1

INTRODUCTION TO COMPLEX NUMBER SYSTEM

Unit Structure

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1.1. Introduction

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1.0 OBJECTIVES

After going through this unit you shall come to know about

- The field of complex numbers denoted by $\mathbb{C}$.
- Representations of complex numbers in polar forms .
- The Euclidean two dimensional plane $\mathbb{R}^2$ along with the point at infinity forms the extended complex plane.
- The extended complex plane is in one to one correspondence with the unit sphere in $\mathbb{R}^3$ and such a correspondence is known to be the stereographic projection.

1.1 INTRODUCTION

Numbers of the form $z = a + bi$, where a and b are real numbers and $\sqrt{i} = -1$ are called as Complex Numbers. The identities involving complex numbers lead to solutions to many problems in the theory of real valued functions .The wider acceptance of complex numbers is because of the geometric representation of complex numbers, which was fully developed and studied by Gauss. The first complete and formal definition of complex numbers was given by William Hamilton. We shall begin with this definition and then consider the geometry of complex numbers.

1.2 THE FIELD OF COMPLEX NUMBERS

A complex number z is an ordered pair (x, y) of real numbers. i.e. $z = (x, y) \quad x \in \mathbb{R},\ y \in \mathbb{R}$. Complex number system, denoted by $\mathbb{C}$ is the set of all ordered pairs of real numbers (i.e. $\mathbb{R} \times \mathbb{R}$) with the two operations of addition and multiplication ($\bullet$ or $\times$) which satisfy :

$$\left.\begin{array}{l}\text{(i)}\ (x_1, y_1) + (x_2, y_2) = (x_1 + x_2 + y_1 + y_2) \\ \text{(ii)}\ (x_1, y_1) . (x_2, y_2) = (x_1 x_2 - y_1 y_2,\ x_1 y_2 + x_2 y_1)\end{array}\right\} \forall (x_1, y_1), (x_2, y_2) \in \mathbb{C}$$

The word <u>ordered pair</u> means (x_1, y_1) and (y_1, x_1) are distinct unless $x_1 = y_1$.

Let $z = (x, y); \quad x \in \mathbb{R},\ y \in \mathbb{R}$. '$x$' is called <u>Real part</u> of a complex number z and it is denoted by $x = \text{Re}\, z$, (Real part of z) and 'y' is called <u>Imaginary part</u> of z and it is denoted by $y = \text{Im}\, z$.

Two complex numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ are said to be <u>equal</u> iff $x_1 = x_2$ and $y_1 = y_2$ i.e. real part and imaginary part both are equal.

<u>About Symbol 'i'</u>:

The complex number $(0, 1)$ is denoted by 'i' and is called the imaginary number.

$i^2 = i . i = (0,1).(0,1)$

$= (0-1,\ 0+0)$ by property (ii) abov

$= (-1, 0)$

$\Rightarrow \quad \boxed{i^2 = -1}$

Similarly,

$i^3 = i^2 . i = (-1,0).(0,1) = (0-0,\ -1+0) = (0,-1)$

$\Rightarrow \quad \boxed{i^3 = -i}$

$i^4 = i^3 . i = (0,-1).(0,1) = (0+1,\ 0+0) = (1,0) \therefore \sqrt{3}$

$\Rightarrow \quad \boxed{i^4 = 1}$

Using this symbol i, we can write a complex number (x, y) as $x + iy$ (Since $x + iy = (x,0) + (0,1)(y,0) = (x,0) + (0,y) = (x,y)$ The complex number $z = (x, y)$ can be written as $z = x + iy$

Note: (The set of all complex numbers) $\mathbb{C}$ forms a field.

Propeties of complex numbers

Let $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$ and $z_3 = (x_3, y_3) \in \mathbb{C}$.

1) **Closure Law :** $z_1 + z_2 \in \mathbb{C}$ and $z_1 . z_2 \in \mathbb{C}$

2) **Commutative Law of addition :** $z_1 + z_2 = z_2 + z_1$

$$z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2,\ y_1 + y_2) = (x_2 + x_1,\ y_2 + y_1)$$
$$= (x_2, y_2) + (x_1, y_1) = z_2 + z_1$$

3) **Associative Law of addition :** $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$

$$z_1 + (z_2 + z_3) = (x_1, y_1) + \left[(x_2, y_2) + (x_3, y_3)\right]$$
$$= (x_1, y_1) + (x_2 + x_3,\ y_2 + y_3) \qquad = (x_1 + x_2 + x_3,\ y_1 + y_2 + y_3)$$
$$= (x_1 + x_2, y_1 + y_2) + (x_3 + y_3) = \left[(x_1, y_1) + (x_2, y_2)\right] + (x_3, y_3)$$
$$= (z_1 + z_2) + z_3$$

4) **Existence of additive Identity :** The Complex Number $0 = (0, 0)$ i.e. $z = 0 + 0i$ is called the identity with respect to addition.

5) **Existence of additive Inverse :**
For each complex number $z_1 \in \mathbb{C}$, $\exists$ a unique complex number $z \in \mathbb{C}$ s.t. $z_1 + z = z + z_1 = 0$ i.e. $z = -z_1$. The complex number z is called the additive inverse of z_1 and it is denoted by $z = -z_1$.

6) **Commutative law of Multiplication :** $z_1 . z_2 = z_2 . z_1$

$$z_1 . z_2 = (x_1, y_1).(x_2, y_2) = (x_1 \cdot x_2 - y_1 \cdot y_2, x_1 \cdot y_2 + x_2 \cdot y_1) \quad \ldots\ldots (1)$$

and

$$z_2 . z_1 = (x_2, y_2).(x_1, y_1) = (x_2 . x_1 - y_2\ y_1, x_2 . y_1 + x_1 . y_2)$$
$$= (x_1 . x_2 - y_2 . y_1,\ x_1 . y_2 + x_2 . y_1) = z_1 . z_2 \qquad \text{from (1)}$$

7) **Associative Law of Multiplication :** $z_1 .(z_2 . z_3) = (z_1 . z_2) . z_3$

$$z_1 .(z_2 . z_3) = (x_1, y_1)\left[(x_2, y_2).(x_3 . y_3)\right]$$
$$= (x_1, y_1)\left[x_2 . x_3 - y_2 . y_3, x_2 . y_3 + x_3 . y_2\right]$$
$$= \left[x_1(x_2 . x_3 - y_2\ y_3) - y_1(x_2 . x_3 - y_2 . y_3),\ x_1(x_2 . x_3 + x_3 . y_2) + y_1(x_2 . x_3 - y_2 . y_3)\right]$$

$$= (x_1 . x_2 . x_3 - x_1 . y_2 . y_3 - x_1 . x_3 . y_1 + y_1 . y_2 . y_3,\ \ x_1 . x_2 . x_3 + x_1 . x_3 . y_2$$
$$+ x_2 . x_3 . y_1 + y_1 . y_2 . y_3) \qquad (*)$$

$$(z_1 . z_2) . z_3 = \left[(x_1, y_1).(x_2, y_2)\right](x_3, y_3) = (x_1\, x_2 - y_1\, y_2,\ \ x_1\, y_2 + x_2\, y_1)(x_3, y_3)$$

$$= x_1\, x_2\, x_3 - x_3\, y_1\, y_2 - x_1\, x_2\, y_3 + y_1\, y_2\, y_3,\ \ x_1\, x_3\, y_2 + x_2\, y_1\, y_3 + x_1\, x_2\, x_3 - y_1\, y_2\, y_3$$

$$= z_1 (z_2 . z_3) \qquad \text{from } (*)$$

8) **<u>Existence of Multiplicative Identity</u> :** $z_1 . 1 = 1 . z_1 = z_1$

The complex number $1 = (1, 0)$ (i.e. $z = 1 + 0i$) is called the identity with respect to multiplication.

9) **<u>Existence of Multiplicative Inverse</u> :** For each complex number $z_1 \neq 0,$ there exists a unique complex number z in $\mathbb{C}$ s.t. $z_1 . z = z . z_1 = 1$ i.e. $z = \dfrac{1}{z_1}$ is called the multiplicative inverse of complex number z_1 and it is denoted by $z = \dfrac{1}{z_1}$ or z^{-1}.

Let $z = (x, y)$ and $z_1 = (x_1, y_1)$

$\therefore\ z_1 . z_2 = 1$

$\therefore\ (x, y)(x_1, y_1) = (1, 0) \quad \Rightarrow (xx_1 - yy_1,\ xy_1 + x_1 y) = (1, 0)$

$\Rightarrow\ x . x_1 - y . y_1 = 1$ ……..(i) and $x . y_1 + x_1 . y = 0$ ……...(ii)

Equation (ii) $\times x_1$ - Equation (i) $\times y_1$, we get

$$\begin{array}{rcccccc} & xx_1 y_1 & + & x_1^2 y & = & 0 \\ - & xx_1 y_1 & - & yy_1^2 & = & y_1 \\ \hline & - & & + & & - \end{array}$$

$$y\left(x_1^2 + y_1^2\right) = -\, y_1$$

$$\therefore\ \boxed{y = \frac{-\, y_1}{x_1^2 + y_1^2}} \qquad \text{(iii)}$$

Substitute equation (iii) in equation (ii) i.e. $x . y_1 + y . x_1 = 0$

$$\therefore\ x . y_1 = -\, y . x_1 = -\left(\frac{-\, y_1}{x_1^2 + y_1^2}\right) x_1 \Rightarrow x = \frac{x_1\, y_1}{x_1^2 + y_1^2} \times \frac{1}{y_1}$$

$$\boxed{x = \frac{x_1}{x_1^2 + y_1^2}}$$

$$\therefore\ z = \left(\frac{x_1}{x_1^2 + y_1^2}, \frac{-\, y_1}{x_1^2 + y_2^2}\right)$$

z is the multiplicative inverse of complex number $z_1 = (x_1, y_1)$.

10) Distributive Law :

$$z_1(z_2 + z_3) = z_1 . z_2 + z_1 . z_3$$

Subtraction: The difference of two complex Numbers $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ is defined as :

$$z_1 - z_2 = (x_1, y_1) - (x_2, y_2) = (x_1 - x_2, \ y_1 - y_2)$$

Division: It is defined by the equality $\frac{z_1}{z_2} = z_1 . z_2^{-1} \quad z_2 \neq 0$

$$= (x_1, y_1)\left(\frac{x_2}{x_2^2 + y_2^2}, \frac{-y_2}{x_2^2 + y_2^2}\right) = \left(\frac{x_1 . x_2 + y_1 . y_2}{x_2^2 + y_2^2}, \frac{-x_1 y_2 + x_2 y_1}{x_2^2 + y_2^2}\right)$$

Geometrical Representation of a Complex Number :

Consider a complex number $z = x + iy$.

∵ Complex number is defined as an ordered paired of real numbers.

i.e. $z = (x, \ y)$

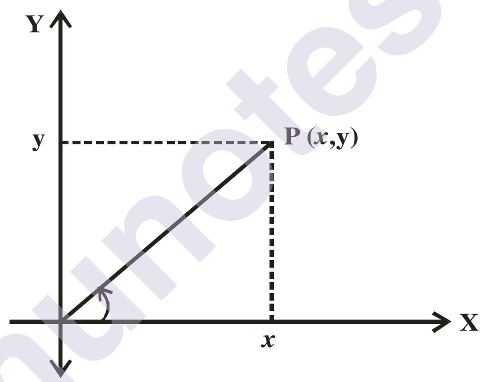


Fig. 1.1

This form of a complex number z suggest that z can be represented by point (say) *P* whose Cartesian co-ordinates are *x* and *y* referred (relating) to rectangular axis *X* and *Y*, usually called the Real and Imaginary axis respectively.

To each complex number there corresponds points in the plane and conversely, one and only one each point in the plane there exist one and only one complex number.

A plane whose points are represented by the complex numbers is called Complex Plane or Gaussian Plane or Argand Plane. Gauss was first who formulated that complex numbers are represented by points in a plane in 1799 then in 1806 it was done by Argand.

Vector Representation of a Complex Numbers :

If P is the point in the Complex Plane corresponding to complex number z can be considered as vector $\overrightarrow{OP}$ whose initial point is the origin 'O' and terminal point is $P = z = (x, y)$ as shown in the figure 1.2.

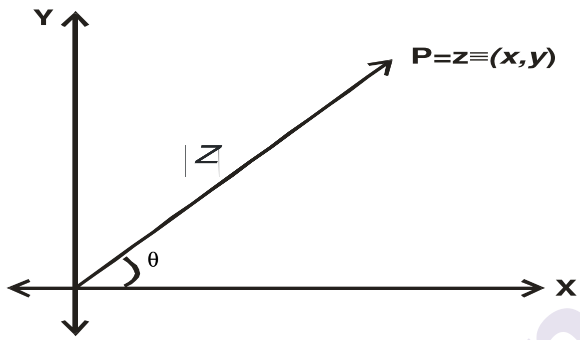


Fig 1.2

Conjugate:

If $z = x + iy \in \mathbb{C}$ then the complex number $x - iy$ is called the conjugate of a complex number z or complex conjugate and it is denoted by $\bar{z}$.

e.g. $z = 4 + 3i \quad \Rightarrow \bar{z} = 4 - 3i$

$w = 4 + 5e^{3i} \quad \Rightarrow \bar{w} = 4 + 5e^{-3i}$

Geometrically:

The complex conjugate of a complex number $z = (x, y)$ is the image or reflection of z in the real axis.

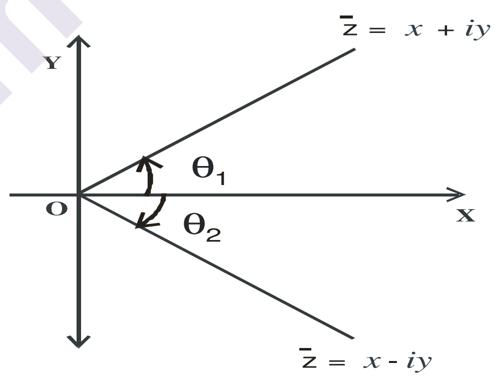


Fig 1.3

Let $z = x + iy$

$\because$ $x = \operatorname{Re} z$ and $z = \operatorname{Im} z$. $\therefore$ $x = \operatorname{Re} z = \dfrac{z + \bar{z}}{2}$ *and* $y = \operatorname{Im} z = \dfrac{z - \bar{z}}{2}$

Definition: The modulus or absolute value of a complex number $z = x + iy$ is defined by $|z| = \sqrt{x^2 + y^2}$.

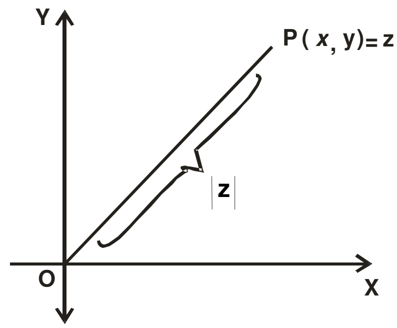


Fig 1.4

The distance between Two Complex Numbers :

Let $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$ in complex plane is given by

$$d(z_1, z_2) = |z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

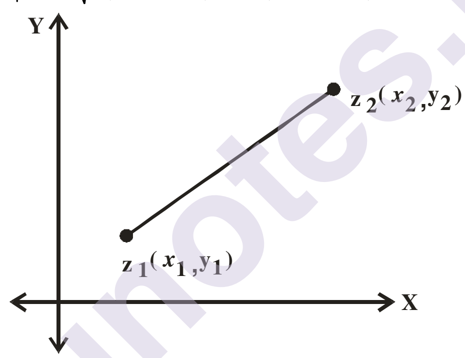


Fig 1.5

Polar form of a Complex Numbers :

If P is a point in the Complex Plane corresponding to complex number $z = x + iy = (x, y)$ and let (r, θ) be the polar co-ordinates of point (x, y) from figure 1.3, $x = r\cos\theta$ and $y = r\sin\theta$, where $r = \sqrt{x^2 + y^2}$ is called the modulus or absolute value of z (denoted by $|z|$ and $\boxed{\theta = \tan^{-1}\left(\frac{y}{x}\right)}$ is called the argument or amplitude of z (denoted by $\theta = \arg z$). Here θ is the angle between the two lines OP and the real axis (x- axis) .

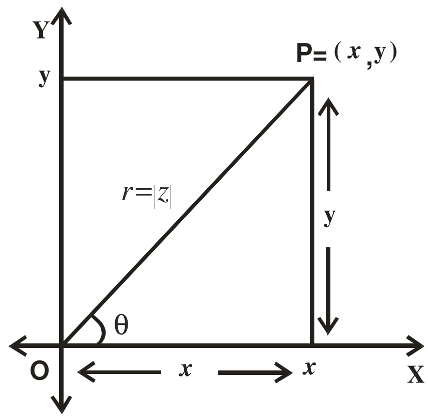


Fig 1.6

$\therefore\ z = x + iy$

$\therefore\ z = r\cos\theta + ir\sin\theta$

$\therefore\ z = r\,(\cos\theta + i\sin\theta)$

This form is called the polar form of a the complex number z.

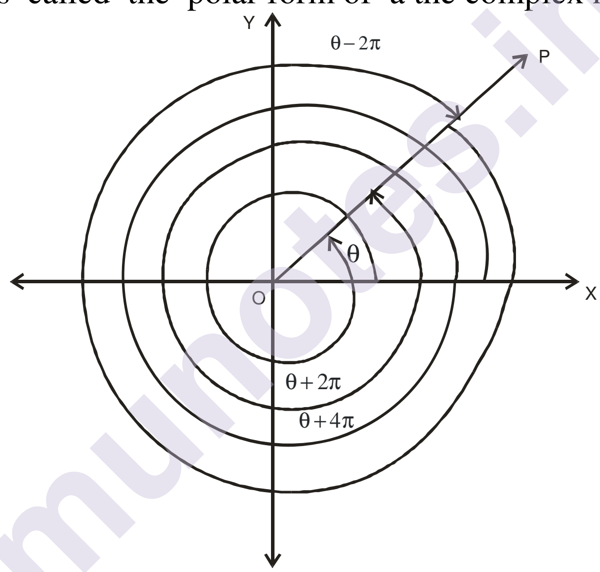


Fig 1.7

Any complex number $z \neq 0$ has an infinite number of distinct arguments.

Any two distinct arguments of z differ each other by an integral multiple of 2π.

If one of the value of argument of z is θ then $\arg z = \theta + 2n\pi$ where $n = 0, \pm 1, \pm 2, \ldots$

The value of θ which lies in the interval $-\pi < \theta \leq \pi$ or $(0 < \theta \leq 2\pi)$ is called the principal value of argument of z and it is denoted by $Arg\ z = \theta$.

The relation between $Arg\ z$ and $\arg z$ is given by $Arg\ z = \arg z + 2n\pi$ where $n = 0,\ \pm 1,\ \pm 2, \ldots$

<u>Exponential form of complex number</u>: A complex number can be written in the form of $z = re^{i\theta}$, where $r = \sqrt{x^2 + y^2}$ and $\boxed{\theta = \tan^{-1}\left(\frac{y}{x}\right)}$ .This is known as the exponential form.

(Note: $e^{i\theta} = \cos\theta + i\sin\theta$,known as Euler's Identity)

Note : 1) $\left| e^{i\theta} \right| = 1$, 2) $\left| e^{\frac{i500}{4}\pi} \right| = 1$

Solved Examples :

1. Let $z_1 = 1 + i, z_2 = 1 - 2i, z_3 = 1 + \sqrt{3}i$. Find i) $z_1.z_2$ ii) z_1/z_2 iii) $\bar{z}_2$ iv) $|z_1|$ v) $arg(z_1)$ vi) Express z_1 in polar and exponential form.

Solution:

i) $z_1 z_2 = (1+i)(1-2i) = 1 - 2i - i - 2i^2 = 1 - 2i - i + 2 = 3 - 3i$

ii) $z_1 / z_2 = \frac{1+i}{1-2i} = \frac{1+i}{1-2i} \times \frac{1+2i}{1+2i} = \frac{1+2i+i-2}{1-4i^2} = \frac{-1+3i}{5}$

iii) $\bar{z}_2 = 1+2i$

iv) $x_1 = 1, y_1 = 1$

$$\therefore |z_1| = \sqrt{x^2 + y^2} = \sqrt{1^2 + 1^2} = \sqrt{2}$$

v) $x_1 = 1, y_1 = 1$

$$\therefore \theta = \tan^{-1}\left(\frac{y_1}{x_1}\right) = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$$

vi) $z_1 = r_1(\cos\theta_1 + \sin\theta_1)$

$$\therefore z_1 = \sqrt{2}\left(\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\right)$$

2. Find the principal value of $\arg 'i'$

$\theta = Arg\ z \qquad -\pi < \theta \le \pi$

$$Arg\ i = \theta = \tan^{-1}\left(\frac{y}{x}\right) \{\because z = i \Rightarrow z = (o.x + iy)\}$$

$$= \tan^{-1}\left(\frac{1}{0}\right) = \tan^{-1}(\infty) = \frac{\pi}{2}$$

3. Find the principal value of $\arg(1+i)$

$\because z = 1 + i = (x + iy)$

$\therefore x = 1,\ y = 1$

$$Arg\ z = \theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{1}{1}\right) = \boxed{\frac{\pi}{4}}$$

4. Express the Complex Number $z=1+\sqrt{3}\,i$ in polar form.

Solution : $\because\ z=r(\cos\theta+i\sin\theta)$ (1)

and $\because\ z=x+iy=1+\sqrt{3}\,i$

$\therefore\ x=1$ and $y=\sqrt{3}$

$\because\ r=\sqrt{x^2+y^2}=\sqrt{1+3}=\sqrt{4}\ \therefore\ \boxed{r=2}$

$\because\ \theta=\tan^{-1}\left(\frac{y}{x}\right)=\tan^{-1}\left(\frac{\sqrt{3}}{1}\right)=\tan^{-1}\left(\sqrt{3}\right)=\tan^{-1}\left(\tan\frac{\pi}{3}\right)$

$\therefore\ \boxed{\theta=\frac{\pi}{3}}\quad \therefore\ z=2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$

Results : $\tan\left(45^\circ\right)=\tan(180+45)=1$

$\tan\left(-45^\circ\right)+\tan(180-45)=-1$

$\tan\left(60^\circ\right)=\tan(180+60)=\sqrt{3}$

$\tan\left(-60^\circ\right)=\tan\left(180-60^\circ\right)=-\sqrt{3}$

5. Express the Complex Number $z=-1+i$ in polar form

Solution : $\because\ z=r(\cos\theta+i\sin\theta)$

$\because\ z=-1+i$ (given)

Comparing with $z=x+iy$

$\therefore\ x=-1,\ y=1$

$\therefore\ r=\sqrt{x^2+y^2}=\sqrt{(-1)^2+(1)^2}=\sqrt{1+1}=\sqrt{2}$

$\because\ \theta=\tan^{-1}\left(\frac{y}{x}\right)=\tan^{-1}\left(\frac{1}{-1}\right)=\tan^{-1}(-1)=\tan^{-1}\left[\tan\left(\pi-\pi/4\right)\right]$

$=\pi-\frac{\pi}{4}=\frac{3\pi}{4}$

$z=\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$

Basic Properties of Complex Numbers :

1) $z=\bar{z}$ iff z is purely a real number

Proof: Let $z=x+iy,\ \bar{z}=x-iy$

Let $z=\bar{z},\ \Rightarrow \cancel{x}+iy=\cancel{x}-iy$

$2iy=0 \iff y=0 \iff z=x \iff z$ is real number.

2) $|z|=0$ iff $z=0$

Proof: $0 = |z| = \sqrt{x^2\ y^2} \iff x^2 = 0\ and\ y^2 = 0 \iff x = 0\ and\ y = 0$

i.e. $z = 0$

3) $|z| = |\bar{z}|$

Proof: $z = x + iy$ and $\bar{z} = x - iy$

$\therefore |z| = \sqrt{x^2 + y^2}$ (1)

$|\bar{z}| = \sqrt{x^2 + (-y)^2} = \sqrt{x^2 + y^2}$ (2)

$\therefore |z| = |\bar{z}|$ from (1) & (2)

4) $\operatorname{Re} z \le |\operatorname{Re} z| \le |z|$

5) $\operatorname{Im} z \le |\operatorname{Im} z| \le |z|$

6) Let $z \in \mathbb{C}$, $w \in \mathbb{C}$ then

$\overline{z + w} = \bar{z} + \bar{w}$

Let $z = x + iy$, $w = u + iv$

$\therefore \overline{z + w} = \overline{(x + iy) + (u + iv)} = \overline{(x + u) + i(y + v)}$

$= (x + u) - i(y + v) = (x - iy) + (u - iv) = \bar{z} + \bar{w}$

$\therefore \overline{z + w} = \bar{z} + \bar{w}$

7) $\overline{zw} = \bar{z}.\bar{w}$

$\overline{zw} = \overline{(x + iy)(u + iv)} = \overline{(xu - yv) + i(xv + yu)}$

$= (xu - yv) - i(xv + yu) = (xu - ixv) - (yv - iyu)$

$= x(u - iv) - iy(u - iv) = (u - iv)(x - iy) = \bar{z}.\bar{w}$

8) $|z|^2 = z.\bar{z}$

$|z|^2 = \left(\sqrt{x^2 + y^2}\right)^2 = x^2 + y^2$ (1)

$z.\bar{z} = (x + iy)(x + iy) = x^2 - \cancel{ixy} + \cancel{ixy} - i^2y^2 = x^2 + y^2$ (2)

$\therefore |z|^2 = z.\bar{z}$ from (1) and (2)

9) $|zw| = |z||w|$

$|zw|^2 = (zw)(\overline{zw}) = (z\bar{z})(w\bar{w}) = |z|^2|w|^2 \Rightarrow |zw| = |z||w|$

10) $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$

$\left|\frac{z}{w}\right| = |z| \times \left|\frac{1}{|w|}\right| = |z| \times \frac{1}{|w|} = \frac{|z|}{|w|}$

11) $\bar{z} = z$

$\bar{z} = \overline{(x+iy)} = \overline{x-iy} = x+iy$

$\because \bar{\bar{z}} = \overline{x-iy} = x - i(-y) = x + iy = z$

Addition of two Complex Numbers :

Let $z = x + iy; w = u + iv$

Now, $z + w = \overline{OA} + \overline{OB} = \overline{OA} + \overline{AC} = \overline{OC}$

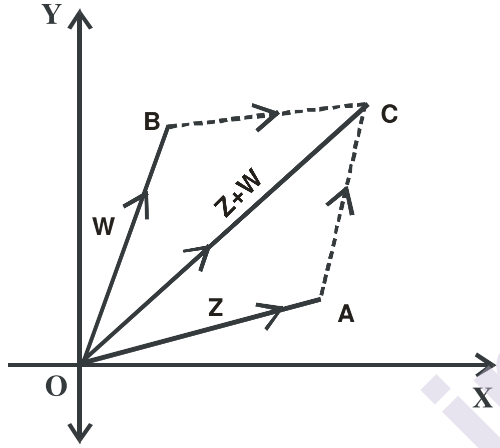


Fig 1.8

Triangle Inequality :

1) $z, w \in \mathbb{C}$ then $|z+w| \le |z| + |w|$

Proof:

$|z+w|^2 = (z+w).\left(\overline{z+w}\right) = (z+w).\left(\bar{z}.\bar{w}\right) = z\bar{z} + z\bar{w} + \bar{z}w + w\bar{w}$

$$|z+w|^2 = |z|^2 + |w|^2 + z\bar{w} + \bar{z}.w \qquad (1)$$

Now,

$z.\bar{w} + \bar{z}.w = (x+iy)(u-iv) + (x-iy)(u+iv) = z.\bar{w} + \overline{z+\bar{w}} = 2\text{Re}\left(z\bar{w}\right)$

$\le 2\left|z.\bar{w}\right| \qquad (\because \text{Re } z \le |z|)$

$= 2|z||w|$

$$z + \bar{w} + \bar{z}.w = 2|z|\left|\bar{w}\right| \qquad (2)$$

Substitute (2) in equation (1), we get

$|z+w|^2 \le |z|^2 + |w|^2 + 2|z||w| \qquad \le (|z| + |w|)^2$

$\Rightarrow \boxed{|z+w| \le |z| + |w|}$

<u>Geometrically</u>, in any triangle, the sum of the two sides of a triangle is greater than or equal to the third side(the points are collinear, in case it is equal).

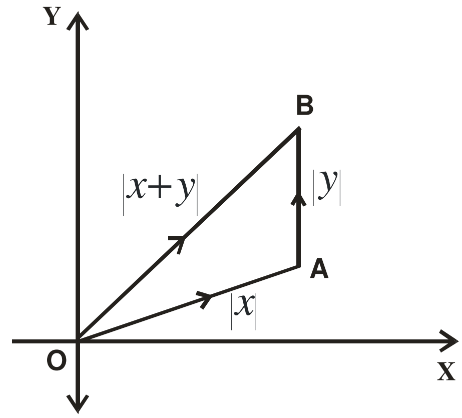


Fig 1.9

2) Let $z, w \in \mathbb{C}$ then $\|z|-|w\| \leq |z-w|$

Proof: Let $z = z + w - w$

Taking mod on $(|\ \ |)$ both the sides

$\therefore \quad |z| = |z+w-w|$

$\qquad \leq |z-w| + |w|$

$\therefore |z| - |w| \leq |z-w|$ (i)

Interchanging z and w, we get

$|w| - |z| \leq |w-z| = |z-w|$ (ii)

$-(|z| - |w|) \leq -|z-w|$

$|z| - |w| \geq -|z-w|$ (iii)

From equation (i) and (ii), we get

$-|z-w| \leq (|z| - |w|) \leq |z-w|$

$\therefore \quad \boxed{\|z|-|w\| \leq |z-w|}$

3) Let $z, w \in \mathbb{C}$ then

$|z+w|^2 = |z|^2 + 2\,\mathrm{Re}\, z\overline{w} + |w|^2$

$|z-w|^2 = |z|^2 - 2\,\mathrm{Re}\, z\overline{w} + |w|^2$

4) Parallelogram Law: The sum of the squares of the lengths of the diagonals of a parallelogram is equal to the sum of squares of lengths of its sides. i.e. prove that $|z+w|^2 + |z-w|^2 = 2\left(|z|^2 + |w|^2\right)$

Proof: Let $z, w \in \mathbb{C}$

$|z+w|^2 = \left|(x+u) + i(y+v)^2\right| = |z|^2 + 2\,\mathrm{Re}\,(z\overline{w}) + |w|^2$

$\because |z-w|^2 = |z|^2 - 2\,\mathrm{Re}\, z\overline{w} + |w|^2$

$\therefore |z+w|^2 + |z-w|^2 = 2\left(|z|^2 + |w|^2\right)$

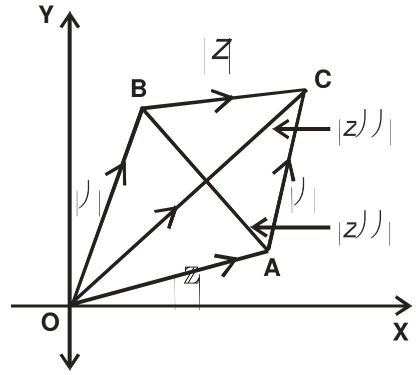


Fig 1.10

5) Let $z, w \in \mathbb{C}$ then $\|z|-|w\| \le |z \pm w| \le |z|+|w|$

Proof: i) T.P.T. $|z \pm w| \le |z|+|w|$

Case (i) $|z+w| \le |z|+|w|$ by triangle inequality

Case (ii) $|z-w| = |z+(-w)| \le |z|+|-w| \le |z|+|w|$

From above both cases,

$|z \pm w| \le |z|+|w|$ (*)

ii) T.P.T. $|z|-|w| \le |z \pm w|$

Consider $|z| = |z+w-w| \le |z+w|+|-w| \le |z+w|+|w|$

$\therefore$ $|z|-|w| \le |z+w|$ (a)

Consider $|w| = |w+z-z|$

$\le |w+z|+|-z| \le |w+z|+|z|$

$\therefore |w|-|z| \le |w+z|$

$\therefore -(|z|-|w|) \le |w+z|$ (b)

$\therefore$ $\|z|-|w\| = |z \pm w|$ (**)

From (*) and (**), we get

$\|z|-|w\| \le |z \pm w| \le |z|+|w|$

Theorem: *The field $\mathbb{C}$ is not a linearly totally ordered field*

OR

The field $\mathbb{C}$ is not partially ordered field (Total ordering or partial ordaring means that if $a \ne b$ then either $a < b$ or $a > b$).

Proof: Suppose that such a total (partial) ordering exists.

Then for i.e. $\mathbb{C}$, we have either $i > 0$ or $i < 0$ if $i > 0$

$\therefore$ $-1 = i \, . \, i > 0$

or if it $i < 0 \, (-i > 0)$

$-1 = (-i)\ (-i) > 0$

We get $-1>0$, which is not true in $\mathbb{R}$.

$\therefore$ Our supposition is not true.

$\therefore$ $\mathbb{C}$ is not linearly totally ordered field.

