$

Let $e_i = (0, \dots .1,0,0 \dots)$

be the unit vector in $\mathbb{R}^n$ with a 1 in the i^{th} slot. Then

$$\frac{u(x+he_i)-u(x)}{h} = \int_{\mathbb{R}^n} \emptyset(y)\left[\frac{f(x+he_i-y)-f(x-y)}{h}\right]dy$$

Now $f \in C^2$implies

$\left[\frac{f(x+he_i-y)-f(x-y)}{h}\right] \to \frac{\partial f}{\partial x_i}(x-y)\text{as} h \to 0$

uniformly on $\mathbb{R}^n$. Therefore,

$$\frac{\partial u}{\partial x_i}(x) = \int_{\mathbb{R}^n} \emptyset(y)\frac{\partial f}{\partial x_i}(x-y)dy$$

Similarly,

$$\frac{\partial^2 u}{\partial x_i \partial x_j}(x) = \int_{\mathbb{R}^n} \emptyset(y)\frac{\partial^2 f}{\partial x_i \partial x_j}(x-y)dy$$

This function is continuous because the right-hand side is continuous.

b) By the above calculations and theorem 1, we see that

$$\Delta_x u(x) = \int_{\mathbb{R}^n} \emptyset(y)\Delta_x f(x-y)dy = -f(x)$$

3.4 GREEN FUNCTIONS:

We are interested in solving the following problem. Let $\Omega \subset \mathbb{R}^n$ be an open, bounded subset of $\mathbb{R}^n$.

Consider $\begin{cases} -\Delta u = f & x \in \Omega \\ u = g & x \in \partial\Omega \end{cases}$ (I)

Suppose we can solve the problem,

$$\begin{cases} -\Delta_y G(x,y) = \delta_x & y \in \Omega \\ G(x,y) = 0 & y \in \partial\Omega \end{cases}$$

for each $x \in \Omega$. Then, formally, we can say that for *u* a solution of (I)

$$u(x) = \int_\Omega \delta_x u(y)dy$$

$$= -\int_\Omega \Delta_y G(x,y)u(y)dy$$

$$= \int_\Omega G(x,y)f(y)dy - \int_{\partial\Omega} \frac{\partial G}{\partial v}(y)g(y)dS(y)$$

Now, we do know that the fundamental solution of Laplace's equation $\emptyset(y)$ satisfies

$-\Delta_y \phi(y) = \delta_0$ and $-\Delta_y \phi(x-y) = \delta_x$

Recalling the definition of distributional derivative, we will start by looking at

$$u(x) = \int_\Omega \phi(x-y)\Delta_y u(y)dy$$

We would like to integrate this term by parts. However, we know that $\phi(x-y)$ has a singularity at $y = x$.

We already find the fundamental solution of Laplace's equation. i.e.

$$\phi(x) = \begin{cases} \dfrac{-1}{2\pi} log|x| & n = 2 \\ \dfrac{1}{n(n-2)\alpha_n}|x|^{2-n} & n \geq 3 \end{cases}$$

Applying the divergence theorem, we have

$$\int_{\Omega} \phi(x-y)\Delta_y u(y)dy$$
$$= -\int_{\partial(\Omega - B(x,\delta))} \frac{\partial \phi(y-x)}{\partial v} u(y)dS(y) + \int_{\partial(\Omega - B(x,\delta))} \phi(y-x)\frac{\partial u(y)}{\partial v} dS(y)$$

Using theorem 1 we conclude that for any$u \in C^2(\overline{\Omega})$

$$u(x) = \int_{\partial(\Omega)} \left[\phi(y-x)\frac{\partial u(y)}{\partial v} - \frac{\partial \phi(y-x)}{\partial v} u(y)\right] dS(y) - \int_{\Omega} \phi(x-y)\Delta_y u(y)dy \quad \text{(III)}$$

We would now like to use the representation above formula to solve (I)

We proceed as follows. For each $x \in \Omega$, we introduce a corrector function $h^x(y)$which satisfies the following boundary-value problem,

$$\begin{cases} \Delta_y h^x(y) = 0 & y \in \Omega, \\ h^x(y) = \phi(x-y) & y \in \partial\Omega \end{cases} \quad \text{(IV)}$$

Now suppose we can find such a (smooth) function h^x which satisfies (IV). Then using the same analysis as above, we have

$$\int_{\Omega} h^x(y)\Delta_y u(y)dy$$
$$= \int_{\Omega} \Delta_y h^x(y)u(y)dy - \int_{\partial\Omega} \frac{\partial h^x(y)}{\partial v} u(y)dS(y) + \int_{\partial\Omega} h^x(y)\frac{\partial u(y)}{\partial v} dS(y)$$

Now using the fact that $h^x$is a solution of (IV), we conclude that

$$\int_{\partial\Omega} \left[\phi(x-y)\frac{\partial u(y)}{\partial v} - \frac{\partial h^x(y)}{\partial v} u(y)\right] dS(y) + - \int_{\Omega} h^x(y)\Delta_y u(y)dy = 0$$
(V)

Now subtracting (V) from (III), we conclude that

$$u(x) = \int_{\partial\Omega} \left[\frac{\partial \phi(x-y)}{\partial v} - \frac{\partial h^x(y)}{\partial v}\right] u(y)dS(y) - \int_{\Omega} [\phi(x-y) - h^x(y)]\Delta_y u(y)dy$$

Let $G(x,y) = \phi(x-y) - h^x(y)$Then, u can be written as

$$u(x) = \int_{\partial\Omega} \left[\frac{\partial G(x,y)}{\partial v}\right] u(y)dS(y) - \int_{\Omega} [G(x,y)]\Delta_y u(y)dy$$

Definition: We define this function *G* as the **Green's function** for **Ω**. That is, the Green's function for a domain **Ω** $\subset \mathbb{R}^n$ is the function defined as

$$G(x,y) = \phi(x-y) - h^x(y) x, y \in \Omega, \qquad x \neq y$$

Where ϕ is the fundamental solution of Laplace's equation and for each *x* $\in$ **Ω**, h x is a solution of (V). We leave it as an exercise to verify that *G(x, y)* satisfies (II) in the sense of distributions.

Polar form of Green's function:

In this case we want to solve

$$\Delta u = f, \qquad \lim_{n\to\infty} (u(r,\theta) - u_r(r,\theta) r log r) = 0$$

In general, solutions to $\Delta u = f$behave like $u \sim A \log r + B$ as $r \to \infty$. The condition just ensures that $B = 0$. Again we look for a Green's function of the form $G = g(|x-y|) = g(r)$so that in polar coordinates

$$\frac{1}{r}\left(rg'(r)\right)' = 0 \quad if\ r \neq 0,$$

$$\lim_{n\to\infty} (g(r) - r log r g'(r)) = 0$$

The general solution is

$$G = C_1 \log r + C_2$$

Where $C_2 = 0$, we get

$$1 = \int_{\partial B} \frac{\partial_x G}{\partial n}(x,y)dx = \int_{\partial B} C_1\, dx = 2\pi C_1$$

Where B is the unit disk, so that$C_1 = \frac{1}{2\pi}$. Thus the Green's function is$G(x,y) = \frac{\log|x-y|}{2\pi}$, and the solution to given equation is

$$u(x) = \int_{\mathbb{R}^2} \frac{log|x-y| f(y)}{2\pi}\, dy^2$$

It is sometimes useful to write G in polar coordinates. Using the law of cosines for the distance $|x-y|$, one gets

$$G(r,\theta;\ r_0,\theta_0)=\frac{1}{4\pi}\log(r^2+r_0^2+2rr_0\cos(\theta-\theta_0))$$

Example : Let $\mathbb{R}_+^2$ be the upper half-plane in $\mathbb{R}^2$. If $\mathbb{R}_+^2=\{(x_1,x_2)\in\mathbb{R}^2: x_2>0\}$ than find Green's function.

Solution: We need to find a corrector functionh^x for each x $\in\mathbb{R}_+^2$, such that

$$\begin{cases}\Delta_y h^x(y)=0 & y\in\mathbb{R}_+^2,\\ h^x(y)=\phi(x-y) & y\in\partial\mathbb{R}_+^2\end{cases}$$

Fix x $\in\mathbb{R}_+^2$ We know$\Delta_y\phi(y-x)=0$ for all$y\neq x$. Therefore, if we choose $z\notin\ \Omega$, then $\Delta_y\phi(y-z)=0$ for all $z\in\ \Omega$. Now, if we choose $z=z(x)$ appropriately $z\notin\ \Omega$ such that

$\phi(y-z)=\phi(y-x)$for $y\in\partial\Omega$, then $h^x(y)=\phi(y-z(x))$.

Recall that for n = 2,

$$\phi(y-z)=\frac{-1}{2\pi}\ log|y-z|$$

Consequently, for$x=(x_1,x_2)\in\mathbb{R}_+^2$,we see that for all $y\in\partial\mathbb{R}_+^2$.

$$|y-x|=|(y_1,0)-(x_1,x_2)|=|(y_1,0)-(x_1,-x_2)|=|y-\tilde{x}|$$

Where $\tilde{x}=(x_1,-x_2)$is the reflection of x in the plane.

Therefore,$h^x(y)=\phi|y-\tilde{x}|$we have found a corrector function for $\mathbb{R}_+^2$,

Therefore, a Green's function for the upper half-plane is given by

$$G(x,y)=\phi(y-x)-\ \phi(y-\tilde{x})=\frac{-1}{2\pi}[log|y-x|-log|y-\tilde{x}|].$$

3.5 GREENS IDENTITIES:

Green's identities provide the main energy estimates for the Laplace and Poisson equations.

Green's first identity:

First recall the Divergence Theorem:

Let D be a bounded solid region with a piecewise C^1 boundary surface ∂D. Let n be the unit outward normal vector on ∂D. Let f be any C^1 vector field on $\overline{D}=D\cup\partial D$. Then

$$\iiint_D \vec{\nabla} .\ f\ dV = \iint_{\partial D} f . n\ dS$$

Where dV is the volume element in D and dS is the surface element on ∂D.

By integrating the identity

$$\vec{\nabla}.\left(v\vec{\nabla}\,u\right) = \vec{\nabla}\,v\,.\vec{\nabla}\,u + v\Delta u$$

Over D and applying the divergence theorem, we gets

$$\iint_{\partial D} v\frac{\partial u}{\partial n}\,dS = \iiint_D \vec{\nabla}\,v\,.\vec{\nabla}\,u\,dV + \iiint_D v\Delta u\,dV$$

Where $\frac{\partial u}{\partial n} = n\,.\,\vec{\nabla}\,u$ is the directional derivative in the outward normal direction.

This is Green's first identity.

Green's second identity:

Switch u and v in Green's first identity, then subtract it from the original form of the identity. The result is

$$\iiint_D (u\Delta v - v\Delta u)dV = \iint_{\partial D} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)\,dS$$

This is Green's second identity. It is valid for any pair of function $u$and v.

Special boundary conditions can be imposed on the functions to make the right hand side of these identity zero, so that

$$\iiint_D u\Delta v\,dV = \iiint_D v\Delta u\,dV$$

Definition: A boundary condition is called symmetric for the operator Δ on D if $\iint_{\partial D}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)\,dS = 0$ for all pairs of functions u and v that satisfy the boundary condtion.

Note: Dirichlet, Neumann, and Robin BCs are symmetric.

Example: Show that Green's functions are symmetric.

Solution: To show that Green's functions are symmetric, i.e.

For all $, y \in \Omega, x \neq y\ , G(x, y) = G(y, x)$

Let $v(z) = G(x.z)$ and $w(z) = G(y,z)$

Now we will show that $v(y) = w(x) \Longrightarrow G(x,y) = G(y,x)$.

By definition of Green's function

$$G(x,y) = \phi(x-y) - h^x(y)x, y \in \Omega, \qquad x \neq y$$

Where$h^x(y)$ satisfies$\begin{cases} \Delta_y h^x(y) = 0 & y \in \Omega, \\ h^x(y) = \phi(x-y) & y \in \partial\Omega \end{cases}$

Therefore, for $z \in \partial\Omega$

$$v(z) = G(x.z) = \phi(z-x) - h^x(z) = \phi(z-x) - \phi(z-x) = 0$$

$$w(z) = G(y.z) = \phi(z-y) - h^y(z) = \phi(z-y) - \phi(z-y) = 0$$

Further, $\Delta_z v = 0$ for $z \neq x$ and $\Delta_z w = 0$ for $z \neq y$.

Now v is smooth, except near $z = x$, while w is smooth, except near $z = y$.

Define the region$V_\delta = \Omega - [B(x,\delta) - B(y,\delta)]$for $\delta > 0$.