Properties of polar form and exponential form

1) Let $z_1 = r_1\, e^{i\theta_1} = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2 e^{i\,\theta_2} = r_2(\cos\theta_1 + i\sin\theta_2)$

then $z_1 . z_2 = r_1 . r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) = r_1 . r_2\, e^{i(\theta_1+\theta_2)}$

$\because\ e^{i\theta} = e^{i(\theta+2n\pi)},\quad e^{2in\pi} = 1,\quad n\in\mathbb{Z}$

$|z_1 . z_2| = \left| r_1 . r_2 . e^{i(\theta_1+\theta_2)} \right| = r_1 . r_2 \qquad \left(\because \left| e^{i(\theta_1+\theta_2)} \right| = 1\right)$

and $\boxed{\arg(z_1 . z_2) = \arg z_1 + \arg z_2 \pmod{2\pi}}$ in the sense that they are same but for an integral multiple of 2π.

Note : $\boxed{\arg z_1 . z_2 = \arg z_1 + \arg z_2 + 2k\pi \qquad \text{where } k = 0, 1 \text{ or } -1}$

2. Let $z_1 = r_1\, e^{i\theta_1}$ and $z_2 = r_2\, e^{i\theta_2}$ and $z_2 \neq 0$

$\therefore\ \dfrac{z_1}{z_2} = \dfrac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \dfrac{r_1}{r_2}\, e^{i(\theta_1-\theta_2)}$

$\therefore\ \arg\left(\dfrac{z_1}{z_2}\right) = \arg z_1 - \arg z_2 \pmod{2\pi}$

Let $z_1 = -1$ and $z_2 = -i$, $\quad(-\pi \le \theta \le \pi)$

$\because\ z_1 = -1 = x+iy \Rightarrow x = -1$ and $y = 0$

$\therefore\quad \arg z_1 = \arg(-1) = \tan^{-1}\left(0/-1\right) = \tan^{-1}(0) = \tan^{-1}(\tan\pi) = \pi$

$\because\quad z_2 = -i = x+iy \quad\Rightarrow\quad x = 0$ and $y = -1$

$\therefore\quad \arg z_2 = \arg(-i) = \tan^{-1}\left(y/x\right) = \tan^{-1}\left(-1/0\right) = \tan^{-1}(\infty) = -\dfrac{\pi}{2}$

$\arg(z_1 . z_2) = \arg(-1.-i) = \arg(i) = \tan^{-1}\left(1/0\right) = \tan^{-1}(\infty) = \dfrac{\pi}{2}$

$\therefore\quad \arg(z_1 . z_2) = \arg z_1 + \arg z_2 + 2k\pi \qquad$ where $k = 0$

Let $z_1 = -1$ and $z_2 = i$

$\because\quad z_1 = -1 \Rightarrow z_1 = x+iy \Rightarrow x = -1,\ y = 0$

$\therefore\quad \arg z_1 = \arg(-1) = \tan^{-1}\left(y/x\right) = \tan^{-1}\left(0/-1\right) = \tan^{-1}(0) = \pi$

$\because\quad z_2 = i = x+iy \Rightarrow x = 0,\ y = 1$

$\therefore\quad \arg z_2 = \arg(i) = \tan^{-1}\left(y/x\right) = \tan^{-1}\left(1/0\right) = \tan^{-1}(\infty) = \dfrac{\pi}{2}$

$\therefore\ \arg(z_1 . z_2) = \arg(-1 . i) = \arg(-i) = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(-\frac{1}{0}\right)$

$= -\tan(\infty) \quad = -\frac{\pi}{2}$

$\therefore \quad \arg(z_1 . z_2) = \arg z_1 + \arg z_2 + 2k\pi \quad \text{where } k = -1$

In this case, we get correct answer by adding -2π to bring within the interval $(-\pi, \pi)$.

When principal argument are added together in multiplication problem, the resulting argument need not be the principle value.

<u>De-Moivre's Theorem</u> **:**

Theorem: *If n is any integer or fraction then* $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$

Proof:

LHS= $(\cos\theta + i\sin\theta)^n = (e^{i\theta})^n = e^{i(\theta n)} = (\cos n\theta + i\sin n\theta)$ **=RHS**

$$\therefore \quad \boxed{(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta}$$

e.g. i) $(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$

ii) $(\cos\theta + i\sin\theta)^{2/5} = \cos\left(\frac{2}{5}\theta\right) + i\sin\left(\frac{2}{5}\theta\right)$

Note: $(\sin\theta + i\cos\theta)^n \neq (\sin\theta + i\cos n\theta)$

But, $(\sin\theta + i\cos\theta)^n = \left[\cos n\left(\frac{\pi}{2} - \theta\right) + i\sin n\left(\frac{\pi}{2} - \theta\right)\right]^n$

$= \cos n\left(\frac{\pi}{2} - \theta\right) + i\sin n\left(\frac{\pi}{2} - \theta\right)$ (by above thm)

e.g. 1) $(\sin\theta + i\cos\theta)^{2/3} = \left[\cos\left(\frac{\pi}{2} - \theta\right) + i\sin\left(\frac{\pi}{2} - \theta\right)\right]^{2/3}$

$= \cos\frac{2}{3}\left(\frac{\pi}{2} - \theta\right) + i\sin\frac{2}{3}\left(\frac{\pi}{2} - \theta\right)$

2) $(\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta$

e.g. $(\cos\theta - i\sin\theta)^{4/5} = \cos\frac{4}{5}\theta - i\sin\frac{4}{5}\theta$

$\because\ z = x + iy$

$x^2 + y^2 = r^2$ is equation of circle with centre at the origin & radius equal to r.

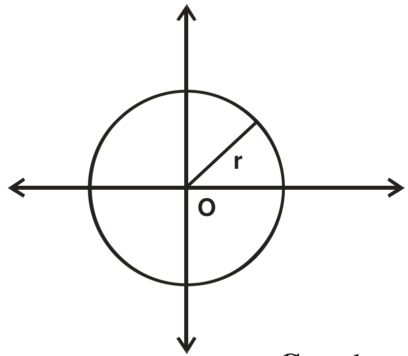


Fig 1.11 $\mathbb{C}-plane$

$\because \quad z = x + iy \therefore \quad |z| = \sqrt{x^2 + y^2} = \sqrt{r^2} = r$

$\therefore \quad x^2 + y^2 = r^2$

The equation of the circle with the centre at $c = a + ib \in \mathbb{C}$ and radius equal to r.

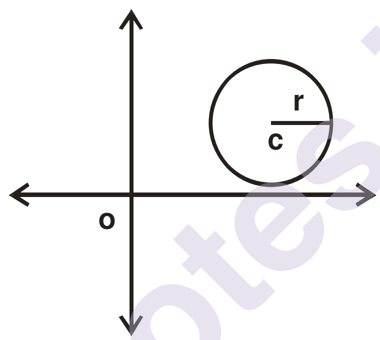


Fig 1.12

$\therefore \quad \boxed{|z - c| = r}$

e.g. 1) $|z - (2+i)| = 1$

This is equation of the circle with centre $(2,1)$ and radius 1.

$|z-1| = 3$, circle with centre $(1,0)$ and radius = 3

$|z+i| = 2$, circle with centre $(0,-1)$ and radius = 2

Roots of Complex Number :

Definition: A number w is called the n[th] root of complex number z if $w^n = z$ or $w = z^{1/n}$.

Theorem: *In* $\mathbb{C}$, *given* $z \neq 0$, *the equation expansion* $w^n = z$ *has n-distinct solution given by* $w_k = \sqrt[n]{r}\,.e^{i\left(\frac{\theta + 2k\pi}{n}\right)}$, $k = 0, 1, ..., n-1$ *where* $r = |z|$ *and* $\theta = \text{Arg}z$.

Proof: Given, $z \in \mathbb{C}$ and $z \neq 0$

$\therefore$ The polar form of complex number z is $z = r(\cos\theta + i\sin\theta)$ where $r = |z|$ and $\theta = \arg z$.

$\therefore$ $w^n = z = r(\cos\theta + i\sin\theta)$ (given)

$= r\left[\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi)\right]$

OR

$$w = z^{1/n} = r^{1/n}\left[\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi)\right]^{1/n}$$

$= \sqrt[n]{r}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$ (by De-Moiver's theorem)

$w = w_k = \sqrt[n]{r}\, e^{i\left(\frac{\theta + 2k\pi}{n}\right)}$ where $k = 0, 1, 2, \ldots, (n-1)$

Note : It is sufficient to take $k = 0, 1, 2, \ldots (n-1)$ since all other values of k lead to repeated roots.

Example : Find all the fourth roots of $z = 1 + i$ and locate these roots in $\mathbb{C}$ plane.

Solution : Let $w^4 = z = 1 + i$

$\therefore$ $x = 1,\ y = 1$

$\therefore$ $r = \sqrt{x^2 + y^2} = \sqrt{1+1}$ $\therefore$ $\boxed{r = \sqrt{2}}$

$\theta = \tan^{-1}\left(y/x\right) = \tan^{-1}\left(1/1\right) = \tan^{-1}(1)$ $\therefore$ $\boxed{\theta = \pi/4}$

$\therefore$ $w^4 = \sqrt{2}\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right]$ (polar form)

$= \sqrt{2}\left[\cos\left(\pi/4 + 2k\pi\right) + i\sin\left(\pi/4 + 2k\pi\right)\right]$

$$w = 2^{1/8}\left[\cos\left(\frac{\pi + 8k\pi}{4}\right) + i\sin\left(\frac{\pi + 2k\pi}{4}\right)\right]^{1/4}$$

$w = 2^{1/8}\left[\cos\left(\frac{\pi + 8k\pi}{16}\right) + i\sin\left(\frac{\pi + 2k\pi}{16}\right)\right]$ where

$\therefore$ Fourth roots of equations are

For $k = 0$, $w_0 = z^{1/8}\left[\cos\left(\frac{\pi}{16}\right) + i\sin\left(\frac{\pi}{16}\right)\right]$

$k = 1$, $w_1 = z^{1/8}\left[\cos\left(\frac{9\pi}{16}\right) + i\sin\left(\frac{9\pi}{16}\right)\right]$

$k=2,\ w_2 = z^{1/8}\left[\cos\left(\frac{17\pi}{16}\right)+i\sin\left(\frac{17\pi}{16}\right)\right]$

$k=3,\ w_3 = z^{1/8}\left[\cos\left(\frac{25\pi}{16}\right)+i\sin\left(\frac{25\pi}{16}\right)\right]$

Which are the required four fourth root of $z=1+i$

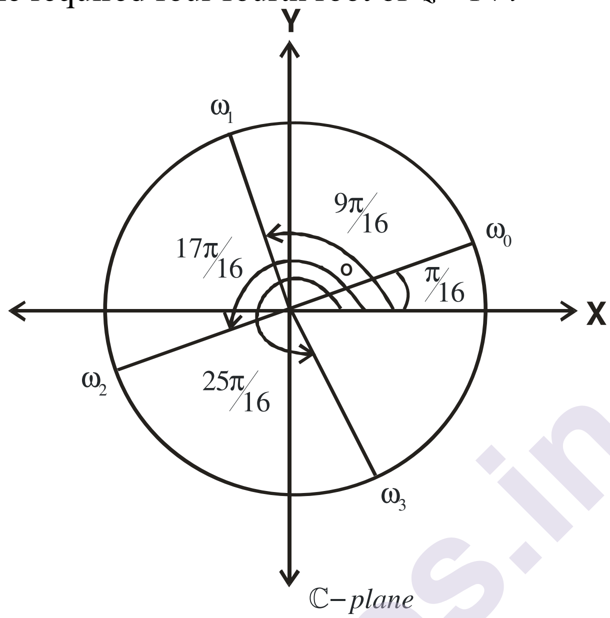


Fig 1.13

Example : Find all the fifth roots of $z=-32$ and locate these roots in $\mathbb{C}$-plane.

Solution : Let $w^5 = z = -32$

$\therefore \quad x=-32$ and $y=0$

$\therefore \quad r=\sqrt{x^2+y^2}=\sqrt{(-32)^2}=32$

$\theta=\tan^{-1}\left(y/x\right)=\tan^{-1}\left(0/32\right)=\tan^{-1}(0) \Rightarrow \boxed{\theta=\pi}$

$\therefore \quad w^5 = 32(\cos\pi + i\sin\pi)$

$\therefore \quad w_k = (32)^{1/5}\left[\cos(\pi+2k\pi)+i\sin(\pi+2k\pi)\right]^{1/5}$

$= 2\left[\cos\left(\frac{\pi+2k\pi}{5}\right)+i\sin\left(\frac{\pi+2k\pi}{5}\right)\right]$

For $\quad k=0,\ w_0 = 2\left[\cos\left(\frac{\pi}{5}\right)+i\sin\left(\frac{\pi}{5}\right)\right]$

$k=1,\ w_1 = 2\left[\cos\left(\frac{3\pi}{5}\right)+i\sin\left(\frac{3\pi}{5}\right)\right]$

$k=2,\ w_2 = 2\left[\cos\left(\frac{5\pi}{5}\right)+i\sin\left(\frac{5\pi}{5}\right)\right] \quad = 2[\cos\pi + i\sin\pi]$

$$k = 3, \quad w_3 = 2\left[\cos\left(\frac{7\pi}{5}\right) + i\sin\left(\frac{7\pi}{5}\right)\right]$$

$$k = 4, \quad w_4 = 2\left[\cos\left(\frac{9\pi}{5}\right) + i\sin\left(\frac{9\pi}{5}\right)\right]$$

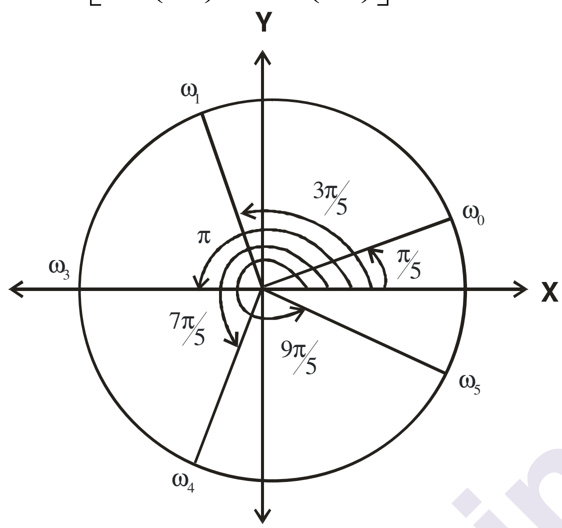


Fig 1.14

Example : Solve $z^8 + z^5 + z^3 + 1 = 0$

$\because \quad z^8 + z^5 + z^3 + 1 = 0$

$\therefore \quad z^5\left(z^3 + 1\right) + 1\left(z^3 + 1\right) = 0$

$\therefore \quad \left(z^5 + 1\right)\left(z^3 + 1\right) = 0$

Consider, $z^3 + 1 = 0$

$\therefore \quad z^3 = -1 \Rightarrow w^{1/3} = z = -1$

$\therefore \quad x = -1, \quad y = 0$

$\therefore \quad r = \sqrt{x^2 + y^2} = \sqrt{1} \quad \Rightarrow \quad \boxed{r = 1}$

and $\theta = \tan^{-1}\left(0/_{-1}\right) = -\tan^{-1}(0)$

$\boxed{\theta = \pi}$

$\therefore \quad w = z^3 = 1(\cos\pi + i\sin\pi)$ in polar

$w = z^3 = \left[\cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi)\right]$

$w_k = z = \left[\cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi)\right]^{1/3}$

by De-Moivre's theorem

$w_k = z = \cos\left(\frac{\pi + 2k\pi}{3}\right) + i\sin\left(\frac{\pi + 2k\pi}{3}\right)$ where $k = 0, 1, 2$

For $k=0$, $w_0=\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right)$

$k=1$, $w_1=\cos\left(\frac{3\pi}{3}\right)+i\sin\left(\frac{3\pi}{3}\right)=\cos(\pi)+i\sin(\pi)$

$k=2$, $w_2=\cos\left(\frac{5\pi}{3}\right)+i\sin\left(\frac{5\pi}{3}\right)$

Now, consider $z^5+1=0$

$\therefore z^5=-1$

$\therefore w^{1/5}=z=-1$

$\therefore$ $\boxed{r=1 \text{ and } \theta=\pi}$

$\therefore w=\cos\pi+i\sin\pi$

$\therefore w=(\cos(\pi+2k\pi)+i\sin(\pi+2k\pi))$

$\therefore w_k=\cos\left(\frac{\pi+2k\pi}{5}\right)+i\sin(\pi+2k\pi)^{1/5}$ where $k=0,1,2,3,4$

$\therefore w_k=\left[\cos\left(\frac{\pi+2k\pi}{5}\right)+i\sin\left(\frac{\pi+2k\pi}{5}\right)\right]$

For $k=0$, $w_0=\cos\left(\frac{\pi}{5}\right)+i\sin\left(\frac{\pi}{5}\right)$

$k=1$, $w_1=\cos\left(\frac{3\pi}{5}\right)+i\sin\left(\frac{3\pi}{5}\right)$

$k=2$, $w_2=\cos(\pi)+i\sin(\pi)$

$k=3$, $w_3=\cos\left(\frac{7\pi}{5}\right)+i\sin\left(\frac{7\pi}{5}\right)$

$k=4$, $w_4=\cos\left(\frac{9\pi}{5}\right)+i\sin\left(\frac{9\pi}{5}\right)$

Example: Find all roots of $\left(8+8\sqrt{3}i\right)^{\frac{1}{4}}$ and represent them graphically. (2009)

Solution: Let $z=8+8\sqrt{3}i$

$r=8\sqrt{1+\left(\sqrt{3}\right)^2}=16, \theta=\tan^{-1}\left(\sqrt{3}\right)=\frac{\pi}{3}$

$\therefore z^{1/4}=\left(16\left(\cos\left(\frac{\pi}{3}+2k\pi\right)+i\sin\left(\frac{\pi}{3}+2k\pi\right)\right)\right)^{1/4}$ $k=0,1,2,3.$

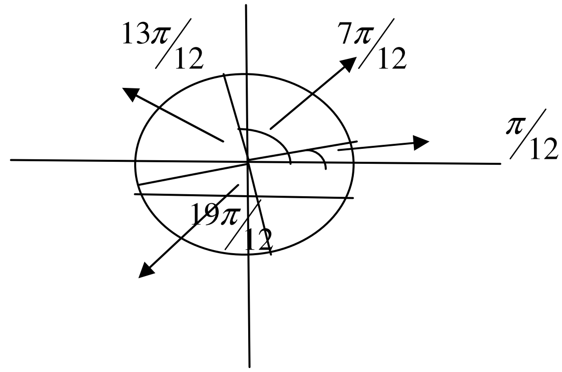
$=2\left(\cos\left(\frac{(6k+1)\pi}{12}\right)+i\sin\left(\frac{(6k+1)\pi}{12}\right)\right)$

$$for \quad k=0 \quad w_0 = 2\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$$

$$for \quad k=1 \quad w_1 = 2\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right)$$

$$for \quad k=2 \quad w_2 = 2\left(\cos\frac{13\pi}{12} + i\sin\frac{13\pi}{12}\right)$$

$$for \quad k=3 \quad w_3 = 2\left(\cos\frac{19\pi}{12} + i\sin\frac{19\pi}{12}\right)$$



Q. Find all the roots of $\left(\sqrt{3}+i\right)^{\frac{1}{3}}$ and locate them graphically.

1.3. EXTENDED COMPLEX PLANE, THE POINT AT INFINITY AND STEREOGRAPHIC PROJECTION

Construction of the Stereographic Projection Map. **(2012)** Let $\mathbb{C}$ be the Complex plane. Consider a unit sphere S (radius 1) tangent to $\mathbb{C}$ at a point $z=0$. The diameter NS is perpendicular to $\mathbb{C}$ and we call points N and S the north and south poles of the sphere S corresponding to any point z on the Complex Plane $\mathbb{C}$, we can construct a straight line *NZ* intersecting sphere S at a point $P(\neq N)$.

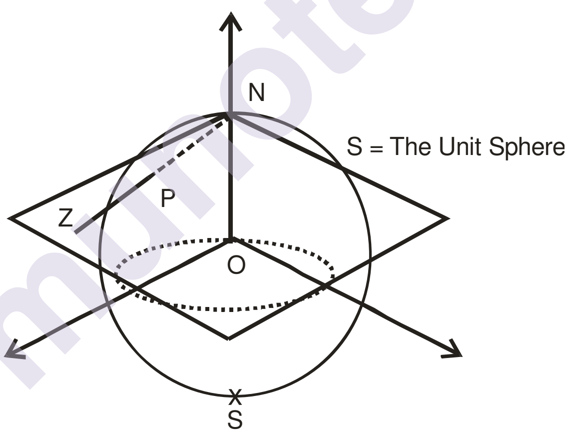


Fig 1.15

Thus to each point of the Complex Plane $\mathbb{C}$, there corresponds one and only one point of the sphere S and conversely, to each point of the sphere S (except N), there corresponds one and only one point on the plane. For completeness, we say that the point N itself corresponds to the point at infinity of the plane $\mathbb{C}$. This one-to-one correspondence between the points of the plane $\mathbb{C}_\infty$ and the points of the sphere S is called the Stereographic Projection. The sphere is called the Riemann Sphere (because Complex Number can also be represented by point on the Sphere.)

Suppose Complex Plane $\mathbb{C}$ passes through centre of the unit sphere S.

Let $x_1^2 + x_2^2 + x_3^2 = 1$ be the equation of unit sphere S.

$N \equiv (0,0,1)$

Also, identify $\mathbb{C}$ with $\{(x_1, x_2, 0) : x_1 \in \mathbb{R},\ x_2 \in \mathbb{R}\}$.

Put $z = (x, y)$ and $p = (x_1, x_2, x_3)$. We will find equations expressing x_1, x_2, x_3 in terms of x and y.

The equation of straight line Nz in $\mathbb{R}^3$ passing through points N and Z is given by

$$\{(1-t)z + tN : t \in \mathbb{R}\} = \{(1-t)(x, y) + t(0,0,1) : t \in \mathbb{R}\}$$

$$= \{((1-t)x, (1-t)y,\ t)\ t \in \mathbb{R}\} \ldots\ldots(1)$$

$\because$ Straight line Nz intersects sphere S.

$$(1-t)^2 x^2 + (1-t)^2 y^2 + t^2 = 1$$

$\therefore$ $(1-t)^2 \left(x^2 + y^2\right) = 1 - t^2$

$\therefore$ $(1-t)^2 \left(x^2 + y^2\right) = (1-t)(1+t)$

$\therefore$

$(1-t)^2 |z|^2 = (1-t)(1+t)$

$\therefore$ $(1-t)|z| = (1+t)$

This equation holds if $P \neq N$ … ($\because$ if $P = N$ then $t = 1$ and $z = \infty$)

$\therefore$ $|z|^2 - t|z|^2 = 1 + t$

$\therefore$ $|z|^2 - 1 = \left(1 + |z|^2\right)t$

$\therefore$ $t = \dfrac{|z|^2 - 1}{1 + |z|^2}$ (for $P \neq N$)

$$1 - t = 1 - \left(\frac{|z|^2 - 1}{1 + |z|^2}\right) = \frac{1 + |z|^2 - |z|^2 + 1}{1 + |z|^2} = \frac{2}{1 + |z|^2}$$

$\because$ Points N, P, Z are collinear.

$\therefore$ From equation (1),

$$\left.\begin{aligned} x_1 &= (1-t)\,x = \frac{2x}{1+|z|^2} = \frac{z+\bar{z}}{1+|z|^2} \\ x_2 &= (1-t)\,y = \frac{2y}{1+|z|^2} = \frac{-i\left(z-\bar{z}\right)}{1+|z|^2} \\ x_3 &= t = \frac{|z|^2-1}{|z|^2+1} \end{aligned}\right\} \qquad (2)$$

$\therefore$ Point $z = x+iy \in \mathbb{C}$ corresponds to point P.

$$P = \left(\frac{z+\bar{z}}{1+|z|^2},\ \frac{-i\left(z-\bar{z}\right)}{1+|z|^2},\ \frac{|z|^2-1}{|z|^2+1} \right)$$

Again from equation (2),

$$x = \frac{x_1}{1-t} = \frac{x_1}{1-x_3}, \qquad y = \frac{x_2}{1-t} = \frac{x_2}{1-x_3}, \qquad z = x+iy = \frac{x_1+i\,x_2}{1-x_3}$$

Point $P \equiv (x_1, x_2, x_3) \in S$ corresponds to point z.

$$z = \left(\frac{x_1+i\,x_2}{1-x_3} \right) \in \mathbb{C}$$

Note : From figure (Fig 1.20)

The straight line Nz in $\mathbb{R}^3$ intersects sphere and in exactly one point $P \neq N$.

If $|z| > 1$, then point P is in the Northern hemisphere and if $|z| < 1$, then point P is in the southern hemisphere. Also, if $|z| = 1$, then $P = z$ and as $z \to \infty$, P approaches N.

Distance function :

Let z and z' be any two points on the Complex Plane $\mathbb{C}$. Suppose point $z(x, y)$ corresponds to point $P \equiv (x_1, x_2, x_3) \in S$. Suppose point $z'(x', y')$ corresponds to point $P' \equiv (x_1', x_2', x_3') \in S$.

We define distance function as

$$d(z, z') = d(P, P') = \sqrt{(x_1-x_1')^2 + (x_2-x_2')^2 + (x_3-x_3')^2}$$

$$\left[d(z, z')\right]^2 = (x_1-x_1')^2 + (x_2-x_2')^2 + (x_3-x_3')^2$$

$$= x_1^2 + x_1'^2 - 2x_1x_1' + x_2^2 + x_2'^2 - 2x_1x_2' + x_3^2 + x_3'^2 - 2x_3x_3'$$

$$\textit{Since } x_1^2 + x_2^2 + x_3^2 = 1 \textit{ and } x'^2_1 + x'^2_2 + x'^2_3 = 1$$

$$\Rightarrow x_1^2 + x_2^2 + x_3^2 + x'^2_1 + x'^2_2 + x_3^2 = 1+1 = 2$$

$$\Rightarrow \left[d(z, z')\right]^2 = 2 - 2\left(x_1x_1' + x_2x_2' + x_3x_3'\right)$$

Put $x_1 = \dfrac{z+\bar{z}}{1+|z|^2}$, $x_2 = \dfrac{-i\left(z-\bar{z}\right)}{1+|z|^2}$, $x_3 = \dfrac{|z|^2-1}{|z|^2+1}$

$$x_1' = \frac{z'+\bar{z}'}{1+|z'|^2},\quad x_2' = \frac{i\left(z'+\bar{z}'\right)}{1+|z'|^2},\quad x_3' = \frac{|z'|^2-1}{|z'|^2+1}$$

$$\left[d\left(z,z'\right)\right]^2 = 2-2\left[\left(\frac{z+\bar{z}}{1+|z|^2}\right)\left(\frac{z'+\bar{z}'^2}{1+|z'|^2}\right)+\left(\frac{-i\left(z'-\bar{z}'^1\right)}{1+|z'|^2}\right)\right.$$

$$\left.\left(\frac{-i\left(z-\bar{z}\right)}{1+|z|^2}\right)+\left(\frac{|z|^2-1}{|z|^2+1}\right)\left(\frac{|z'|^2-1}{|z'|^2+1}\right)\right]$$

1.4 SUMMARY

1) A Complex Number Z is an ordered pair (x, y) of real numbers.

2) The distance between Two Complex Numbers : Let $Z_1 = (x_1, y_1)$, $Z_2 = (x_2, y_2)$ be two complex numbers. The distance between them in complex plane is given by $d(Z_1, Z_2) = |Z_1 - Z_2| = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$

3) If P is a point in the Complex Plane corresponding to Complex Number $Z = x+iy = (x,\ y)$ and let (r, θ) be the polar co-ordinates of point (x, y) from figure $x = r\cos\theta$ and $y = r\sin\theta$ where

$\boxed{r = \sqrt{x^2+y^2}}$ is called the modulus or absolute value of Z (denoted by $|Z|$ and

$\boxed{\theta = \tan^{-1}\left(\dfrac{y}{x}\right)}$ is called the argument or amplitude of Z (denoted by $\theta = \arg Z$)

Here θ is the angle between the two lines OP & the real axis (axis – X)

4) The modulus or absolute value of a Complex Number $Z = x+iy$ is defined by $|Z| = \sqrt{x^2+y^2}$.

5) De-Moivre's Theorem :

If n is any integer or fraction then $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

6)Theorem : In $\mathbb{C}$, given $z \neq 0$, the equation expansion $w^n = z$ has n-distinct solution given by $w_k = \sqrt[n]{r}\,.\,e^{i\left(\frac{\theta+2k\pi}{n}\right)}$, $k = 0, 1, \ldots, n-1$ where $r = |z|$ and $\theta = \arg z$.

1.5 UNIT END EXERCISES

1) Find two square roots of $2i$.

(Hint: Let $x+iy$ be a square root of 2i $\Rightarrow$ $(\text{x+iy})^2 = zi$.

$\Rightarrow \left(x^2 - y^2\right) + (2xy)i = 2i$, comparing real and imaginary parts on both the sides, we get two equations in x,y.

$x^2 - y^2 = 0;\ 2xy = 2 . \Rightarrow (x+iy) = 1+i$ or $(x+iy) = -1-i$

2) Describe the set $\{z : |z+1| < 1\}$ in the Complex plane $\mathbb{C}$.

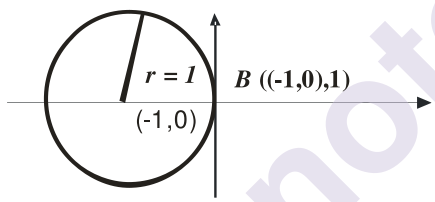
Solution: Let $z = x+iy; x = \text{Re}(z),\ y = \text{Im}(z)$

$\Rightarrow |z+1| = |x+iy+1| = \sqrt{(x+1)^{2+} y^2}$.

Hence $|z+1| < 1$ describes all real number pairs (x, y) in $\mathbb{R}^2$ such that $\sqrt{(x+1)^2 + y^2} < 1$.

$\Rightarrow (x+1^2) + y^2 < 1$ This is an equation of the open disc with centre

at and radius equal to 1, which can be described as follows :



3) Find polar form of the Complex Number $1+i$.

4) Show that the n th roots of 1 satisfy the " cyclotomic " equation

$z^{n-1} + z^{n-2} + \cdots + z + 1 = 0$.

(Hint : Use the identity $z^n - 1 = (z-1)\left(z^{n-1} + z^{n-2} + ... + z + 1\right)$.

(2009)

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2

SEQUENCES OF COMPLEX NUMBERS

Unit Structure

2.0 Objectives
2.1. Introduction
2.2. Convergent Sequences
2.3. Topological Aspects of the Complex Plane (Limits, Continuity, Uniform Continuity)
2.4. Summary
2.5. Unit End Exercises

2.0 OBJECTIVES

This unit shall make you understand :

- Cauchy and convergent sequences of complex number z .
- The connection between the convergence of real and imaginary parts of a sequence $z_n = x_n + y_n$, namely x_n and y_n with the convergence of z_n in $\mathbb{C}$. We shall also see that under what conditions a given sequence of complex number $z_n = x_n + y_n$ is a Cauchy sequence. Can we relate to fdour findings for real values sequence x_n and y_n .

2.1. INTRODUCTION

We have already associated the meaning to a sequence of real numbers as a function , $a : Z^+ \to \mathbb{R}$, denoted by $(a(n))_{n \in N}$. On a similar line , we shall define a sequence of complex numbers , where each term of a sequence is a complex number . For example $z_n = \frac{1}{z^n}$ is a sequence of Complex Numbers with terms $\frac{1}{z^1}, \frac{1}{z^2}, \frac{1}{z^3}, \ldots$ etc. In this Unit, we shall consider the topological aspects of the Complex plane. The concept of absolute value can be used to define the notion of a limit of a sequence of complex numbers . We shall begin with the definition of a complex valued sequence .

Definition: A function whose domain is a set of natural number $(\mathbb{N})$ and range is a subset of $\mathbb{R}$, is said to be Real sequence.

Any function whose domain is a set of nature numbers $(\mathbb{N})$ and range is subset of complex numbers $\mathbb{C}$, is said to be complex sequence.

Generally, we denote it by $\{z_n\}$. z_n is the n^th^ term of the sequence.

e.g. 1) The set of numbers $i, i^2, i^3, ..., i^{200}$. This is finite sequence and its n^th^ term is $z_n = i^n$, $n = 1, 2, ..., 200$

2) The set of numbers $\frac{2+i}{1}, \frac{(2+i)^2}{2}, \frac{(2+i)^3}{3}, ...$

It is the infinite sequence and its n^th^ term is $z_n = \frac{(2+i)^n}{n}$.

Sequences : Definition : A function whose domain is a set of natural number $(\mathbb{N})$ and range is a subset of $\mathbb{R}$, is said to be Real sequence.

Any function whose domain is a set of nature numbers $(\mathbb{N})$ and Range is subset of $\mathbb{R}$, is said to be Complex sequence.

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It is the infinite sequence and its n^th^ term is $z_n = \frac{(2+i)^n}{n}$.