Our functions are smooth. Therefore, integration by parts as follows,

$$\int_{V_\delta} \Delta v w \, dz = \int_{V_\delta} v \Delta w \, dz - \int_{\partial V_\delta} v \frac{\partial w}{\partial \vartheta} \, dS(z) + \int_{\partial V_\delta} \frac{\partial v}{\partial \vartheta} w \, dS(z)$$

Using the fact that $\Delta v = 0 = \Delta w$ onV_δ , we conclude that

$$\int_{\partial V_\delta} v \frac{\partial w}{\partial \vartheta} \, dS(z) = \int_{\partial V_\delta} \frac{\partial v}{\partial \vartheta} w \, dS(z)$$

Using the fact that $v = 0 = w$ on $\partial \Omega$, we conclude that

$$\int_{\partial B(x,\delta)} \left[\frac{\partial v}{\partial \vartheta} w - v \frac{\partial w}{\partial \vartheta}\right] dS(z) = \int_{\partial B(y,\delta)} \left[v \frac{\partial w}{\partial \vartheta} - \frac{\partial v}{\partial \vartheta} w\right] dS(z)$$

whereϑ denotes the inward pointing unit vector field on $\partial B(x, \delta) \cup \partial B(y, \delta)$. Now we claim that as $\delta \to 0^+$, the left-hand side converges to $w(x)$, while the right-hand side converges to $v(y)$.

For the terms on the left-hand side, we first look at$\int_{\partial B(x,\delta)} \left[v \frac{\partial w}{\partial \vartheta}\right] dS(z)$

Now w is smooth near x. Therefore,$\frac{\partial w}{\partial \vartheta}$is bounded near $\partial B(x, \delta)$.

$$v(z) = G(x.z) = \phi(z-x) - h^x(z)$$

Therefore, on$\partial B(x, \delta)$, $v(z) \approx \frac{1}{\delta^{n-2}}$

$$\left|\int_{\partial B(x,\delta)} \left[v \frac{\partial w}{\partial \vartheta}\right] dS(z)\right| \leq C \underset{\partial B(x,\delta)}{sup} |v| \int_{\partial B(x,\delta)} dS(z) = C\delta^{n-1}$$

$$\underset{\partial B(x,\delta)}{sup} |v| \to 0 \ as \ \ \delta \to 0$$

Now $\int_{\partial B(x,\delta)} \left[\frac{\partial v}{\partial \vartheta} w\right] dS(z) = \int_{\partial B(x,\delta)} \left[\frac{\partial \emptyset}{\partial \vartheta}(z-x) - \frac{\partial h^x}{\partial \vartheta}(z)\right] w \, dS(z)$

First, using the fact that h^x is smooth and *w* is smooth near *x*, we see that

$$\left|\int_{\partial B(x,\delta)} \left[\frac{\partial h^x}{\partial \vartheta}(z)\right] w \, dS(z)\right| \leq C \int_{\partial B(x,\delta)} dS(z) \leq C\delta^{n-1}$$

Therefore, $\int_{\partial B(x,\delta)} \left[\frac{\partial h^x}{\partial \vartheta}(z)\right] w \, dS(z) \to 0 \ \ as \ \delta \to 0.$

For the other term, we see that

$$\int_{\partial B(x,\delta)} \left[\frac{\partial \emptyset}{\partial \vartheta}(z-x)\right] w(z) \, dS(z) = \frac{1}{n\alpha(n)} \int_{\partial B(x,\delta)} \frac{1}{|z-x|^{n-1}} w(z) \, dS(z)$$

$$= \frac{1}{n\alpha(n)\delta^{n-1}} \int_{\partial B(x,\delta)} w(z) \, dS(z)$$

$$\int_{\partial B(x,\delta)} w(z) \, dS(z) \to w(x) \ \ \ as \ \ \delta \to 0$$

Hence the left-hand side converges to $w(x)$.

Similarly, the right-hand side converges to *v(y)*.

Hence prove,

Proposition:A Green function has the following property. In the case $n = 2$ we assume*diam* $\Omega < 1$. $0 < G(x,y) < s(|x-y|), \quad x, y \in \Omega, x \neq y.$

Proof. Since $G(x,y) = s(|y-x|) + \emptyset(x,y)$

and$G(y, x) = 0$ if $y \in \partial\Omega$ and $x \in \Omega$ we have for $y \in \partial\Omega$

$$\emptyset(x,y) = -s(|y-x|)$$

From the definition of $s(|y-x|)$it follows that $\emptyset(x,y) < 0$ if $x \in \partial\Omega$. Thus, since $\Delta_x \emptyset = 0$ in Ω, the maximum-minimum principle implies that $\emptyset(x,y) < 0$ for all $y, x \in \Omega$. Consequently

$$G(x,y) < s(|x-y|), \quad x, y \in \Omega, x \neq y$$

It remains to show that

. $0 < G(x,y), \quad x, y \in \Omega\,, x \neq y$

Fix $x \in \Omega$ and let $B\rho(y)$ be a ball such that $B\rho(y) \subset \Omega$ for all $0 < \rho < \rho_0$. There is a sufficiently small $\rho_0 > 0$ such that for each ρ, $0 < \rho < \rho_0$,

$G(x, y) > 0$ for all $y \in B\rho(y)$, $x \neq y$

see property (iii) of a Green function.

Since $\Delta_x G(x,y) = 0$in $\Omega \setminus B\rho(y)$

$G(x, y) > 0$ *if* $x \in \partial B\rho(y)$

$G(x, y) = 0$ *if* $x \in \partial\Omega$

it follows from the maximum-minimum principle that

$G(x, y) > 0$ *on* $\Omega \setminus B\rho(y)$.

Hence prove.

Example: consider a sphere with center at origin and radius 'a' apply the divergence theorem to the sphere and show that $\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta(r)$. where$\delta(r)$ is a Dirac delta function.

Solution: Applying divergence theorem to

$$grad\left(\frac{1}{r}\right) = \nabla\left(\frac{1}{r}\right)$$

$$\iiint_V \nabla.\nabla\left(\frac{1}{r}\right) dV = \iint_S \nabla\left(\frac{1}{r}\right).\hat{n}\ dS$$

Where $\hat{n}$ is an outward drawn normal. If $u = u(r, \theta, \emptyset)$, then

$$grad\, u = \hat{e_r}\frac{\partial u}{\partial r} + \widehat{e_\theta}\frac{1}{r}\frac{\partial u}{\partial \theta} + \widehat{e_\emptyset}\frac{1}{r}\ sin\theta\ \frac{\partial u}{\partial \emptyset},$$

Hence $\iint_S \nabla\left(\frac{1}{r}\right).\hat{e_r}\, dS = \iint_S \frac{\partial}{\partial r}\left(\frac{1}{r}\right)\ dS = \iint_S \left(\frac{-1}{r^2}\right)\ dS = \left(\frac{-1}{a^2}\right) \times 4\pi a^2 = -4\pi$

By properties of $\nabla^2\left(\frac{1}{r}\right)$, Its integral over any sphere with center at the origin is -4π.

Hence we say that $\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta(r)$.

Theorem: If G is continuous and $\frac{\partial G}{\partial n}$ has discontinuity at 'r'. than show that

$$\lim_{\varepsilon \to 0} \iint_{\partial V} \frac{\partial G}{\partial n}\, dS = 1.$$

Proof: Let V be a sphere with radius ε bounded by ∂V.

We already know that G satisfies $\nabla^2 G = \delta(x - y)$.

Integrating both sides over the sphere V, we get

$$\iiint_V \nabla^2 G\, dV = 1$$

Which can be written as$\lim_{\varepsilon \to 0} \iiint_V \nabla^2 G\, dV = 1$

Applying divergence theorem we get,

$$\lim_{\varepsilon \to 0} \iint_{\partial V} \frac{\partial G}{\partial n}\, dS = 1.$$

Hence prove.

3.6 LETS SUM UP

In this chapter we have learnt the following:

- Singularity functions.
- The fundamental solution of Laplace equation.
- Definition of Green functions using fundamental solution.
- Green's first identity.
- Green's second identity.
- The use of Green's function to solve partial differential equations.

3.7 UNIT END EXERCISE

1. Find the fundamental solution of Laplace equation.

2. State and prove Green's first identity.

3. Find the Green's function for the first quadrant in XY-plane.

4. State and prove symmetric property of Green's function.

5. Show that Green's function is unique.

6. Find the Fundamental Solution of the Laplace Operator for n = 3.

3. Find the Green's function for the first quadrant of $\mathbb{R}^2$, namely the domain

$$\Omega = \{(x, y) \in \mathbb{R}^2 | \ x > 0, y > 0\}$$

8. Find the Green's function for the upper half ball$B^+(0, r)$ in $\mathbb{R}^3$.

9. Show that the Fundamental Solution of the Laplace Operator is given by

$$u(x) = \begin{cases} \frac{1}{2\pi} \log r & if \ n = 2 \\ \frac{1}{(2-n)w_n} r^{2-n} & if \ n \geq 3 \end{cases}$$

10.Use the method of images to find the Green's function for Laplace's equation to infinite

strip $a < x < b$ in the (x, y)-plane.

3.8 REFERENCE

- Phoolan Prasad &Renuka Ravindran, Partial Differential Equations, Wiley Eastern Limited, India
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4

GREEN'S FUNCTION II

Unit Structure:

4.1 Objectives

4.2 Introduction

4.3 Green's function for m-dimensions sphere of radius R

4.4 Green's functions Dirichlet problem in the plane,

4.5 Neumann's function in the plane.

4.6 Lets sum up

4.7 Unit End exercise

4.8 Reference

4.1 OBJECTIVES:

After going through this chapter students will be able to:

- To provide an understanding of, and methods of solution for, the most important types of partial differential equations that arise in Mathematics.
- Use Green's functions to solve Laplace's equation.
- Use Green's functionto solve Laplace's equation for m-dimensions sphere of radius R.
- Use Green's functions to solve Dirichlet problem in the plane.
- Use Green's functions to solve Neumann's problem in the plane.

4.2 INTRODUCTION:

In general the type of conditions that may be applied depends on the applications that are involved. In practices two types of boundary conditions are commonly considered. The first one is known as the homogeneous Dirichlet boundary condition which states that u is zero on S and second one is known as the homogeneous Neumann condition which is stats that ∇u is zero on S. When u satisfies these homogeneous boundary condition. We obtained representation formula for problems on $\mathbb{R}^n$. We now fix Ω to bean bounded open domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$.We will try to build Green's function using the ideas developed so far. Later, we will check directly that the derived representation formula gives the solution. We will also use the reflection idea about the boundary of the domain.

4.3 GREEN'S FUNCTION FOR N-DIMENSIONS SPHERE OF RADIUS R:

Let $Bn(0, 1)$ be the unit ball in $\mathbb{R}^n$. We look for a formula for the solution of Laplace's equation in $Bn(0, 1)$ with Dirichlet boundary conditions,

$$\begin{cases} \Delta u = 0 & x \in B_n(0,1) \\ u = g & x \in \partial B_n(0,1) \end{cases} \qquad \text{(I)}$$

if u is a solution of (I), then u will have the form

$$u(x) = -\int_{\partial B_n(0,1)} g(y)\frac{\partial G}{\partial v}(x,y)dS(y)$$

Now we just need to calculate $\frac{\partial G}{\partial v}$on $\partial Bn(0, 1)$ where G is a Green's function for $Bn(0, 1)$. As shown above,

$$G(x,y) = \emptyset(y-x) - \emptyset(|x|(y-x^*))$$

is a Green's function for the unit ball in $\mathbb{R}^n$ where

$$x^* = \frac{x}{|x|^2}$$

is the point dual to x. We consider the case when $n \geq 3$. The case $n = 2$ can be handled similarly. For $n \geq 3$, we have

$\emptyset(y) = \frac{1}{n\alpha(n)}\frac{1}{|y|^{n-2}} \Longrightarrow \quad \nabla\emptyset(y) = \frac{-y}{n\alpha(n)|y|^n}$

Therefore,$\nabla\emptyset(y-x) = \frac{-(y-x)}{n\alpha(n)|y-x|^n}$

While$\emptyset\big(|x|(y-x^*)\big) = \frac{1}{n\alpha(n)}\frac{1}{\big||x|(y-x^*)\big|^{n-2}} = \frac{1}{|x|^{n-2}}\ \emptyset(y-x^*)$