2.2 CONVERGENT SEQUENCES

A sequence $\{z_n\}$ is said to converge to a point z_0 [or a sequence $\{z_n\}$ has to limit z_0] if for every $\varepsilon > 0$, there is an N s.t. $|z_n - z_0| < \varepsilon \quad \forall\ n \geq N$ and we write $\lim_{n\to\infty} z_n = z_0$.

Geometrically, $z_n \to z_0$ if every ε-nbd of z_0 contains almost all terms of the sequence $\{z_n\}$.

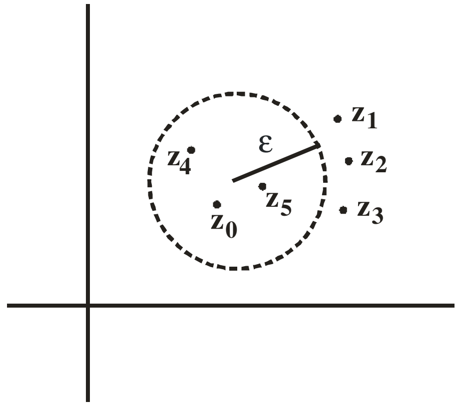

Fig 2.1

Divergent Sequences: A sequence is said to be divergent if it is not convergent.

Theorem: *Prove that any convergent sequence has a unique limit.*

Solution : Let $\lim_{n\to\infty} z_n = z_1$ and $\lim_{n\to\infty} z_n = z_0$

If $z_1 \neq z_0$ then for $\in = \frac{|z_1 - z_0|}{2} > 0$

$\exists N_1$ s.t. $n \geq, \mathrm{N}_1 \Rightarrow |z_n - z_1| < \varepsilon/2$

and $\exists N_2$ s.t. $n \geq, \mathrm{N}_2 \Rightarrow |z_n - z_0| < \varepsilon/2$

choose N = max $\{N_{1,} N_2\} \Rightarrow N \geq N_1$ and $N \geq N_2$

$\Rightarrow |z_N - z_1| < \varepsilon_2$ and $|z_N - z_0| < \varepsilon/2$

$\Rightarrow |z_1 - z_2| \leq |z_1 - z_N| + |z_N - z_0| < \varepsilon$

$\Rightarrow 0 < |z_1 - z_0| < \in$ a contradiction.

$\therefore z_1 = z_0$.

Theorem : *Suppose* $z_n = x_n + iy_n$ *and* $z_0 = x_0 + iy_0$ *then* $\lim_{n\to\infty} z_n = z_0$ *iff* $\lim_{n\to\infty} x_n = x_0$ *and* $\lim_{n\to\infty} y_n = y_0$.

Proof : Suppose $\lim_{n\to\infty} z_n = z_0 \Rightarrow \varepsilon > 0$, $\exists$, an integer N s.t. $|\, z_n - z_0 \,| < \varepsilon \;\; \forall\, n \geq N$.

Now, $|\, z_n - z_0 \,| = |\, x_n + iy_n - x_0 - iy_0 \,| \leq |\, x_n - x_0 \,| + i |\, y_n - y_0 \,|$

$$\therefore \left.\begin{array}{l} |\, x_n - x_0 \,| < \varepsilon \\ \text{and } |\, y_n - y_0 \,| < \varepsilon \end{array}\right\} \forall\, n \geq N$$

$\Rightarrow \lim\limits_{n\to\infty} x_n = x_0$ and $\lim\limits_{n\to\infty} y_n = y_0$

Conversely, suppose $\lim\limits_{n\to\infty} x_n = x_0$ and $\lim\limits_{n\to\infty} y_n = y_0 \quad \forall \ \varepsilon > 0, \exists \ N_1$ and N_2 s.t. $|x_n - x_0| < \varepsilon/2 \qquad \forall \ n \geq N_1$

and $|y_n - y_0| < \varepsilon/2 \qquad \forall \ n \geq N_2$

Choose $N = \max\{N_1, N_2\}$

if $n \geq N$ then

$\therefore |z_n - z_0| = |x_n + iy_n - x_0 - iy_0| \leq |x_n - x_0| + |y_n - y_0|$

$(\because \operatorname{Re} z \leq |z|$ and $\operatorname{Im} z \leq |z|)$

$< \varepsilon/2 + \varepsilon/2 = \varepsilon$

$\therefore |z_n - z_0| < \varepsilon$

$\lim\limits_{n\to\infty} z_n = z_0$

Theorem : *If* $\lim\limits_{n\to\infty} z_n = z_0$, *then* $\lim\limits_{n\to\infty} |z_n| = |z_0|$ *and the sequence* $\{z_n\}$ *is bdd.*

Proof : Suppose $\lim\limits_{n\to\infty} z_n = z_0 \quad \forall \ \varepsilon > 0, \quad \exists$ an integer N s.t. $|z_n - z_0| < \varepsilon \quad \forall \ n \geq N$.

$\because \ |z_n| - |z_0| \leq |z_n - z_0|$

$\therefore \ |z_n| - |z_0| < \varepsilon \qquad \forall \ n \geq N \qquad (1)$

$\therefore \lim\limits_{n\to\infty} |z_n| = |z_n|$

$\therefore$ from equation (1)

$|z_0| - \varepsilon < |z_n| < |z_0| + \varepsilon$

$\therefore$ Sequence $\{z_n\}$ is bounded.

Example : If $\lim\limits_{n\to\infty} z_n = z_0$ and $\lim\limits_{n\to\infty} w_n = w_0$ prove that

i) $\lim\limits_{n\to\infty} [z_n + w_n] = z_0 \pm w_0$

ii) $\lim\limits_{n\to\infty} [z_n \,.\, w_n] = z_0 \, w_0$

iii) $\lim\limits_{n\to\infty} \left[\dfrac{z_n}{w_n}\right] = \dfrac{z_0}{w_0}$ provided $w_0 \neq 0$

Definition : Cauchy Sequence : A sequence $\{z_n\}$ is said to be a Cauchy sequence for every $\varepsilon > 0$ there is an integer N s.t. $|z_n - z_m| < \varepsilon \ \forall \ n \geq N$, and $m \geq N$.

Note : From equation (1) $|z_m - z_n| < \varepsilon, \quad \forall\ n \geq N$

Put $m = n + p$ for $p = 1, 2, 3, ...$

$\therefore \quad \left| z_{n+p} - z_n \right| < \varepsilon \qquad \forall\ n \geq N$ and $p \geq 1$.

***Theorem**: Every convergent sequence is a Cauchy Sequence.*

Proof : Suppose $\{z_n\}$ is a convergent Sequence.

$\therefore$ A sequence $\{z_n\}$ has a limit of z_0.

$\lim\limits_{n\to\infty} z_n = z_0$

$\therefore$ For every $\varepsilon > 0$, there is an integer N s.t. $|z_n - z_0| < \varepsilon/2 \qquad \forall\ n \geq N$

If $m \geq N$ and $n \geq N$ then

$|z_m - z_0| < \varepsilon/2 \qquad \forall\ m \geq N$

$|z_n - z_0| < \varepsilon/2 \qquad \forall\ n \geq N$

$\therefore\ |z_n - z_m| = |z_n - z_0 + z_0 - z_m| \leq |z_n - z_0 + z_0 - z_m| < \varepsilon/2 + \varepsilon/2 = \varepsilon$

$\therefore\ |z_n - z_m| < \varepsilon \qquad$ if $n \geq N$ and $m \geq N$

$\Rightarrow\ \{z_m\}$ is a Cauchy sequence.

***Theorem :** $\mathbb{C}$ is complete. [i.e. T.P.T. every Cauchy sequence in $\mathbb{C}$ is convergent.]*

Proof : Let $\{z_n\} = \{x_n + iy_n\}$ be a Cauchy sequence in $\mathbb{C}$.

$\Rightarrow \{x_n\}$ and $\{y_n\}$ are Cauchy sequence in $\mathbb{R}$.

$\because\ \mathbb{R}$ is complete.

$\therefore\ x_n \to x_0$ and $y_n \to y_0$ for $x_0, y_0 \in \mathbb{R}$

$\therefore\ \lim\limits_{n\to\infty} z_n = \lim\limits_{n\to\infty} (x_n + iy_n) = \lim\limits_{n\to\infty} x_n + i \lim\limits_{n\to\infty} y_n = x_0 + iy_0 = z_0$

$\Rightarrow$ sequence $\{z_n\}$ is convergent.

Hence $\mathbb{C}$ is complete.

Note : A sequence is convergent iff it is a Cauchy sequence (Cauchy Criteria for convergence of a sequence.)

***Theorem**: Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers.* If $\lim\limits_{n\to 0} \dfrac{a_{n+1}}{a_n} = l < 1$, *then* $\lim\limits_{n\to 0} a_n = 0$.

Proof: We have $l < \frac{l+1}{2} < 1$, and by data there exist $m \in \mathbb{N}$ such that $l < \frac{a_{n+1}}{a_n} < \frac{l+1}{2}$ for all $n \geq m$

Put $r = \frac{l+1}{2}$. Then $0 < r < 1$. Then

$a_{m+1} < a_m, a_{m+2} < a_{m+r} r < (a_m r) r = a_m r^2$ and so on. We get

$a_{m+k} < a_m r^k \; \forall\, k \in \mathbb{N}$. Put $c = \frac{a_m}{r^m}$. Then $0 < a_n < cr^n \; \forall\, n \in \mathbb{N}$. Since

$0 < r < 1$, $cr^n \to 0$ as $n \to \infty$. So $a_n \to 0 \; as \; n \to \infty$.

2.3 TOPOLOGICAL ASPECTS OF THE COMPLEX PLANE

Topology in the $\mathbb{C}$-plane :

A function $\mathbb{C} \times \mathbb{C} \to \mathbb{R}$, $(z, z') \to |z - z'|$ has the following properties.

i) $|z - z'| \geq 0$, if $z \neq z'$ and $|z - z'| = 0$ if $z = z'$

ii) $|z - z'| = |z' - z|$

iii) $|z - z'| \leq |z - w| + |w - z'| \qquad z, z', w \in \mathbb{C}$

Thus, $\mathbb{C}$ is a metric space with Euclidean metric (distance) defined by $d(z_1, z_2) = |\, z_1 - z_2 \,|$, $z_1, z_2 \in \mathbb{C}$

1) Let $z_0 \in \mathbb{C}$ and $\varepsilon > 0$, then the set $B(z_0, \varepsilon) = \{z \in \mathbb{C} : |\, z - z_0 \,| < \varepsilon\}$ is called an <u>open disk</u> or <u>open ball</u> with centre at z_0 and radius ε (This is also called the ε-nbd of z_0 or nbd z_0).

Geometrically, $B(z_0, \varepsilon)$ is an open disk, consisting of all points at a distance less than ε from the point z_0.

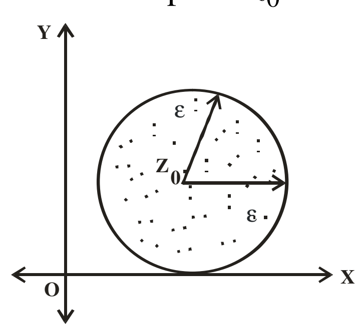

Fig 2.2

2) A set of the form, $B(z_0, \varepsilon) \left| \{z_0\} = \{z \in \mathbb{C} : 0 < | z - z_0 | < \varepsilon\}\right.$ is called <u>the deleted neighbourhood of</u> z_0 or punctured disk.

3) The set of the form $\partial B(z_0, r) = \{z \in \mathbb{C} : | z - z_0 | = r\}$ is the circle with centre at z_0 and radius r and is called the <u>boundary of circle</u>.

4) Let $G \in \mathbb{C}$, A set of G is said to be <u>open</u> in $\mathbb{C}$ if for every $z_0 \, \varepsilon \, G$, $\exists \;\; r > 0$, s.t. $B\left(z_{0,} r\right) c \quad G$

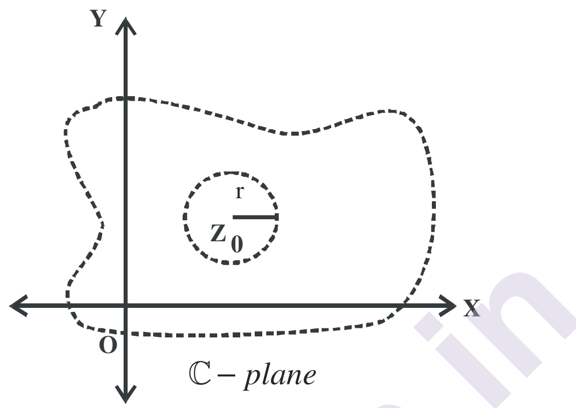

Fig 2.3

e.g. i) Interior of circle is an open set

ii) The entire plane $\mathbb{C}$ is an open set

iii) Half planes: $\operatorname{Re} z > 0, \operatorname{Re} z < 0, \operatorname{Im} z > 0, \operatorname{Im} z < 0$ are open set.

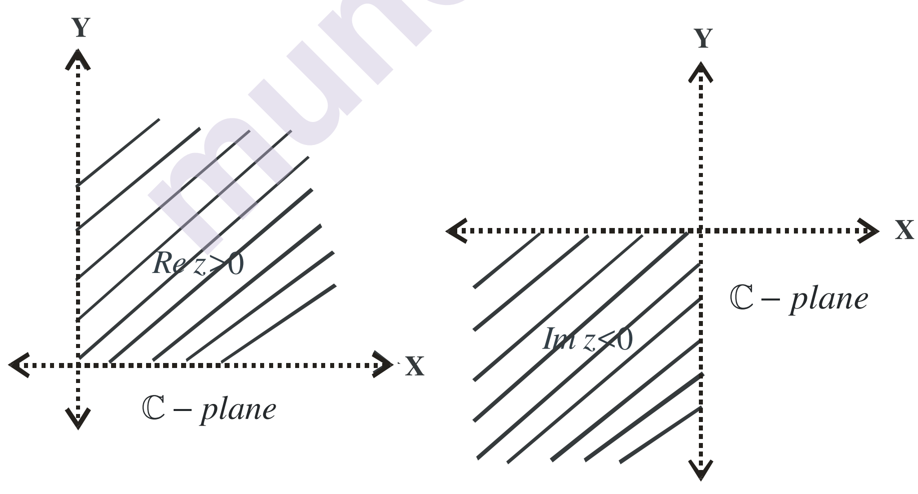

Fig 2.4

Thoerem :.*Any open disk is an open set*

Proof : Let $z_0 \in \mathbb{C}$, $r > 0$ and $B(z_0, r) = \{z \in \mathbb{C} : | z - z_0 | < r\}$ be an open disk.

Let $a \in B(z_0, r) \Rightarrow |\, a - z_0 \,| < r$ (i)

If $|z-a| < d$ then

$$|\, z - z_0 \,| = |\, z - a + a - z_0 \,| \le |\, z - a \,| + |\, a - z_0 \,|$$
$$< \delta + r - \delta \quad \text{(From (i))}$$
$$< r$$

$\therefore$ $|\, z - z_0 \,| < r$

i.e. $z \in B(a, \delta) \Rightarrow z \in B(z_0, r)$

$\Rightarrow$ $B(a, \delta) \subset B(z_0, r)$

$\Rightarrow$ Any open disk is an open set

5) The complement of a set $S \subset \mathbb{C}$ is denoted by S^c, and defined by $S^c = \{z \in \mathbb{C} : z \notin S\}$

6) A set $F \subset \mathbb{C}$ is said to be closed if its complement i.e. F^c is open.

OR

A set *F* is said to be closed if it contains all its limit points.

7) A set of the form $\overline{B}(z_0, r) = \{z \in \mathbb{C} : |\, z - z_0 \,| \le r\}$ is called the <u>closed disk or closed ball</u>.

e.g. i) $\mathbb{C}$ is closed set

ii) $\varnothing$ is closed set

iii) $E = \{z \in \mathbb{C} : \operatorname{Im} z = 4\}$

iv) $S = \{z \in \mathbb{C} : |\, z \,| \le 2\}$

v) $S = \{z \in \mathbb{C} : |\, z - 2 \,| \le |\, z \,|\}$

8) Interior point : Let $S \subset \mathbb{C}$, then the point $z \in S$ is said to be an <u>interior point</u> of set *S* if $\exists\ r > 0$ s.t. $B(z, r) \subset S$.

9) The point $C \in S$ is said to be exterior point of the set *S* if $\exists$ a $B(c, r)$ which does not contain any point of set *S*.

10) A point $p \in S$ is said to be a boundary point of set *S* if it is a neither a interior point nor an exterior point.

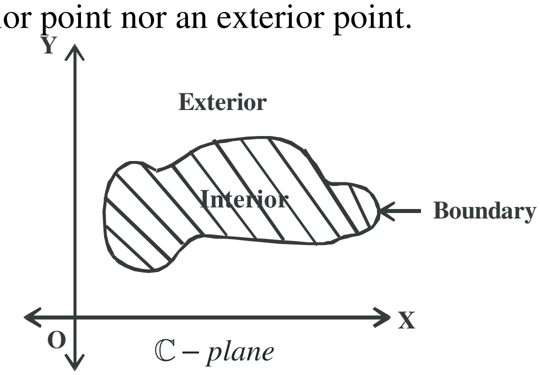

Fig 2.5

11) The set of all interior point of the set S is said to be <u>interior of S</u>.

12) A set $G \subset \mathbb{C}$ is said to be <u>open</u> if each point of G is an interior point of G.

13) <u>Closure set</u> : The closure of the set $S \subset \mathbb{C}$, denoted by $Cl(S)$

$Cl(S) = S \cup \delta S$ (where δS is boundary element is always closed.)

14) A subset S of $\mathbb{C}$ is said to be <u>Dense</u> if $Cl(S) = \mathbb{C}$

e.g. i) $\mathbb{Q}$ is dense in $\mathbb{R}$.

ii) $\{x + iy \,/\, x \in \mathbb{Q},\ y \in \mathbb{Q}\}$ is dense in $\mathbb{C}$.

15) An open set G is said to be <u>connected</u> if for any two points z_1 and z_2 can be joined by a curve that lies entirely in G.

OR

A metric space (X, d) is said to be <u>connected</u> if the only subset of X which are both open and closed are X and ($\varnothing = \textit{the empty set}$).

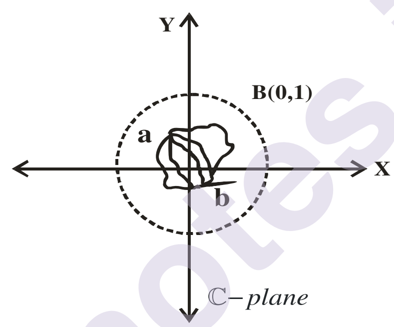

Fig 2.6

e.g. 1) Open disk is a connected set.

2) The unit disk $B(0,1) = \{z \in \mathbb{C} \,||\, z \,| < 1\}$ is a connected set.

3) The annulus $B = \{z \in \mathbb{C} : 1 \le | \, z \,| < 2\}$ is connected Fig. 2.6

4) The set $S = \{z \in \mathbb{C} : | \, z - 2 \,| < 1\}$ or $|z + 2| < 1$ is not connected Fig. 2.6(b).

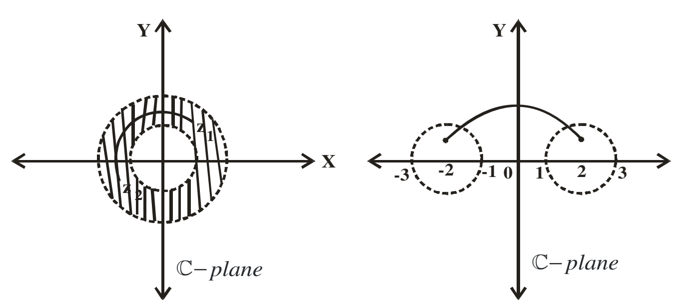

fig (a) $|z - 2| < 1$ fig (b) $|z + 2| < 1$

Fig 2.7

16) A domain is an open connected set.

17) A domain together with some none or all of its boundary point is referred to as a region.

18) Bounded Set : A set is said to be bounded if $\exists\ R > 0$ s.t. $S \subset B(O, R) = \{z \in \mathbb{C} : |z| \le R\}$.

19) A set which is closed as well as bounded is called compact set.

20) A set that cannot be enclosed by any closed disk is called unbounded set.

21) Let $z_1 \in \mathbb{C}$ and $z_2 \in \mathbb{C}$. These we denote the line segment from z_1 to z_2 by $[z_1, z_2] = \{(1-t) z_1 + t z_2 : 0 \le t \le 1\}$

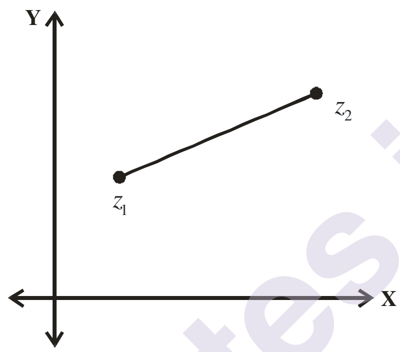

Fig 2.8

Function, limits and continuity :

Definition : Let A and B be two non-empty subset of complex numbers. A function from A to B is a rule, f, which associates to each $z_0 = x_0 + iy_0 \in A$ a unique $w_0 = u_0 + i\, v_0 \in B$

The number w_0 is the value of f at z_0 and we write $f(z_0) = w_0$. If z varies in A then $f(z) = w$ varies in B. We say that f is a complex valued function of a complex variable.

Here w is the dependent and z is the independent variable.

Let $f : A \to B$ be a function and $S \subset A$ then $f(S) = \{f(z)/, z \in S\}$ where $f(S)$ is called the image of S under 'f' and the set $R = \{f(z) / z \in A\}$ is called range of 'f'.

Single and Multiple Valued Function :

Let $z \in \mathbb{C} - \{0\}$, then we write the polar form of a complex number z is $z = re^{i\theta}$ where $r = |z|$ and $\theta \in [-\pi, \pi]$ i.e. $z = z(r, \theta) = re^{i\theta}$.

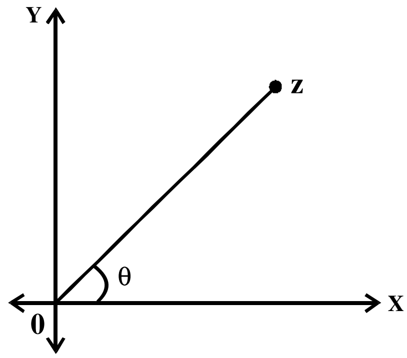

Fig 2.9

If we increase θ to $\theta+2\pi$

$\therefore \quad z(r,\theta+2\pi)=re^{i(\theta+2\pi)}=re^{i\theta}.e^{2\pi i}=re^{i\theta}=z(r,\theta)$ returning to its original value.

Definition : A function *f* is said to be a single valued if *f* satisfies $f(z)=f(z(r,\theta))=f(r,\theta+2\pi)$.

Otherwise, *f* is said to be a <u>multiple valued function</u>.

e.g. $f(z)=z^n, \quad n\in\mathbb{Z}$ is a single valued function.

Solution : $\because \; f(z)=f(z(r,\theta))=\left(re^{i\theta}\right)^n$

$$f(z(r,\theta+2\pi))=\left[r\,e^{i(n\theta+2\pi)}\right]^n=r^n e^{i(n\theta+2n\pi)}=r^n e^{in\theta}.e^{2in\pi}$$

$$=r^n e^{in\theta} \qquad \left\{\because e^{2in\pi}=1, n\in\mathbb{Z}\right\}$$

$$=\left(re^{i\theta}\right)^n=f(z(r,\theta))$$

Note : If $n\notin\mathbb{Z}$ then $f(z)=z^n$ is a multiplied valued function.

$\because \; e^{2i\pi n}\neq 1$, when $n\notin\mathbb{Z}$

Let $f:A\to B$ be a function.

i) If the elements of A are complex numbers and those of *B* are Real Numbers then we say that *f* is a <u>real valued function</u> of complex variable.

ii) If the elements of A are Real Numbers and those of *B* are complex numbers then we say that *f* is a <u>complex valued function</u> of real variable.

Let $f:\mathbb{R}\to\mathbb{R}$ be a function then the graph of *f* is a subset of $\mathbb{R}\times\mathbb{R}$ and it is two dimensional object and we can represent it very well on the two dimensional page. However the graph of the $f:\mathbb{C}\to\mathbb{C}$ is a subset of $\mathbb{C}\times\mathbb{C}$ ($\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$ Cartesian product) i.e. a four dimensional object and we cannot represent it on two dimensional plane. In this case we consider two plane, one plane is z-plane and other one is w-plane.

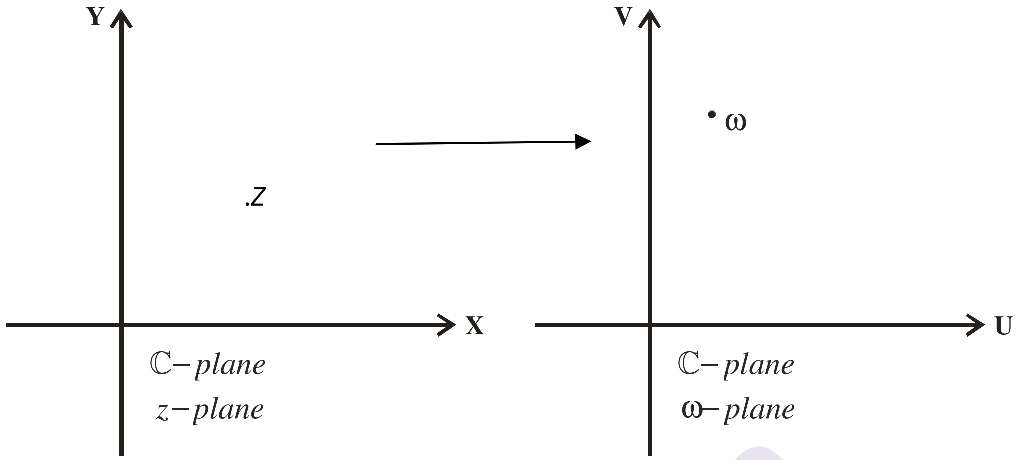

Fig 2.10

<u>Limit Point</u> :

Let D be a subset of $\mathbb{C}$ i.e. $D\subset\mathbb{C}$ then we say that a point z_0 is a limit point of D if every neighbourhood of z_0 contains a point of D other than z_0 i.e. $(B(z_0,r)-\{z_0\})$ for any $r>0$.

Definition : Let f be a complex valued function defined on D and let $z_0\in Cl(D)$. We say that a number ℓ is a <u>limit</u> of $f(z)$ as $z\to z_0$ and we write $\lim\limits_{z\to z_0} f(z)=\ell$.

$\Leftrightarrow$ iff $\forall\,\varepsilon>0,\ \exists\ \delta>0$ s.t. $|f(z)-\ell|<\varepsilon$ whenever $z\in D$ and $0<|z-z_0|<\delta$.

$\Leftrightarrow f(z)\in B(\ell,\varepsilon)$ wherever $z\in D\cap\left[B(z_0,\delta)-\{z_0\}\right]$

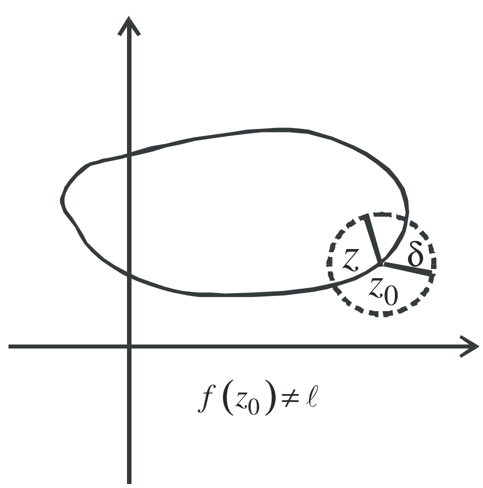

Fig 2.11

Note : 1) *f may* not be defined at $z = z_0$

2) z_0 need not be in *D*.

3) even if $z_0 \in D$, $f(z_0) \neq \ell$

4) In real variable theory, if $x_0 \in \mathbb{R}$ then $x \to x_0$ has only two possible ways, either from left or from right. In complex case, $z \to z_0$, in any manner in the Complex Plane.

Theorem : *Let f be a complex valued function defined on D and let* $z_0 \in Cl(D)$. *If* $\lim\limits_{z \to z_0} f(z)$ *exists, then this limit is unique.*

Proof : Let $\lim\limits_{z \to z_0} f(z) = \ell_1$ and $\lim\limits_{z \to z_0} f(z) = \ell_2$

T.P.T. $\ell_1 = \ell_2$

By definition for a given $\varepsilon > 0$, $\exists\ \delta_1 > 0,\ \delta_2 > 0$ s.t. $|f(z) - \ell_1| < \varepsilon/2$, whenever $z \in D \cap [B(z_0, \delta_1) - \{z_0\}]$ and $|f(z) - \ell_2| < \varepsilon/2$, whenever $z \in D \cap [B(z_0, \delta_2) - \{z_0\}]$.

Let $\delta = \min\{\delta_1, \delta_2\}$.

If $z \in D \cap \{B(z_0, \delta) - \{z_0\}\}$ then

$|\ell_1 - \ell_2| = |\ell_1 - f(z) + f(z) - \ell_2| \leq |\ell_1 - f(z)| + |f(z) - \ell_2|$

$< \varepsilon/2 + \varepsilon/2 = \varepsilon$

$\because$ ε is arbitrary.

$\therefore$ $\ell_1 = \ell_2$

i.e. limit is unique.

Theorem : *Let f be a complex valued function defined on D. suppose,* $f(z) = u(x, y) + iv(x, y)$, $z_0 = x_0 + i\,y_0$, $w_0 = u_0 + iv_0$ *and* $z_0 \in Cl(D)$.

Then $\lim\limits_{z \to z_0} = w_0$ *iff* $\lim\limits_{z \to z_0} u(x, y) = u_0$ *and* $\lim\limits_{z \to z_0} v(x, y) = v_0$.

Proof : Direct part –

Let $\lim\limits_{z \to z_0} f(z) = w_0$ and $w_0 = u_0 + iv_0$

By definition, $\forall\ \varepsilon > 0,\ \exists\ \delta > 0$ s.t. $|f(z) - w_0| < \varepsilon$ whenever $z \in D \cap [B(z_0, \delta) - \{z_0\}]$.

Now,

$|f(z) - w_0| = |u(x, y) + iv(x, y) - (u_0 + iv_0)|$

$= |[u(x, y) - u_0] + i[v(x, y) - v_0]|$

$$\geq \begin{cases} |u(x,y)-u_0| \\ |v(x,y)-v_0| \end{cases} \qquad \begin{pmatrix} \because \operatorname{Re} z \leq |\operatorname{Re} z| \leq |z| \\ \operatorname{Im} z \leq |\operatorname{Im} z| \leq |z| \end{pmatrix}$$

If $z \in D \bigcap \left[B(z_0, \delta) - \{z_0\}\right]$ then $|u(x,y)-u_0| < \varepsilon$ and $|v(x,y)-v_0| < \varepsilon$

$\Rightarrow \lim_{z \to z_0} u(x,y) = u_0$ and $\lim_{z \to z_0} v(x,y) = v_0$

Conversely, assume that $\lim_{z \to z_0} u(x,y) = u_0$ & $\lim_{z \to z_0} v(x,y) = v_0$

$\because$ By definition given $\in > 0 \ \exists \ \delta_1, \delta_2 > 0$.

s.t. $|u(x,y)-u_0| < \in/2$ and $|v(x,y)-v_0| < \in/2$ whenever $|z-z_0| < \delta_1$ and $|z-z_0| < \delta_2$.

let $\delta = \min\{\delta_1, \delta_2\}$.

$\because$ whenever $|z-z_0| < \delta$,

Consider $|f(z)-w_0 = |u(x,y)+i\,v(x,y)-u_0-iv_0||$

$\leq |u(x,y)-u_0| + |v(x,y)-v_0| < \frac{\in}{2} + \frac{\in}{2} = \in$

$\Rightarrow \mathop{lin}_{z \to z_0} f(z) = w_0$

Examples : If $f(z) = \frac{iz}{2}$ in the open disk $B(0,1)$, prove that $\lim_{z \to 1} \frac{iz}{2} = \frac{i}{2}$.

Solution : Given $f(z) = \frac{iz}{2}$

$\therefore$ We must prove that for every $\varepsilon < 0$, $\exists \ \delta > 0$, s.t. $\left|\frac{iz}{2} - \frac{i}{2}\right| < \varepsilon$ whenever $z \in B(0,1)$ and $0 < |z-1| < \delta$, $f(z) \neq \ell$

if $0 < |z-1| < \delta$, then $\left|\frac{iz}{2} - \frac{i}{2}\right| = \frac{|i|}{2}|z-1| < \frac{\delta}{2}$

choosing $\delta = 2\varepsilon$, we see that $\left|\frac{iz}{2} - \frac{i}{2}\right| < \varepsilon$ whenever $z \in B(0,1)$ and $0 < |z-1| < \delta$.

$$\therefore \boxed{\lim_{z \to 1} \frac{iz}{2} = \frac{i}{2}}$$

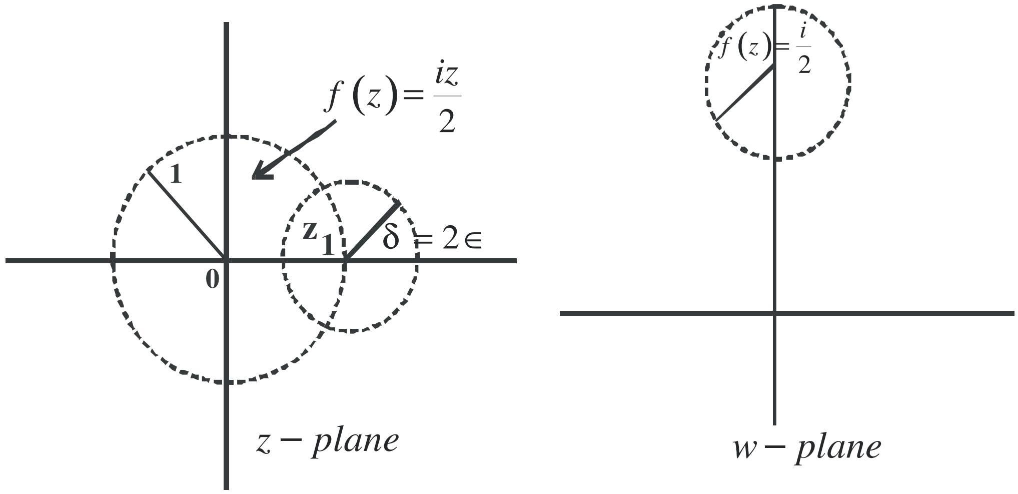

Fig 2.12

Problem : If $f(z)-\frac{zi}{2}$ in the open disk $|z|<1$ prove that $\lim_{z\to 1} f(z)=\frac{i}{2}$.

Solution : Given $f(z)=\frac{iz}{2}$

We must prove that for every $\varepsilon>0$, for given any $\varepsilon>0$ we can find $\delta>0$ s.t. $\left|\frac{iz}{2}-\frac{i}{2}\right|<\varepsilon$ whenever $0<|z-1|<\delta$.

If $0<|z-1|<\varepsilon$

$\Rightarrow \frac{|iz-i|}{2}<\varepsilon \quad \Rightarrow |i||z-1|<2\varepsilon \qquad \Rightarrow |z-1|<2\varepsilon$

$\because |i|=1$

Choosing, $\delta=2\varepsilon$, we see that $\left|\frac{iz}{2}-\frac{i}{2}\right|<\varepsilon$, whenever $0<|z-1|<\delta$

$\therefore \lim_{z\to 1}\frac{iz}{2}=\frac{i}{2}$

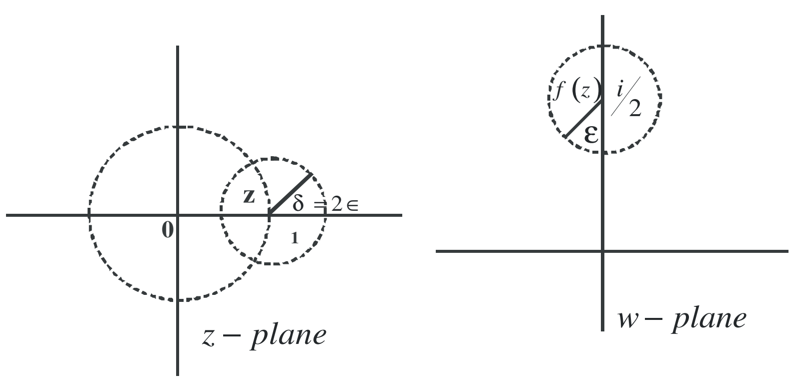

Fig 2.13

Problem : Prove that $\lim_{z\to 0}\frac{\bar{z}}{z}$ does not exist.

Solution : We know that the function $f(z)\to \ell$ (a unique limit) as $z\to z_0$ in any manner in the $\mathbb{C}$-plane.

Let $f(z)=\frac{\bar{z}}{z}$

Let $z\to 0$, along the real axis.

$\therefore\ y=0,\ z=x \qquad (\because z=x+iy)$

$\therefore\ \lim_{z\to 0} f(z)=\lim_{z\to 0}\frac{\bar{z}}{z}=\lim_{z\to 0}\frac{x}{x}=1$

Let $z\to 0$, along the imaginary axis.

$\therefore\ x=0,\ z=iy \qquad (\because z=x+iy)$

$\therefore\ \lim_{z\to 0} f(z)=\lim_{z\to 0}\frac{\bar{z}}{z}=\lim_{z\to 0}\frac{-iy}{iy}=-1$

$\Rightarrow$ limit is not unique along real and imaginary axis.

$\therefore\ \lim_{z\to 0}\frac{\bar{z}}{z}$ does not exist.

Problem : If $f(z)=z^2$, prove that $\lim_{z\to z_0} f(z)=z^2$.

Solution : Let $\in>0$ given, to find $\delta>0$ s.t. $\left|z^2-z^2\right|<\varepsilon$ whenever $0<\left|z-z_0\right|<\delta$

consider $\left|z^2-z^2\right|=\left|(z+z_0)(z-z_0)\right|$

$\left|z+z_0\right|\left|z-z_0\right|<\delta\left|z+z_0\right|$

$=\delta\left|z-z_0+2z_0\right|\le\delta\left|z-z_0\right|+2\delta\left|z_0\right|<\delta.\delta+2\delta\left|z_0\right|=\in$

$\because$ Choose $\delta>0$ s.t. min $\left\{\frac{\in}{1+2\left|z_0\right|},1\right\}$

$\Rightarrow\left|z^2-z0^2\right|<\in$.

$\Rightarrow \lim_{z\to z_0} f(z)=zo^2$.

Theorem : *Let f and g be defined in the neighbourhood of z_0 except possibly at $z=z_0$.*

If $\lim_{z\to z_0} f(z)=\ell$ and $\lim_{z\to z_0} g(z)=m$

Then 1) $\lim_{z\to z_0}\left[f(z)\pm g(z)\right]=\ell\pm m$

2) $\lim_{z\to z_0}\left[f(z).g(z)\right]=\ell m$

3) $\lim_{z \to z_0} \left[\frac{f(z)}{g(z)} \right] = \frac{\ell}{m}$

Continuity

Definition : A function $f : D \to \mathbb{C}$ is said to be continuous at a point $z_0 \in D$, iff $\lim_{z \to z_0} f(z)$ exists and $\lim)_{z \to z_0} f(z) = f(z_0)$.

OR

Definition :A function $f : D \to \mathbb{C}$ is said to be continuous at a point $z_0 \in D$ iff $\forall\ \varepsilon > 0,\ \exists\ \delta > 0$ s.t. $|f(z) - f(z_0)| < \varepsilon$ whenever $z \in D$ and $|z - z_0| < \varepsilon$

OR

Definition : A function *f* is said to be continuous at a point $z_0 \in D$ iff the following 3 conditions hold true :

i) f is defined at z_0 i.e. $f(z_0)$ exists.

ii) $\lim_{z \to z_0} f(z)$ exists

iii) $\lim_{z \to z_0} f(z) = f(z_0)$

OR

Definition : A function $f : D \to \mathbb{C}$ is continuous or *f* is continuous on *D* if *f* is continuous at every point of *D*.

Example : If $f(z) = z^2$ then prove that *f* is continuous at a point $z = i \in \mathbb{C}$.

Solution : Given, $f(z) = z^2, \ z_0 = i$

$\therefore \quad f(i) = i^2 = -1$

$\therefore \quad \lim_{z \to i} z^2 = \ell^2 = -1$

$\therefore \quad \lim_{z \to i} z^2 = -1 = f(i)$

$\Rightarrow$ *f* is continuous at a point $z = i$.

Example : Let $f(z) = \begin{cases} z^2 & z \neq i \\ 0 & z = i \end{cases}$ prove that *f* is not continuous at a point $z = i$.

Solution : $f(i)=0$ (given)

$\therefore \lim_{z\to i} f(z)=\lim_{z\to i} \ell^2=-1$

$\therefore \lim_{z\to i} f(z)=-1\neq f(i)$

$\therefore$ f is not continuous at $z=z_0$

Problem : Discuss the continuity of $f(z)=\dfrac{z^2}{z^4+3z^2+1}$ at $z=e^{i\frac{\pi}{4}}$

Solution : $z=e^{i\frac{\pi}{4}}\Rightarrow z^2=e^{i\frac{\pi}{2}}=i\Rightarrow z^4=-1$

$\therefore f(z)=\dfrac{i}{-1+3i+1}=\frac{1}{3}$

$\therefore$ the limit exist $z=e^{i\frac{\pi}{4}}$.

$\therefore$ $f(z)$ is continuous at $z=e^{i\frac{\pi}{4}}$

Uniformly Continuous : A function $f:D\to\mathbb{C}$ is said to be uniformly continuous on D iff the following conditions holds for every $\varepsilon>0$, $\exists$ $\delta>$ s.t. for any two points z_1 and z_2 in *D*.

$|z_1-z_2|<\delta\Rightarrow|f(z_1)-f(z_2)|<\varepsilon$

Example : Let $f(z)=z^2$ in the open disk $B(0,1)$, prove that f is uniformly continuous on $B(0,1)$.

Solution : Given, $f(z)=z^2$.

$\therefore$ We must prove that for a given $\varepsilon>0$, we can find $\delta>0$, s.t. for any two points z_1 and $_2$ in $B(0,1)$ and

$|z_1-z_2|<\delta\Rightarrow|f(z_1)-f(z_2)|=\left|z_1^2-z_2^2\right|=|z_1-z_2|\,|z_1+z_2|$

$<\delta(|z_1|+|z_2|)$ by triangle inequality

$<2\delta$ $(\because|z_1|<1$ and $|z_2|<1)$

Choosing $\delta=\dfrac{\varepsilon}{2}$, we see that $\left|z_1^2-z_2^2\right|<\varepsilon$ whenever $|z_1-z_2|<\delta$, $\forall\, z_1,z_2\in B(0,1)$.

$\therefore$ f is uniformly continuous on $B(0,1)$.

Definition : <u>Unbounded set</u> :

A set E is said to be unbounded if $\exists$ $R>0$ s.t. $z\in E$ $\forall z\in E$.

Definition : Limit at Infinity

Let f be defined on an unbounded set E. If for each $\varepsilon > 0$, $\exists\ R > 0$ s.t. $|f(z) - \ell| < \varepsilon$ whenever $z \in E$ and $|z| > R$ then we say that $f(z) \to \ell$ as $z \to \infty$ and we write $\lim_{z \to \infty} f(z) = \ell$.

e.g. $\lim_{z \to \infty z} \frac{1}{z} = 0$ for given $\in > 0$ above $R > 0$ s.t. $R > \frac{1}{\in}$

$\Rightarrow \left|\frac{1}{z} - 0\right| = \frac{1}{|z|} < \varepsilon \left(|z| > R > \frac{1}{\in}\right)$

Infinite Limit :

Let f be defined out D except possible at $z_0 \in D$. If for every $R > 0,\ \exists\ \delta > 0$ s.t. $|f(z)| > R$ whenever $0 < |z - z_0| < \delta$ then we say that $f(z) \to \infty$ as $z \to z_0$ and we write $\lim_{z \to z_0} f(z) = \infty$.

e.g. $\lim_{z \to 1} \frac{1}{z^2 - 1} = \infty$

2.4 SUMMARY

1) Let f be a complex valued function defined on D and let $z_0 \in Cl(D)$. If $\lim_{z \to z_0} f(z)$ exists, then this limit is unique.

2) Let f and g be defined in the neighbourhood of $z_0 e$ except possibly at

If $\lim_{z \to z_0} f(z) = \ell$ and $\lim_{z \to z_0} g(z) = m$

Then 1) $\lim_{z \to z_0} [f(z) \pm g(z)] = \ell \pm m$

2) $\lim_{z \to z_0} [f(z) . g(z)] = \ell m$

3) $\lim_{z \to z_0} \left[\frac{f(z)}{g(z)}\right] = \frac{\ell}{m}$

3) A function f is said to be continuous at a point $z_0 \in D$ iff the following 3 conditions holds

i) f is defined at z_0 i.e. $f(z_0)$ exists.

ii) $\lim_{z \to z_0} f(z)$ exists

iii) $\lim_{z \to z_0} f(z) = f(z_0)$

4) A function $f: D \to \mathbb{C}$ is said to be Uniformly continuous iff the following conditions holds for every $\varepsilon > 0$, $\exists$ $\delta >$ s.t. for any two points z_1 and z_2 in D then $|z_1 - z_2| < \delta \Rightarrow |f(z_1) - f(z_2)| < \varepsilon$

5) Every convergent sequence is a Cauchy Sequence.

6) $\mathbb{C}$ is complete.

2.5 UNIT END EXCERCISES

1) Find the limit of a sequence $z_n = z^n$ for $|z| < 1$.

Solution : Consider $|z_n - 0| = |z^n - 0| = |z^n| \to 0$ as $n \to \infty$, for $|z| < 1$.

2) Check whether the sequence $z_n = \dfrac{n}{n+i}$ is convergent or not.

Solution : $\because z_n = \dfrac{n}{n+i}$, then $a_n = \left(\dfrac{i}{3+4i}\right)^n$ $|z-2| < 5$, because

$$\left|\frac{n}{n+i} - 1\right| = \left|-\frac{i}{(n+i)}\right| = \frac{1}{\sqrt{n^2+1}} \to 0 \text{ as } n \to \infty .$$

3) Which of the following subsets of $\mathbb{C}$ are connected , if not connected then what are it's components ?

(a) $X = \{z : |z| \le 1\}$ Ans : X is connected .

(b) $X = \{z : |z| \le 1\} \cup \{z : |z-2| \le 1\}$

Ans: X is not connected , because $X = \{z : |z| \le 1\} \cup \{z : |z-2| \le 1\}$ is a disjoint union of nonempty closed subsets (Components) of X.

4) Let z_n, z be points in $\mathbb{C}$ and let d be the metric on $\mathbb{C}_\infty$. Show that $|z_n - z| \to 0$ if and only if $d(z_n, z) \to 0$ as $n \to \infty$.

(Hint: For $z, z' \in \mathbb{C}$, $d(z, z') = \dfrac{2|z - z'|}{\sqrt{((1+|z|^2)(1+|z'|^2))}}$ and

$$d(z, \infty) = \frac{2}{(1+|z|^2)^{\frac{1}{2}}}$$

5) Let $P(z)$ be a nonconstant polynomial in z. Show that $P(z) \to \infty$ as $z \to \infty$.

6) Suppose $f : X \to \Omega$ is uniformly continuous , show that if $\{x_n\}$ is a Cauchy sequence in , then $\{f(x_n)\}$ is a Cauchy sequence in Ω.

7) Show that if f and g are bounded uniformly continuous functions from X into $\mathbb{C}$ then fg is also bounded and uniformly continuous function from X into $\mathbb{C}$.

(Hint:

$$\left|(fg)(x)-(fg)(y)\right| = \left|f(x)g(x)-f(y)g(y)\right|$$
$$\le \left|f(x)-f(y)\right|\left|g(x)\right| + \left|f(y)\right|\left|g(x)-g(y)\right|.)$$

8) Verify the continuity of the following function f of the extended complex plane $\mathbb{C}\cup\{\infty\}$ at the point $a=\frac{-3}{4}$ (2012)

$$f(z)=\infty \qquad \text{if } z \ne \frac{-3}{4}$$
$$= \frac{z+1}{4z+3} \qquad \text{if } z \ne \frac{-3}{4}$$

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3

SERIES OF COMPLEX NUMBERS

Unit Structure

3.0. Objectives
3.1. Introduction
3.2. Convergence of Series
3.3. Tests for determining the Convergence of Power Series
3.4. Summary
3.5. Unit End Exercises

3.0 OBJECTIVES

This unit shall make you construct a series of complex numbers by understanding the definition of a series of real's . Basically, we are going to define a power series of the form $\sum_{n=0}^{\infty} a_n z^n$. We shall check for the conditions , under which the given power series is convergent or not. Hence, we shall employ certain tests in order to determine the convergence of the given power series.

3.1 INTRODUCTION

An infinite series of real's is the expression of the form $\sum_{k=1}^{\infty} a_k$, where a_k is a real number for all $k \geq 1$. Similarly we construct an infinite series of complex numbers as $\sum_{n=1}^{\infty} z_n$, where $z_n's$ are complex numbers for all $n \geq 1$. For example $\sum_{k=1}^{\infty} \frac{i^k}{k^2+i}$ is an infinite series of complex numbers. To check whether the sum exists or not, in other words whether a given series of complex numbers is convergent or not, we employ certain tests for convergence and we shall convert the given problem of checking convergence of the series of complex numbers to checking convergence of the series of real numbers. For example, $\sum_{k=1}^{\infty} \left|\frac{i^k}{k^2+i}\right|$ is a convergent series of real numbers, because $\left|\frac{i^k}{k^2+i}\right| = \frac{1}{\sqrt{k^4+1}}$ and we know that $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k^4+1}}$

converges. Let us start with defining a series of complex numbers.

3.2 COVERGENCE OF SERIES

Definition: Let $\{z_n\}$ be a sequence of complex numbers, Form a new sequence defined by $S_1 = z_1,\ S_2 = z_1 + z_2, ...,\ S_n = z_1 + z_2 + ... z_n$, where S_n is called the sequence of nth partial sums of sequence $\{z_n\}$.

The sequence $\{S_n\}$ is symbolized by $z_1 + z_2 + ... = \sum_{n=1}^{\infty} z_n$ called an Infinite series.

If $\lim_{n\to\infty} S_n = S$ exists then the series is said to be convergent and S is its sum i.e. $\sum_{n=1}^{\infty} z_n = \lim_{n\to\infty} \sum_{k=1}^{\infty} z_k = \lim_{n\to\infty} S_n = S$

A series is said to be divergent if it is not convergent sequence. (The necessary condition for the convergence of the series.)

Theorem : *If the series* $\sum_{n=1}^{\infty} z_n$ *is convergent then* $\lim_{n\to\infty} z_n = 0$.

Proof : Given series is $\sum_{n=1}^{\infty} z_n$.

Let $S_n = z_1 + z_2 + ... + z_{n-1} + z_n$ ....(1)

be the n^{th} partial sum of series.

Given that the series is convergent

Let S be the sum $\sum_{n=1}^{\infty} z_n$

$\therefore \lim_{n\to\infty} S_n = S$

$\therefore$ from equation (1)

$z_n = S_n - S_{n-1}$ $\quad (\because z_1 + z_2 + ... + z_{n-1} = S_{n-1})$

Taking limit on both sides,

$\therefore \lim_{n\to\infty} z_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S$

$\therefore \lim_{n\to\infty} z_n = 0$

Consider the infinite series $\sum_{n=0}^{\infty} z_n = z_0 + z_1 + ...$

If $R_n = z_n + z_{n+1} + z_{n+2} + ...$ then R_n is called remainder of infinite series. If S is sum of infinite series then $S = S_n + R_n$ or $R_n = S - S_n$.

Theorem : *A series* $\sum_{n=0}^{\infty} z_n$ *of complex terms is convergent iff for every* $\varepsilon > 0$*,* $\exists$ *an integer N s.t.* $\left| z_n + z_{n+1} + ... + z_{n+p} \right| < \varepsilon \ \forall \ n \geq N$ *and* $p \geq 0$*. (Cauchy criteria for convergence of series)*

Proof : Suppose $\sum_{n=0}^{\infty} z_n$ is convergent

Let $S_n = z_0 + z_1 + z_2 + ... + z_{n-1}$ be the n[th] partial sum of series and let S be a sum of series.

$\therefore \lim_{n \to \infty} S_n = S$

$\forall \ \varepsilon > 0, \ \exists$ an integer N s.t. $|S_n - S| < \varepsilon$ (1) $\forall \ n \geq N$

Let $R_n = z_n + z_{n+1} + z_{n+2} + ...$ be the remainder of an infinite series.

$\therefore S = S_n + R_n$ or $R_n = S - S_n$

$\therefore$ From equation (1)

$|S_n - S| = |S - S_n| = |R_n| < \varepsilon \quad \forall \ n \geq N$

i.e. $|z_n + z_{n+1} + z_{n+2} + ...| < \varepsilon \quad \forall \ n \geq N$

$\therefore \left| z_n + z_{n+1} + z_{n+2} + ... + z_{n+p} \right| < \varepsilon \quad \forall n \geq N$ and $p \geq 0$

Converse :

Given $\varepsilon > 0$, there is an integer N s.t.

$\left| Z_n + z_{n+1} + z_{n+2} + ... + z_{n+p} \right| < \varepsilon$ ....(2)

$\forall n \geq N$ and $\forall \ p \geq 0$

We know that, $\sum_{n=0}^{\infty} z_n = z_0 + z_1 + ... + z_n + ...$

If S is its sum then we write $S = S_n - R_n$

$\therefore R_n = S_n - S \Rightarrow |R_n| < |S_n - S|$

But $|R_n| = \left| z_n + z_{n+1} + ... + z_{n+p} \right| < \varepsilon$ given from (2)

$\forall \ n \geq N$ and $\forall \ p > 0$

$\Rightarrow \quad |S_n - S| < \varepsilon \quad \forall \ n \geq N$

$\Rightarrow \quad \lim_{n \to \infty} S_n = S$

$\therefore \sum_{n=1}^{\infty} z_n$ is convergent.

Definition : Let $z_n \in \mathbb{C}$. For every $n \geq 0$ the series $\sum_{n=1}^{\infty} z_n$ converges to z_0 iff for every $\varepsilon > 0$, $\exists$ an integer N s.t. $\left|\sum_{k=1}^{n} z_k - z_0\right| < \varepsilon \quad \forall n \geq N$.

Definition : A series $\sum_{n=1}^{\infty} z_n$ converges <u>absolutely</u> if $\sum_{n=1}^{\infty} |z_n|$ converges.

Proposition : If the series $\sum_{n=1}^{\infty} z_n$ converges absolutely then $\sum_{n=1}^{\infty} z_n$ converges.

Proof : Let $\varepsilon > 0$, consider an infinite series $\sum_{n=1}^{\infty} z_n$.

Let $S_n = z_1 + z_2 + ... + z_n$ be the partial sum of series given that $\sum_{n=1}^{\infty} z_n$ convergent absolutely.

$\therefore$ For a given $\varepsilon > 0$, $\exists$ an integer N s.t.

$$\sum_{k=n+1}^{\infty} |z_k| < \varepsilon \qquad \forall\, n \geq N \qquad (1)$$

If $n > m \geq N$ then

$\therefore |S_n - S_m| = |z_{m+1} + z_{m+2} + ... + z_n| = \left|\sum_{k=m+1}^{n} z_k\right| \leq \sum_{k=n+1}^{n} |z_k|$

$\leq \sum_{k=n+1}^{\infty} |z_k| < \varepsilon$ from (1)

$\Rightarrow \{S_n\}$ is a Cauchy sequence.

$\Rightarrow \{S_n\}$ is a convergent Sequence. ($\because$ by Cauchy criteria)

$\therefore \exists\ z_0 \in \mathbb{C}$ s.t. $\lim_{n \to \infty} S_n = z_0$

Thus, $\sum_{n=1}^{\infty} z_n$ is convergent.

Examples :

1) Prove that $\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + ... = \frac{1}{1-z}$ if $|z| < 1$

Solution : Given $\sum_{n=0}^{\infty} z_n = 1 + z + z^2 + ...$

Let $S_n = 1 + z + z^2 + ... + z^{n-1}$

$z\, S_n = z + z^2 + z^3 + ... + z^n$ (multiplied by z, $z \neq 0$)

$\therefore\ S_n - z S_n = 1 - z^n \quad \Rightarrow \ S_n(1-z) = 1 - z^n \qquad \Rightarrow \qquad S_n = \dfrac{1-z^n}{1-z}$

$(z \neq 1)$

T.P.T. $S_n = \dfrac{1}{1-z}$ ……..(1)

i.e. T.P.T. $\lim\limits_{n\to\infty} z^n = 0$

Given any $\varepsilon > 0$, we must find integer N s.t. $\left| z^n \right| < \varepsilon \quad \forall\ n \geq N$ [If $z = 0$, then the result is true].

Let $z \neq 0$

$\therefore \quad \left| z^n \right| = |z|^n < \varepsilon$

$\Rightarrow \quad n \log|z| < \log \varepsilon$

$\Rightarrow \quad n > \dfrac{\log \varepsilon}{\log|z|}$ $\{\because \log|z|$ is negative when $|z| < 1\}$

Choosing $N = \dfrac{\log \varepsilon}{\log|z|}$, we see that $\left| z^n \right| < \varepsilon \quad \forall\ n > N$

$\therefore \quad \lim\limits_{n\to\infty} z^n = 0$

$\therefore$ From equation (1)

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[\frac{1-z^n}{1-z}\right] = \frac{1}{1-z} \qquad \left(\because z^n = 0 \text{ as } n \to \infty\right)$$

Hence, $\sum\limits_{n=0} z^n = \dfrac{1}{1-z}$ when $|z| < 1$

Note : 1) Geometric series (G.S.) $\sum\limits_{n=0}^{\infty} z^n$ is cgt when $|z| < 1$ and divergent when $|z| \geq 1$.

<u>Uniformly converges for series</u> :

For each $n \in \mathbb{N}$, let $f_n(z)$ be a complex function of complex variable.The series $\sum f_n(z)$ converges to $f(z)$ <u>point wise</u> for each $z \in D$ iff $\sum f_n(z) = f(z)$ and for each $z \in D$ [This means that for each $z \in D$ and for each $\varepsilon > 0$, $\exists$ an integer N (depends on z) and ε], s.t. $\left| S_n(z) - f(z) \right| < \varepsilon \quad \forall\ n \geq N$.

Definition : The series $\sum f_n(z)$ is said to be <u>uniformly convergent</u> on D to *f(z)* if for every $\varepsilon > 0$, $\exists$ an integer N (depends only on ε) s.t. $\left| S_n(z) - f(z) \right| < \varepsilon\ \forall\ z \in D$, and $\forall\ n \geq N$. A <u>power series</u> about

z_0 is an infinite series of the form $\sum_{n=0}^{\infty} a_n (z-z_0)^n = a_0 + (z-z_0)a_1 + (z-z_0)^2 a_2 + \ldots\ldots\ldots$, where constants a_n and z_0 are called complex numbers and z is a complex variable.

Note : If $z_0 = 0$ then $\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 + \ldots$

This is power series about origin (i.e. $z = 0$)

e.g. Geometric Series (G.S.) $\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \ldots$

i) If $|z| < 1$ then $\lim_{n\to\infty} S_n = \frac{1}{1-z}$ and the G.S. converges with $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$.

ii) If $|z| > 1$ then $\lim_{n\to\infty} S_n = \infty$ and the G.S. diverges.

3.3 TESTS FOR DETERMINING THE CONVERGENCE OF A POWER SERIES

Weierstrass M-test :

Statement : Let $f_n : D \subset \mathbb{C} \to \mathbb{C}$ be a complex function defined on D s.t. $|f_n(z)| < M_n \;\forall\; z \in D$ and $n \in N$. If $\sum_{1}^{\infty} M_n$ is convergent series of positive real numbers then series $\sum_{1}^{\infty} f_n$ is uniformly convergent.

Proof : Given $|f_n(z)| \le M_n \;\forall z \in D$ and $n \in N$ (1)

Let $\varepsilon > 0$

Given that $\sum_{1}^{\infty} M_n$ is convergent

$\therefore$ $\exists$ an integer N s.t. $\sum_{k=n+1}^{\infty} M_n < \varepsilon \quad \forall\; n \ge N$ (2)

Given series $\sum_{1}^{\infty} fn(z)$

Let $S_n(z) = f_1(z) + f_2(z) + \ldots + f_n(z)$ if $n > m \ge N$, then

$$\left| S_n(z) - S_m(z) \right| = \left| f_{m+1}(z) + f_{m+2}(z) + \ldots + f_n(z) \right|$$

$$= \left| \sum_{k=m+1}^{n} f_k(z) \right| \leq \sum_{k=m+1}^{n} \left| fn(z) \right|$$

$$\leq \sum_{k=m+1}^{n} M_k \qquad \text{from (1)}$$

$$\leq \sum_{k=m+1}^{n} M_k < \varepsilon \qquad \text{from (2)}$$

$\therefore$ $\left| S_n(z) - S_m(z) \right| < \varepsilon, \quad \forall\ n, m \geq N$

$\Rightarrow$ $\{S_n(z)\}$ is a Cauchy sequence.

$\Rightarrow$ Sequence $\{S_n(z)\}$ is a convergent sequence

$\therefore$ $\exists\ w \in \mathbb{C}$ s.t. $\lim\limits_{n \to \infty} S_n(z) = w$

Define $w = f(z)$ this gives a function $f : D \to \mathbb{C}$ for each $z \in D$ and for each $n \geq N$ $\left| S_n(z) - f(z) \right| = \left| f(z) - S_n(z) \right|$.

$\left| S_n(z) - f(z) \right| = \left| f(z) - S_n(z) \right| \qquad = \left| S_{n+1}(z) + f_{n+2}(z) + \ldots \right|$

$$= \left| \sum_{k=n+1}^{\infty} fk(z) \right| \leq \sum_{k=n+1}^{\infty} \left| f_k(z) \right|$$

$$\leq \sum_{k=n+1}^{\infty} M_k < \varepsilon \qquad \text{from (1) and (2)}$$

Hence $\left| S_n(z) - f(z) \right| < \varepsilon \quad \forall\ z \in D$ where $n \geq \mathbb{N}$

$\Rightarrow$ Series $\sum\limits_{1}^{\infty} f_n$ is uniformly convergent on *D*.

Examples : 1) Prove that the series $\sum\limits_{n=1}^{\infty} \dfrac{z^n}{\sqrt[n]{n+1}}$ is uniformly convergent on a set $D = \{z \in \mathbb{C} : |z| \leq 1\}$.

Solution : Given series $\sum\limits_{n=1}^{\infty} \dfrac{z^n}{\sqrt[n]{n+1}}$

Let $f_n(z) = \dfrac{z^n}{\sqrt[n]{n+1}}$

$$\therefore \left| f_n(z) \right| = \frac{\left| z^n \right|}{\left| n\sqrt{n+1} \right|} \leq \frac{1}{\sqrt[n]{n+1}} \qquad \{\because |z| \leq 1\}$$

$$\leq \frac{1}{n^{3/2}} = M_n \qquad \left(\because \sqrt{n}\ \ \sqrt{n+1}\right)$$

$\therefore \quad \sum M_n = \sum \frac{1}{n^{3/2}}$ is a p-series and it is convergent $\left(\because p = \frac{3}{2} > 1\right)$

$\therefore$ By Weierstrass M-test,

The given series $\sum_{n=1}^{\infty} \frac{z^n}{n\sqrt{n+1}}$ is uniformly convergent.

2) Given series $\sum_{n=1}^{\infty} z^n (1-z)$

i) Prove that the series converges for $|z| < 1$ and find its sum.

ii) Prove that the series converges uniformly to the sum z for $|z| \le 1/2$.

iii) Does the series converges uniformly for $|z| \le 1$? Explain.

Solution :

i) The given series is $\sum_{n=1}^{\infty} z^n (1-z)$

$$\therefore \sum_{n=1}^{\infty} z^n (1-z) = z(1-z) + z^2(1-z) + z^3(1-z) + \dots$$

$$= z - z^2 + z^2 - z^3 + z^3 - z^4 + \dots + z^n - z^{n+1}$$

$$= z - z^{n+1} = z\left(1 - z^n\right)$$

$\therefore$ Let $S_n(z) = z - z^{n+1}$

We must prove that given any $\varepsilon > 0$, we can find an integer N s.t. $|S_n(z) - z| < \varepsilon \quad \forall\ n \ge N$

$$\therefore |S_n(z) - z| = \left|z - z^{n+1} - z\right| < \varepsilon \Rightarrow \left|-z^{n+1}\right| < \varepsilon \Rightarrow |z|^{n+1} < \varepsilon$$

$$\Rightarrow (n+1)\log|z| < \log\varepsilon$$

$$\Rightarrow n+1 > \frac{\log\varepsilon}{\log|z|}, \quad z \ne 0 \quad \Rightarrow n > \frac{\log\varepsilon}{\log|z|} - 1 = N$$

Choosing, $N = \frac{\log\varepsilon}{\log|z|} - 1$ (1)

$\therefore \quad |S_n - Z| < \varepsilon \qquad \forall\ n \ge N$

$\therefore \quad \lim_{n\to\infty} S_n = \lim_{n\to\infty} z - z^{n+1} = z$

Hence, the series converges for $|z| < 1$ and z is its sum.

ii) Since from (i) the series converges to some z for $|z|<1$ and hence it converges for $|z| \le \frac{1}{2}$

$$N = \frac{\log \varepsilon}{\log |z|} - 1 \qquad \text{from (1)}$$

If $|z| = \frac{1}{2}$, then $N = \frac{\log \varepsilon}{\log \frac{1}{2}} - 1$ is the largest value of $\frac{\log \varepsilon}{\log |z|} - 1$.