Therefore,

$$\nabla_y\emptyset\big(|x|(y-x^*)\big) = \frac{-1}{|x|^{n-2}}\frac{(y-x^*)}{n\alpha(n)|y-x^*|^n}$$

$$= \frac{-y|x|^2 - x}{n\alpha(n)\big||x|(y-x^*)\big|^n}$$

$$= \frac{-y|x|^2 - x}{n\alpha(n)|y-x|^n}$$

Now, the unit normal to *Bn*(0, 1) is given by

$$v = \frac{y}{|y|} = y$$

Therefore, the normal derivative of *G(x, ·)* on ∂*Bn*(0, 1) is given by

$$\frac{\partial G}{\partial v}(x,y) = \frac{\partial \emptyset(y-x)}{\partial v} - \frac{\partial \emptyset(|x|(y-x^*))}{\partial v}$$

$$= \frac{-(y-x)}{n\alpha(n)|y-x|^n}\,.y + \frac{y|x|^2 - x}{n\alpha(n)|y-x|^n}\,.y$$

$$= \frac{|x|^2 - 1}{n\alpha(n)|y-x|^n}$$

Therefore, the solution formula for (I) is given by

$$u(x) = -\int_{\partial B_n(0,1)} g(y)\frac{\partial G}{\partial v}(x,y)dS(y)$$

$$= \frac{1-|x|^2}{n\alpha(n)}\int_{\partial B_n(0,1)} \frac{g(y)}{|y-x|^n}dS(y)$$

We can use this formula to derive the solution formula for Laplace's equation on the ball of radius *r* with Dirichlet boundary conditions,

$$\begin{cases} \Delta u = 0 & x \in B_n(0,r) \\ u = g & x \in \partial B_n(0,r) \end{cases} \qquad \text{(II)}$$

Suppose u is a solution of (II), then$\tilde{u}(x) = u(rx)$is a solution of (I) with boundary data

$\tilde{g}(x) = g(rx)$. Therefore, by our work above, we see the formula for $\tilde{u}$ is given by

$$\tilde{u}(x) = \frac{1-|x|^2}{n\alpha(n)}\int_{\partial B_n(0,1)} \frac{\tilde{g}(y)}{|y-x|^n}dS(y)$$

$$= 1-|x|^2 \int_{\partial B_n(0,1)} \frac{g(ry)}{|y-x|^n}dS(y)$$

$$= 1 - |x|^2 \int_{\partial B_n(0,r)} \frac{g(\tilde{y})}{\left|\frac{\tilde{y}}{r} - x\right|^n} dS(\tilde{y})$$

$$= r^n(1 - |x|^2) \int_{\partial B_n(0,r)} \frac{g(\tilde{y})}{|\tilde{y} - rx|^n} dS(\tilde{y})$$

$$u(rx) = \frac{(r^2 - |rx|^2)}{n\alpha(n)r} \int_{\partial B_n(0,r)} \frac{g(y)}{|y - rx|^n} dS(y)$$

which implies the solution formula for (II) is given by

$$u(x) = \frac{(r^2 - |x|^2)}{n\alpha(n)r} \int_{\partial B_n(0,r)} \frac{g(y)}{|y - x|^n} dS(y)$$

This representation formula is called **Poisson's formula** for the ball. The function

$$K(x, y) = \frac{(r^2 - |x|^2)}{n\alpha(n)r|y - x|^n}$$

is called **Poisson's kernel** for the ball.

Example 1: Let $\mathbb{R}^n_+$ be the upper half-space in $\mathbb{R}^n$,

$$\mathbb{R}^n_+ = \{(x_1, x_2, \dots, x_n) \in R^2 : x_n > 0\}$$

Find Poisson's formula and Poisson's kernel be the upper half-space in$\mathbb{R}^n_+$.

Solution:G is a Green's function for $\mathbb{R}^n_+$. As shown above,

$$G(x, y) = \emptyset(y - x) - \emptyset(y - \tilde{x})$$

Where $\tilde{x} = (x_1, x_2, x_3, \dots \dots x_{n-1,} - x_n)$ and $\emptyset$ is the fundamental solution of Laplace's equation in $\mathbb{R}^n$. Our proposed solution has the form

$$u(x) = -\int_{\partial \mathbb{R}^n_+} g(y) \frac{\partial G}{\partial v}(x, y) dS(y) + \int_{\mathbb{R}^n_+} f(y) G(x, y) dy$$

Now, we calculate $\frac{\partial G}{\partial v}$ on $\{y_n = 0\}$ to find an explicit formula for solutions to

$$\begin{cases} \Delta u = 0 & x \in \Omega \\ u = g & x \in \partial\Omega \end{cases}$$

Now $\frac{\partial \emptyset}{\partial y_n}(y) = \frac{-y_n}{n\alpha(n)|y|^n}$

Therefore, the normal derivative of G on $\{y_n = 0\}$ is given by

$$\frac{\partial G(x,y)}{\partial v} = \frac{\partial \emptyset(y-x)}{\partial y_n} - \frac{\partial \emptyset(y-\tilde{x})}{\partial y_n}$$

$$= \frac{y_n - x_n}{n\alpha(n)|y-x|^n} - \frac{y_n - \tilde{x}_n}{n\alpha(n)|y-\tilde{x}|^n}$$

$$= \frac{-2x_n}{n\alpha(n)|y-x|^n}$$

Therefore, if u is the solution of Laplace's equation on the upper half-space Ω with Dirichlet boundary conditions, then we suspect that u will have the form

$$u(x) = \frac{2x_n}{n\alpha(n)} \int_{\partial\mathbb{R}^n_+} \frac{g(y)}{|y-x|^n} dS(y)$$

This is called **Poisson's formula** for the half-space$\mathbb{R}^n_+$. The function

$$K(x,y) = \frac{2x_n}{n\alpha(n)|y-x|^n}$$

is called Poisson's kernel for the half-space $\mathbb{R}^n_+$.

4.4 GREEN'S FUNCTIONS DIRICHLET PROBLEM IN THE PLANE:

Consider the Dirichlet problem for the Poisson equation

$$\Delta u = f \qquad x \in D$$

$$u = g \qquad x \in \partial D$$

where D is a bounded planar domain with a smooth boundary ∂D. The fundamental solution of the Laplace equation plays an important role in our discussion. Recall that this fundamental solution is defined by

$$\emptyset(x,y) = \frac{-1}{2\pi} \log r = \frac{-1}{4\pi} \log(x^2 + y^2)$$

The fundamental solution is harmonic in the punctured plane, and it is a radially symmetric function with a singularity at the origin. Fix a point $(a,b) \in \mathbb{R}^2$. Note that if $u(x, y)$ is harmonic, then $u(x - a, y - b)$ is also harmonic for every fixed pair (a, b). We use the notation

$$\emptyset(x,y;a,b) = \emptyset(x-a, y-b)$$

We call $\emptyset(x, y; a, b)$ the fundamental solution of the Laplace equation with a pole at (a, b).

The function$\emptyset(x, y; a, b)$ is harmonic for any point (x, y) in the plane such that $(x, y) \neq (a, b)$For ε > 0, set.

$$B_\varepsilon = \left\{(x, y) \in D, \sqrt{(x-a)^2 + (y-b)^2} < \varepsilon\right\}, \qquad D_\varepsilon = D \backslash B_\varepsilon$$

Let $u \in C^2(\bar{D})$. We use the second Green identity in the domain D_ε where the function $v(x, y) = \emptyset(x, y; a, b))$ is harmonic to obtain

$$\int_{D_\varepsilon} (\emptyset\Delta u - u\Delta\emptyset)dxdy = \int_{\partial D_\varepsilon} (\emptyset\partial_n u - u\partial_n \emptyset)dS$$

Therefore,

$$\int_{D_\varepsilon} (\emptyset\Delta u)dxdy = \int_{\partial D} (\emptyset\partial_n u - u\partial_n \emptyset)dS + \int_{\partial B_\varepsilon} (\emptyset\partial_n u - u\partial_n \emptyset)dS$$

Let ε tend to zero, recalling that the outward normal derivative (with respect to the domain D_ε) on the boundary of B_ε is the inner radial derivative pointing towards the pole *(a,b)*.

We obtain$\left|\int_{\partial B_\varepsilon} (\emptyset\partial_n u)dS\right| \leq C\varepsilon|log\varepsilon| \to 0 \qquad as\ \varepsilon \to 0$

$$\left|\int_{\partial B_\varepsilon} (u\partial_n \emptyset)dS\right| = \frac{1}{2\pi\varepsilon} \int_{\partial B_\varepsilon} (u)dS \to u(a, b) \qquad as\ \varepsilon \to 0$$

Therefore

$$u(a, b) = \int_{\partial D} \big(\emptyset(x-a, y-b)\partial_n u - u\partial_n \emptyset(x-a, y-b)\big)dS$$
$$- \int_D \big(\emptyset(x-a, y-b)\big)\Delta u\, dxdy$$

is called Green's representation formula for Dirichlet problem in the plane.

The function$\emptyset[f](a, b) = -\int_D \big(\emptyset(x-a, y-b)\big)f(x, y)dxdy$.is called the Newtonian potential of*f*.

Example2 : Determine the Green's function for Dirichlet problem for a circle given by

$$\nabla^2 u = 0, \quad r < a$$

$$u = f(\theta) \ \ on \ \ r = a.$$

Solution: Let $P(r, \theta)$ and Q(r', θ') have position vectors r and r'.

Let P' be the inverse of P with respect to the circle so that $OP.OP' = a^2$ and P' has coordinate $\left(a^2/_r, \ \theta\right)$.

Now we construct the Green's Function G such that

$$G = log\frac{1}{|r - r'|} + H$$

Let $H = log\left(r.P'Q/_a\right)$ so that it can be verified that $\nabla^2 u = 0$.

$$G = log\frac{r.P'Q}{a.PQ}$$

On the circle $r = a$,

$$G = log\frac{P'Q}{PQ} = \log 1 = 0$$

However $PQ^2 = r^2 + r'^2 - 2rr'\cos(\theta' - \theta)$

$$P'Q^2 = \frac{a^4}{r^2} + r'^2 - 2\frac{a^2}{r}r'\cos(\theta' - \theta)$$

Replace value of PQ and $P'Q$ in $G = log\frac{r.P'Q}{a.PQ}$ we get,

$$G = \frac{1}{2}\ log\left[\frac{\frac{r^2}{a^2}\left(\frac{a^4}{r^2} + r'^2 - 2rr'\cos(\theta' - \theta)\right)}{r^2 + r'^2 - 2rr'\cos(\theta' - \theta)}\right]$$

$$= \frac{1}{2}\ log\left[\frac{\left(a^2 + \frac{r^2}{a^2}\ r'^2 - 2\frac{a^2}{r}r'\cos(\theta' - \theta)\right)}{r^2 + r'^2 - 2rr'\cos(\theta' - \theta)}\right]$$

But On the circle $r = a$,

$$\left(\frac{\partial G}{\partial r'}\right)_{r'=a} = \frac{-(a^2 - r^2)}{a[a^2 - 2arcos(\theta' - \theta) + r^2]}$$

Therefore, $u(r,\theta) = \frac{a^2-r^2}{2\pi a}\int_0^{2\pi}\frac{f(\theta')d\theta\prime}{[a^2-2arcos(\theta'-\theta)+r^2]}$.

The Eigen function method:

Consider the eigen value problem associated with the operator ∇^2 in the domain $\mathbb{R}$. i.e.

$$\Delta^2\emptyset + \lambda\emptyset = 0 \qquad \text{in}\mathbb{R}$$

$$\emptyset = 0\text{in } \partial\mathbb{R}$$

Let λ_{mn} be the eigen values and $\emptyset_{mn}$ be the corresponding eigen functions. Suppose we give furior series expression to G and δ in terms of the eigen functions $\emptyset_{mn}$ in the following form:

$$G(x,y,\xi,\eta) = \sum_m\sum_n a_{mn}(\xi,\eta)\emptyset_{mn}(x,y)$$

$$\delta(x-\xi,y-\eta) = \sum_m\sum_n b_{mn}(\xi,\eta)\emptyset_{mn}(x,y)$$

Where $b_{mn} = \frac{1}{\|\emptyset_{mn}\|^2}\iint_{\mathbb{R}}\ \delta(x-\xi,y-\eta)\emptyset_{mn}(x,y)dxdy = \frac{\emptyset_{mn}(\xi,\eta)}{\|\emptyset_{mn}\|^2}$

$$\|\emptyset_{mn}\|^2 = \iint_{\mathbb{R}} \emptyset_{mn}^2\, dxdy$$

$$\therefore\ \ \nabla^2\emptyset_{mn} + \lambda_{mn}\emptyset_{mn} = 0$$

We obtain $\quad \nabla^2\sum_m\sum_n a_{mn}(\xi,\eta)\emptyset_{mn}(x,y) = \sum_m\sum_n b_{mn}(\xi,\eta)\emptyset_{mn}(x,y)$

$$-\sum_m\sum_n \lambda_{mn}a_{mn}(\xi,\eta)\emptyset_{mn}(x,y) = \frac{\sum_m\sum_n \emptyset_{mn}(\xi,\eta)\emptyset_{mn}(x,y)}{\|\emptyset_{mn}\|^2}$$

From which we get

$$a_{mn}(\xi,\eta) = \frac{-\emptyset_{mn}(\xi,\eta)}{\lambda_{mn}\|\emptyset_{mn}\|^2}$$

Hence the required Green's function for the Dirichlet problem in the form

$$G(x,y,\xi,\eta) = \frac{\sum_m \sum_n \phi_{mn}(\xi,\eta)\phi_{mn}(x,y)}{\lambda_{mn}\|\phi_{mn}\|^2}.$$

If eigen value $\lambda_{mn} = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$ and corresponding eigen function $\phi_{mn} = Sin\left(\frac{m\pi x}{a}\right) Sin\left(\frac{n\pi y}{b}\right)$ where $m, n = 1,2, \dots \dots \dots \dots$

Therefore Green's function for the Dirichlet problem in the form can be written as,

$$G(x,y,\xi,\eta) = \frac{-4}{ab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{Sin\left(\frac{m\pi\xi}{a}\right)Sin\left(\frac{n\pi\eta}{b}\right)}{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2} Sin\left(\frac{m\pi x}{a}\right) Sin\left(\frac{n\pi y}{b}\right).$$

Example 3: Using Green's function method Solve PDE,

$\nabla^2 u = -\pi^2 \sin(\pi x)\sin(2\pi y)$ in $0 < x < 1,\ \ 0 < y < 2$

With the initial boundary condition $u(x,0) = 0;\ u(x,2) = 0,\ 0 < x < 1$

$$u(0,y) = 0;\ \ u(1,y) = 0,\ \ 0 < y < 2.$$

Solution: Here $a = 1, b = 2$ and $f(x,y) = -\pi^2 \sin(\pi x)\sin(2\pi y)$

We have Green's function for the Dirichlet problem in the form can be written as,

$$G(x,y,\xi,\eta) = \frac{-4}{ab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{Sin\left(\frac{m\pi\xi}{a}\right)Sin\left(\frac{n\pi\eta}{b}\right)}{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2} Sin\left(\frac{m\pi x}{a}\right) Sin\left(\frac{n\pi y}{b}\right).$$

$$G(x,y,\xi,\eta) = -2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{Sin\left(\frac{m\pi\xi}{1}\right)Sin\left(\frac{n\pi\eta}{2}\right)}{\left(\frac{m\pi}{1}\right)^2+\left(\frac{n\pi}{2}\right)^2} Sin\left(\frac{m\pi x}{1}\right) Sin\left(\frac{n\pi y}{2}\right).$$

By definition of Green's function,

$$u(x,y) = \int_0^1\int_0^2 -2\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{Sin\left(\frac{m\pi\xi}{1}\right)Sin\left(\frac{n\pi\eta}{2}\right)}{\left(\frac{m\pi}{1}\right)^2+\left(\frac{n\pi}{2}\right)^2} Sin\left(\frac{m\pi x}{1}\right) Sin\left(\frac{n\pi y}{2}\right)$$
$$\times -\pi^2 \sin(\pi x)\sin(2\pi y)$$

$$u(x,y)$$
$$= 8\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{4m^2+n^2}\left(\int_0^1 \sin(\pi\xi)\sin(m\pi\xi)\,d\xi\right)\left(\int_0^2 \sin(2\pi\eta)\sin\left(\frac{n\pi y}{2}\right)d\eta\right)$$
$$\times \sin(\pi x)\sin(2\pi y)$$

$$u(x,y) = \frac{1}{2}\left(\frac{8}{4(1^2)+(4^2)}\right)\sin(\pi x)\sin(2\pi y)$$

$$u(x,y) = \frac{1}{5}\sin(\pi x)\sin(2\pi y)$$

Example 4: Let Ω be the triangle on $\mathbb{R}^2$ with vertices (-1, 0), (1, 0) and (0, $\sqrt{3}$). Solve the following Dirichlet problem

$$-\Delta u = 2 \text{ in } \Omega$$

$$u = 0 \text{ on } \partial\Omega.$$

Solution: We first need to find equations of sides of triangle.

Equations of sides of triangle are $y = 0$, $y + \sqrt{3}\,x - \sqrt{3} = 0$, $y - \sqrt{3}\,x - \sqrt{3} = 0$.

Thus the solution has the following form

$$u(x,y) = cy(,\ \ y + \sqrt{3}\,x - \sqrt{3})(,\ \ y - \sqrt{3}\,x - \sqrt{3})$$

Now we need to determine constant $'c'$ which fulfilled the boundary condition.

A direct calculation is given by

$$-\Delta u = 4\sqrt{3}c = 2$$

$$\therefore c = \frac{1}{2\sqrt{3}}$$

Therefore the solution is

$$u(x,y) = \frac{y}{2\sqrt{3}}(,\ \ y + \sqrt{3}\,x - \sqrt{3})(,\ \ y - \sqrt{3}\,x - \sqrt{3}).$$

4.5 NEUMANN'S FUNCTION IN THE PLANE:

We move on to present an integral representation for solutions of the Neumann problem for the Poisson equation:

$$\Delta u = f \qquad x \in D$$

$$\partial_n u = g \qquad x \in \partial D$$

Where D is a bounded planar domain with a smooth boundary ∂D. The fundamental solution of the Laplace equation plays an important role in our discussion. Recall that this fundamental solution is defined by

$$\emptyset(x, y) = \frac{-1}{2\pi} \log r = \frac{-1}{4\pi} \log(x^2 + y^2)$$

for any closed curve that is fully contained in D.

The fundamental solution is harmonic in the punctured plane, and it is a radially symmetric function with a singularity at the origin. Fix a point $(a, b) \in \mathbb{R}^2$. Note that if $u(x, y)$ is harmonic, then $u(x - a, y - b)$ is also harmonic for every fixed pair (a, b). We use the notation

$$\emptyset(x, y; a, b) = \emptyset(x - a, y - b)$$

We call $\emptyset(x, y; a, b)$ the fundamental solution of the Laplace equation with a pole at (a, b).

The function $\emptyset(x, y; a, b)$ is harmonic for any point (x, y) in the plane such that $(x, y) \neq (a, b)$ For ε > 0, set.

$$B_\varepsilon = \left\{(x, y) \in D, \sqrt{(x-a)^2 + (y-b)^2} < \varepsilon\right\}, \qquad D_\varepsilon = D \backslash B_\varepsilon$$

Let $u \in C^2(\bar{D})$. We use the second Green identity in the domain D_ε where the function $v(x, y) = \emptyset(x, y; a, b)$) is harmonic to obtain

$$\int_{D_\varepsilon} (\emptyset \Delta u - u \Delta \emptyset) dxdy = \int_{\partial D_\varepsilon} (\emptyset \partial_n u - u \partial_n \emptyset) dS$$

Therefore,

$$\int_{D_\varepsilon} (\emptyset \Delta u) dxdy = \int_{\partial D} (\emptyset \partial_n u - u \partial_n \emptyset) dS + \int_{\partial B_\varepsilon} (\emptyset \partial_n u - u \partial_n \emptyset) dS$$

Let ε tend to zero, recalling that the outward normal derivative (with respect to the domain D_ε) on the boundary of B_ε is the inner radial derivative pointing towards the pole (a,b).

We obtain$\left|\int_{\partial B_{\varepsilon}}(\emptyset \partial_n u)dS\right| \leq C\varepsilon|log\varepsilon| \to 0 \quad as\ \varepsilon \to 0$

$$\left|\int\limits_{\partial B_{\varepsilon}}(u\partial_n\emptyset)dS\right| = \frac{1}{2\pi\varepsilon}\int\limits_{\partial B_{\varepsilon}}(u)dS \to u(a,b) \qquad as\ \varepsilon \to 0$$

Therefore

$$u(a,b) = \int\limits_{\partial D}\big(\emptyset(x-a,y-b)\partial_n u - u\partial_n\emptyset(x-a,y-b)\big)dS - \int\limits_{D}\big(\emptyset(x-a,y-b)\big)\Delta u\ dxdy$$

Enables us to reproduce the value of an arbitrary smooth function u at any point (*a,b*) in *D* provided that Δu is given in *D*, and u and $\partial_n u$are given on ∂D. For the Neumann problem, *u* is not known on *∂D.*

Let *h*(*x, y; a, b*) be a solution (depending on the parameter (*a, b*)) of the following Neumann problem:

$$\Delta h(x,y;a,b) = 0(x,y) \in D$$

$$\partial_n h(x,y;a,b) = \partial_n\emptyset(x,y;a,b) + \frac{1}{L} \qquad (x,y) \in \partial D$$

Where L is the length of *∂D.* Substituting *u* = *1* into the Green representation formula we get

$$\int\limits_{\partial D}\partial_n\emptyset(x,y;a,b)\ ds = -1$$

Therefore, the above boundary condition a sufficient condition for the solvability of the problems to $\int_{\emptyset}\partial_n u\ ds = 0$.

Definition:A Neumann function for a domain *D* and the Laplace operator is the function

$$N(x,y;a,b) = \emptyset(x,y;a,b) - h(x,y;a,b)(x,y),(a,b) \in D, \\ (x,y) \neq (a,b)$$

where$h(x,y;a,b)$ is a solution of

$$\Delta h(x,y;a,b) = 0(x,y) \in D$$

$$\partial_n h(x,y;a,b) = \partial_n\emptyset(x,y;a,b) + \frac{1}{L} \qquad (x,y) \in \partial D$$

i.e. a Neumann function satisfies

$$\Delta N(x, y; a, b) = -\delta(x-a, y-b)(x, y) \in D$$

$$\partial_n N(x, y; a, b) = \frac{-1}{L} \qquad\qquad (x, y) \in \partial D$$

Therefore

$$u(a, b) = \int_{\partial D} N(x, y; a, b)\ \partial_n u(x, y) ds$$
$$- \int_{D} N(x, y; a, b) \Delta u(x, y) dx dy + \frac{1}{L} \int_{\partial D} u\ ds$$

We obtain the following representation formula for solutions of the Neumann problem.

Note:

i) The kernel N is not called the Green function of the problem, since N does not satisfy the corresponding homogeneous boundary condition. There is no kernel function that satisfies

$$\Delta G(x, y; a, b) = -\delta(x-a, y-b)(x, y) \in D$$
$$\partial_n N(x, y; a, b) = 0 \qquad\qquad (x, y) \in \partial D$$

ii) The Neumann function is determined up to an additive constant. In order to uniquely define N it is convenient to use the normalization

$$\int_{\partial D} N(x, y; a, b)\ ds = 0$$

iii) The third term in the representation formula (4.32) is $\frac{1}{L} \int_{\partial D} u\ ds$, the average of u on the boundary, which is not given. But since the solution is determined up to an additive constant, it is convenient to add the condition

$$\int_{\partial D} u(x, y) ds = 0$$

and then the problem is uniquely solved, and the corresponding integral representation uniquely determines the solution.

Example5 : Consider the Neumann boundary value problem for Laplace's equation in the upper half plane $\nabla^2 u = 0$ in $y > 0$ with $\frac{\partial u}{\partial n} = -\frac{\partial u}{\partial y} = f(x)$ on $y = 0$.