$\therefore \quad |S_n - z| < \varepsilon, \ \forall \ n \ge N = \frac{\log \varepsilon}{\log \frac{1}{2}} - 1$ where N depends only on ε and not on z.

$\therefore$ The given series converges uniformly to sum z for $|z| \le \frac{1}{2}$.

iii) If $|z| \le 1$

$$\therefore \quad N = \frac{\log \varepsilon}{\log 1} - 1 = \infty \qquad \text{from (1)}$$

Hence, the series does not converges uniformly for $|z| \le 1$.

Ratio Test for series (2012)

Statement: *Let $\sum z_k$ be an infinite series for non-zero complex term s.t.*

$$|z_n| = |z_N| \cdot \left|\frac{z_{N+1}}{z_N}\right| \cdot \left|\frac{z_{N+2}}{z_{N+1}}\right| \cdots\cdots \left|\frac{z_n}{z_{n-1}}\right| < |z_N| \lambda^{n-N} \qquad = L \textit{ then}$$

$$\sum_{p \ge 1} |z_N| \lambda^p \therefore z_n \not\to 0 \sum z_n$$

i) If $L < 1$, the series converges absolutely.
ii) If $L > 1$, the series diverges.
iii) If $L = 1$, the series may converge or diverge.

Proof: Suppose *L<1*. Then for λ with L<λ<1, there exist an integer N s.t.

$$\left|\frac{z_{n+1}}{z_n}\right| < \lambda \quad \forall n \ge N \text{ so that,}$$

$$|z_n| = |z_N| \cdot \left|\frac{z_{N+1}}{z_N}\right| \cdot \left|\frac{z_{N+2}}{z_{N+1}}\right| \cdots\cdots \left|\frac{z_n}{z_{n-1}}\right| < |z_N| \lambda^{n-N}$$

$\therefore |z_{N+p}| < |z_N| \lambda^p$ for $p \ge 1$.

Since $\sum_{p \ge 1} |z_N| \lambda^p$ is a convergent (geometric series), $\sum_{n \ge 1} |z_N|$ is covergent by comparision test. This proves (i)

If $L>1$, then there an integer N s.t. $L>k>1$ and $\left|\frac{z_{n+1}}{z_n}\right| > k$ for all $n \geq N$

$\therefore$ for all $n > N$ $|z_n| = |z_N| . \left|\frac{z_{N+1}}{z_N}\right| . \left|\frac{z_{N+2}}{z_{N+1}}\right| .... \left|\frac{z_n}{z_{n-1}}\right| > |z_N| k^{n-N} \to \infty$

Hence, $\therefore z_n \not\to 0$ as $n \to \infty$ and so $\sum z_n$ diverges. Hence (ii).

Example:

1) Prove that the power series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges for all values of z.

Solution : Given power series is $\sum_{n=1}^{\infty} \frac{z^n}{n!}$

Here, $a_n = \frac{1}{n!} \Rightarrow a_{n+1} = \frac{1}{(n+1)!}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{1/(n+1)!}{1/n!}\right| = \lim_{n\to\infty} \left|\frac{1}{n+1}\right| = 0 < 1$$

Therefore the the series is convergent.

<u>Comparison Test</u> :

If the series $\sum |v_n|$ converges and $|u_n| \leq |v_n|$ then $\sum u_n$ converges absolutely. Also, $\sum |u_n|$ converges.

<u>Abels' theorem</u> :

Statement : *If the power series $\sum a_n z^n$ converges to particular value $z_0 \neq 0$ of z then it converges absolutely $\forall$ z s.t. $|z| < |z_0|$.*

Proof : Given that the power series $\sum a_n z^n$ us converges for a particular value $z_0 \neq 0$ of z.

$\therefore \sum a_n z_0^n$ converges.

$\therefore \lim_{n\to\infty} a_n z_0^n = 0$

$\Rightarrow$ sequence $\{a_n z_0^n\}$ is bounded.

$\therefore \exists$ positive number M s.t. $|a_n {z_0}^n| \leq M$

$|a_n| \leq \frac{M}{|z_0|^n}$

or $\left| a_n z^n \right| \le M \left| \frac{z}{z_0} \right|^n \quad \forall\ n$

$\because$ $M . \sum \left| \frac{z}{z_0} \right|^n$ is a geometric series and convergent for $\left| \frac{z}{z_0} \right| < 1$

i.e. $|z| < |z_0|$

$\therefore$ By comparison test $\sum \left| a_n z^n \right|$ converges absolutely for $|z| < |z_0|$.

Cauchy-Hadaward Theorem :

Statement: *For a given power series* $\sum_{n=0}^{\infty} a_n z^n$ *define a number R,* $0 \le R \le \infty$, *by* $\frac{1}{R} = \lim . \sup |a_n|^{1/n}$ *then*

i) If $|z| < R$, *then the series converges absolutely*

ii) If $0 < r < R$, *then the series converges uniformly on* $\{z \in \mathbb{C} : |z| \le r\}$

iii) If $|z| > R$, *then the series diverges [Here R is radius of converges of power series.]* (2008)

Proof : Given, $\frac{1}{R} = \lim \sup |a_n|^{1/n}$ (1)

[**Note :** A number L is said to be a limit superior of the sequence $\{u_n\}$ if infinitely many terms of the sequence u_n are greater than $L - \varepsilon$, while finite number of terms greater than $L + \varepsilon$ where $\varepsilon > 0$.]

i) Let $|z| < R$, then $\exists\ r > 0$ s.t. $|z| < r < R$

$\therefore$ $\frac{1}{r} > \frac{1}{R}$

$\therefore$ By definition of the lim sup and from equation (1)

$\Rightarrow$ $|a_n| < \frac{1}{r^n}$

$|a_n z_n| < \left(\frac{|z|}{r} \right)^n \quad \forall\ n \ge N$

$\because$ $\sum \left(\frac{|z|}{r} \right)^n$ is a G.S. and it is convergent for $\frac{|z|}{r} < 1$ i.e. $|z| < r$.

$\therefore$ By comparison test,

$\therefore$ $\sum a_n z^n$ converges for $|z| < R$.

$\therefore \sum_{n=0}^{\infty} a_n z^n$ converges absolutely for $|z| < R$.

ii) Let $0 < r < R$ choosing $r',\exists \quad 0 < r < r' < R$.

By using part (1), we have

$|a_n|^{1/n} |a_n|^{1/n} < \frac{1}{r'} \qquad \forall n \geq N$

$\Rightarrow \quad |a_n| < \frac{1}{(r')^n}$

$\therefore \quad |a_n z^n| < \left(\frac{|z|}{r'}\right)^n \leq \left(\frac{r}{r'}\right)^n$

$\because \sum \left(\frac{r}{r'}\right)^n$ is a G.S. of positive real numbers and it convergent for $r < r'$.

$\therefore$ By Weierstrass M-test, the power series $\sum_{n=0}^{\infty} a_n z^n$ converges uniformly on $\{z \in \mathbb{C} : |z| \leq r\}$.

iii) Let $|z| > R$, then $\exists \quad r > 0 \ni |2| > r > R$

$\Rightarrow \quad \frac{1}{r} < \frac{1}{R}$.

$\therefore$ By definition $\limsup$ and from equation (1)

$|a_n|^{1/n} > \frac{1}{r} \qquad \forall \; n \geq N$

$\Rightarrow \quad |a_n| > \frac{1}{r^n}$

$\Rightarrow \quad |a_n z^n| > \frac{|z|^n}{r^n} > \frac{r^n}{r^n} = 1$

$\therefore \quad |a_n z^n| > 1 \qquad \forall \; n \geq N$

$\Rightarrow \quad a_n z^n \not\to 0$ as $n \to \infty \{ \because \sum z_n$ converges then $\lim_{n\to\infty} z_n = 0, \; z_n \to 0$ as $n \to \infty \}$

$\Rightarrow$ Power series $\sum_{n=0}^{\infty} a_n z^n$ is divergent for $|z| > R$.

Definition : The radius of convergence R of the power series $\sum_{n=0}^{\infty} a_n z^n$ is defined as $R = \sup \{ r'$ the series converges $\forall z$ satisfying $|z| \le r \}$.

i) If $R = 0$, then power series $\sum a_n z^n$ cgs only for $z = 0$.

ii) If $R = \infty$, then the power series converges , for all values of z.

iii) If $0 < R < \infty$, the power series converges for all $z, f_n(z) < R$ and diverges $\forall z, f_n(z) > R$.

The power series may converge or diverge on the circle $|z| = R$. The circle $|z| = R$ is then called the circle of convergence.

Note : If $\sum_{n=0}^{\infty} a_n z^n$ is power series with radius of converges R then

$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$,provided this limit exist.

Theorem : *If $a_n \neq 0$ for all but finitely many values of n then the radius of convergence R of $\sum_{n=0}^{\infty} a_n z^n$ is related by following,*

$$\liminf \left| \frac{a_{n+1}}{a_n} \right| \le \frac{1}{R} \le \limsup \left| \frac{a_{n+1}}{a_n} \right|.$$

In particular, if $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$ exists, then

$$\frac{1}{R} = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \limsup |a_n|^{1/n}. \quad (2007,2008,)$$

Proof : Given R is radius of convergence of the series $\sum_{n=0}^{\infty} a_n z^n$.

Suppose, $\limsup \left| \frac{a_{n+1}}{a_n} \right| = L$ and $\liminf \left| \frac{a_{n+1}}{a_n} \right| = \ell$

$\therefore$ By the definition of limit sup, $\exists$ an N s.t.

$$\left| \frac{a_{n+1}}{a_n} \right| < L + \varepsilon \qquad \forall\ n \ge N$$

$$\therefore \quad \left| \frac{a_{N+1}}{a_N} \right| < L + \varepsilon, \left| \frac{a_{N+2}}{a_{N+1}} \right| < L + \varepsilon, ..., \left| \frac{a_n}{a_{n-1}} \right| < L + \varepsilon$$

Multiplication of these inequalities gives

$$\left|\frac{a_{N+1}}{a_N}\right|\left|\frac{a_{N+2}}{a_{N+1}}\right|\cdots\left|\frac{a_n}{a_{n-1}}\right|<(L+\varepsilon)^{n-N}$$

$\therefore \quad \left|\frac{a_n}{a_N}\right|<(L+\varepsilon)^{n-N}$

$\therefore \quad |a_n|<|a_N|(L+\varepsilon)^{n-N}$

$\therefore \quad |a_n|^{1/n}<\left[|a_N|(L+\varepsilon)^{n-N}\right]^{1/n}$

$\therefore \quad |a_n|^{1/n}<\left[|a_N|(L+\varepsilon)^{-N}\right]^{1/n}(L+\varepsilon)$

$\because \quad \varepsilon$ is arbitary as $n\to\infty$, we get

$$\lim_{n\to\infty}\sup|a_n|^{1/n}\le L \tag{1}$$

$\left(\because \lim\sqrt[n]{p}=1 \quad n\to a \quad \text{if} \quad p>0\right)$

Similarly, by the definition of lim in f

$$\ell \le \lim\inf|a_n|^{1/n} \tag{2}$$

From equation (1) and (2), we get

$\therefore \quad \ell\le\lim\inf|a_n|^{1/n}\le\lim\sup|a_n|^{1/n}\le L$

$\Rightarrow \quad \ell\le\frac{1}{R}\le L \qquad \left\{\because \lim\sup|a_n|^{1/n}=\frac{1}{R}\right\}$

If $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ exists then $\ell=L$

$\therefore \quad \frac{1}{R}=\lim\inf|a_n|^{1/n}=\lim\sup|a_n|^{1/n}$

$\therefore \quad \frac{1}{R}=\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim\limits_{n\to\infty}\sup|a_n|^{1/n}$

Note : $R=\lim\limits_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$

Theorem: *Let $\sum a_n z^n$ be a power and $\sum na_n z^{n-1}$ be the power series obtained by differentiating $\sum a_n z^n$ term by term. Then the derived series has same radius of convergence as the original series.*

(2009)

Proof: Suppose for R and R' be the radii of the convergenve of the series $\sum a_n z^n$ and $\sum na_n z^{n-1}$ respectively.

Then we have,

$$\frac{1}{R}=\overline{\lim}|a_n|^{\frac{1}{n}} \quad and \quad \frac{1}{R'}=\overline{\lim}\, n^{\frac{1}{n}}|a_n|^{\frac{1}{n}}$$

In order the desired result we have to show that $\lim_{n\to\infty} n^{\frac{1}{n}}=1$

Suppose $n^{\frac{1}{n}}=1+h$. Then we have

$n = (1+h)^n=1+nh+\frac{n(n-1)}{2!}h^2+.........h^n > \frac{1}{2}n(n-1)h^2$ i.e. $h^2 < \frac{2}{n-1}$

Thus $=\ln 1+i\frac{\pi}{2}=i\frac{\pi}{2}$ so that $\lim_{n\to\infty} n^{\frac{1}{n}}=1$.

Hence $R=R'$

Proposition : *Let* $f(z)=\sum_{n=0}^{\infty} a_n z^n$ *have radius of convergence* $R>0$ *then for each* $K\geq 1$, *the series* $\sum_{n=K}^{\infty} n(n-1)(n-2)...(n-K+1)\,a_n\, z^{n-K}$ *has radius of convergence R*. (2009)

Proof : Let R be a radius of convergence of the power series $f(z)=\sum_{n=0}^{\infty} a_n z^n$.

Let R' be a radius at cgs. of a power series $\sum_{n=K}^{\infty}\frac{n!}{(n-K)!}$.

T.P.T. $R=R'$

$$\therefore \frac{1}{R'}=\lim\sup\left|\frac{n!}{(n-k)!}a^n\right|^{1/n}$$

$$=\lim\sup\left|\frac{n!}{(n-k)!}\right|^{1/n}.\lim\sup|a_n|^{1/n}$$

$$\frac{1}{R'}=\lim\sup\left|\frac{n!}{(n-k)!}\right|^{1/n}.\frac{1}{R} \qquad ....(1)$$

$$\left\{\because \lim\sup|a_n|^{1/n}=1/R\right\}$$

Now, $\lim\limits_{n\to\infty} \sup |a_n|^{1/n} = \lim\limits_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

$\therefore$

$$\lim\sup \left| \frac{n!}{(n-k)!} \right|^{1/n} = \lim_{n\to\infty} \left| \frac{(n+1)!}{(n+1-k)} \times \frac{(n-k)!}{n!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)\,n!}{(n+1-k)} \cdot \frac{(n-k)!}{(n-k)!\;n!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n+1}{n+1-k} \right| = \lim_{n\to\infty} \left| \frac{1+\frac{1}{n}}{1+\frac{1}{n}-\frac{k}{n}} \right|$$

Substituting in equation (1), we get

$$\frac{1}{R'} = 1 . \frac{1}{R}$$

$$\boxed{R = R'}$$

Example: 1) Prove that the power series $\sum\limits_{n=0}^{\infty} \frac{z^n}{n!}$ converges for all values of z.

Solution : Given power series is $\sum \frac{z^n}{n!}$.

Here, $a_n = \frac{1}{n!} \Rightarrow a_{n+1} = \frac{1}{(n+1)!}$

$$\therefore\; R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{\frac{1}{n!}}{\frac{1}{(n+1)!}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)!}{n!} \right| = \lim_{n\to\infty} \left| \frac{(n+1)\,n!}{n!} \right| = \lim_{n\to\infty} |n+1| = \infty$$

$\therefore \quad R = \infty$

$\therefore \quad \sum\limits_{0}^{\infty} \frac{z^n}{n!}$ converges for all values of z.

2) Prove that the power series $\sum\limits_{0}^{\infty} n!\,z^n$ converges only for $z = 0$.

Solution : Given the power series $\sum\limits_{0}^{\infty} n!\,z^n$

$\therefore \quad a_n = n! \;\Rightarrow\; a_{n+1} = (n+1)!$

$\therefore \quad R = \lim_{n\to\infty}\left|\dfrac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\dfrac{n!}{(n+1)!}\right|$

$= \lim_{n\to\infty}\left|\dfrac{\cancel{n!}}{(n+1)\cancel{n!}}\right| = \lim_{n\to\infty}\left|\dfrac{1}{n+1}\right| = \dfrac{1}{\infty} = 0$

$\therefore \quad R = 0$

$\therefore \quad \sum_{0}^{\infty} n!z^n$ converges only for $z = 0$.

3) Find the radius of convergence of series $\dfrac{\sum(n+2)}{(3n+5)}z^n$.

Solution : The given series is $\dfrac{\sum(n+2)}{(3n+5)}z^n$.

$\therefore \quad a_n = \dfrac{n+2}{3n+5} \Rightarrow a_{n+1} = \dfrac{(n+1)+2}{[3(n+1)+5]} = \dfrac{n+3}{3n+8}$

$\therefore \quad R = \lim_{n\to\infty}\left|\dfrac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\dfrac{(n+2)}{(3n+5)} \Big/ \dfrac{(n+3)}{(3n+8)}\right|$

$= \lim_{n\to\infty}\left|\dfrac{(n+2)}{(3n+5)}\ \dfrac{(3n+8)}{(n+3)}\right| = \lim_{n\to\infty}\left|\dfrac{n\left(1+\frac{2}{n}\right)n\left(3+\frac{8}{n}\right)}{n\left(3+\frac{5}{n}\right)n\left(1+\frac{3}{n}\right)}\right|$

$= \lim_{n\to\infty}\dfrac{n^2}{n^2}\ \dfrac{\left(1+\frac{2}{n}\right)\left(3+\frac{8}{n}\right)}{n\left(3+\frac{5}{n}\right)\left(1+\frac{3}{n}\right)}$

$= \dfrac{\left(1+\frac{2}{\infty}\right)\left(3+\frac{8}{n}\right)}{\left(3+\frac{5}{\infty}\right)\left(1+\frac{3}{\infty}\right)} = \dfrac{(1+0)(3+0)}{(3+0)(1+0)} = \dfrac{3}{3} \qquad \therefore R = 1$

4) Find the radius of convergence of the power series $\sum\left(\dfrac{2n+1}{3n+5}\right)z^n$.

Solution : The given series is $\sum\left(\dfrac{2n+1}{3n+5}\right)z^n$

$\therefore \quad a_n = \dfrac{2n+1}{3n+5} \Rightarrow a_{n+1} = \dfrac{2(n+1)+1}{3(n+1)+5} = \dfrac{2n+3}{3n+8}$

$\therefore$

$R = \lim_{n\to\infty}\left|\dfrac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\dfrac{(2n+1)}{(3n+5)}\cdot\dfrac{(3n+8)}{(2n+3)}\right|$

$= \lim_{n\to\infty}\left|\dfrac{n^2\left(2+\frac{1}{n}\right)\left(3+\frac{8}{n}\right)}{n^2\left(3+\frac{5}{n}\right)\left(2+\frac{3}{n}\right)}\right|$

$$= \frac{\left(2+\frac{1}{\infty}\right)\left(3+\frac{8}{\infty}\right)}{\left(3+\frac{5}{\infty}\right)\left(2+\frac{3}{\infty}\right)} = \frac{(2+0)(3+0)}{(3+0)(2+0)} = 1 \qquad \therefore R = 1$$

5) Find the radius of convergence of the series $\sum (c+id)^n z^n$ where $c,\ d \in \mathbb{R}$.

Solution : The given series $\sum (c+id)^n z^n$

$\therefore \quad a_n = (c+id)^n \Rightarrow a_{n+1} = (c+id)^{n+1}$

$\therefore$

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \left| \frac{(c+id)^n}{(c+id)^{n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(c+id)^n}{(c+id)^n (c+id)} \right| = \lim_{n\to\infty} \left| \frac{1}{c+id} \right|$$

By Rationalizing

$$= \lim_{n\to\infty} \left| \frac{c-id}{c^2+d^2} \right| = \frac{1}{\sqrt{c^2+d^2}}$$

6) Find the radius of convergence of the power series $\sum \frac{1}{4^n+1} z^n$.

Solution : The given power series is $\sum \frac{1}{4^n+1} z^n$

$\therefore \quad a_n = \frac{1}{4^n+1} \Rightarrow a_{n+1} = \frac{1}{4^{n+1}+1}$

$$\therefore \quad R = \lim_{n\to\infty} \left| \frac{4^n\left(4+\frac{1}{4^n}\right)}{4^n\left(1+\frac{1}{4^n}\right)} \right| = \lim_{n\to\infty} |4| = 4$$

7) Find the radius of convergence of the series $\sum \left(1+\frac{1}{n}\right)^{n^2} z^n$.

Solution : The given series is $\sum \left(1+\frac{1}{n}\right)^{n^2} z^n$. Let $a_n = (1+\frac{1}{n})^{n^2}$

$$\frac{1}{R} = \lim\sup |a|^{1/n} = \lim\sup \left| \left(1+\frac{1}{n}\right)^{n^{\not{2}}} \right|^{1/\not{n}} = \lim\sup \left| \left(1+\frac{1}{n}\right) \right|^n$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e \qquad \boxed{R = \frac{1}{e}}$$

8) Find the domain of region of convergence of the power series $\sum_{n=1}^{\infty} \frac{1.3.5....(2n-1)}{n!} \left(\frac{1-z}{z}\right)^n$ and show the domain or region graphically.

Solution : The given power series is $\sum_{n=1}^{\infty} \frac{1.3.5....(2n-1)}{n!} \left(\frac{1-z}{z}\right)^n$.

Put $\frac{1-z}{z} = \xi$, we get

$$\Rightarrow \quad \sum_{n=1}^{\infty} \frac{1.3.5....(2n-1)}{n!} \xi^n$$

Now, $a_n = \frac{1.3.5....(2n-1)}{n!}$

$$\Rightarrow a_{n+1} = \frac{1.3.5....(2n-1)(2n+1)}{(n+1)!}$$

$$\therefore \; R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{\dfrac{1.3.5....(2n-1)}{n!}}{\dfrac{1.3.5....(2n-1)(2n+1)}{(n+1)!}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{1.3.5....(2n-1)(n+1)!}{1.3.5....(2n-1)(2n+1)\,n!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)!}{(2n+1)\,n!} \right| = \lim_{n\to\infty} \left| \frac{(n+1)\cancel{n!}}{(2n+1)\cancel{n!}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n+1}{2n+1} \right| = \lim_{n\to\infty} \left| \frac{\cancel{n}\left(1+\frac{1}{n}\right)}{\cancel{n}\left(2+\frac{1}{n}\right)} \right|$$

$$= \lim_{n\to\infty} \left| \frac{1+\frac{1}{n}}{2+\frac{1}{n}} \right| = \frac{1+\frac{1}{\infty}}{2+\frac{1}{\infty}}$$

$$\boxed{R = \frac{1}{2}}$$

$\therefore$ Domain of convergence of power series $|\xi| < \frac{1}{2}$

i.e. $\left|\frac{1-z}{z}\right| < \frac{1}{2} \Rightarrow |1-z| < \frac{1}{2}|z|$

Taking square on both sides.

$$|1-z|^2 < \frac{1}{4}|z|^2$$

$$\therefore \quad (1-z)\left(1-\bar{z}\right) < \frac{1}{4} z\bar{z} \qquad \left\{\because |z|^2 = z.\bar{z}\right\}$$

$$\therefore \quad 4\left(1-\bar{z}-z+z\bar{z}\right) < z\bar{z}$$

$$\Rightarrow \quad 4-4\bar{z}-4z+4z\bar{z} < z\bar{z} \Rightarrow \quad 4-4\bar{z}-4z+3z\bar{z} < 0$$

$$\Rightarrow \quad z\bar{z} - \frac{4}{3}\left(z+\bar{z}\right) + \frac{4}{3} < 0$$

Put $z = x+iy \qquad \Rightarrow \bar{z} = x+iy$

$$\Rightarrow \quad x^2 + y^2 - \frac{8}{3} + \frac{4}{3} < 0$$

$$\Rightarrow \quad x^2 - \frac{8}{3}x + \frac{16}{9} + y^2 + \frac{4}{3} - \frac{16}{9} < 0$$

$$\Rightarrow \quad \left(x - \frac{4}{3}\right)^2 + y^2 - \frac{4}{9} < 0 \quad \Rightarrow \quad \left(x - \frac{4}{3}\right) + y^2 = \frac{4}{9}$$

$$\therefore \quad C = \left(4/3,\ 0\right) \text{ and } r = \frac{2}{3}$$

Centre = $\left(\frac{4}{3},\ 0\right)$, Radius = $\frac{2}{3}$

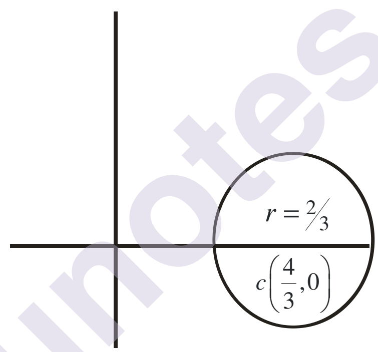

Fig 3.1

Given series converges inside the circle.

9) Find the domain of convergence of the power series $\sum_{n=0}^{\infty}\left[\frac{(iz-1)}{3+4i}\right]^n$.

Solution : The given power series is

$$\sum_{n=0}^{\infty}\left[\frac{iz-1}{3+4i}\right]^n = \sum_{n=0}^{\infty}\left[\frac{i(z+i)}{3+4i}\right]^n \qquad \left(\because i^2 = -1\right)$$

$$= \sum_{n=0}^{\infty}\left(\frac{i}{3+4i}\right)^n (z+i)^n$$

$$\therefore \quad a_n = \left(\frac{i}{3+4i}\right)^n$$

$\because \ \frac{1}{R} = \limsup |a_n|^{1/n} = \limsup \left| \left(\frac{i}{3+4i} \right)^n \right|^{1/n} = \limsup \left| \frac{i}{3+4i} \right|$

$= \left| \frac{i}{3+4i} \right| = \frac{|i|}{|3+4i|}$

$\therefore \quad z = x+iy \Rightarrow |z| = \sqrt{x^2+y^2}$

$i = 0+iy \Rightarrow |i| = \sqrt{0+1} = \sqrt{1} = 1$ and

$|3+4i| = \sqrt{3^2+4^2} = \sqrt{9+16} = \sqrt{25} = 5$

$\therefore \ \frac{1}{R} = \frac{|i|}{|3+4i|} = \frac{1}{5}$

$\therefore$

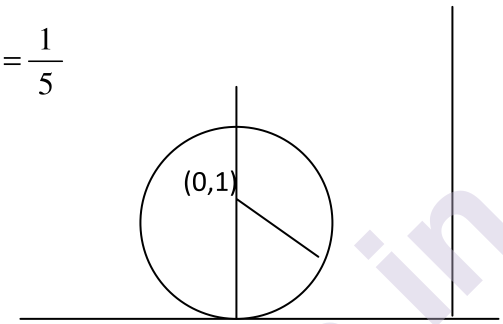

Fig 3.2

$\therefore \quad \boxed{R=5}$

$\therefore$ Domain of convergence of power series is $|z+i|<5$

$\therefore$ Centre =(0,1) , Radius = 5

$\therefore$ The given series converge inside the circle.

10) Find the radius of converges of the series $\sum (3n+2)(z-2)^n$.

Solution : The given series is $\sum (3n+2)(z-2)^n$.

Put $(z-2) = \xi$

$\Rightarrow \quad \sum (3n+2)\xi^n$

$\therefore \quad a_n = 3n+2 \quad \Rightarrow a_{n+1} = 3(n+1)+2 = 3n+5$

Now, $R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{3n+2}{3n+5} \right| = \lim_{n\to\infty} \left| \frac{n\left(3+2/n\right)}{n\left(3+5/n\right)} \right| = 1$

$\therefore \ \boxed{R=1}$

$\because$ Domain of convergence of power series is $|z-2|<5$

i.e a circle with C=(2,0) and r =5

11) Find the region of convergence of power series $\sum_{n=1}^{\infty} \frac{(z+2)^{n-1}}{(n+1)^3 4^n}$.

Solution : The given series is $\sum_{n=1}^{\infty} \frac{(z+2)^{n-1}}{(n+1)^3 4^n}$

$\therefore \quad a_n = \frac{1}{(n+1)^3 4^n} \Rightarrow a_{n+1} = \frac{1}{(n+2)^3 4^{n+1}}$

Now, $\frac{1}{R} = \lim\sup |a_n|^{1/n} = \lim\sup \left| \frac{1}{(n+1)^3 4^n} \right|^{1/n}$

$= \lim\sup \left| \frac{1}{(n+1)^{3/n} 4} \right| = \frac{1}{4}$

$\boxed{R = 4}$

$\therefore$ The region of convergence of the series is $|z+2| < R$

i.e. $|z+2| < 4 \qquad \Rightarrow |z+2|^2 < 16$

$\Rightarrow \quad |z+2| \, |\bar{z}+2| < 16 \qquad (\because |z|^2 = z.\bar{z})$

$\Rightarrow \quad (z.\bar{z} + 2z + 2\bar{z} + 4) < 16 \qquad \Rightarrow z\bar{z} + 2(z+\bar{z}) + 4 < 16$

Put $z = x + iy \Rightarrow \bar{z} = x - iy$

$\Rightarrow \quad x^2 + y^2 + 2x.2 - 12 < 0 \qquad (\because z.\bar{z} = x^2 + y^2 \text{ and } z + \bar{z} = 2x)$

$\Rightarrow \quad x^2 + y^2 + 4x - 12 < 0 \qquad \Rightarrow x^2 + 4x + 4 + y^2 - 12 - 4 < 0$

$\Rightarrow \quad (x+2)^2 + y^2 - 16 < 0 \qquad \Rightarrow (x+2)^2 + y^2 < 16$

$\therefore \quad c = (-2, 0) \qquad r = 4$

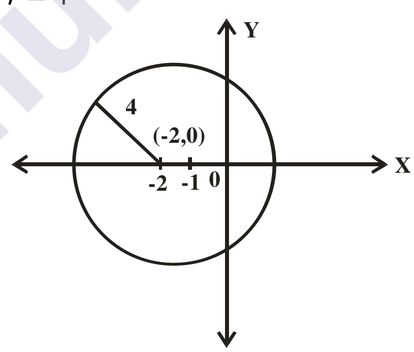

Fig 3.3

$\therefore$ The given series converges inside the circle.

(12) Find the radius of convergence of (i) $\sum \frac{(-1)^n}{(n!)^2}(z/2)^{2n}$ (2009)

(ii) $\sum_{j=2}^{\infty} \frac{z^{2j}}{j(j-1)}$ (2009) (iii) $\sum n^n z^n$ (2008) (iv) $\sum \frac{z^n}{n}$ (2008)

(13) Find the power series for the function $f(z)=\dfrac{1}{z}$ about the point $z=2$ and find its radius of convergence. (2007)

14) Check for the convergence of the series $\sum\limits_{n=0}^{\infty} nz^{n}$.

Solution : Here after comparison with $\sum\limits_{n=0}^{\infty} a_n z^n$,

$\because a_n = n \quad \forall \quad n \geq 1$.

$\therefore \quad \lim_{n\to\infty} \sup |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} n^{\frac{1}{n}} = 1 \qquad \therefore L = 1.$

$\Rightarrow \quad R = \dfrac{1}{L} = 1. \quad \therefore$ The series $\sum_{n}^{\infty} = 0^{n\,z^{n}}$ converges for $|z|<1$ and diverges for $|z|>1$.

For $|z|=1, |n\, z^n| = n \to \infty$, as $n \to \alpha$.

$\sum_{n=0}^{\infty} nz^n$ diverges for $|z|=1$.

15) Find the radius of convergence of the following series .

(i) $\sum\limits_{n=1}^{\infty} \dfrac{z^n}{n}$ (ii) $\sum\limits_{n=1}^{\infty} \dfrac{z^n}{n^2}$ (iii) $\sum\limits_{n=1}^{\infty} \dfrac{z^n}{n!}$ (iv) $\sum\limits_{n=1}^{\infty} z^{n^2}$

Solution : (i) $\because \; a_n = \dfrac{1}{n^2} \Rightarrow L = \lim_{n\to\infty} \sup |a_n|^{\frac{1}{n}} = 1.$

$\therefore R = \dfrac{1}{L} = 1.$ The radius of convergence of the power series $\sum_{n=1}^{\infty} \dfrac{z^n}{n^2}$ is equal to 1.

(iv) $\because \; a_n = 1$ if $\because \; n = k^2$ for some integer $k = 0$ otherwise.

$\therefore$ Consider $\because \; L = \lim_{n\to\infty} \sup |a_n|^{\frac{1}{n}} = \sup\{1,0\} = 1.$

$\therefore \; R = \dfrac{1}{L} = 1.$ The radius of convergence of the power series $\sum_{n=0}^{\infty} z^{n^2}$ is equal to 1.

3.5 SUMMARY

1) If the series $\sum_{n=1}^{\infty} z_n$ is convergent then $\lim_{n\to\infty} z_n = 0$.

2) A series $\sum_{n=0}^{\infty} z_n$ of complex terms is convergent iff for every $\varepsilon > 0$, $\exists$ an integer N s.t. $\left| z_n + z_{n+1} + ... + z_{n+p} \right| < \varepsilon \ \ \forall\ n \geq N$ and $p \geq 0$. (Cauchy criteria of convergence of series)

3) The series $\sum f_n$ is said to be uniformly converges on D to f if for every $\varepsilon > 0$, $\exists$ an integer N (depends only on ε) s.t. $\left| S_n(z) - f(z) \right| < \varepsilon \quad \forall\ z \in D$, and $\forall\ n \geq N$.