Solution: Draw boundary value condition and add image to make $\frac{\partial G}{\partial y} = 0$ on the boundary condition:

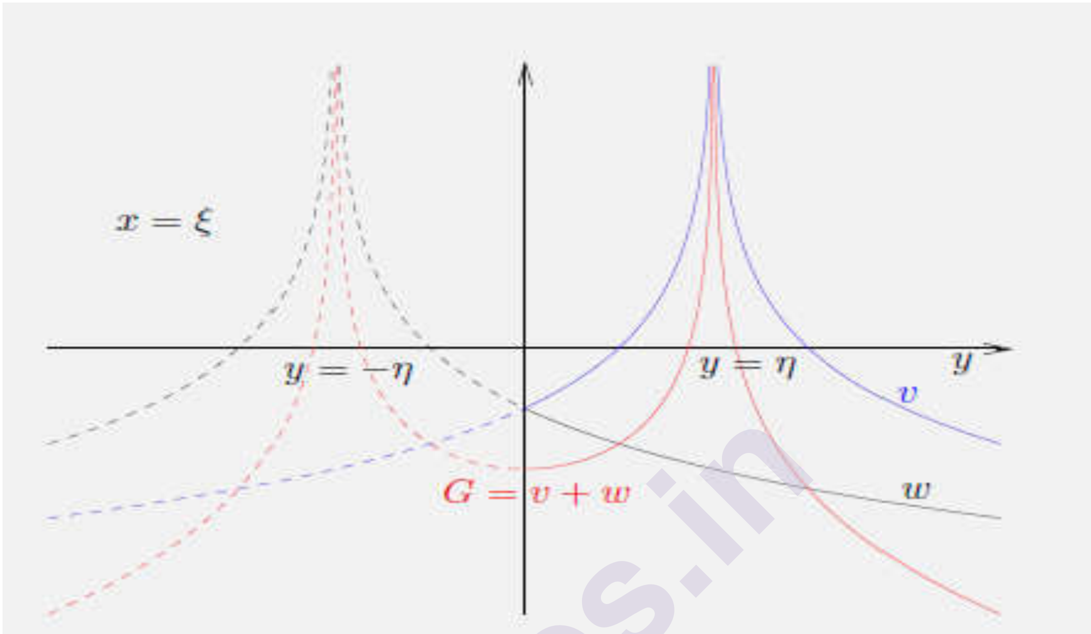

Figure 1

$$G(x, y, \xi, \eta) = -\frac{1}{4\pi}\log[(x-\xi)^2 + (y-\eta)^2]$$
$$+ -\frac{1}{4\pi}\log[(x-\xi)^2 + (y+\eta)^2]$$

Note that $\frac{\partial G}{\partial y} = \frac{-1}{4\pi}\left(\frac{2(y-\eta)}{(x-\xi)^2+(y-\eta)^2} + \frac{2(y+\eta)}{(x-\xi)^2+(y+\eta)^2}\right)$

And as required Neumann boundary value problem,

$$\frac{\partial G}{\partial n_S} = \frac{-\partial G}{\partial y}_{y=0} = \frac{1}{4\pi}\left(\frac{-2\eta}{(x-\xi)^2 + (\eta)^2} + \frac{2\eta}{(x-\xi)^2 + (\eta)^2}\right) = 0.$$

Then, since $G(x, 0, \xi, \eta) = -\frac{1}{2\pi}\log[(x-\xi)^2 + (\eta)^2]$

$$u(\xi, \eta) = \int_{-\infty}^{+\infty} f(x)\log[(x-\xi)^2 + (\eta)^2]\,dx$$

Therefore we can write,

$$u(x,y) = \int_{-\infty}^{+\infty} f(\lambda)\log[(x-\lambda)^2 + (y)^2]\, d\lambda$$

We have not given condition on G and $\frac{\partial G}{\partial n}$ at infinity. For instance we can think of the boundary of the upper half plane as a semi-circle with $R \to +\infty$.

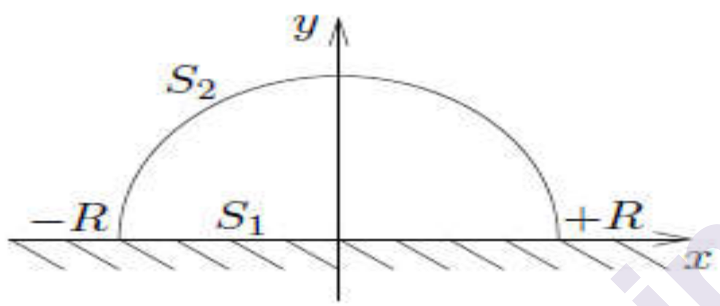

Figure 2

Green's theorem in the half-disc for u and G is

$$\int_V (G\nabla^2 u - u\nabla^2 G)dV = \int_S \left(G\frac{\partial u}{\partial n} - u\frac{\partial G}{\partial n}\right) ds\,.$$

Example 6: Interior Neumann problem for Laplace's equation in a disc,

$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 r}{\partial \theta^2} = 0$in $r < a$, $\qquad \frac{\partial u}{\partial n} = f(\theta)$ on $r = a$.

Solution: Here we need,

$\Delta^2 G = -\delta(x-\xi)\delta(y-\eta) + \frac{1}{\vartheta}$with $\left.\frac{\partial G}{\partial r}\right|_{r=a} = 0.$

Where $\vartheta = \pi a^2$ is the surface area of the disc. In order to deal with this term we solve the equation $\nabla^2 k(r) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial k}{\partial r}\right) = \frac{1}{\pi a^2}$

$$\Rightarrow k(r) = \frac{r^2}{4\pi a^2} + C_1 logr + C_2$$

And take the particular solution with $C_1 = C_2 = 0$. Then add in source at inverse point and an arbitrary function h to fix the symmetry and boundary condition of G.

$$G(r,\theta,\rho,\emptyset) = \frac{-1}{4\pi} log[r^2 + \rho^2 - 2r\rho\cos(\theta - \emptyset)] \frac{-1}{4\pi} log\left[\frac{a^2}{\rho^2}\left(a^2 + \frac{r^2\rho^2}{a^2} - 2r\rho\cos(\theta - \emptyset)\right)\right] + \frac{r^2}{4\pi a^2} + h$$

So, $\frac{\partial G}{\partial r} = \frac{-1}{4\pi}\frac{2r - 2\rho\cos(\theta-\emptyset)}{r^2+\rho^2-2r\rho\cos(\theta-\emptyset)} - \frac{1}{4\pi}\left(\frac{2r - \frac{2a^2}{\rho}\cos(\theta-\emptyset)}{r^2+\frac{a^4}{\rho}-\frac{2ra^2}{\rho}\cos(\theta-\emptyset)}\right) + \frac{r}{2\pi a^2} + \frac{\partial h}{\partial r}$

$$\left.\frac{\partial G}{\partial r}\right|_{a=r} = \frac{-1}{2\pi}\frac{a - \rho\cos(\theta-\emptyset) + \frac{\rho^2}{a} - \rho\cos(\theta-\emptyset)}{a^2+\rho^2-2a\rho\cos(\theta-\emptyset)} + \frac{1}{2\pi a} + \frac{\partial h}{\partial r}$$

$$\left.\frac{\partial G}{\partial r}\right|_{a=r} = \frac{-1}{2\pi a} + \frac{1}{2\pi a} + \frac{\partial h}{\partial r}$$

And $\left.\frac{\partial h}{\partial r}\right|_{r=a} = 0$ implise $\frac{\partial G}{\partial r} = 0$ on the boundary ,

then put $h = \frac{1}{2\pi} log\left({}^{a}/_{\rho}\right)$

So,

$G(r,\theta,\rho,\emptyset) = \frac{-1}{4\pi} log[r^2 + \rho^2 - 2r\rho\cos(\theta-\emptyset)] \frac{-1}{4\pi} log\left[\frac{a^2}{\rho^2}\left(a^2 + \frac{r^2\rho^2}{a^2} - 2r\rho\cos(\theta-\emptyset)\right)\right] + \frac{r^2}{4\pi a^2}$ on $r = a.$

$$G|_{r=a} = \frac{-1}{4\pi} log[a^2 + \rho^2 - 2a\rho\cos(\theta-\emptyset)^2] + \frac{1}{4\pi}$$

$$G|_{r=a} = \frac{-1}{2\pi}\left(log[a^2 + \rho^2 - 2a\rho\cos(\theta-\emptyset)] - \frac{1}{2}\right)$$

Then $u(\rho,\emptyset) = \bar{u} + \int_0^{2\pi} f(\theta)\, G|_{r=a}\, ad\theta$

$$u(\rho,\emptyset) = \bar{u} - \frac{a}{2\pi}\int_0^{2\pi}\left(log[a^2 + \rho^2 - 2a\rho\cos(\theta-\emptyset)] - \frac{1}{2}\right) f(\theta)d\theta$$

Now recall the Neumann problem compatibility condition;

$$\int_0^{2\pi} f(\theta)d\theta = 0$$

Indeed $\int_V \nabla^2 u \, dV = \int_S \frac{\partial u}{\partial n} \, dS$

From divergence theorem,

$$\int_0^{2\pi} f(\theta) d\theta = 0$$

So the term involving $\int_0^{2\pi} f(\theta) d\theta$in the solution $u(\rho, \emptyset)$ vanishes, hence

$$u(\rho, \emptyset) = \bar{u} - \frac{a}{2\pi} \int_0^{2\pi} log[a^2 + \rho^2 - 2a\rho \cos(\theta - \emptyset)] f(\theta) d\theta$$

Or

$$u(r, \theta) = \bar{u} - \frac{a}{2\pi} \int_0^{2\pi} log[a^2 + r^2 - 2ar \cos(\theta - \emptyset)] f(\emptyset) d\emptyset$$

4.6 LETS SUM UP:

In this chapter we have learnt the following:

- Use Green's functions to solve Laplace's equation.
- Use Green's function to solve Laplace's equation for m-dimensions sphere of radius R.
- Use Green's functions to solve Dirichlet problem in the plane.
- Use Green's functions to solve Neumann's problem in the plane.

4.7 UNIT END EXERCISE:

1. Prove that the Neumann function for the Poission equation is symmetric.
2. Find the Green's function for the Dirichlet problem on the rectangle $R: 0 \leq x \leq a, 0 \leq y \leq b$ described by the PDE $(\nabla^2 + \lambda)u = 0$ in R and the initial boundary condition $u = 0$ on ∂R.
3. Use Green's function technique to solve the Dirichlet problem for a semi-infinite space.
4. Find the Green's function for Boundary value problem $\nabla^2 u = F$ in the quadrant

$$x > 0, y > 0.$$

5. Prove that Exterior Neumann problem for Laplace's equation in a disc,

$$u(r,\theta) = \bar{u} - \frac{a}{2\pi}\int_0^{2\pi} log[a^2 + r^2 - 2ar\cos(\theta - \emptyset)]f(\emptyset)d\emptyset$$

6. Solve the Neumann problem in the quarter-plane $\{x > 0, y > 0\}$.

7. Use the Green's function method to find the solution of the Neumann boundary value problem :

$$\nabla^2 u = 0, \quad 0 < x < 1, \quad 0 < y < 1$$
$$u(x,0) = u(x,1) = 0, \quad 0 < x < 1,$$
$$u(0,y) = u(1,y) = 0, \quad 0 < y < 1.$$

8. Solve the following Dirichlet problem,

   $-\Delta u = 0 \text{in} D(0,1)$

   $u(\rho,\theta) = Asin^2\theta + Bcos^2\theta \text{on} \rho = 1$

   Where $x = (x_1, x_2) = (\rho cos\theta, \rho sin\theta)$ and A and B are constants.

9. Find a bounded solution to the following Dirichlet problem outside a unitball in $\mathbb{R}^3$:

$$-\Delta u = 0, \quad r < 1$$
$$u|_{r=1} = \frac{2}{5 + 4x_2}$$

   Where $r = |x|$.

10. Let u be the solution of

   $-\Delta u = 0 \text{in} \mathbb{R}^n_+$

   $u = g \ \text{on} \partial\mathbb{R}^n_+$

   Given by the Poisson formula for the half-space. Assume g is bounded and $g(x) = |x|$for $x \in \partial\mathbb{R}^n_+$, $|x| \leq 1$. Show that Du is not bounded near $x = 0$.

4.8 REFERENCE

- Phoolan Prasad & Renuka Ravindran, Partial Differential Equations, Wiley Eastern Limited, India
- Yehuda Pinchover and Jacob Rubistein, An Introduction to Partial Differential Equations, Cambridge University Press

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5

THE DIFFUSION EQUATION& PARABOLIC DIFFERENTIAL EQUATIONS

Unit Structure:

5.1 Objectives

5.2 Introduction

5.3 Existence and Uniqueness theorem for initial value problem in an infinite domain

5.4 Existence and Uniqueness theorem for initial value problem in semi-infinite domain

5.5 One dimensional Heat equation

5.6 Maximum and Minimum Principle for the Heat equation

5.7 One dimensional wave equation

5.8 Lets sum up

5.9 Unit End exercise

5.10 Reference

5.1 OBJECTIVES:

After going through this chapter students will be able to:

- Existence and Uniqueness theorem for initial value problem in an infinite domain and semi-infinite domain.
- One dimensional Heat equation and also solve its initial value problem.
- Maximum and Minimum principle for the heat equation.
- One dimensional wave equation and also solves its initial value problem.
- Solve one dimensional PDE by method of separation of variables.

5.2 INTRODUCTION:

In this chapter we are going to look at one of the more common methods for solving simple partial differential equations. The method we will be taking a look at is that of Separation of Variables.

We will do a partial derivation of the heat equation that can be solved to give the temperature in a one dimensional rod of length L. In addition, we give several possible boundary conditions that can be used in this situation. We do a partial derivation of the wave equation which can be used to find the one dimensional displacement of a vibrating string.

Model heat flow in a one-dimensional object (thin rod). Place rod along x-axis, and let

$u(x,t)$be a temperature in rod at position x, time t.

Under ideal conditions (e.g. perfect insulation, no external heat sources, uniform rod material), one can show the temperature must satisfy

$$u_t = C^2 \nabla u \qquad \text{(theone-dimensional heat equation)}$$

The constant C^2 is called the thermal diffusivity of the rod.

Now we will discuss existence and uniqueness theorem for IVP in infinite and semi-infinite domain.