4) Let $fn : D \subset \mathbb{C} \to \mathbb{C}$ be a complex function defined on D s.t. $\left| f_n(z) \right| < M_n \ \ \forall\ z \in D$ and $n \in N$. If $\sum_{1}^{\infty} M_n$ is convergent series of positive Real numbers then series $\sum_{1}^{\infty} f_n$ is uniformly convergent.

5) Let $\sum u_n$ be an infinite series for non-zero complex term s.t. $\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = L$ then

   i) If $L < 1$, the series converges absolutely.
   ii) If $L > 1$, the series diverge.
   iii) If $L = 1$, the series may converges or diverge.

6) A power series about z_0 is an infinite series of the form $\sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + (z - z_0)a_1 + (z - z_0)^2 a_2 + ...$ where constants a_n and z_0 are called complex numbers and z is a complex variable.

7) The radius of convergence R of the power series $\sum_{n=0}^{\infty} a_n z^n$ is defined as $R = \sup\{ r : \text{the series cgs } \forall\ z \text{ satisfying } |z| \leq r \}$.

3.6 UNIT END EXCERCIES

1) Check for the convergence of the series $\sum_{n=0}^{\infty} z^n$.

Solution: If $|z|<1$ then $1+z+...+z^n = \frac{1-z^{n+1}}{1-z} \to \frac{1}{1-z}$ as $n \to \infty$

$\therefore \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$.

If $|z|>1$, then $\lim_{n\to\infty} |z|^n = \infty$. $\therefore$ The series $\sum_{n=0}^{\infty} z^n$ diverges.

2) Show that the radius of convergence of the power series $\sum_{n\geq 1}^{\infty} z^{n(n+1)}$ is equal to 1.

Solution : $\because$ $\sum_{n\geq 1}^{\infty} z^{n(n+1)} = -z^2 + \frac{1}{2}z^6 - \frac{1}{3}z^{12} + ..$

$\therefore a_n = 0$ if $n \neq k(k+1)$ for some integer k

$a_n = \frac{(-1)^n}{n}$ otherwise.

$\therefore L = \lim_{n\to\infty} \sup |a_n|^{\frac{1}{n}} = \sup\{0,1\} = 1$. $\because$ $R = \frac{1}{L} = 1$

$\therefore$ The series has the radius of convergence equal to 1.

3) Find whether $\sum_{k\geq 1}^{\infty} \frac{i^k}{k^2+i}$ converges or not.

Solution : $\therefore$ $\left|\frac{i^k}{k^2+i}\right| = \frac{1}{\sqrt{k^2+1}}$ and we know that $\sum_{k\geq 1}^{\infty} \frac{1}{\sqrt{k^2+1}}$ converges. $\therefore$ The series $\sum_{k\geq 1}^{\infty} \frac{i^k}{k^2+1}$ converges.

4) Check whether $\sum_{k=1}^{\infty} \frac{1}{k+i}$ is convergent or not. (Hint : Check whether $\sum_{k=1}^{\infty} \text{Re}\left(\frac{1}{k}+i\right)$ is convergent or not .)

5) Show that $f(z) = \sum_{\kappa=1}^{\infty} kz^k$ is continuous in $|z|<1$.(Hint : Here the convergence is uniform . Show that $\sum_{k=1}^{\infty} kz^k$ is convergent in $|z|<1$. Let $f_k(z) = kz^k$ which is uniformly continuous for all $k \geq 1$)

6) Show that the series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ is convergent everywhere in the complex plane.

7) Show that the functions $f(z)=\cos(z), g(z)=\sin(z)$ are analytic in the whole complex plane. (Hint: Show that each of the series $f(z)=\cos(z)=\sum_{n=0}^{\infty}\frac{(-1)^n z^{zn}}{(2n)1}$ and $g(z)=\sin(z)=\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{(2n+1)!}$ have infinite radius of convergence.)

8) Let (a_n) be a sequence of positive real's and $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=L$. Show that $\lim_{n\to\infty} a_n^{\frac{1}{n}}=L$.

9) Find the radius of convergence of the series $\sum_{n=0}^{\infty}\frac{z^{2n+1}}{(2n+1)}$.

Solution: $\sum_{n=0}^{\infty}\frac{z^{2n+1}}{(2n+1)!}=z+\frac{z^3}{3!}+\frac{z^5}{5!}+...+...$

$\therefore$ Comparing with $\sum_{n=0}^{\infty}a_n z^n$, we get

$a_n=0$ if $n=2,4,6,...$

$=\frac{1}{n!}$ if $n=3,5,7...$

$\therefore L=\lim_{n\to\infty}\sup|a_n|^{\frac{1}{n}}=\sup\{0,0\}=0$.

$\therefore R=\frac{1}{0}=\infty$. The power series $\sum_{n=0}^{\infty}\frac{z^{2n+1}}{(2n+1)}$ has infinite radius of convergence.

10) Find the domain of convergence of the series $\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}(z+1)^n$.

Solution: Put $\theta=z+1$.

$\therefore \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}(z+1)^n=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\theta^n$. $a_n=\frac{(-1)^n}{n!}$.

$\therefore L=\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{1}{n+1}\right|=0$

$\therefore R=\frac{1}{0}=\infty$. The given power series converges for all θ.

But $\theta=z+1$. As θ varies over $\mathbb{C}$, $\therefore$ z also varies over $\mathbb{C}$.

$\therefore$ The given power series converges for all complex numbers.

11) Find the domain of convergence of the power series $\sum_{n=0}^{\infty}(z-i)^n$

(Ans : The series converges for $z \in \mathbb{C}$ such that $|z-i|<1$.

12) Show that $n^{\frac{1}{n}} \to 1$ as $n \to \infty$. (Hint: Put $a_n = \log\left(n^{\frac{1}{n}}\right)$ for $n \geq 1$.)

13) If $\sum a_n z^n$ has radius of convergence *R*, what is the radius of convergence of

$\sum a_n z^{2n}$ and of $\sum a_n^2 z^n$?

14) Prove that the series $\sum\limits_{n=1}^{\infty} \frac{(z+2)^{n-1}}{(n+1)^3 4^n}$ converges for $|z+2| \leq 4$.

15) Find the radius of convergence of the power series $\sum\limits_{n=0}^{\infty} a^{n^2} z^n, \quad a \in \mathbb{C}$.

16) Find the domain of convergence of the series $\frac{1}{2}z + \frac{1.3}{2.5}z^2 + \frac{1.3.5}{2.5.8}z^3 + \ldots\ldots\ldots\ldots\ldots$

17) Find the radii of convergence of the following power series $\sum n^2 \left(\frac{z+1}{1+i}\right)^n$

18) Show that the ROC for the power series $\sum \frac{z^n}{n}$ is 1. Discuss the convergence of this series of the points on the boundary or the disc $\{z \in \mathbb{C} / |z| < 1\}$

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4

DIFFRENTIABILITY

Unit Structure

4.0. Objectives

4.1. Introduction

4.2 . Differentiability in complex

4.3. Summary

4.4. Unit End Exercises

4.0. OBJECTIVES

After going through this chapter you shall come to know about :

- Defining a polynomial with complex coefficients and in an indeterminant z, which can take any complex number value.
- An infinite series of the form $\sum_n^\infty = 0 a_n z^n$ is called a power series.

We shall investigate for the differentiability of a power series as a function of a complex variable z, at the same time we shall also check for the condition , under which two power series are one and the same , that is both the power series represent the same complex valued function.

4.1. INTRODUCTION

Through this Unit , we shall examine the notion of " a function of z ", where z is a Complex Number of the form $z = x + iy$. A Complex Number z can be viewed as an ordered pair of real numbers x and y as $z = (x, y)$. The point of view taken in this Unit is to understand some functions , which are direct functions of $z = x + iy$ and not simply functions of the separate parts x and y Consider for example the function $x^2 - y^2 + 2ixy$ is a direct function of $x + iy$, since $x^2 - y^2 + 2ixy = (x + iy)^2$.

$\left(f(z)=z^2\right)$ but x^2+y^2-2ixy is not expressible as a polynomial in variable $x+iy$. Therefore we are compelled to consider a special class of functions , given by direct/ analytic expressions in $x+iy$. We shall name such direct functions as the analytic functions . Let us start this Unit by defining an c polynomial $p(z)$ in a Complex variable z.

Definition : A polynomial P(x,y) in a Complex variable z =(x, y) is an expression of the form $P(x,y)=\alpha_0+\alpha_1(x+iy)+...+\alpha_n(x+iy)^n$ where $\alpha_0,\alpha_1..,\alpha_n$ are complex constants e.g. $(i)\ P(x,y)=x^2-y^2+2ixy$ $(ii)\ x^2+y^2-2ixy$ is not a polynomial in $z=x+iy$.

4.2 DIFFERENTIABILITY IN COMPLEX

Differentiation: Let G be an open set in $\mathbb{C}$ and $f:G\to\mathbb{C}$ can be a function, we say that *f* is differentiable at a point z_0 in *G* if the limit

$$\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}\ \ldots\ldots(1)$$

exists, this limit is denoted by $f'(z_0)$ and is called derivative of *f* at z_0.

Put $z=z_0+h$, (complex number) then equation (1) becomes

$$\Rightarrow\qquad f'(z_0)=\lim_{z\to z_0}\frac{f(z_0+h)-f(z_0)}{h}.$$

In terms of '$\varepsilon-\delta$' notation limit in equation (1) exists iff

$$\forall\ \varepsilon>0,\ \exists\ \delta>0,\ \ni\left|\frac{f(z)-f(z_0)}{z-z_0}-f'(z_0)\right|<\varepsilon \text{ whenever } 0<|z-z_0|<\delta.$$

If *f* is differentiable at each point of *G* then *f* is differentiable on *G*. Notice that if *f* is differentiable on *G*, $f'(z_0)$ defines a function $f':G\to\mathbb{C}$.

If f' continuous then we say that *f* is continuously differentiable. If f' is differentiable, then *f* is twice differentiable continuing, a different function $\ni$ each successive derivative is differentiable is called infinitely differentiable.

Proposition: *If $f:G\to\mathbb{C}$ is differentiable at $z_0\in G$ then f is continuous at z_0* *(2012)*

Proof : Given $f:G\to\mathbb{C}$ is differentiable at $z_0\in G$.

$\therefore \quad \lim\limits_{z\to z_0}\dfrac{f(z)-f(z_0)}{z-z_0}=f'(z_0)$ exists.

$\therefore$

$$\lim_{z\to z_0}|f(z)-f(z_0)|=\lim_{z\to z_0}\frac{|f(z)-f(z_0)|}{|z-z_0|}.|z-z_0|$$

$=f'(z_0).0=0$

$\therefore \quad \lim\limits_{z\to z_0} f(z)=f(z_0)$

$\therefore$ f is continuous at z_0.

Theorem : *If* f *and* g *are differentiable at* $z_0\in G$ *then* $f\pm g,\ f.g,\ f/g,\ (g\neq 0)$ *are also differentiable at* $z_0\in G$.

The Increment Theorem: *Let* $f:G\to\mathbb{C}$ *be a complex valued function* $z_0\in G$ *and* $r>0,\ \exists\ B(z_0,\ r)\subset G$. *Then f is differentiable at* z_0 *iff* $\exists$ *a complex number* α *and a function* $\eta: B(o;s)\to\mathbb{C}(o,s,r)$ *such that* $\forall\ h\in B(o;s)$,

$f(z_0+h)=f(z_0)+h\alpha+h\eta(h)$ *and* $\lim\limits_{h\to 0}\eta(h)=0$.

Proof : Let *f* is differentiable at point z_0.

(Let) put $\eta(h)=\dfrac{f(z_0+h)-f(z_0)}{h}-\alpha$

So that, $f(z_0+h)=f(z_0)+h\alpha+h\eta(h)$

Let $f'(z_0)=\alpha$ (*)

$\because$ f is differentiable at z_0.

$$\therefore \quad \lim_{h\to 0}\eta(h)=\lim_{h\to 0}\left[\frac{f(z_0+h)-f(z_0)}{h}-\alpha\right]$$

$=f'(z)-\alpha=\alpha-\alpha$ from (*)

$=0$

Conversely,

Let $\lim\limits_{h\to 0}\eta(h)=0$ and (**)

$f(z_0+h)=f(z_0)+h\alpha+h\eta(h)$

$$\therefore \quad \frac{f(z_0-h)-f(z_0)}{h}=\alpha+\eta(h)$$

Taking $\lim\limits_{n\to\infty}$ on both sides.

$$\lim_{h\to 0}\left[\frac{f(z_0-h)-f(z_0)}{\eta}\right]=\lim_{h\to 0}[\alpha+0]$$

$\therefore \quad \lim_{h\to 0}\left[\frac{f(z_0+h)-f(z_0)}{n}\right]=\alpha+0$ from (**)

$$f(z)=\sum a_n z^n \frac{dw}{dz}\mathbb{C}\to\mathbb{C}$$

$\therefore$ f is differentiable at point z_0 and $f'(z_0)=\alpha$.

Composite Function : Let $G\in\mathbb{C}$ and $\Omega\in\mathbb{C}$ be open sets. Let $f:G\to\mathbb{C}$ and $g:\Omega\to\mathbb{C}$ be functions $\ni f(G)\subset\Omega$. Then for each $Z\in G$, the association $g\circ f$ defined by $[g\circ f](z)=g[f(z)]$ is a function called of <u>composite function</u>.

Note : In general $f\circ g\neq g\circ f$

Chain Rule :

Theorem: *Let $G\in\mathbb{C}$, $\Omega\in\mathbb{C}$ be open sets and let f and g be differentiable on G and Ω (respectively). Suppose $f(G)\subset\Omega$ then $g\circ f$ is differentiable on G and*

$(g\circ f)(z)=g'[f(z)].f'(z) \quad \forall\ z\in G$

Proof : Fix a point $z\in G$, choose $r>0 \ni B(z,r)\subset G$.

Let $0\neq h\in\mathbb{C}$ and $|h|<r\ (z\neq h)$

Given that *f* is differentiable on *G.*

$\therefore$ f is differentiable at a point $Z\in G$.

$\therefore$ By increment theorem,

$f(z+h)-f(z)=hf'(z)+h\eta(h)$ where $\eta(h)$ is continuous function and $\lim_{h\to 0}\eta(h)=0$

Put $K=f(z+h)-f(z)$, where $K=hf'(z)+h\eta(h)$

Also g is differentiable at $f(z)\in\Omega$

$\therefore$ by increment them,

$$g[f(z+h)]=g(f(z)+K)=g[f(z)]+K\,g'[f(z)]+K\,\psi(K)$$

where $\psi(K)$ is continuous function and $\lim_{k\to 0}\psi(K)=0$.

$\therefore \ g[f(z+h)]=g[f(z)]+[hf'(z)+h\eta(h)].[g'[f(z)]+\psi(K)]$

$=g[f(z)]+hf'(z).g[f(z)]+h\eta(h).g[f(z)]$

$+hf'(z).\psi(K)+h\eta(h).\psi(K)$

$= g\left[f(z)\right] + hg'\left[f(z)\right]f'(z) + h\,\delta(h)$

Where, $\delta(h) = \eta(h)\,g'\left[f(z)\right] + f'(z)\psi(K) + \eta(h)\psi(K)$

T.P.T. $\lim_{h\to 0}\delta(h) = 0.$

$\because$ $\eta(h) \to 0$ as $h \to 0$

As $h \to 0$, $K = f(z+h) - f(z) \to 0$

$\therefore$ $\psi(K) \to 0$ as $h \to 0$

Hence, $\lim_{n\to\infty} \delta(h) = 0$

$\therefore$ by increment theorem,

$(g \circ f)$ is differentiable at $z \in G$

$\because$ Z was arbitrary .

$\therefore$ $g \circ f$ is differentiable on G and $(g \circ f)(z) = g'\left[f(z)\right]f'(z) \;\; \forall\, z \in G$

Let $z = x + iy \in G$ and $f : G \to \mathbb{C}$ be defined by, $f(z) = u(z) + iv(z)$, where u and v are real valued function

OR

$f(x, y) = u(x, y) + iv(x, y)$.

Definition : If $\lim_{h\to 0} \frac{u(x+h, y) - u(x, y)}{h}$ exists then it is called partial derivative of u w.r.t x as the point (x, y) and is denoted by $\frac{\partial u}{\partial x}(x, y)$ or $u_x(x, y)$.

Theorem : *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *have the radius of convergence* $R > 0$ *then*

1) The function f is infinitely differentiable on $B(0; r)$ *and*

$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)(n-2)...(n-k+1)a_n z^{n-k}$ *for* $|z| < R$ *and* $\forall\, k \geq 1.$

2) If $n \geq 0$ *then* $a_n = \frac{f^{(n)}(0)}{n!}$

Proof 1) For $|z| < R$, we will write $f(z) = \sum_{n=0}^{\infty} a_n z^n = S_n(z) + R_n(z)$

where $S_n(z) = \sum_{n=0}^{\infty} a_n z^n$ and $R_n(z) = \sum_{K=n+1}^{\infty} a_K z^K$

Put $g(z)=\sum_{n=1}^{\infty} na_{n-1}z^{n-1} = \lim_{n\to\infty} S'_n(z)$

Fix a point z_0 in $B(0;R)$

(Choose $r>0$, $\ni |z_0|<r<R$ and $|z|<r<R$ $\quad (z\neq z_0)$

We will prove that $f'(z_0)=g(z_0)$

Let $\delta>0$ be arbitary $\ni \overline{B}(z_0;\delta)\subset B(0,r)$

Let $z\in B(z_0,\delta)$ then

$$\frac{f(z)-f(z_0)}{z-z_0}-g(z_0)=\frac{S_n(z)+R_n(z)-\left[S_n(z_0)+R_n(z_0)\right]}{z-z_0}-g(z_0)$$

$$=\frac{S_n(z)-S_n(z_0)}{z-z_0}-S'_n(z_0)+S'_n(z_0)+\frac{R_n(z)-R_n(z_0)}{z-z_0}$$

Taking modulus on both the sides.

$$\therefore \left|\frac{f(z)-f(z_0)}{z-z_0}-g(z_0)\right|\le\left|\frac{S_n(z)-S_n(z_0)}{z-z_0}-S_n(z_0)\right|+\left|S'_n(z_0)-g(z_0)\right|$$

$$+\left|\frac{R_n(z)-R_n(z_0)}{z-z_0}\right| \qquad (1)$$

Let $\varepsilon>0$, be given

Now, $$\frac{R_n(z)-R_n(z_0)}{z-z_0}=\frac{1}{z-z_0}\sum_{K=n+1}^{\infty} a_K\left(z^K-z_0^K\right)$$

$$=\frac{1}{z-z_0}\sum_{K=n+1}^{\infty} a_K(z-z_0)\left[z^{K-1}+z_0\,z^{K-2}+\ldots+z\,z_0^{K-2}+z_0^{K-1}\right]$$

$$\left|\frac{R_n(z)-R_n(z_0)}{z-z_0}\right|\le\sum_{K=n+1}^{\infty}|a_K|\left|z^{K-1}+z_0\,z^{K-2}+\ldots+z\,z_0^{K-2}+z_0^{K-1}\right|$$

$$\le\sum_{K=n+1}^{\infty}|a_K|\left|r^{K-1}+r^{K-1}+\ldots+r^{K-1}\right| = \sum_{K=n+1}^{\infty}|a_K|.K\,r^{K-1}$$

$\because$ The derived series $\sum_{n=1}^{\infty} n\,a_n\,z^{n-1}$ is convergent at $z=r$.

$\therefore$ The power series $\sum_{K=1}^{\infty}|a_K|.K\,r^{K-1}$ converges for $r<R$.

$\therefore$ For the above $\varepsilon > 0$, $\exists$ an integer $N_1 \ni$

$\sum_{K=n+1}^{\infty} |a_K|.K.r^{K-1} < \varepsilon/3 \quad \forall\ n \geq N_1$. (by Cauchy criteria)

Thus, $\left|\dfrac{R_n(z)-R_n(z_0)}{z-z_0}\right| < \dfrac{\varepsilon}{3} \quad \forall\ n \geq N_1$ (2)

$\because \lim_{n\to\infty} S_n'(z_0) = g(z_0)$

For the above $\varepsilon > 0$, $\exists$ an integer N_2 s.t.

$$|S_n'(z_0) - g(z_0)| < \varepsilon/3 \qquad (3)$$

Choose $N = \max\{N_1, N_2\}$. $for\ n \geq \text{N}$

$\therefore$ For this n, we can find $\delta > 0$ s.t.

$$\left|\frac{S_n(z)-S_n(z_0)}{z-z_0} - S_n'(z_0)\right| < \frac{\varepsilon}{3} \qquad (4)$$

whenever $0 < |z - z_0| < \delta$.

From equation (1), (2), (3), (4) we get

$$\left|\frac{f(z)-f(z_0)}{z-z_0} - g(z_0)\right| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}$$

$$= \varepsilon \qquad \text{whenever } 0 < |z - z_0 < \delta|$$

$\Rightarrow$ f is differentiable at $z_0 \in B(0, R)$

$\because$ z is arbitary.

$\therefore$ f is differentiable on $B(0;\ R)$

A repeated application of this argument shows that the heigher derivatives $f', f'', ..., f^{(K)}$... exists, so that

$f^{(K)}(z) = \sum_{n=K}^{\infty} n(n-1)...(n-K+1)\, a_n\, z^{n-K}$ exists for $|z| < R$ and $\forall\ K \geq 1$.

$\Rightarrow$ f is infinitely differentiable on $B(0;\ R)$.

2) Since

$$f^{(K)}(z) = \sum_{n=K}^{\infty} n(n-1)...(n-K+1)\, a_n\, z^{n-K} = \sum_{n=K}^{\infty} \frac{n!}{(n-K)!} a_n\, z^{n-K}$$

$$= K!a_k + \sum_{K=n+1}^{\infty} \frac{n!}{(n-K)!} a_n z^{n-K}$$

Put $z = 0$

$\therefore$ $f^{(K)}(0) = K!a_K + 0$

$$a_k = \frac{f^{(K)}(0)}{K!}$$

Replace K by n

$\therefore$ $$a_n = \frac{f^{(n)}(0)}{n!}$$

Corollary : *If the power series* $\sum_{n=0}^{\infty} a_n z^n$ *has radius of convergence* $R > 0$, *then* $f(z) = \sum a_n z^n$ *is analytic on* $B(0; R)$.

Theorem : *If G is an open connected set and* $f : G \to \mathbb{C}$ *is differentiable with* $f'(z) = 0 \ \forall\ z \in G$, *then f is constant.*

Proof : Fix a point $z_0 \in G$ and let $w_0 = f(z_0)$.

Let $A = \{z \in G;\ f(z) = w_0\}$

T.P.T. $A = G$.

[i.e. by showing A is both open and closed and $A \neq \varnothing$]

T.P.T. A is closed.

Let $z \in G$ and $\{z_n\}$ be a sequence in $A \ni \lim_{n\to\infty} z_n = z$

$\therefore$ $f(z_n) = w_0$, for each $n \in \mathbb{N}$

f is differentiable on G (given)

$\therefore$ f is continuous on *G*.

$\therefore$ $f(z) = f\left[\lim_{n\to\infty} z_n\right] = \lim_{n\to\infty} f(z_n) = w_0$

$\Rightarrow$ $z \in A$

$\therefore$ A contain its limit point.

$\Rightarrow$ A is closed.

Now, T.P.T. A is open.

Fix $a \in A$, since *G* is open.

$\therefore$ $\exists\ r > 0 \ \ni B(a; r) \subset G$

Let $z \in B(a;r)$ and set $g(t) = f\left[tz + (1-t)a\right], \qquad 0 \le t \le 1$

$$\therefore \frac{g(t)-g(s)}{t-s} = \frac{f\left[tz+(1-t)a\right]+f\left[S_z+(1-S)a\right]}{(t-S)z+(S-t)a} \times \frac{(t-S)z+(S-t)a}{(t-S)}$$

$$\therefore \lim_{t \to S}\left[\frac{g(t)-g(S)}{t-S}\right] = \lim_{t \to S}\left[\frac{f\left[tz+(1-t)a\right]-f\left[sz+(1-S)a\right]}{(t-S)z+(S-t)a} \times (z-a)\right]$$

$\therefore \qquad g's = f'\left[sz+(1-s)a\right] \times (z-a)$

$\therefore \qquad g'(s) = 0, \; 0 \le s \le 1 \left(\because f'(z) = 0, \; \forall \; z \in G\right)$

$\Rightarrow \qquad g(s) = \text{constant}, \;\; 0 \le S \le 1$

$\Rightarrow \qquad g(1) = \text{constant} = g(0)$

$\therefore \qquad f(z) = g(1) = \text{constant} = g(0) \; = f(a) = w_0$

$\Rightarrow \qquad z \in A$

$\therefore \qquad z \in B(a;R) \Rightarrow z \in A$

$\Rightarrow \qquad B(a;r) c A$

$\Rightarrow$ A is open and $A \neq \varnothing \qquad (\because z \in A)$

Hence, by the connectedness of G

$A = G$

$\therefore$ f is constant on G.

4.3 SUMMARY

1) If $f : G \to \mathbb{C}$ is differentiable at a point z_0 in G, then f is continuous at z_0.

2)**The Increment Theorem :** Let $f : G \to \mathbb{C}$ be a complex valued function $z_0 \in G$ and $r > 0$, $\exists \; B(z_0, r) \subset G$. Then f is differentiable at z_0 iff $\exists$ a complex number α and a function $\eta : B(0;s) \to \mathbb{C}(0,S,r) \ni \forall \; h \in B(0;S)$
$f(z_0+h) = f(z_0) + h\alpha + h\eta(h)$ and $\lim_{h \to 0} \eta(h) = 0$.

3) **Chain Rule :** Let $G \in \mathbb{C}$, $\Omega \in \mathbb{C}$ be open sets and let f and g be differentiable on G and Ω (respectively). Suppose $f(G) \subset \Omega$ then $g \circ f$ is differentiable on G and $(g \circ f)(z) = g'\left[f(z)\right].f'(z) \qquad \forall \; z \in G$

4) If $\lim_{h\to 0}\frac{u(x+h, y)-u(x, y)}{h}$ exists then it is called partial derivative of u w.r.t x as the point (x, y) and is denoted by $\frac{\partial u}{\partial x}(x, y)$ or $u_x(x, y)$.

5) Let $f(z)=\sum_{n=0}^{\infty} a_n z^n$ has the radius of convergence $R>0$ then

i) The function *f* is infinitely differentiable on $B(0; r)$ and

$f^{(k)}(z)=\sum_{n=k}^{\infty} n(n-1)(n-2)...(n-k+1)a_n z^{n-k}$ for $|z|<R$ and $\forall$ $K\geq 1$.

ii) If $n\geq 0$ then $G_n=\frac{f^{(n)}(0)}{n!}$

6) If G is an open connected set and $f: G\to\mathbb{C}$ is differentiable with $f'(z)=0$ $\forall$ $z\in G$, then *f* is constant.

4.4. UNIT END EXERCISES

1) Check for the differentiability of the power series $f(z)=\sum_{n=0}^{\infty}\frac{z^n}{n!}$.

Solution: We know that the series $\sum_{n=1}^{\infty}\frac{z^n}{n!}$ converges for all complex numbers.

$\therefore$ $f'(z)$ exist for all $z\in\mathbb{C}$ and

$$f'(z)=\sum_{n=1}^{\infty}\frac{nz^{n-1}}{n!}=\sum_{n=1}^{\infty}\frac{z^{n-1}}{(n-1)!}=\sum_{n=1}^{\infty}\frac{z^n}{n!}=f(z).$$

$f'(z)=f(z)$ for all $z\in\mathbb{C}$.

2) If the series $\sum_{n=0}^{\infty}a_n(z-a)^n$ has the radius of convergence $R>0$, then show that $f(z)=\sum_{n=0}^{\infty}a_n(z-a)^n$ is analytic in $B(a; R)$.

(Hint : Use the fact that *f* is infinitely differentiable on $B(a; R)$ and $a_n=\frac{1}{n!}f^n(a)\forall n\geq 1$.)

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5

COMPLEX LOGARITHM

Unit Structure

5.0. Objectives
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5.3. Branches of Logarithmic Function
5.4. Properties of Logarithmic Function
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5.0. OBJECTIVES

We are already familiar with a logarithm function, defined for positive real *x*. In the same manner one can define a complex logarithm *Log (z)* of a complex number $z \in \mathbb{C}$. We shall study the branches of this complex logarithm function . The complex logarithm *Log (z)* posseses some branches , which we shall try to investigate. We shall also study the properties of a complex logarithm in detail.

5.1. INTRODUCTION

With the help of order completeness property of, we proved in our earlier course that if $y > 0$ and $n \geq 2$ is any integer, then there is a unique positive number *x* such that $x^n = y$. x is called n^{th} root of *y*, since there is a unique positive number *x* satisfying this, defining $y^{\frac{1}{n}}$ is justified. We proved that, for $a > 1$ and $x \in \mathbb{R}$, $a^x.a^y = a^{x+y}$ $\forall\, x, y \in \mathbb{R}$ and $\left(a^x\right)^y = a^{xy}$.

$f : \mathbb{R} \to (0,\infty)$ defined by $f(x) = a^x$ is a bijective function and it's inverse is called as the logarithm of *y* to the base *a*, denoted by $\log_a(y)$. We want to discuss these concepts once again but we consider the logarithm of complex numbers with base e, hence we try to identify the nature of inverse of the exponential function of complex variable z, namely $f(z)=e^{z}$ on some domain $D\subseteq\mathbb{C}$. Here we shall start defining $\log(z)$ for $z\in\mathbb{C}$.

5.2 LOGARITHMIC FUNCTION

Definition : For $z\neq 0$, the logarithmic function of a complex variable z, denoted by $\log z$, is defined as $\log z=\ln|z|+i(\arg z+2n\pi)$ where $\theta=\arg z\in[-\pi,\pi]$ or $[0,2\pi]$ and $n\in\mathbb{Z}$.
Here, $\log z$ is a single valued function.

5.3 BRANCHES OF LOGARITHMIC FUNCTION

Definition : If $0\notin G$ is an open connected set in $\mathbb{C}$ and $f:G\to\mathbb{C}$ is a continuous function such that $e^{f(z)}=z,\ \forall\ z\in G$, then *f* is branch of logarithm.

Theorem: *A branch of the logarithm is analytic and its derivative is* $\dfrac{1}{z}$.

Proof: Let $f(z)=\log z=\ln|z|+i\arg z$ be a branch of logarithm, where $z\neq 0$, $\arg(z)=[-\pi,\pi]$.

Let $f(z)=u(z)+iv(z)$ and $z=x+iy$

$\therefore\quad u+iv=\ln\sqrt{x^2+y^2}+i\tan^{-1}\left(y/x\right)$ where $|z|=\sqrt{x^2+y^2}$ and $\theta=\tan^{-1}\left(y/x\right)$

$\therefore\quad u(x,y)=\ln\sqrt{x^2+y^2}$ and $v=\tan^{-1}\left(y/x\right)$

$u(x,y)=\ln\sqrt{x^2+y^2}$

$\therefore\ \dfrac{\partial u}{\partial x}(x,y)=\dfrac{2x}{2\sqrt{x^2+y^2}}=\dfrac{x}{\sqrt{x^2+y^2}}$ and $\dfrac{\partial u}{\partial y}(x,y)=\dfrac{y}{\sqrt{x^2+y^2}}$

$\therefore\quad v=\tan^{-1}\left(y/x\right)$

$\therefore \quad \dfrac{\partial v}{\partial x} = \dfrac{1}{1+\left(y/x\right)^2} \times \left(\dfrac{-y}{x^2}\right) = \dfrac{-y}{x^2\left(\dfrac{x^2+y^2}{x^2}\right)} = \dfrac{-y}{x^2+y^2}$ and

$$\frac{\partial v}{\partial y} = \frac{-x}{x^2+y^2}$$

Therefore C-R equations are satisfied

$$\frac{\partial}{\partial z} f(z) = \frac{\partial}{\partial z}(\log z)$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{x}{x^2+y^2} + i\left(\frac{-y}{x^2+y^2}\right) = \frac{\bar{z}}{|z|^2} = \frac{\bar{z}}{z.\bar{z}} = \frac{1}{z}$$

Theorem : *Let $0 \notin G$ be an open connected set in $\mathbb{C}$ and suppose that $f: G \to \mathbb{C}$ is analytic. Then f is a branch of logarithm iff $f'(z) = \frac{1}{z}$, $\forall z \in G$ and $e^{f(a)} = a$ for atleast one $a \in G$.*

Proof : Suppose f is a branch of a logarithm.

$\therefore \quad e^{f(z)} = z \quad \forall\ z \in G$ (1)

Differentiate w.r.t. to z on both sides.

$\therefore \quad e^{f(z)}.f'(z) = 1$

$\therefore \quad f'(z) = \dfrac{1}{e^{f(z)}}$

$\therefore \quad f'(z) = \dfrac{1}{z} \quad \forall\ z \in G$

Clearly, from equation (1), $e^{f(a)} = a$ for atleast one $a \in G$.

Conversely,

Suppose $f'(z) = \frac{1}{z}$ $\forall\ z \in G$ and $e^{f(a)} = a$ for atleast one $a \in G$.

T.P.T. f is a branch of the logarithm.

Define, $g(z) = z.e^{-f(z)}$ (2)

$\because$ g is analytic.

$\therefore$ g is differentiable.

$\therefore \quad g'(z) = \dfrac{d}{dz}\left[z\,e^{-f(z)}\right] = e^{-f(z)}.\dfrac{dz}{dz} + z\dfrac{d}{dz}\left[e^{-f(z)}\right]$

$= e^{-f(z)} + \left[-f'(z).z\,e^{-f(z)}\right]$

$= e^{-f(z)} - z.e^{-f(z)} \times \dfrac{1}{z} \qquad \left(\because f'(z) = \dfrac{1}{z}\right)$

$= e^{-f(z)} - e^{-f(z)}$

$g'(z) = 0 \qquad \forall\ z \in G$

$\Rightarrow\ g(z) = \text{constant} = K$ (Say) (3)

To find *K*, put $z = a$ in equation (2) and (3)

$\therefore\quad g(a) = a\,.\,e^{-f(a)}$ and $g(a) = K$.

$\Rightarrow\quad K = a\,.\,e^{-f(a)} = a\,.\,\frac{1}{a} \qquad \left\{\because e^{-f(a)} = \frac{1}{a}\right\}$

$= 1$

$\Rightarrow\quad \boxed{K = 1}$

Put $K = 1$ in equation (3), we get $g(z) = 1$

Put $g(z) = 1$ in equation (2), we get $1 = z\,.\,e^{-f(z)}$

$e^{f(z)} = z \quad \forall\ z \in G$

$\Rightarrow f$ is branch of logarithm (by definition)

A single valued function (is branch of logarithm) $\log z = \ln|z| + i \arg z \quad (z \neq 0 \text{ and } \theta = \arg z \in [-\pi, \pi])$ is continuous function in the region or a Domain $D = \mathbb{C} \,|\, \{z = x + iy \in \mathbb{C};\ y = 0,\ x \leq 0\}$

$\log z = \ln|z| + i \arg z \quad (z \neq 0 \text{ and } \theta = \arg z \in [-\pi, \pi])$

$\therefore \log z$ is not defined at the point $z = 0$.

Theorem : *Prove that* $\log z$ *is not continuous on the negative real axis.*

Proof : Let $z_0 = x_0 < 0$ be any point on the negative real axis.

For $z = x + iy$ with $x < 0,\ y < 0$,

we have, $\lim\limits_{z \to z_0} \arg z = \lim\limits_{\substack{x \to x_0 \\ y > 0}} \arg(x + iy) = \pi \ = \pi$

For $z = x + iy$ with $x < 0$ and $y < 0$,

We have $\lim\limits_{z \to z_0} \arg z = \lim\limits_{\substack{x \to x_0 \\ y < 0}} \arg(x + iy) = -\pi$

$\therefore$ Two limits obtained are different.

i.e. $\arg z$ fails to possess a limit every point of the negative real axis.

$\therefore \log z$ is not continuous along the negative real axis.

Theorem: *Let* $0 \notin G$ *be an open connected set in* $\mathbb{C}$. *If a branch of the logarithm* $f : \to \mathbb{C}$ *is related by* $g(z) = f(z) + 2\pi i n$ *[for some integer* $n \in \mathbb{Z}$*] with* $g : G \to \mathbb{C}$ *then g is branch of logarithm.(2008)*

Proof : Given that f is a branch of the logarithm.

$\therefore\quad e^{f(z)} = z \quad \forall\ z \in G$

Given, $g(z) = e(z) + 2\pi in$ (for some int $n \in \mathbb{Z}$)

$\therefore \quad e^{g(z)} = e^{f(z)+2\pi in} = e^{f(z)} . e^{2\pi in}$

$e^{g(z)} = z \quad \forall \; z \in G \qquad \left\{\because e^{f(z)} = z \text{ and } e^{2\pi in} = 1\right\}$

$\therefore$ g is a branch of logarithm.

5.4 PROPERTIES OF LOGARITHM FUNCTION

Theorem:

1) $\log(z_1 . z_2) = \log z_1 + \log z_2 + 2\pi in$, *where* $n = 1, 0$ or -1 *by definition.*

Proof : $\log z = \ln|z| + i \arg z$ where $z \neq 0$ and $\theta = \arg z \in [-\pi, \pi]$.

$\therefore \log(z_1 . z_2) = \ln(z_1 . z_2) + i \arg(z_1 . z_2)$

$= \ln|z_1| + \ln|z_2| + i[\arg z_1 + \arg z_2 + 2\pi in]$

$(\because \arg(z_1 . z_2) = \arg z_1 + \arg z_2 + 2\pi in)$

$= (\ln|z_1| + i \arg z_1) + (\ln|z_2| + i \arg z_2) + 2\pi in = \log z_1 + \log z_2 + 2\pi in$

2) $\log\left(\frac{z_1}{z_2}\right) = \log z_1 - \log z_2 + 2\pi in$

3) $\log\left(\frac{1}{z}\right) = -\log z$

Proof: $\log\left(\frac{1}{z}\right) = \ln\left|\frac{1}{z}\right| + i \arg\left(\frac{1}{z}\right)$

$= -\ln|z| + i \arg \bar{z} \qquad \left(\because \arg\left(\frac{1}{z}\right) = \arg \bar{z}\right)$

$= -\ln|z| - i \arg z = -[\ln|z| + i \arg z] = -\log z$

Evaluate :-

1) $\log i$

$\because \quad z = x + iy = i \quad \Rightarrow \quad x = 0$ and $y = 1$

$\therefore \quad \log z = \ln|z| + i \arg|z|$ (by definition)

$\therefore \quad \log i = \ln|i| + i \arg|i| = \ln 1 + i\frac{\pi}{2} = i\frac{\pi}{2}$

2) $\log(1-i)$

$\because \quad z = 1 - i \Rightarrow x = 1$ and $y = -1$,

$|z| = \sqrt{x^2 + y^2} = \sqrt{1+1} = \sqrt{2}$

$$\therefore \quad \log(1-i) = \ln(1-i) + i \arg|1-i| = \ln\left(\sqrt{2}\right) + i\tan^{-1}\left(-1/i\right)$$

$$= \ln\left(\sqrt{2}\right) - i\tan^{-1}(1) = \ln\left(\sqrt{2}\right) - i\frac{\pi}{4}$$

3) $\log(1+i)$

$$z = 1+i \quad \Rightarrow \quad x = 1,\ y = 1 \quad \Rightarrow \quad |z| = \sqrt{2}$$

$$\therefore \quad \log(1+i) = \ln|1+i| + i\arg|1+i| = \ln\left(\sqrt{2}\right) + i\tan^{-1}\left(1/1\right)$$

$$= \ln\left(\sqrt{2}\right) + i\tan^{-1}(1) = \ln\left(\sqrt{2}\right) + i\frac{\pi}{4}$$

4) In unit disk $B(0,1) = \{z \in \mathbb{C} : |z| < 1\}$ prove that power series $\sum_{n=1}^{\infty} \frac{z^n}{n} = \log\left(\frac{1}{1-z}\right)$ where $\log\left(\frac{1}{1-z}\right)$ is a branch of the logarithm $\log\left(\frac{1}{1-z}\right)$.

Solution : Let $f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n}$ (1)

and $g(z) = \log\left(\frac{1}{1-z}\right)$ (2)

$\Rightarrow$ g is differentiable.

$$g'(z) = \frac{d}{dz}\left[\log\left(\frac{1}{1-z}\right)\right] = \frac{1}{\cancel{1}\big/(1-z)^2}\left[\frac{-1}{(1-z)^2}(-1)\right] = \frac{(1-z)}{(1-z)^2}$$

$$g'(z) = \frac{1}{1-z} \qquad (3)$$

Given, the power series $f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n}$

Here, $a_n = \frac{1}{n}$, $a_{n+1} = \frac{1}{n+1}$

$$\therefore \quad R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{1}{n}\times(n+1)\right| = \lim_{n\to\infty}\left|\frac{\cancel{n}\left(1+1/n\right)}{\cancel{n}}\right|$$

$$= 1 \qquad \left\{\because \frac{1}{\infty} = 0\right\}$$

$\Rightarrow$ f is analytic in open disk $B(0,1)$ (using corollary)

$\Rightarrow$ f is differentiable in $B(0;1)$

$$\therefore \quad f'(z) = \sum_{n=1}^{\infty} \frac{n \cdot z^{n-1}}{n} = \sum_{n=1}^{\infty} z^{n-1} = \sum_{n=0}^{\infty} z^n$$

$\therefore \quad f'(z) = \frac{1}{1-z}$ (4)

$\left(\because \sum_{n=0}^{\infty} z^n \text{ is a G.S. and cgs to} |z| < 1, \ \because \sum_{n=0}^{\infty} G^n = \frac{1}{1-z}\right)$

From equation (3) and (4), we get

$$f'(z) - g'(z) = 0 \quad \Rightarrow \quad [f(z) - g(z)]^1 = 0$$

$\Rightarrow \quad f(z) - g(z) =$ constant $= K$ (5)

To find k, put $z = 0$ in equation (1), (2) and (3)

$$f(0) = 0, \quad g(0) = 0$$

$\therefore \quad f(0) - g(0) = K$

$\Rightarrow \quad K = 0$

Put $K = 0$ in equation (3), we get

$$f(z) - g(z) = 0$$

$\therefore \quad f(z) = g(z)$

$\therefore \sum_{n=1}^{\infty} \frac{z^n}{n} = \log\left(\frac{1}{1-z}\right)$ for all $z \in B(0;1)$

Definition : Given $0 \notin \mathbb{C}$, the principal value of z^b (i.e. the b^{th} power of z) is defined by $z^b = e^{b \cdot \log z}, \ b \in \mathbb{C}$

Here, z^b is analytic. ($\because$ $\log z$ is analytic)

Consider $z^b = e^{b \cdot \log z}, \quad b \in \mathbb{C}$

Here, z^b is multivalued function.

$\because$ $\arg z$ (and hence $\log z$) is a multiple valued function.

Case I : If b is an integer then $z^b = e^{b \cdot \log z}$ is a single valued function.

Proof : Let $b = K \in \mathbb{Z}$.

$$\therefore \quad z^b = e^{b \cdot \log z} = e^{k[\ln|z| + i(\arg z + 2\pi n)]}$$

$$= e^{k[\ln|z| + i \arg z]} \cdot e^{i(+2\pi Kn)}$$

$$= e^{k \log z}(i) \qquad \left(\because e^{i(2\pi Kn)} = 1, \ n \in \mathbb{Z}\right)$$

$$= e^{b \log z}$$

$z^b = e^{b \cdot \log z}$ is a single valued function.

Case II : If $b = p/q$ (real rational) then z^b has produces exactly q values.

Case III : If b is an irrational number or imaginary number then z^b is infinite valued function.

Example 1 : Find the principal value of i^i.

Solution : $z^b = e^{b \cdot \log z}, \qquad b \in \mathbb{C} \qquad$ (by definition)

$$\therefore \quad i^i = e^{i \cdot \log i} = e^{i\left[\ln|i| + i \arg i\right]} = e^{i\left[0 + i\pi/2\right]} = e^{i^2 \cdot \pi/2}$$

$$\boxed{i^i = e^{-\pi/2}}$$

Example 2 : Find all the values of i^{-2i}.

Solution : $z^b = e^{b \cdot \log z} \qquad b \in \mathbb{C} \qquad$ (by definition)

$$\therefore \quad i^{-2i} = e^{-2i \cdot \log i} = e^{-2i\left[\ln|i| + i(\arg i + 2n\pi)\right]} = e^{-2i\left[0 + i\left(\frac{\pi}{2} + 2n\pi\right)\right]}$$

$$= e^{-\cancel{2}i\left(i\left(\frac{\pi + 4n\pi}{\cancel{2}}\right)\right)} = e^{-i^2\left[\pi(1+4n)\right]}$$

$$= e^{-(-1)\left[\pi(1+4n)\right]} \qquad n \in \mathbb{Z}$$

$$= e^{\pi(4n+1)}$$

Here, the principal value of i^{-2i} is e^{π} .(All values are not found.)

Example 3 : Find the value of i^2.

Solution : $i^2 = e^{2\log i} = e^{2\left[\ln|i| + i(\arg i + 2\pi n)\right]} = e^{2\left[\ln 1 + i(\arg i + 2\pi n)\right]}$

$$= e^{2\left[0 + 1\pi/2\right]} \cdot e^{4\pi ni} = e^{i\pi} \qquad \left(\because e^{4\pi ni} = 1,\ n \in \mathbb{Z}\right)$$

$$= \cos\pi + i\sin\pi = -1 + 0$$

$$\boxed{i^2 = -1}$$

Example 4 : Find all the values of $(1+i)^{(1+i)}$.

5.5 TRIGNOMETRIC AND HYPERBOLIC FUNCTIONS

Trigonometric Function : The Complex trigonometric functions sin and cos are defined by

$$\sin z = \frac{e^{iz} - e^{-iz}}{2!} \quad \text{and} \quad \cos z = \frac{e^{-z} + e^{-iz}}{2} \qquad (2008)$$

Similarly, $\tan z = \dfrac{e^{iz} - e^{-iz}}{i\left(e^{iz} + e^{-iz}\right)}$, $\sec z = \dfrac{2}{e^{iz} + e^{-iz}}$

$$\operatorname{cosec} z = \frac{2i}{e^{iz} - e^{-iz}}, \qquad \cot z = \frac{i\left(e^{iz} + e^{-iz}\right)}{e^{iz} - e^{-iz}}$$

Note :

1) $\sin^2 z + \cos^2 z = 1$
2) $\dfrac{d}{dz}(\sin z) = \cos z$ and $\dfrac{d}{dz}(\cos z) = -\sin z$
3) $\sin(-z) = -\sin z$ and $\cos(-z) = \cos z$
4) $\sin(z+w) = \sin z . \cos w + \cos z . \sin w$
5) $\cos(z+w) = \cos z . \cos w - \sin z . \sin w$
6) $\sin 2z = 2 \sin z . \cos z$

<u>Hyperbolic Function</u> : The complex hyperbolic functions $\sin h$ and $\cos h$ are defined by

$\sin hz = \dfrac{e^{z} - e^{-z}}{2}$ and $\cos h = \dfrac{e^{z} + e^{-z}}{2}$.

Similarly, $\tan hz = \dfrac{e^{z} - e^{-z}}{e^{z} + e^{-z}}$, $\cot hz = \dfrac{e^{z} + e^{-z}}{e^{z} - e^{-z}}$

$$\sec hz = \frac{2}{e^{z} + e^{-z}}, \qquad \operatorname{cosec} hz = \frac{2}{e^{z} - e^{-z}}$$

Note :

1) $\cos^2 z - \sin^2 z = 1$
2) $\dfrac{d}{dz}(\sin hz) = \cos hz$
3) $\dfrac{d}{dz}(\cos hz) = \sin hz$
4) $\sin h(z+w) = \sin hz . \cos hw + \cos hw . \sin hz$
5) $\cos h(z+w) = \cos hz . \cos hw + \sin hw . \sin hz$

<u>Relation between Trigonometric and Hyperbolic Function</u> :

1) $\sin iz = i \sin hz$

Proof :

$$\sin iz = \frac{e^{i(iz)} - e^{-i(iz)}}{2i} = \frac{e^{-z} - e^{z}}{2!} = \frac{i^2\left(e^{z} - e^{-z}\right)}{2i} = \frac{i\left(e^{z} - e^{-z}\right)}{2}$$

$= i \sin hz$

2) $\cos iz = \cos hz$

Proof : $\cos(z) = \dfrac{e^{iz} + e^{-iz}}{2} \;\; = \dfrac{e^{-z} + e^{z}}{2} = \cos hz$

3) $\tan iz = i \tan hz$

4) $\cos hiz = \cos z$

5) $\sin hiz = i \sin z$

6) $\tan hiz = i \tan z$

<u>Periodic Function</u> : A function $f : G \to \mathbb{C}$ is said to be periodic if $\exists$ a non-zero complex number $T \ni f(z+T) = f(z) \quad \forall\, z \in G$.
Here T is a period of the function *n*.

Periodicity of e^z :

Let T be the period of e^z

$\therefore \quad e^{z+T} = e^{z} \quad \forall\, z \in \mathbb{C}$.

To find T, put $z = 0$

$\therefore \quad e^{T} = e^{0} = 1 = e^{2\pi in}$

Let $T = \alpha + i\beta$

$\therefore \quad e^{\alpha + i\beta} = e^{2\pi in}$

$\quad e^{\alpha} = 1$ and $e^{i\beta} = e^{2\pi in}$

$\Rightarrow \quad \alpha = 0$ and $\beta = 2\pi n$

$\therefore \quad T = \alpha + i\beta = 0 + 2i\,\pi n$ is a period of e^z.

OR

$T = \log 1 = 0$ (is not possible by definition)

Let $T = \alpha + i\beta$

$\therefore \quad e^{\alpha + i\beta} = 1$

$\quad e^{\alpha}.\cos\beta = 1$ and $\sin\beta = 0$

$\Rightarrow \quad e^{\alpha} = 0$ and $\beta = 2n\pi \quad \Rightarrow \quad T = \alpha + i\beta = 0 + 2\,in\,\pi = 2\,in\,\pi$

$2\,in\,\pi$ is a period of e^z

$\therefore \; e^{z + 2\,ni\,\pi} = e^{z}$

Periodicity of $\sin z$:

Let T be the period of $\sin z$.

$\therefore \quad \sin(z+T) = \sin z$

Put $z = 0$

$\therefore \quad \sin(0+T)=\sin 0$

$\Rightarrow \quad \sin T = 0$

$\Rightarrow \quad T = n\pi$, where $n = 0, \pm 1, \pm 2, ...$

$\therefore \quad \sin(z+2\pi) = \sin z . \cos 2\pi + \cos z . \sin 2\pi = (-1)^n \sin z$

$= \sin z$ if n is even.

$\therefore$ The period of $\sin z$ is $2n\pi$ where $n \in \mathbb{Z}$.

Periodicity of $\cos z$:

Let T be the period of $\cos z$.

$\therefore \quad \cos(z+T) = \cos z$

Put $z = 0$

$\therefore \quad \cos(0+T) = \cos 0$

$\Rightarrow \quad \cos T = 1$

$\Rightarrow \quad T = 2\pi n \quad \forall \ n \in \mathbb{Z}$

$\therefore$ The period of $\cos z$ is $2n\pi$, where $n \in \mathbb{Z}$.

5.6 SUMMARY

1) For $z \neq 0$, the logarithmic function of a complex variable z, denoted by $\log z$, is defined as $\log z = \ln|z| + i(\arg z + 2n\pi)$ where $\theta = \arg z \in [-\pi, \pi]$ or $[0, 2\pi]$ and $n \in \mathbb{Z}$.

2) If $0 \notin G$ is an open connected set in $\mathbb{C}$ and $f : G \to \mathbb{C}$ is a continuous function $\ni \ e^{f(z)} = z, \ \forall \ z \in G$, then f is a <u>branch of the logarithm.</u>

3) Given $0 \notin c$, the <u>principal value</u> of z^b (i.e. the b^th^ power of z) is defined by

$$z^b = e^{b . \log z}, \ \ b \in \mathbb{C}$$

4) The Hyperbolic functions *sinh* and *cosh* are defined by

$$\sin hz = \frac{e^z - e^{-z}}{2} \ \text{ and } \ \cos h = \frac{e^z + e^{-z}}{2}.$$

5.7 UNIT END EXERCISES

1) Suppose that $f : G \to \mathbb{C}$ is a branch of the logarithm and n is any integer . Prove that $z^n = \exp(nf(z))$ for all $z \in G$.

Solution: Since $f:G\to\mathbb{C}$ is a branch of the logarithm ($G\subseteq\mathbb{C}$ is an open connected set .)

$\therefore z=\exp(f(z))$ for all $z\in G$.

$\therefore z^2=\exp(f(z).\exp(f(z))=\exp(f(z)+f(z))=\exp(2f(z))$.

$\therefore z^n=\exp(nf(z))$ for all $z\in G$. (By induction on power of z.

2) Describe the branches of an analytic function $f(z)=\sqrt{z}$.

Solution: $\therefore f(z)=\sqrt{z}=\exp\left(\frac{1}{2}\log(z)\right)$,since

$$\left(\exp\left(\frac{1}{2}\log(z)\right)\right)^2=\exp\left(\frac{1}{2}\log(z)+\frac{1}{2}\log(z)\right)=\exp\left(\log(z)\right)=z.$$

$\therefore$ This defines $\sqrt{z}$ and it is analytic , where the $\log(z)$ is analytic.

$\therefore$ Different branches of $\log(z)$ yield different branches of $\sqrt{z}$.
$\log(z)$ has infinitely many different branches $\log(z)+2\pi ki$ for any integer k but there are only two different branches of $\sqrt{z}$.
Since $\exp\left(\frac{1}{2}\log(z)\right)=\exp\left[\frac{1}{2}\left(\log z+2\pi ki\right)\right]$ whenever k is an even integer .

3) Find all values of the complex number i^i.

Solution: $\therefore i^i=e^{i\log(i)}=e^{\log(i)+i\arg(i)}=e^{-\arg(i)}$.
Here we know that

$$\arg(i)=\left\{\ldots,-\frac{3\pi}{2},\frac{\pi}{2},\frac{5\pi}{2},\frac{7\pi}{2},\ldots\right\}=\left\{\frac{\pi}{2}+2n\pi:n\in\mathbb{Z}\right\}.$$

$$\therefore i^i=\left\{\ldots,e^{\frac{3\pi}{2}},e^{-\frac{\pi}{2}},e^{-\frac{5\pi}{2}},\ldots\right\}.$$

4) Find all values of $(1+i)^{1+i}$ (Hint : $(1+i)^{1+i}=e^{(1+i)\log(1+i)}$).

5) Let $f:G\to\mathbb{C}$ and $g:G\to\mathbb{C}$ be branches of z^a and z^b respectively. Show that fg is a branch of z^{a+b} and $\frac{f}{g}$ is a branch of z^{a-b}.

Solution: $\therefore\ f(z)=z^a$ and $g(z)=z^b$ for all $z\in G$.

$\therefore\ fg:G\to\mathbb{C}$ defined by $(fg)(z)=f(z).g(z)=z^a.z^b=z^{a+b}$

$(fg)(z)=z^{a+b}$ for all $z\in G$.

$\therefore\ z^{a+b}$ has a branch *fg* on *G*.

Similarly, $\dfrac{f}{g}$ is a branch of z^{a-b}.

6) Let $z_1,z_2,...,z_n$ be complex numbers such that $\text{Re}(z_k)>0$ and $\text{Re}(z_1z_2...z_k)>0$, for $1\le k\le n$. Then show that $\log(z_1z_2...z_k)=\log(z_1)+...+\log(z_n)$.

Solution : Let $f(z)=\log(z)$ be the principle branch of the logarithm function $\therefore\ e^{f(z)}=e^{\log(z)}=z$.

Take $a=z_1...z_n$. Since the arguments of each z_k and that of $z_1z_2...z_k$ lies between $-\dfrac{n}{2}$ to $\dfrac{\pi}{2}$ for all $1\le k\le n$. Therefore $\log(z_1z_2...z_k)=\log|z_1...z_k|+i\,Arg(z_1z_2...z_k)$ and $\log(z_k)=\log|z_k|+iArg(z_k)$, for $1\le k\le n$.

$\therefore\ Arg(z_1z_2...z_n)=Arg(z_1)+Arg(z_2)+...+Arg(z_n)+2k\pi$ where k is any integer .

$\therefore\ f(a)=\log|z_1...z_n|+i\,\text{Arg}(z_1z_2...z_n)$

$=\sum_{k=1}^{n}\log|z_k|+i\left[Arg(z_1)+Arg(z_2)+...+Arg(z_n)+2k\pi\right]$

$=\sum_{k=1}^{n}\left[\log|z_k|+iArg(z_k)\right]+2k\pi i$

$=\sum_{k=1}^{n}\log(z_k)+2k\pi i$

$\therefore\ e^{f(a)-\sum_{k=1}^{n}\log(z_k)=e^{2k\pi i}=1}$

$\Rightarrow f(a)-\sum_{k=1}^{n}\log(z_k)=0$

$\therefore\ f(a)=\sum_{k=1}^{n}\log(z_k)$

$\therefore\ \log(z_1z_2...z_n)=\log(z_1)+...+\log(z_n)$.

7) Give the principal branch of $\sqrt{1-z}$ (Hint : $e^{\log\left(\sqrt{1-z}\right)}=\dfrac{1}{2}\log(1-z)$)

8) Prove that there is no branch of the logarithm defined on $G = \mathbb{C} - 0$.

(Hint: Assume the existence of a continuous function $L(z)$ defined on a connected open set G of the complex plane such that $L(z)$ is a logarithm of z for each z in G, compare $L(z)$ with the Principal branch of $\log(z)$. As α goes from 0 to 2π, since $L(e^{ia}) = ia$ and L being continuous function of α, $\therefore L\left(e^{2\pi i}\right) = 2\pi i = L(1) = 0$, a contradiction.)

9) Evaluate i^i by taking the logarithm in its principal branch.

10) Prove that $|\sin z|^2 = \sin^2 x + \sinh^2 y$

Solution:

$$|\sin z|^2 = |\sin(x+iy)|^2 = |\sin x \cos iy + \cos x \sin iy|^2 = |\sin x \cosh y + i \cos x \sinh y|^2$$
$$= (\sin x \cosh y + i \cos x \sinh y)\overline{(\sin x \cosh y + i \cos x \sinh y)}$$
$$= (\sin x \cosh y + i \cos x \sinh y)(\sin x \cosh y - i \cos x \sinh y)$$
$$= \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y = \sin^2 x(1+\sinh^2 y) + (1-\sin^2 x)\sinh^2 y$$
$$= \sin^2 x + \sin^2 x \sinh^2 y + \sinh^2 y - \sinh^2 y \sin^2 x = \sin^2 x + \sinh^2 y$$

11) Find the principal value of i^{-2i}

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6

ANALYTIC FUNCTIONS

Unit Structure

6.0. Objectives
6.1. Introduction
6.2. Analytic Functions
6.3. Cauchy Riemann equations
6.4. Harmonic Functions
6.5. The Functions $e^z, \sin(z), \cos(z)$ etc.
6.6. Summary
6.7. Unit End Exercises

6.0. OBJECTIVES

In this unit we shall characterise the differentiability of a complex valued function in terms of it's power series expansion , in this case the function is said to be an analytic function about some point $z_0 \in \mathbb{C}$. An analytic function $f(z)$ satisfies some properties, among these one important property is to satisfy Cauchy-Riemann equations . Further we shall also see the term by term differentiation of a power series function, provided that such term by term differentiation is possible. We shall also study the inverse function theorem then we shall define a class of functions called as harmonic functions .We shall also discuss the differentiability of a complex valued functions like $e^z, \sin(z), \cos(z)$ etc.

6.1 INTRODUCTION

Given a function of the complex variable z, we wish to examine if f is a differentiable function of z or not. As we saw in the case real valued functions, we look for existence of the limit $\lim_{n \to 0} \frac{f(z+h) - f(z)}{n}$ which should exist regardless of the manner in which h approaches 0 through complex values. An immediate consequence is that the partial derivatives of f, considered as a function of two real variables x and $y(f(z) = f(x+iy) = f(x, y)$ must satisfy the Cauchy Riemann equations. Let us define the derivative of a function of complex variable z at the point $z = z_0 \in \mathbb{C}$.

Let $z = x+iy \in G$ and $f : G \to \mathbb{C}$ be defined by, $f(z) = u(z) + iv(z)$, where u and v are real valued function

OR

$$f(x, y) = u(x, y) + iv(x, y).$$

Definition : If $\lim_{h \to 0} \frac{u(x+h, y) - u(x, y)}{h}$ exists then it is called partial derivative of u w.r.t x as the point (x, y) and is denoted by $\frac{\partial u}{\partial x}(x, y)$ or $u_x(x, y)$.

6.2 ANALYTIC FUNCTIONS

A function f is said to be analytic (or holomorphic or regular) at a point $z = z_0$ if f is differentiable at every point of some nbd. of z_0.

Definition : A function $f : G \to \mathbb{C}$ is analytic if f is continuous differentiable on *G*.

A function f is analytic on a closed set *S* if f is differentiable at every point of some open set containing *S*.

Theorem : *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *have the radius of converges* $R > 0$ *then*

1) *The function f is infinitely differentiable on* $B(0; r)$ *and*

$$f^{(K)}(z) = \sum_{n=K}^{\infty} n(n-1).......(n-K+1)a_n z^{n-K} \text{ for } |z| < R \text{ and } \forall\, K \geq 1.$$

2) *If* $n \geq 0$ *then* $a_n = \frac{f^n(0)}{n!}$

Proof : 1) For $|z| < R$, we will write $f(z) = \sum_{n=0}^{\infty} a_n z^n = S_n(z) + R_n(z)$ where $S_n(z) = \sum_{n=0}^{\infty} a_n z^n$ and $R_n(z) = \sum_{K=n+1}^{\infty} a_K z^K$

Put $g(z) = \sum_{n=1}^{\infty} n a_n z^{-1} = \lim_{n \to \infty} S_n'(z)$

Fix a point z_0 in $B(0; R)$

(Choose $r > 0, \ni |z_0| < r < R$ and $|z| < r < R$ $\qquad (z \neq z_0)$

We will prove that $f'(z_0) = g(z_0)$

Let $\delta > 0$ be arbitrary $\ni \overline{B}(z_0; \delta) \subset B(0, r)$

Let $z \in B(z_0, \delta)$ then

$$\frac{f(z) - f(z_0)}{z - z_0} - z(z_0) = \frac{S_n(z) - R_n(z) - [S_n(z_0) + R_n(z_0)]}{z - z_0} - g(zl0)$$

$$= \frac{S_n(z) - S_n(z_0)}{z - z_0} - S_n'(z_0) + S_n'(z_0) + \frac{R_n(z) - R_n(z_0)}{z - z_0}$$

$$\therefore \left| \frac{f(z) - f(z_0)}{z - z_0} - g(z_0) \right| \leq \left| \frac{S_n(z) - S_n(z_0)}{z - z_0} - S_n(z_0) \right| + |S_n'(z_0) - g(z_0)|$$

$$+ \left| \frac{R_n(z) - R_n(z_0)}{z - z_0} \right| \ldots .(1)$$

Let $\varepsilon > 0$, be given

Now, $\frac{R_n(z) - R_n(z_0)}{z - z_0} = \frac{1}{z - z_0} \sum_{K=n+1}^{\infty} a_K \left(z^K - z_0^K \right)$

$$= \frac{1}{z - z_0} \sum_{K=n+1}^{\infty} a_K (z - z_0) \left[z^{K-1} + z_0 z^{K-2} + \ldots + z z_0^{K-2} + z_0^{K-1} \right]$$

$$\left| \frac{R_n(z) - R_n(z_0)}{z - z_0} \right| \leq \sum_{K=n+1}^{\infty} |a_K| \left| z^{K-1} + z_0 z^{K-2} + \ldots + z z_0^{K-2} + z_0^{K-1} \right|$$

$$\leq \sum_{K=n+1}^{\infty} |a_K| \left| r^{K-1} + r^{K-1} + \ldots + r^{K-1} \right|$$

$$= \sum_{K=n+1}^{\infty} |a_K| . K r^{K-1}$$

$\because$ The derived series $\sum_{n=1}^{\infty} n\, a_n\, z^{n-1}$ is convergent at $z = r$.

$\therefore$ The power series $\sum_{K=1}^{\infty} |\, a_K \,|.\, K\, r^{K-1}$ converges for $r < R$.

For the above $\varepsilon > 0$, $\exists$ an integer $N_1 \ni$

$\sum_{K=n+1}^{\infty} |\, a_K \,|.\, K\,.\, r^{K-1} < \varepsilon/3 \quad \forall\ n \geq N_1$. ($\because$ by Cauchy criteria)

Thus, $\left| \dfrac{R_n(z) - R_n(z_0)}{z - z_0} \right| < \dfrac{\varepsilon}{3} \qquad \forall\ n \geq N_1$ (2)

$\because \lim_{n \to \infty} S_n'(z_0) = g(z_0)$

For the above $\varepsilon > 0$, $\exists$ an integer N_2 s.t.

$$|\, S_n'(z_0) - g(z_0) \,| < \varepsilon/3 \qquad (3)$$

Choose $N = \max\{N_1, N_2\}$

$\therefore$ For this *n*, we can find $\delta > 0$ s.t.

$$\left| \frac{S_n(z) - S_n(z_0)}{z - z_0} - S_n'(z_0) \right| < \frac{\varepsilon}{3} \qquad (4)$$

whenever $0 < |\, z - z_0 \,| < \delta$.