5.3 EXISTENCE AND UNIQUENESS THEOREM FOR INITIAL VALUE PROBLEM IN AN INFINITE DOMAIN:

We will start out by considering the temperature in a 1-D rod of length L. What this means is that we are going to assume that the bar starts off at $x = 0$ and ends when we reach $= L$. We are also going to so assume that at any location, x the temperature will be constant at every point in the cross section at that x.

We have learn the Green's function, using Green's function for the problem of heat flow in an infinite rod, the position of the rod coincide with X-axis and rod is homogeneous. Also heat is uniformly supply to it in cross section area in time $t > 0$. $u(x,t)$ is the temperature at the point x at time t without loss of heat through boundary condition. Then the problem can be described by PDE,

$$u_t = \alpha \Delta_x u \qquad -\infty < x < \infty, \quad t > 0$$

Initial boundary condition$u\ (x,t) = f(x) \qquad -\infty < x < \infty.$

Suppose the Fourier transform of $u(x,t)$ is $U(k,t)$.

i.e. $F[u(x,t)] = U(k,t) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t) e^{i\,kx}\, dx$

Taking the Fourier transform of $u_t = \alpha \Delta_x u$ and summing that $u,\ u_x \to 0$, as $|x| \to \infty$ we get

$$U_t + \alpha k^2 U = 0.$$

Its solution is given by

$$U(k,t) = A(k)e^{-k^2\alpha t}$$

When $A(k)$ is an arbitrary function to be determined from the initial condition as follows.

$$U(k,0) = F[u(x,0)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,0)e^{-i\,kx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-i\,kx}\,dx = F(k).$$

Hence, $$U(k,t) = F(k)e^{-k^2\alpha t}$$

Hence by convolution theorem,

$$u(x,t) = f(x) \times F^{-1}(e^{-k^2\alpha t})$$

$$= \frac{1}{2\sqrt{\pi\alpha t}} \int_{-\infty}^{\infty} f(\xi)\exp\left(-\frac{(x-\xi)^2}{4\alpha t}\right)d\xi$$

$$= \int_{-\infty}^{\infty} G(x-\xi,\alpha t)\,f(\xi)d\xi$$

is called the Green's function for heat transfer in infinite domain of road.

Now we consider the case $\alpha = 1$ and

$$f(x) = \begin{cases} 0 & x < 0 \\ a & x > 0 \end{cases}$$

$$u(x,t) = \frac{a}{2\sqrt{\pi t}} \int_{0}^{\infty} exp\left(-\frac{(x-\xi)^2}{4\alpha t}\right)d\xi$$

Put $\eta = \left(\frac{\xi - x}{2\sqrt{t}}\right)$

$$u(x,t) = \frac{a}{\sqrt{\pi}} \int_{-x/2\sqrt{t}}^{\infty} e^{-\eta^2}\,d\eta$$

$$= \frac{a}{\sqrt{\pi}}\left[\int_{-x/2\sqrt{t}}^{0} e^{-\eta^2}\,d\eta + \int_{0}^{\infty} e^{-\eta^2}\,d\eta\right]$$

$$= \frac{a}{2} + \frac{a}{\sqrt{\pi}} \int_0^{x/2\sqrt{t}} e^{-\eta^2}\, d\eta$$

$$= \frac{a}{2}\left[1 + erf\left(\frac{x}{2\sqrt{t}}\right)\right]$$

Where erf is the error function.

5.4 EXISTENCE AND UNIQUENESS THEOREM FOR INITIAL VALUE PROBLEM IN SEMI-INFINITE DOMAIN:

Now using Green's function for the problem of heat flowin semi-infinite rod, the position of the rod coincide with X-axis and rod is homogeneous. Also heat is uniformly supply to it in cross section area in time $t > 0$. A boundary condition at the finite end $x = 0$ and other end condition ∞. The initial condition on the temperature distribution $u(x, t)$ can be described by,

$$u(x, 0) = u_0(x) \qquad 0 < x < \infty$$

There are various boundary conditions that can be prescribed at the end $x = 0$.

Ist condition: The temperature is prescribed at $x = 0$ for all time $u(0, t) = f(t)$.

IInd condition: The flux of heat across $x = 0$ is prescribed for all time. i.e. $u_x(0, t) = g(t)$.

IIIrd condition: The flux of heat across $x = 0$ is propositional to the difference between the temperature at $x = 0$ and the surrounding medium. i.e.

$$u_x(0, t) + \alpha\, u(0, t) = C$$

We define a function $U(x, t)$ called the derived singularity function

$$U(x, t) = \frac{x}{t} k(x, t) = -2k_x(x, t). \quad t > 0,\ \ x > 0$$

The properties of $U(x, t)$ are given by

$\int_0^{\infty} U(x, t)dt = 1and\lim_{x\to 0^+} \int_0^{c} U(x, t)dt = 1\ ,\ c > 0$

Using these propertieswe get the relation,

$$\int_0^{\infty} U(x, t)dt = \mathrm{erf}c\left(2x\sqrt{c}\right) = \frac{2}{\sqrt{\pi}} \int_{x/2\sqrt{c}}^{\infty} e^{-y^2}\, dy$$

Where the complementary error function, $erfc\ (\xi)$ is defined by $\mathrm{erf}c\ (\xi) = 1 - \mathrm{erf}(\xi)$.

Theorem : If $f(t) \in C$ for $0 < t < \infty$ and $f(0) = 0$ than $u(x,t) = \int_0^t U(x, t-y) f(y) dy$ satisfies the heat equation in $0 < x < \infty,\ o < t < \infty$ and

Case I) $\lim_{x \to 0^+} u(x,t) = f(t) \qquad 0 < x < c$

Case II) $\lim_{t \to 0^+} u(x,t) = 0 \qquad 0 < x < \infty$

Both the case is uniformly continuous. Where c is a constant.

Proof: $U(x,t)$ satisfies the heat equations for $t > 0$.

We shall first prove case II:

Let $\in > 0$ such that we can determine a δ,

$|f(t)| < t \qquad 0 < t < \delta.$

Hence for $0 < t < \delta$,

$$u(x,t) = \left| \int_0^t U(x, t-y) f(y) dy \right|$$

$$\leq \int_0^t U(x,\xi) |f(t-\xi)| d\xi < \in \int_0^\infty U(x,\xi) d\xi = \in$$

For a specific value of δ, its holds for all x. Hence proved.

To prove case I:

Let $\in > 0$ such that we can determine a η which is independent to t on $0 \leq t \leq c$ and $\quad 0 < x < \eta$,

$$|u(x,t) - f(t)| < \in$$

For any $\in > 0$, let δ_1 be such that $|f(\xi)| < \frac{\in}{2}$

Whenever $|\xi| < \delta_1$ due to uniform continuity of $f(t)$ in $0 \leq t \leq c$,

We can choose δ_1, so that we also have

$|f(t-\xi) - f(t)| < \frac{\in}{2}$ Whenever $|\xi| < \delta_1,\ 0 \leq t \leq c$.

Now divide $0 \leq t \leq c$ into two sub-interval $0 \leq t \leq \delta_1$ and $\delta_1 \leq t \leq c$, we have

$|u(x,t) - f(t)| < \in$ for all $x > 0$.

In $\delta_1 \leq t \leq c$,

$$u(x,t) - f(t) = \int_0^t U(x,t-y)f(y)dy - \int_0^\infty U(x,y)f(t)dy$$

$$= \int_0^t U(x,\xi)f(t-\xi)d\xi - \int_0^\infty U(x,y)f(t)dy$$

$$= \int_0^t U(x,y)[f(t-y) - f(t)]dy - f(t)\int_t^\infty U(x,y)dy$$

$$\therefore \; |u(x,t) - f(t)| < \frac{\in}{2}\int_0^\infty U(x,y)dy + 3M\int_{\delta_1}^\infty U(x,y)dy$$

Where $M = \text{Sup}_{0 \le t \le c}|f(t)|$.

We know that $\lim_{x \to 0^+} \int_0^c U(x,t)dt = 1$, $c > 0$ and $\int_{\delta_1}^\infty U(x,y)dy < \frac{\in}{6M}$

But for sufficiently small x,we have the result

$|u(x,t) - f(t)| < \in$for$0 < x < \eta$ and $0 \le t \le c$

Hence the solution is given by

$$u(x,t) = \int_0^t U(x,t-y)f(y)dy.$$

This result gives solution of I[st] condition that the problem of finding the temperature of semi-infinite rod whose initial temperature $t = 0$ is everywhere zero and whose temperature at finite end $x = 0$ is prescribed by all the t as $f(t)$.

For II[nd]condition : The temperature of semi-infinite rod arises when initial temperature $t = 0$ is everywhere zero and whose temperature at finite end $x = 0$ is prescribed by all the t as $g(t)$ and $g(0) = 0$.

Here $u(x,t) = -2\int_0^t K(x,t-y)g(y)dy$ is the solution for $0 < x < \infty$, $0 < t < c$ where c is a constant. Here $K(x,t)$ is satisfies the heat equation.

We have $\lim_{t \to 0^+} u(x, \;\; 0^+) = 0$ for $0 \le x \le \infty$

Also $u_x(x,t) = -2\int_0^t K_x(x,t-y)g(t)dy = \int_0^t U(x,t-y)g(y)dy$

$u_x(x,t) \to g(t)$as$x \to 0^+, 0 < t < c$.

IIIrd condition: The flux of heat across $x = 0$ is propositional to the difference between the temperature at $x = 0$ and the surrounding medium. i.e.

$$u_x(0, t) + \alpha\, u(0, t) = C$$

Here $u(x, t)$ is the solution of the diffusion equation satisfying the condition

$$u(x, 0) = 0, \qquad x \geq 0; \quad u_x(0, t) + \alpha u(0, t) = \emptyset(t), \qquad t \geq 0\,.$$

Where α is positive constant and $\emptyset(t)$ is continuous.

Therefore, $u_x - \alpha u = \int_0^t \emptyset(t)\ U(x, t - \tau) d\tau$

Till that we have consider initial temperature distribution is Zero.

Now we have to consider the case of non-zero initial distribution and zero boundary condition at $x = 0$. i.e.

$$u(x, 0) = u_0(x) \qquad 0 \leq x < \infty$$

$$|u_0(x)| < Me^{Ax^2}$$

And one of the following boundary conditions:

IVth condition: $u(0, t) = 0 \qquad 0 < t < \infty.$

Vth condition: $u_x(0, t) = 0 \qquad 0 < t < \infty.$

VIth condition: $u_x(0, t) - \alpha u(0, t) = 0 \quad 0 < t < \infty.$

In IVth condition we extend $u_0(x)$ as an odd function, we get

$u_0(x) = -u_0(-x)$for$x < 0$.

Then the solution of the initial value problem in $-\infty < x < \infty$ is given by

$$u(x, t) = \int_{-\infty}^{\infty} K(x - y, t) u_0(y) dy$$

$$= \int_0^{\infty} [K(x - y, t) - K(x + y, t)]\, u_0(y) dy$$

Since $K(0 - y, t) - K(0 + y, t) = 0$ the boundary condition $u(0^+, t) = 0$ is automatically satisfies.

In Vth condition we extend $u_0(x)$ as an even function of x for $x < 0$,

$u_0(x) = u_0(-x)$for$x < 0$.

Then the solution is given by

$$u(x,t) = \int_0^{\infty} [K(x-y,t) - K(x+y,t)]\, u_0(y) dy$$

Since $K_x(0-y,t) - K_x(0+y,t) = 0$ the boundary condition $u(0^+,t) = 0$ is automatically satisfies.

In VIth condition we extend $u_0(x)$ as

$u_0(-x) = u_0(x) + 2\alpha\, e^{\alpha x} \int_0^x e^{-\alpha\xi} u_0(\xi) d\xi$ for $x \geq 0$.

In order to satisfies boundary condition

$$u_x(0,t) - \alpha u(0,t) = 0 \quad 0 < t < \infty.$$

Than the solution is given by

$$u(x,t) = \int_0^{\infty} [K(x-y,t) - K(x+y,t)]\, u_0(y) dy + 2\alpha \int_0^{\infty} K(x+y,t) e^{\alpha y} \int_0^{x} e^{-\alpha\xi} u_0(\xi) \;\; dy\, d\xi$$

A linear combination of solution of one of the problem condition I, II, and III with $u_0(x) = 0$ and one of the conditions IV, V, VI lead us to the general mixed initial boundary value problems for the heat equation for semi-infinite rod.

5.5 ONE DIMENSIONAL HEAT EQUATION:

We now begin to study finite difference method for time-dependent PDE where variations in space are related to variation in time.

The diffusion equation is of the form

$$u_t = k\Delta_m u$$

Taking $k = 1$ by suitable change in x or t.This is called as heat equation.

The one-dimensional diffusion equation for $u(x,t)$ is

$$u_t = k u_{xx}$$

Where k is diffusion constant.

Now solving heat equationby separation of variable method

Let $\quad u(x,t) = X(x)\, T(t) \neq 0 \qquad\qquad (*)$

i.e. $u = XT$

Differentiate the separated solution (*) once with respect to t and twice with respect to x and substitute these derivatives into the PDE. We then obtain

$$XT' = kX''T$$

Now, usingthe separation of variables step.

$$\frac{X''}{X} = \frac{T'}{kT} = -\alpha^2$$

Where α is positive constant.

Now we get following ODE's

i) $X'' + \alpha^2 X = 0 \qquad 0 < x < L$
ii) $T' + \alpha^2 kT = 0 \qquad t > 0$

Solution of (i) is $X = A\cos\alpha x + B\sin\alpha x$

Solution of (ii) is $T = ce^{-\alpha^2 kt}$

Therefore the general solution is

$u = (A\cos\alpha x + B\sin\alpha x)(ce^{-\alpha^2 kt})$

If the boundary conditions$u(0,t) = 0 \qquad t \geq 0$

$u(l,t) = 0 \qquad t \geq 0$

With initial condition $u(x,0) = f(x) \qquad 0 \leq x \leq l.$

Then $X(0) = 0 \implies A = 0.$

$X(l) = 0 \implies B\sin\alpha l = 0asB \neq 0$

$$\therefore \quad \sin\alpha l = 0$$

Thus $\alpha = \frac{n\pi}{l}$ for $n = 1,2,3, \dots..$

Substituting these α in solution we get

$$X_n(x) = B_n \sin\left(\frac{n\pi x}{l}\right)$$

$$T_n(t) = C_n e^{-\left(n\pi/l\right)^2 kt}$$

Hence the non-trivial solution of the heat equation which satisfies the two boundary condition.

$$u_n(x,t) = X_n(x)T_n(t)$$

$u_n(x,t) = a_n e^{-(n\pi/l)^2 kt} \sin\left(\frac{n\pi x}{l}\right)$ for $n = 1, 2, 3, \ldots\ldots$

Where $a_n = B_n C_n$ is an arbitrary constant.

By the principle of superposition implies that any linear combination

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

$$= \sum_{n=1}^{\infty} a_n e^{-(n\pi/l)^2 kt} \sin\left(\frac{n\pi x}{l}\right)$$

Which satisfies the initial condition if

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right).$$

This hold true if $f(x)$ can be represented by a Fourier Sine series with coefficient

$a_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx.$

Hence,

$$u(x,t) = \sum_{n=1}^{\infty} \left[\frac{2}{l}\int_0^l f(\tau) \sin\left(\frac{n\pi x}{l}\right) d\tau\right] e^{-(n\pi/l)^2 kt} \sin\left(\frac{n\pi x}{l}\right)$$

is the general solution of the heat conduction equation of the function f with respect to the eigenfunctions of the problem, and a_n, $n = 1, 2 \ldots$ are called Fourier coefficientsof the series.

Example 1: Solve the heat problem

$$u_t = u_{xx} \qquad 0 < x < \pi, \qquad t > 0$$

$$u(0,t) = u(\pi,t) = 0 \qquad t \geq 0$$

$$u(x,0) = f(x) = \begin{cases} x & 0 \leq x \leq \pi/2 \\ \pi - x & \pi/2 \leq x \leq \pi \end{cases}$$

Solution: The formal solution of heat equation is

$$u(x,t) = \sum_{n=1}^{\infty} \left[\frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx\right] e^{-(n\pi/l)^2 kt} \sin\left(\frac{n\pi x}{l}\right)$$

$$A_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx = \frac{2}{\pi}\int_0^\pi f(x)\sin nx\, dx$$

$$A_n = \frac{2}{\pi}\int_0^{\pi/2} x\sin nx\, dx + \frac{2}{\pi}\int_{\pi/2}^{\pi}(\pi - x)\sin nx\, dx$$

$$A_n = \frac{4}{\pi n^2}\sin\left(\frac{n\pi}{2}\right)$$

Thus the formal solution is

$$u(x,t) = \sum_{n=1}^{\infty}\left[\frac{4}{\pi n^2}\sin\left(\frac{n\pi}{2}\right)\right]e^{-(n)^2 t}\sin(nx)$$

But

$$\sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0 & n = 2m \\ (-1)^{m+1} & n = 2m-1 \end{cases}$$

Where $m = 1,2,3, \ldots..$

Therefore we can write solution as

$$\sum_{n=1}^{\infty} u_n(x,t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^{m+1}}{(2m-1)^2}\sin\left(\frac{(2m-1)\pi}{2}\right)\right]e^{-(2m-1)^2 t}\sin((2m-1)x)$$

Example 2: Solve the heat problem

$$u_t = 3u_{xx} \qquad 0 < x < 2,\ \ t > 0$$

$$u(0,t) = u(2,t) = 0 \qquad t > 0$$

$$u(x,0) = 50 \qquad 0 < x < 2.$$

Solution: Comparing with heat equation we get,

$$c = \sqrt{3}\ \ and\ L = 2$$

$$f(x) = 50 = \frac{200}{\pi} \sum_{n=0}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi x}{2}\right)$$

Since,

$$\alpha_n = \frac{cn\pi}{L} = \frac{\sqrt{3}\, n\pi}{2}$$

We obtained the solution

$u(x,t) = \frac{200}{\pi} \sum_{n=0}^{\infty} \frac{1}{n} e^{-3n^2\pi^2 t/4} \sin\left(\frac{n\pi x}{2}\right).$

5.6 MAXIMUM AND MINIMUM PRINCIPLE FOR THE HEAT EQUATION:

We shall prove the maximum and minimum properties of the heat equation. These properties can be used to prove uniqueness and continuous dependence on data of the solutions of these equations. To begin with, we shall first prove the maximum principle for the inhomogeneous heat equation (F □= 0).

Theorem: (The maximum principle) : Let $R: 0 \leq x \leq L, 0 \leq t \leq T$ be a closed region and let $u(x,t)$ be a solution of

$$u_t - \alpha^2 u_{xx} = F(x,t), \qquad (x,t) \in R \tag{I}$$

Which is continuous in the region R. if $F < 0$ in R, then $u(x,t)$ attains its maximum valus on $t = 0, x = 0$ or $x = L$ and not in the interior of the region or at $t = T$. If $F > 0$ in R, then $u(x,t)$ attains its minimum values on $t = 0, x = 0$ or $x = L$ and not in the interior of the region or at $t = T$.

Proof: We shall show that if a maximum or minimum occurs at an interior point $0 < x_0 < l$and $0 < t_0 < T$, then we will arrive at contradiction. Let us consider the following cases:

Case-I: first, consider the case with $F < 0$. Since $u(x,t)$ is continuous in a closed and bounded region in R, $u(x,t)$ must attain its maximum in R. Let (x_0, t_0) be the interior maximum point. Then we must have

$$u_{xx}(x_0,t_0) \leq 0, \qquad u_t(x_0,t_0) \geq 0 \tag{II}$$

Since $u_x(x_0,t_0) = 0 = u_t(x_0,t_0)$, we have $u_t(x_0,t_0) = 0$if $t_0 < T$.

If $t_0 = T$, the point $(x_0,t_0) = (x_0,T)$ is on the boundary of R, then we claim that

$u_t(x_0,t_0) \geq 0.$

As u may be increasing at (x_0, t_0). Substituting II in I , we find that the left side the equation I is non-negative while the right side is strictly negative. This leads to a contradiction and hence, the maximum must be assumed on the initial line or on the boundary.

Case –II : Consider the case with $F > 0$. Let there be an interior minimum point (x_0, t_0) in R such that

$$u_{xx}(x_0, t_0) \geq 0, \qquad u_t(x_0, t_0) \leq 0 \qquad \text{(III)}$$

Note that the inequalities III is same as II with the signs reversed. Again arguing as before, this leads to a contradiction, hence the minimum must be assumed on the line or on the boundary.

Note : when $F = 0$i.e. for homogeneous equation, the inequalities II at a maximum or III at a minimum do not leads to a contradiction when they are inserted into I as u_{xx} andu_t may both vanish at (x_0, t_0).

Below, we present a proof of the maximum principle for the homogeneous heat equation.

Theorem : (The maximum principle): Let $R: 0 \leq x \leq L, 0 \leq t \leq T$ be a closed region and let $u(x, t)$ be a solution of

$$u_t = \alpha^2 u_{xx}, \qquad (x, t) \in R \qquad \text{(IV)}$$

Which is continuous in the closed region *R*. The maximum and minimum values of *u(x, t)*are assumed on the initial line *t = 0*or at the points on the boundary *x = 0*or *x = L*.

Proof. Let us introduce the auxiliary function

$$v(x, t) = u(x, t) + \in x^2 \qquad \text{(V)}$$

Where $\in > 0$ is a constant and *u*satisfies IV. Note that $v(x, t)$is continuous in *R* and

hence it has a maximum at some point (x_1, t_1) in the region *R*.

Assume that (x_1, t_1) is an interior point with $0 < x_1 < L$and$0 < t_1 < T$. Then we

find that

$$v_t(x_1, t_1) \geq 0, \qquad v_{xx}(x_1, t_1) \leq 0 \qquad \text{(VI)}$$

Since *u* satisfies IV, we have

$$v_t - \alpha^2 v_{xx} = u_t - \alpha^2 u_{xx} - 2\alpha^2 \in = -2\alpha^2 \in < 0 \qquad \text{(VII)}$$

Substituting VI into IV and using VII now leads to

$$0 \leq v_t - \alpha^2 v_{xx} < 0,$$

which is a contradiction since the left side is non-negative and the right side is strictly

negative. Therefore, $v(x,t)$assumes its maximum on the initial line or on the boundary

since$v$satisfies I with F <0.

Let$M = \max\{u(x,t)\}$ on $t = 0, x = 0$ and $x = L$.

i.e. $M$is the maximum value of u on the initial line and boundary lines. Then

$$v(x,t) = u(x,t) + \in x^2 \leq M + \in L^2, \quad \text{for} 0 \leq x \leq L, 0 \leq t \leq T \qquad \text{(VIII)}$$

Since v has its maximum on $t = 0$, $x = 0$, or $x = L$, we obtain

$$u(x,t) = v(x,t) - \in x^2 \leq v(x,t) \leq M + \in L^2 \qquad \text{(IX)}$$

Since □ is arbitrary, letting □ → 0, we conclude that

$$u(x,t) \leq M \text{for all } (x,t) \in R, \qquad \text{(X)}$$

Hence proof.

As a consequence of the maximum principle, we can show that the heat flow problem has

a unique solution and depend continuously on the given initial and boundary data.

Theorem : (uniqueness) Let $u_1(x,t)$ and $u_2(x,t)$ be the solution of the following problem

$$\begin{aligned} u_t &= \alpha^2 u_{xx} \quad 0 < x < L,\ t > 0, \\ u(0,t) &= g(t),\ u(L,t) = h(t), \qquad \text{(XI)} \\ u(x,0) &= f(t), \end{aligned}$$

Where $f(t), g(t)$ and $h(t)$ are given function. then$u_1(x,t) = u_2(x,t)$, for all $0 \leq x \leq L$ and $t \geq 0$.

Proof: Let $u_1(x,t)$ and $u_2(x,t)$ be the solution of the given XI problem.

Set $w(x,t) = u_1(x,t) - u_2(x,t)$. Then w satisfies

$$\begin{aligned} u_t &= \alpha^2 w_{xx} \quad 0 < x < L,\ t > 0, \\ w(0,t) &= ,\ w(L,t) =, \qquad \text{(XI)} \\ w(x,0) &= 0, \end{aligned}$$

By the maximum principal we have,

$w(x,t) \leq 0, \Longrightarrow\ u_1(x,t) \leq u_2(x,t)$for all $0 \leq x \leq L$ and $t \geq 0$.

Similarly we have $\bar{w} = u_2 - u_1$, for that we get

$\Longrightarrow\ u_2(x,t) \leq u_1(x,t)$for all $0 \leq x \leq L$ and $t \geq 0$

Therefore we have

$\Longrightarrow\ u_1(x,t) = u_2(x,t)$for all $0 \leq x \leq L$ and $t \geq 0$

Hence it has unique solution.

5.7 ONE DIMENSIONAL WAVE EQUATION:

We write the wave equation as

$$u_{tt} = C^2 \nabla u \text{for} -\infty < x < \infty.$$

This is the simplest second order equation.

$$u_{tt} - C^2 \nabla u = \left(\frac{\partial}{\partial t} - C\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + C\frac{\partial}{\partial x}\right) = 0.$$

This means that starting from a function $u(x,t)$ you compute $u_t + Cu_x$ call the result for $v(x,t)$ than you compute $v_t - Cv_x$ and you get zero function.

The general solution is

$$u(x,t) = f(x + Ct) + g(x - Ct). \qquad \text{(XII)}$$

Where f and g are two arbitrary function of a single variable.

Initial value problem is to solve the wave equation

$$u_{tt} = C^2 \nabla u \text{for} -\infty < x < \infty.$$

With the initial conditions $u(x,0) = \emptyset(x)$, $u_t(x,0) = \varphi(x)$

Where $\emptyset$ and φ are arbitrary functions of x. There is one and only one solution of this problem.

Proof: Let if $\emptyset(x) = \sin x$ and $\varphi(x) = 0$ then $u(x,t) = \sin x\ \cos Ct$.

The solution of above IVP is easily found from the general solution formula (XII) replacing$t = 0$, we get

$$\emptyset(t) = f(x) + g(x) \qquad \text{(XIII)}$$

Then using the chain rule we differentiate (XII) with respect to t and put $t = 0$, to get

$$\varphi(x) = Cf'(x) - Cg'(x) \qquad \text{(XIV)}$$

Lets regards XIII and XIV as two equations for the two unknown functions $f$and g. To solve them change the name of variable as same neutral $x$to s.

Now we differentiate XIII and divide XIV by C to get

$\emptyset' = f' + g'$and$\frac{1}{C}\varphi = f' - g'$

Adding and subtracting the last pair of equations, we get

$f' = \frac{1}{2}\left(\emptyset' + \frac{\varphi}{C}\right)andg' = \frac{1}{2}\left(\emptyset' - \frac{\varphi}{C}\right)$

Integrating we get,

$$f(s) = \frac{1}{2}\emptyset(s) + \frac{1}{2C}\int_0^s \varphi + A$$

And

$$g(s) = \frac{1}{2}\emptyset(s) - \frac{1}{2C}\int_0^s \varphi + A$$

Where A and B are constant because of XIII we have A + B = 0. This tells us what f and g are in general formula XII. Substituting $s = x + Ct$ into f and $s = x - Ct$ into g we get

$$u(x,t) = \frac{1}{2}\emptyset(x+Ct) + \frac{1}{2C}\int_0^{x+Ct} \varphi + \frac{1}{2}\emptyset(x-Ct) - \frac{1}{2C}\int_0^{x-Ct} \varphi$$

This simplifies to

$$u(x,t) = \frac{1}{2}[\emptyset(x+Ct) + \emptyset(x-Ct)] + \frac{1}{2C}\int_{x-Ct}^{x+Ct} \varphi(s)ds$$

This is called D'Alembert's the solution for initial value problem of one dimension wave equation.

Example 3: Consider the Cauchy problem

$$u_{tt} = u_{xx} \qquad -\infty < x < \infty, \quad t > 0$$

With boundary condition:

$$u(x,0) = f(x) = \begin{cases} 0 & -\infty < x < -1 \\ x+1 & -1 \le x < 0 \\ 1-x & 0 \le x \le 1 \\ 0 & 1 < x < \infty \end{cases}$$

$$u_t(x,0) = g(x) = \begin{cases} 0 & -\infty < x < -1 \\ 1 & -1 \le x \le 1 \\ 0 & 1 < x < \infty \end{cases}$$

a) Evaluate $u(1, {}^1/_2)$.
b) Discuss the smoothness of the solution u.

Solution: a) Using D'Alembert formula of one dimension wave equation we get

$$u(x,t) = \frac{1}{2}[\emptyset(x+Ct) + \emptyset(x-Ct)] + \frac{1}{2C}\int_{x-Ct}^{x+Ct} \varphi(s)ds$$

$$u\left(1, \frac{1}{2}\right) = \frac{f\left(\frac{3}{2}\right) + f\left(\frac{1}{2}\right)}{2} + \frac{1}{2}\int_{1/2}^{3/2} g(s)ds$$

Since $\frac{3}{2} > 1$, it follows that $f\left(\frac{3}{2}\right) = 0$. On other hand $0 \le \frac{1}{2} \le 1$, therefore $f\left(\frac{1}{2}\right) = \frac{1}{2}$.

$$\int_{1/2}^{3/2} g(s)ds = \int_{1/2}^{1} 1\, ds = \left[1 - \frac{1}{2}\right] = \frac{1}{2}.$$

Hence

$$u\left(1, \frac{1}{2}\right) = \frac{0 + \frac{1}{2}}{2} + \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

b) The solution is not classical, since $u \notin C^1$. Yet $u$is a generalized solution of the problem. Note that although g is not continuous, nevertheless the solution $u$is a continuous function. The singularities of the solution propagate along characteristics that intersect the initial line $t = 0$ at the singularities of the initial conditions. Theseare exactly the characteristics $x \pm t = -1, 0, 1$. Therefore, the solution is smooth in a neighborhood of the point $\left(1, \frac{1}{2}\right)$ which does not intersect thesecharacteristics.

Method of Separation of Variables for one-dimensional Wave equation:

PDE $u_{tt} - c^2 u_{xx} = 0 \quad 0 \le x \le L, \qquad t > 0$

Boundary condition: $u(0,t) = 0 \qquad t > 0$

To obtained separation of variables solution we assume

$$XT'' = c^2TX''$$

i.e. $\frac{X''}{X} = \frac{T''}{c^2T} = k$

Case-I: When $k > 0$, taking $k = \alpha^2$ we get

$$X'' - \alpha^2X = 0$$

$$T'' - c^2\alpha^2T = 0$$

The solution in the form

$$X = c_1e^{\alpha x} + c_2e^{-\alpha x}$$

$$T = c_3e^{c\alpha t} + c_4e^{-c\alpha t}$$

Therefore, $u(x,t) = (c_1e^{\alpha x} + c_2e^{-\alpha x})(c_3e^{c\alpha t} + c_4e^{-c\alpha t})$

Now using boundary condition's

$$u(0,t) = (c_1 + c_2)(c_3e^{c\alpha t} + c_4e^{-c\alpha t})$$

This implies that $c_1 + c_2 = 0$, also

$u(L,t) = 0 \Longrightarrow c_1e^{\alpha L} + c_2e^{-\alpha L} = 0.$

This gives non-trivial solution if and only if

$$\begin{vmatrix} 1 & 1 \\ e^{\alpha L} & e^{-\alpha L} \end{vmatrix} = 0$$

$\Longrightarrow e^{2\alpha L} = 1$ or $\alpha L = 0$

This implies that $\alpha = 0$, since L cannot be zero which is against the case-I assumption.

Hence solution is not acceptable.

Case-II: When $k = 0$, we get

$X'' = 0$ and $T'' = 0.$

Their solutions are formed to be

$X = Ax + B, \qquad T = Ct + D.$

Therefore required solution of the PDE

$$u(x,t) = (Ax + B)(Ct + D)$$

Using boundary conditions we get,

$u(0,t) = 0 = B(Ct + D) \Longrightarrow B = 0.$

$$u(L,t) = 0 = AL(Ct + D) \Longrightarrow A = 0.$$

Hence only trivial solution is possible.

Case-III: When $k < 0$, taking $k = -\alpha^2$, we get

$$X'' + \alpha^2 X = 0; \qquad T'' + c^2\alpha^2 T = 0.$$

Their general solution is given by

$$u(x,t) = (c_1 \cos \alpha x + c_2 \sin \alpha x)(c_3 \cos c\alpha t + c_4 \sin c\alpha t)$$

Using the boundary condition :$u(0,t) = 0$ we get $c_1 = 0$.

$u(L,t) = 0$we get $\sin \alpha L = 0 \Longrightarrow \alpha_n = \frac{n\pi}{L}, \quad n = 1,2,3, \dots \dots \dots$

Hence the possible solution

$$u_n(x,t) = \sin\left(\frac{n\pi x}{L}\right)\left(A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)\right) \qquad n = 1,2,3, \dots \dots \dots \dots$$

Using superposition principle, we get

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left(A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)\right)$$

The initial condition gives

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$$

Which is half-range of Fourier sine series, where

$$A_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

Also we get

$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)\left(\frac{n\pi c}{L}\right)$$

Which is also half-range of Fourier sine series, where

$$B_n = \frac{2}{n\pi c}\int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

Hence the required solution is obtained.

Example 4: Solve the one-dimensional wave equation $u_{tt} = 16u_{xx}$ for $0 < x < 2, \quad t > 0$.

The boundary conditions: $u(0,t) = u(2,t) = 0$.

The initial conditions: i) $u(x,0) = 6\sin\pi x - 3\sin 4\pi x$. ii) $u_t(x,0) = 0$.

Solution: The general solution of 1-dimensional wave equation is given by

$$u(x,t) = (c_1\cos\alpha x + c_2\sin\alpha x)(c_3\cos c\alpha t + c_4\sin c\alpha t)$$

Using boundary condition :$u(0,t) = 0$ for all t gives

$$c_1(c_3\cos c\alpha t + c_4\sin c\alpha t) = 0$$

Which implies that $c_1 = 0$.

$u(2,t) = 0$ for all t gives

$$c_2\sin\alpha x\,(c_3\cos c\alpha t + c_4\sin c\alpha t) = 0$$

For non-trivial solution $\sin 2\alpha = 0 \Longrightarrow \alpha = \frac{n\pi}{2}$ for some integer n.

$$u_n(x,t) = \sin\left(\frac{n\pi x}{2}\right)(A_n\cos(2n\pi t) + B_n\sin(2n\pi t))$$

Now insert the initial condition $u_t(x,0) = 0$ for all $0 < x < 2$.

$$u_t = \sin\left(\frac{n\pi x}{2}\right)(-2n\pi A_n\sin(2n\pi t) + 2n\pi B_n\cos(2n\pi t))$$

$$u_t(x,0) = \sin\left(\frac{n\pi x}{2}\right)(2n\pi B_n) = 0 \Longrightarrow B = 0.$$

Finally using the initial condition $u(x,0) = 6\sin\pi x - 3\sin 4\pi x$

We get $u(x,t) = \sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{2}\right)$

Hence we get,

$$u(x,t) = 6\sin\pi x\cos 4\pi t - 3\sin 4\pi x\cos 16\pi t.$$

5.8 LETS SUM UP:

In this chapter we have learnt the following:

- Definition diffusion equation.
- 1-dimensional heat equation and its solution by separation of variable method:

- Existence and Uniqueness theorem for initial value problem for infinite and semi-infinite domain.
- Maximum and Minimum principle for heat equation.
- 1- dimensional wave equation and its solution by separation of variable method

5.9 UNIT END EXERCISE:

1. Let $u(x,t)$ be the solution of the Cauchy problem

$$u_{tt} = 9u_{xx} \qquad -\infty < x < \infty, \quad t > 0$$

$$u(x,0) = f(x) = \begin{cases} 1 & |x| \le 2 \\ 0 & |x| > 2 \end{cases}$$

$$u_t(x,0) = g(x) = \begin{cases} 1 & |x| \le 2 \\ 0 & |x| > 2 \end{cases}$$

   a) Evaluate$u(0, \frac{1}{2})$.
   b) Discuss the large time behavior of the solution.
   c) Find the maximum value of $u(x,t)$ and point when this maximum is achieved.
   d) Find all the point when $u \in C^2$.

2. Obtain the solution of the wave equation

$$u_{tt} = c^2 u_{xx}$$

Under the following conditions:

i) $u(0,t) = u(2,t) = 0$.
ii) $u(x,0) = sin^2\left(\frac{\pi x}{2}\right)$.
iii) $u_t(x,0) = 0$.

3. Solve the following heat problem:

$$u_t = \frac{1}{4}u_{xx} \qquad 0 < x < 1, \quad t > 0.$$
$$u_x(0,t) = u_x(1,t) = 0 \qquad t > 0$$
$$u(x,0) = 100x(1-x), \qquad 0 < x < 1$$

4. Use the maximum/minimum principle to show that the solution u of the problem:

$$u_t = u_{xx} \qquad 0 < x < \pi, \quad t > 0.$$
$$u_x(0,t) = u_x(1,t) = 0 \qquad t > 0$$
$$u(x,0) = \sin x + \frac{1}{2}\sin 2x, \qquad 0 < x < \pi$$

Satisfies $0 \le u(x,t) \le \frac{3\sqrt{3}}{4}, \quad t > 0$.

5. Solve the one-dimensional wave equation $u_{tt} = 4u_{xx}$ for $0 < x < 1, \quad t > 0$.

$$u_x(0,t) = u_x(1,t) = 0 \qquad t \geq 0$$
$$u(x,0) = f(x) = cos^2 \pi x \qquad 0 \leq x \leq 1.$$
$$u_t(x,0) = g(x) = sin^2 \pi x \cos \pi x \qquad 0 \leq x \leq 1.$$

6. Prove that the solution we found by separation of variables for the vibration of a free
string can be represented as a superposition of a forward and a backward wave.
7. Show that the solution of the 1-dimensional wave problem if it is exists, is unique.
8. State and prove maximum and minimum principle.

5.10 REFERENCE

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