From equation (1), (2), (3), (4) we get

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - g(z_0) \right| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \text{ whenever } 0 < |\, z - z_0 < \delta |$$

$\Rightarrow$ f is differentiable at $z_0 \in B(.;\ R)$

$\because$ z is arbitrary.

$\therefore$ f is differentiable on $B(0;\ R)$

A repeated application of this argument shows that the heigher derivatives $f', f'', ..., f^{(K)}$... exists, so that

$f^{(K)}(z) = \sum_{n=K}^{\infty} n(n-1)...(n-K+1)\, a_n\, z^{n-K}$ exists for $|\, z \,| < R$ and $\forall\ K \geq 1$.

$\Rightarrow$ f is infinitely differentiable on $B(0;\ R)$.

2) Since

$$f^{(K)}(z)=\sum_{n=K}^{\infty} n(n-1)\ldots(n-K+1)\,a_n\,z^{n-K} = \sum_{n=K}^{\infty} \frac{n!}{(n-K)!}\,a_n\,z^{n-K}$$

$$=K!\,a_k+\sum_{K=n+1}^{\infty} \frac{n!}{(n-K)!}\,a_n\,z^{n-K}$$

Put $z=0$

$\therefore\ f^{(K)}(0)=K!\,a_K+0$

$$a_k=\frac{f^{(K)}(0)}{K!}$$

Replace K by n

$$\therefore\qquad a_n=\frac{f^{(n)}(0)}{n!}$$

Corollary : *If the power series* $\sum_{n=0}^{\infty} a_n z^n$ *has radius of cgs.* $R>0$*, then* $f(z)=\sum a_n z^n$ *is analytic on* $B(0;R)$.

Theorem : *If G is an open connected set and* $f:G\to\mathbb{C}$ *is differentiable with* $f'(z)=0\ \forall\ z\in G$*, then f is constant.*

Proof : Fix a point $z_0\in G$ and let $w_0=f(z_0)$.

Let $A=\{z\in G;\ f(z)=w_0\}$

T.P.T. $A=G$.

[i.e. by showing A is both open and closed and $A\neq\varnothing$]

T.P.T. A is closed.

Let $z\in G$ and $\{z_n\}$ be a sequence in $A\ni \lim_{n\to\infty} z_n=z$

$\therefore\qquad f(z_n)=w_0$, for each $n\in\mathbb{N}$

f is differentiable on G (given)

$\therefore\qquad$ f is continuous on *G*.

$$\therefore\qquad f(z)=f\left[\lim_{n\to\infty} z_n\right]=\lim_{n\to\infty} f(z_n)=w_0$$

$\Rightarrow\qquad z\in A$

$\therefore\qquad$ A contain its limit point.

$\Rightarrow\qquad$ A is closed.

Now, T.P.T. A is open.

Fix $a\in A$, since *G* is open.

$\therefore\qquad \exists\ r>0\ \ni B(a;r)\subset G$

Let $z \in B(a; r)$ and set $g(t) = f\left[tz + (1-t)a\right], \quad 0 \le t \le 1$

$$\therefore \quad \frac{g(t)-g(s)}{t-s} = \frac{f\left[tz+(1-t)a\right]+f\left[S_z+(1-S)a\right]}{(t-S)z+(S-t)a} \times \frac{(t-S)z+(S-t)a}{(t-S)}$$

$$\therefore \quad \lim_{t \to S}\left[\frac{g(t)-g(S)}{t-S}\right] = \lim_{t \to S}\left[\frac{f\left[tz+(1-t)a\right]-f\left[sz+(1-S)a\right]}{(t-S)z+(S-t)a} \times (z-a)\right]$$

$$\therefore \quad g'S = f'\left[Sz+(1-S)a\right] \times (z-a)$$

$$\therefore \quad g'(S) = 0, \quad 0 \le S \le 1 \qquad (\because f'(z) = 0, \ \forall \ z \in G)$$

$$\Rightarrow \quad g(s) = \text{constant}, \ \ 0 \le S \le 1$$

$$\Rightarrow \quad g(1) = \text{constant} = g(0)$$

$$\therefore \quad f(z) = g(1) = \text{constant} = g(0) \ = f(a) = w_0$$

$$\Rightarrow \quad z \in A$$

$$\therefore \quad z \in B(a; R) \Rightarrow z \in A$$

$$\Rightarrow z \in b(a; r) \subseteq A$$

$\Rightarrow$ A is open.

Hence, by the connectedness of *G*.

$$A = G$$

- f is constant on *G*.

6.3 CAUCHY RIEMANN EQUATIONS (C-R Eq.)

Theorem : *Let u and v be real valued function defined on the domain $G \subset \mathbb{C}$ and suppose that u and v have continuous partial derivatives then $f : G \to \mathbb{C}$ defined by $f(z) = u(z) + iv(z)$ is analytic iff u and v satisfy Cauchy Riemann equation. i.e. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$*

(2006, 2007, 2008, 2009)

Proof : Let $z = x + iy \in G$ and $\Delta z = \Delta x + i\Delta y$.

Given, $f : G \to \mathbb{C}$ is defined by $f(\ ) = u(z) + iv(z)$ OR $f(x, y) = u(x, y) + iv(x, y)$

$\because$ f is analytic on *G*.

$\therefore$ f is differentiable at $z \in G$.

$$\therefore \quad \frac{f(z+\Delta z) - f(z)}{\Delta z} \to f'(z_0) \qquad \text{(a unique limit)}$$

as $\Delta z \to 0$ in any manner in $\mathbb{C}$.

Now,

$$\frac{f(z+\Delta z)-f(z)}{\Delta z}=\frac{\left[u(x+\Delta x, y+\Delta y)+iv(x+\Delta x, y+\Delta y)\right]-\left[u(x, y)+iv(x, y)\right]}{\Delta x+i\Delta y}$$

$$=\frac{u(x+\Delta x, y+\Delta y)-u(x, y)}{\Delta x+i\Delta y}+i\left[\frac{v(x+\Delta x . y+\Delta y)-v(x, y)}{\Delta x+i\Delta y}\right]$$

…………………(I)

Suppose $z \to 0$, along the real axis (x-axis)

$\therefore \quad \Delta z = \Delta x$ and $\Delta y = 0$

$$\therefore \quad \lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z}=\lim_{\Delta x\to 0}\left[\frac{u(x+\Delta x, y)-u(x, y)}{\Delta x}+\frac{i\{v(x+\Delta x, y)-v(x, y)\}}{\Delta x}\right]$$

$$\therefore \quad f'(z)=\frac{\partial u}{\partial x}(x, y)+i\frac{\partial v}{\partial x}(x, y) \qquad \text{…………………(II)}$$

Suppose $\Delta z \to 0$, along the imaginary axis (y-axis).

$\therefore \quad \Delta z = i\Delta y$ and $\Delta x = 0$

$$\lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z}=\lim_{\Delta y\to 0}\left[\frac{u(x, y+\Delta y)-u(x, y)}{i\Delta y}+\frac{i\{v(x, y+\Delta y)-v(x, y)\}}{i\Delta y}\right]$$

$$f'(z)=\frac{1}{i}\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}=-i\left(\frac{\partial u}{\partial y}\right)+\frac{\partial v}{\partial y}$$

$$\left\{\because \frac{1}{i}=\frac{i}{i^2}=\frac{i}{-i}=-i\right\}$$

$$=-i\frac{\partial u}{\partial y}(x, y)+\frac{\partial v}{\partial y}(x, y) \qquad \text{………………..(III)}$$

From equation (II) and (III)

$$\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=-i\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}$$

Equating Real and imaginary part on both sides.

i.e. $$\boxed{\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}}$$

which are Cauchy Riemann equations.

Conversely,

Let $z \in G$

$\because$ G is open $\Rightarrow \ \exists \ r > 0, \ni \ B(z, r) \subset G$

Let $\Delta z = \Delta x + i\Delta y \in B(0; r)$.

Given, u and v have continuous partial derivatives.

$\therefore \ u, v, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial x'}{\partial x}, \frac{\partial v}{\partial y}$ are continuous on G.

$\therefore$ The expression $u(x + \Delta x, \ y + \Delta y) - u(x, y)$ can be written as (by definition of partial derivative.)

$$u(x + \Delta x, \ y + \Delta y) - u(x, y) = \Delta x . u_x(x, y) + \Delta y \ u_y(x, y) + \varnothing(\Delta x, \Delta y)$$

where, $\lim\limits_{\Delta x \to \Delta y \to 0} \frac{\varnothing(\Delta x, \Delta y)}{\Delta x + i\Delta y} = 0$ OR $\lim\limits_{\Delta z \to 0} \frac{\varnothing(\Delta z)}{\Delta z} = 0$

…………………(IV)

Similarly,

$$v(x + \Delta x, \ y + \Delta y) - v(x, y) = \Delta x \ v_x(x, y) + \Delta y \, v_y(x, y) + \psi(\Delta x, \Delta y)$$

where $\lim\limits_{\Delta x + i\Delta y \to 0} \frac{\psi(\Delta x, \Delta y)}{\Delta x + i\Delta y} = 0$ ……………..(V)

$\therefore$

$$f(z + \Delta z) - f(z) = u(x + \Delta x, \ y + \Delta y) - u(x, y)$$

$+i\left[v(x + \Delta x, y + \Delta y) - v(x, y)\right]$ from equation (I)

$= \Delta x . u_x + \Delta y \ u_y + \varnothing(\Delta x, \Delta y) + i\left[\Delta x \ v_x + \Delta y \ v_y + \psi(\Delta x, \Delta y)\right]$

from equation (IV) and (V)

$$= \Delta x(u_x + iv_x) + \Delta y(u_y + iv_y) + \varnothing(\Delta x, \Delta y) + i\psi(\Delta x, \Delta y)$$

By Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$$

i.e. $u_x = v_y$ and $u_y = -v_x = i^2 v_x$ $\left(\because i^2 = -1\right)$

$$\therefore \quad f(z + \Delta z) - f(z) = \Delta x(u_x + i\,v_x) + \Delta y\left(i^2 v_x + iu_x\right) + \varnothing + i\psi$$

$$= \Delta x(u_x + iv_x) + i\Delta y(iv_x + u_x) + \varnothing + i\psi$$

$$= (\Delta x + i\Delta y)(u_x + iv_x) + \varnothing + i\psi$$

$$\therefore \ \frac{f(z + \Delta z) - f(z)}{\Delta z} = \frac{\Delta x + i\Delta y}{\Delta z}\left(u_x + i\,v_y\right) + \frac{\varnothing + i\psi}{\Delta z}$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + \frac{\varnothing + i\psi}{\Delta x + i\Delta y} \qquad \{\because \Delta z = \Delta x + i\Delta y \text{ where}$$

$$\lim_{\Delta x + i\Delta y \to 0} \frac{\varnothing + i\psi}{\Delta x + i\Delta y} = 0\}$$

$\Rightarrow$ f is differentiable at z and

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$\Rightarrow$ f' is continuous ($\because \frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$ are continuous)

$\Rightarrow$ f is continuously differentiable.

$\Rightarrow$ f is analytic.

Note: If f(z) is analytic then it can be differentiated directly

Example :

(1) Prove that the function $f(z) = e^z$ is analytic in $\mathbb{C}$. Also find its derivative.

Solution : Let $f(z) = u(z) + iv(z)$ and $z = x + iy$

Given that, $f(z) = e^z$

$$u + iv = e^{x+iy} = e^x . e^{iy} = e^x(\cos y + i \sin y)$$

Equating real and imaginary parts on both sides.

T.P.T. f is analytic.

By previous theorem, we see that in order to prove and is analytic we have to verify that u and v are satisfy Cauchy-Riemann equations.

i.e. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$

Now, $\because$ $u = e^x \cos y$ and $v = e^x \sin y$

$$\frac{\partial u}{\partial x} = e^x . \cos y, \quad \frac{\partial v}{\partial y} = e^x \sin y$$

$\Rightarrow$ $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$

$\Rightarrow$ u and v satisfies the C-R equation.

$\Rightarrow$ f is analytic.

$$\sin i\theta = \frac{e^{-\theta} + e^{\theta}}{2} = \sin h\theta$$

and $\cos i\theta = \dfrac{e^{-\theta} + e^{\theta}}{2} = \cos h\theta$

$f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x} = e^x \cos y + i\, e^x \sin y = e^x(\cos y + i \sin y) = e^x . e^{iy}$

$= e^{x+iy} = e^z$

(Or $\because f(z)$ is analytic it can be differentiated directly i.e. $f'(z) = e^z$)

2) Show that the function $f(z) = w = \sin z$ is analytic and also find $\dfrac{dw}{dz}$.

Solution : Let $z = x + iy$ and $u + iv = w$

Given that $w = \sin z$

$u + iv = \sin(x + iy) = \sin x + \cos iy + \cos x . \sin iy$

$= \sin x . \cos hy + i \cos x . \sin hy$

Comparing real and imaginary parts

$\therefore$ $u = \sin x . \cos hy$, $v = \cos x . \sin hy$

$\Rightarrow$ u and v are real valued function of x and y.

$\therefore$ $\dfrac{\partial u}{\partial x} = \cos x . \cos hy, \ \dfrac{\partial v}{\partial x} = -\sin x . \sin hy$

and $\dfrac{\partial u}{\partial y} = \sin hy . \sin x, \ \dfrac{\partial v}{\partial y} = \cos x . \cos hy$

$\therefore$ $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = \dfrac{-\partial v}{\partial x}$

$\therefore$ Cauchy-Riemann equations are satisfied.

$\therefore$ $f(z) = \sin z$ is an analytic function.

$\therefore$ $\dfrac{\partial w}{\partial z} = f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x} = \cos x + \cos hy - i \sin x . \sin hy$

(Or $\because f(z)$ is analytic it can be differentiated directly i.e. $f'(z) = \sin z$)

3) Using the Cauchy- Riemann equations , verify that $x^2 + y^2 + 2ixy$ is not analytic.

Solution : $\because$ $P(x, y) = x^2 + y^2 + 2ixy$

$\Rightarrow P_y = -2y + 2ix, P_x = 2x + 2iy \Rightarrow P_y \neq iP_x$

$\Rightarrow P(x, y)$ is analytic.

4) Using Cauchy-Riemann equations, verify that $x^2 + y^2 - 2ixy$ is not analytic.

Solution : $\because P(x, y) = x^2 + y^2 - 2ixy$

$\Rightarrow P_y = 2y - 2ix, P_x = 2x - 2iy \Rightarrow iP_x = 2ix + 2y$

$\Rightarrow P_y = 2y - 2ix, P_x = 2x - 2iy \Rightarrow iP_x = 2ix + 2y \Rightarrow P_y \neq iP_x \Rightarrow P(x, y)$

is not analytic.

(5) Give an example of function which is continuous everywhere but not analytic

Solution: Let $f(z) = xy + iy$

$\therefore u = xy, v = y$. Since u and v are polynomials, they are continuous everywhere.

Now, $\dfrac{\partial u}{\partial x} = y, \dfrac{\partial u}{\partial y} = x, \dfrac{\partial y}{\partial x} = 0, \dfrac{\partial y}{\partial x} = 1$

$\therefore \dfrac{\partial u}{\partial x} \neq \dfrac{\partial v}{\partial y}, \dfrac{\partial u}{\partial y} \neq -\dfrac{\partial v}{\partial x}$

Therefore f(z) is not analytic

6.4 HARMONIC FUNCTIONS

If G is an open subset of $\mathbb{C}$, then the function $U : G \to \mathbb{R}$ (i.e. Real valued function of complex variable) is harmonic if, it has continuous second order partial derivatives and

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \qquad \text{(This is called Laplace's equation)}$$

e.g. $u(x, y) = e^x . \cos y$ is harmonic function ?

$$\frac{\partial u}{\partial x} = e^x . \cos y \qquad \frac{\partial u}{\partial y} = -e^x \sin y$$

$$\frac{\partial^2 u}{\partial x^2} = e^x . \cos y \qquad \frac{\partial^2 u}{\partial y^2} = -e^x \cos y$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^x \cos y - e^x \cos y = 0$$

$\therefore$ Above function is Harmonic function.

Proposition : Let f be a analytic function in a region and $f(z)=u(z)+iv(z)$. If u and v have continuous second partial derivatives then *u, v* are harmonic function.

OR

1) If $f:G\to\mathbb{C}$ defined by $f(z)=u(z)+iv(z)$ is analytic then, $u=\operatorname{Re} f$ and $v=\operatorname{Im} f$ are harmonic functions.

Proof : Given that $f(z)=u(z)+iv(z)$ is analytic.

$\therefore$ Cauchy Riemann equations are satisfied.

i.e. $\dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial y}$ (I)

and $\dfrac{\partial u}{\partial y}=\dfrac{-\partial v}{\partial x}$ (II)

Differentiate equation (I) partially w.r.t. x and (II) w.r.t. y.

$\therefore$ $\dfrac{\partial^2 u}{\partial x^2}=\dfrac{\partial^2 v}{\partial x.\partial y}$ and $\dfrac{\partial^2 u}{\partial y^2}=\dfrac{-\partial^2 v}{\partial x.\partial y}$

$\therefore$ $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 v}{\partial y^2}=\dfrac{\partial^2 v}{\partial x.\partial y}-\dfrac{\partial^2 v}{\partial x.\partial y}=0$

$\therefore$ v is harmonic function.

Differentiate equation (I) and (II) partially w.r.t. y and x respectively.

$\therefore$ $\dfrac{\partial^2 u}{\partial x.\partial y}=\dfrac{\partial^2 v}{\partial y^2}$ and $\dfrac{\partial^2 u}{\partial x.\partial y}=\dfrac{-\partial^2 v}{\partial y^2}$

Consider, $\dfrac{\partial^2 v}{\partial x^2}+\dfrac{\partial^2 v}{\partial y^2}=\dfrac{\partial^2 u}{\partial x.\partial y}-\dfrac{\partial^2 u}{\partial x.\partial y}=0$

$\therefore$ v is harmonic function.

Definition : If $f:G\to\mathbb{C}$ is analytic and $f(z)=u(z)+iv(z)$ then $u=\operatorname{Re} f$ and $v=\operatorname{Im} f$ are harmonic conjugate i.e. u and v are harmonic conjugate then u and v are harmonic function and *u, v* are satisfied C – R equations.

Example : If $f:G\to\mathbb{C}$ is analytic and $f(z)=u(z)+iv(z)$ then prove that harmonic function u satisfies the partial differential equations $\dfrac{\partial^2 u}{\partial \bar{z}.\partial z}=0$.

Solution : Given f is analytic.

Let $z = x + iy \Rightarrow \bar{z} = x - iy$.

Here, $u = u(x, y)$ where $x = \frac{z + \bar{z}}{2}$ and $y = \frac{z - \bar{z}}{2}$.

$$\therefore \quad \frac{\partial u}{\partial z} = \frac{\partial}{\partial z}\left[u(x, y)\right] = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial z} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial z} = \frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{2i}\frac{\partial u}{\partial y}$$

$$= \frac{1}{2}\left[\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right] \qquad \left\{\frac{1}{i} = \frac{i}{i^2} = \frac{i}{-1} = -i\right.$$

$$\frac{\partial u}{\partial \bar{z}} = \frac{\partial}{\partial \bar{z}}\left[u(x, y)\right] = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \bar{z}} \quad = \frac{1}{2}\frac{\partial u}{\partial x} - \frac{1}{2i}\frac{\partial u}{\partial y}$$

$$= \frac{1}{2}\left[\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right]$$

OR

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left[\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right]$$

$$\frac{\partial^2 u}{\partial \bar{z} . \partial z} = \frac{\partial}{\partial \bar{z}}\left[\frac{\partial u}{\partial z}\right] = \frac{1}{2}\left[\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right] . \frac{1}{2}\left[\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right]$$

$$= \frac{1}{4}\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + i\cancel{\frac{\partial^2 u}{\partial y . \partial x}} - i\cancel{\frac{\partial^2 u}{\partial y . \partial x}}\right] = \frac{1}{4}\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] = 0$$

$$\boxed{\frac{\partial^2 u}{\partial \bar{z} . \partial z} = 0}$$

Example: Prove that the function $u(x, y) = x^3 - 3xy^2 + 3x^2 - 3y^2 + 2$ is harmonic. Find its harmonic conjugate and corresponding analytic function $f(z) = u(z) + iv(z)$.

Solution : Given function $u(x, y) = x^3 - 3xy^2 + 3x^2 - 3y^2 + 2$

$$\therefore \quad \frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x \text{ and } \frac{\partial u}{\partial y} = -6xy - 6y$$

$$\frac{\partial^2 u}{\partial x^2} = 6x + 6 \qquad \text{and } \frac{\partial^2 u}{\partial y^2} = -6x - 6$$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial^2 y} = 6x + 6 - 6x - 6 = 0$$

$\therefore$ u is a harmonic function.

To find *v,* we use Cauchy Riemann equation.

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y}=\frac{-\partial v}{\partial x}$$

$$\because \quad \frac{\partial u}{\partial x}=3x^2-3y^2+6x \Rightarrow \frac{\partial v}{\partial y}=3x^2-3y^2+6x$$

By integrating $\int \partial v=\int\left(3x^2-3y^2+6x\right)\partial y$

$$v=3x^2y-\frac{3y^3}{3}+6xy+\varnothing(x) \;=3x^2y-y^3+6xy+\varnothing(x) \qquad (1)$$

where $\varnothing(x)$ is an arbitrary function of x. To find $\varnothing(x)$, we use another equation of Cauchy-Riemann.

$$\frac{\partial u}{\partial y}=\frac{-\partial v}{\partial x}$$

$$\therefore -6xy-6y=-\frac{\partial}{\partial x}\left[3x^2y-y^3+6xy+\varnothing(x)\right]$$

$$=-\left[6xy+6y+\frac{\partial\varnothing(x)}{\partial x}\right]=-6xy-6y-\varnothing'(x)$$

$$\therefore \quad \varnothing'(x)=0$$

Integrating, we get $\varnothing(x)=c$, where c is constant.

$$\therefore \quad v=3x^2y-y^3+6xy+c$$

i.e. the required harmonic conjugate.

$\therefore$ Analytic function

$$f(z)=u(x,y)+iv(x,y)$$

$$=x^3-3xy^2+3x^2-3y^2+2+i\left(3x^2y-y^3+6xy+c\right)$$

Put $x=z$ and $y=0$

$$f(z)=z^3+3z+2+c$$

(Alternate method to find harmonic conjugate using Milne Thompson method)

Given function $u(x,y)=x^3-3xy^2+3x^2-3y^2+2$

$\therefore$ let $\phi_1(x,y)=\frac{\partial u}{\partial x}=3x^2-3y^2+6x$ and $\phi_2(x,y)=\frac{\partial u}{\partial y}=-6xy-6y$

Now,

$$f'(z)=\phi_1(z,0)-i\phi_2(z,0)=(3z^2+6z)+i(0) \quad \left(\textit{putting} \quad x=z,\ y=0\right)$$

$$\therefore f(z)=\int(3z^2+6z)dz=z^3+3z^2$$

Put $z = x+iy$,

$\therefore f(z)=(x+iy)^3+3(x+iy)^2$

$=x^3+3x^2yi+3xy^2i^2+y^3i^3+3x^2+6xyi+3y^2i^2$

Separating real and imaginary parts we get $v=3x^2y-y^3+6xy$

Example : If $f:G\to\mathbb{C}$ defined by $f(z)=u(z)=u(z)+iv(z)$ is analytic and $u-v=e^x(\cos y-\sin y)$ then find the function $f(z)$ in terms of z.

Solution :

$f(z)=u+iv$

$if(z)=ui-v$

$\therefore (1+i)f(z)=u+iv+ui-v=u-v+i(u+v)=U+iV(say)$

$\therefore U_x=e^x(\cos y-\sin y)=\phi_1(x,y)$

$U_y=e^x(-\sin y-\cos y)=\phi_2(x,y)$

$\therefore (1+i)f'(z)=U_x-iU_y=\phi_1(z,0)-\phi_2(z,0)$

$\therefore (1+i)f(z)=\int(e^z+ie^z)dz=(1+i)\int e^z dz$

$\therefore f(z)=e^z+c$

Proposition: Suppose that *f* is analytic in a region *G*. If $|f(z)|=\text{constant}$.

Proof : Let $z=x+iy\in G$ and $f(z)=u(z)+iv(z)$.

Given that *f* is analytic.

$\Rightarrow$ Cauchy-Riemann equations are satisfied.

$$\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y}=\frac{-\partial v}{\partial x}$$

Here, we given that $|f(z)|=\text{constant}=K$ (say).

Let $k\neq 0$ [If $k=0$, it is obvious that $f(z)=0$].

$\therefore \quad |u+iv|=K \qquad \therefore \quad \sqrt{u^2+v^2}=K$

$\therefore \quad u^2+v^2=K^2$ …….(I)

Differentiate equation (I) partially w.r.t. x

$$2u\frac{\partial u}{\partial x}+2v\frac{\partial v}{\partial x}=0$$

$$\therefore \quad u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}=0$$

$$\therefore \quad u\frac{\partial u}{\partial x}-v\frac{\partial u}{\partial y}=0 \qquad \left\{\because \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}\right\} \qquad \text{…………(II)}$$

Again, differentiate equation (I) partially w.r.t y

$$2u\frac{\partial u}{\partial y}+2v\frac{\partial v}{\partial y}=0 \Rightarrow u\frac{\partial u}{\partial y}+v\frac{\partial v}{\partial y}=0$$

$$u\frac{\partial u}{\partial y}+v\frac{\partial u}{\partial x}=0 \qquad \{\because \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\} \qquad \ldots\ldots\ldots\ldots(III)$$

Multiplying equation (II) by u and equation (III) by v and add

$$u^2\frac{\partial u}{\partial x} - uv\frac{\partial u}{\partial y} = 0$$

$$+\ v^2\frac{\partial u}{\partial x} + uv\frac{\partial u}{\partial y} = 0$$

$$\left(u^2+v^2\right)\partial u/dx = 0$$

$$\Rightarrow \quad K^2\frac{\partial u}{\partial x}=0 \Rightarrow \frac{\partial u}{\partial x}=0$$

Multiply equation II by v and III by u and subtract.

$$uv\frac{\partial u}{\partial x} - v^2\frac{\partial u}{\partial y} = 0$$

$$-\ uv\frac{\partial u}{\partial x} + u^2\frac{\partial u}{\partial y} = 0$$

$$-\left(u^2+v^2\right)\partial u/dy = 0$$

$$\Rightarrow \quad -K^2\frac{\partial u}{\partial y}=0$$

$$\frac{\partial u}{\partial y}=0$$

Using Cauchy-Riemann equation

$$\frac{\partial u}{\partial x}=0=\frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y}=0=\frac{-\partial v}{\partial x}$$

$$\Rightarrow \quad \frac{\partial v}{\partial x}=0=\frac{\partial v}{\partial y}$$

$\therefore$ f is analytic at z.

$\therefore$ f is differentiable at z and

$$f'(z)=\frac{\partial y}{\partial x}+i\frac{\partial v}{\partial x}=0+i\,0=0$$

$\therefore$ $f'(z)=0, \qquad z\in G$

$\Rightarrow$ $f(z)$ = constant.

Theorem : *Suppose that f is analytic in a domain (region) D then*

a) If $f(z)=0, \forall z\in D$, *f is constant.*

b) If any one of $|\,f\,|$, $\operatorname{Re} f$, $\operatorname{Im} f$, *are f is constant in D* (2008)

Proof : Let $z = x + iy \in D$ and $f(z) = u(z) + iv(z)$

Given that *f* is analytic in *D*.

$\therefore$ Cauchy-Riemann (C.R) equations are satisfied.

i.e. $$\left\{ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \; and \; \frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x} \right\} \quad (1)$$

a) If $f'(z) = 0, \; \forall \; z \in D$ then f is constant. (Already done, last proposition)

b) i) Let $|f|$ = constant.

$\therefore \sqrt{u^2 + v^2} = K$ (say)

$\Rightarrow u^2 + v^2 = K \quad (2)$

where $u(x, y)$ and $v(x, y)$ are real valued function. Let $K \neq 0$ [if $K = 0$ then nothing to prove.]

Differentiate equation (2) w.r.t. *x*

$$2u \frac{\partial u}{\partial x} + 2v \frac{\partial v}{\partial x} = 0$$

$$\therefore \quad u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial x} = 0$$

$$\therefore \quad u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial y} = 0 \qquad \left\{ \because \frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x} \right\} \quad (3)$$

Again differentiate equation (2) w.r.t *y*

$$2u \frac{\partial u}{\partial y} + 2v \frac{\partial v}{\partial y} = 0$$

$$u \frac{\partial u}{\partial y} + v \frac{\partial v}{\partial y} = 0$$

$$v \frac{\partial u}{\partial x} + u \frac{\partial u}{\partial y} = 0 \qquad \left\{ \because \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \right\} \quad (4)$$

Multiplying equation (2) by *u* and (4) by *v* and adding

$$\begin{array}{rcl} u^2 \frac{\partial u}{\partial x} - uv \frac{\partial u}{\partial y} & = & 0 \\ + \; v^2 \frac{\partial u}{\partial x} + uv \frac{\partial u}{\partial y} & = & 0 \\ \hline \left(u^2 + v^2\right) \partial u / dx & = & 0 \end{array}$$

$$\Rightarrow \quad K^2 \frac{\partial u}{\partial x} = 0$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = 0$$

Multiply equation (3) by *v* and (4) by *u* and subtracting

$$uv\frac{\partial u}{\partial x} - v^2\frac{\partial u}{\partial y} = 0$$

$$- uv\frac{\partial u}{\partial x} + u^2\frac{\partial u}{\partial y} = 0$$

$$-\left(u^2+v^2\right)\partial u/dy = 0$$

$\Rightarrow$ $$-K^2\frac{\partial u}{\partial y}=0$$

$\therefore$ $$\frac{\partial u}{\partial y}=0$$

From equation (1), $\frac{\partial v}{\partial x}=0$ and $\frac{\partial v}{\partial y}=0$

$$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=0+i\,0=0$$

$\because$ z is arbitrary.

$\therefore$ $f'(z)=0 \quad \forall\, z\in D$

$\therefore$ by part (a)

f is constant.

ii) Let $\operatorname{Re} f = \text{constant} = K$ (say)

$\therefore$ $u(x, y)=K$ (say)

$\frac{\partial u}{\partial x}=0$ and $\frac{\partial u}{\partial y}=0$

$\therefore$ from equation (1)

$\frac{\partial v}{\partial x}=0$ and $\frac{\partial v}{\partial y}=0$

$\because$ f is analytic.

f is differentiable and $f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$

$f'(z)=0+i\,.\,0=0$

$\therefore$ $f'(z)=0 \Rightarrow f(z)=$ constant by (a)

iii) Let $\operatorname{Im} f =$ constant $= K$

$\therefore$ $v(x, y)=K$ (say)

$\therefore$ $\frac{\partial v}{\partial x}=0$ and $\frac{\partial v}{\partial y}=0$

$\because$ f is analytic.

$\therefore$ f is differentiate and $f'(z)=\frac{\partial v}{\partial x}+i\frac{\partial y}{\partial y}$

By Cauchy Riemann equation $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0$

$\therefore\ f'(z) = 0 + i\,.\,0 = 0 \ \Rightarrow f(z) =$ constant

by (a)

iv) Let $\arg f(z) =$ constant

$$\arg(f(z)) = \theta = \tan^{-1}\left(\frac{v}{u}\right) = K \quad \text{(say)}$$

$$\tan^{-1}\left(\frac{v}{u}\right) = K$$

$$\frac{u}{v} = \tan K \qquad \Rightarrow \quad u = \frac{v}{\tan K} = v\cot K$$

$\therefore\ u - v\cot K = 0$

Put $cut\ K = C$

$\Rightarrow u - v\,c = 0$ unless $v\,.\,c = 0$

But $u - cv$ is a real part of $(1+ic)\,f(z)$

$\Rightarrow (1+ic)\,f(z) =$ constant by (a)

$$f(z) = \frac{\text{constant}}{(1+ic)} = \text{constant} \qquad \{\because 1+ic = \text{constant}\}$$

$\Rightarrow f$ is constant.

Example 1 : Prove that the function $f(z) = \bar{z}$ is not differentiable anywhere in the complex plane.

Solution : We know that, f is differentiable at z if

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} \to f'(z) \text{ (a unique limit)}$$

as $\Delta z \to 0$ in any manner in $\mathbb{C}$-plane.

$\therefore \quad f(z+\Delta z) = \overline{z+\Delta z} = \bar{z} + \Delta\bar{z}$

$$\therefore \quad f'(z) = \lim_{\Delta z\to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z\to 0} \frac{\bar{z}+\Delta\bar{z}}{\Delta z} = \lim_{\Delta z\to 0} \frac{\bar{z}+\Delta\bar{z}-\bar{z}}{\Delta z} = \lim_{\Delta z\to 0} \frac{\Delta\bar{z}}{\Delta z}$$

$$= \lim_{\Delta z\to 0} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} \ \ldots\ldots\ldots\ldots(1)$$

Let $\Delta z \to 0$ along the Real axis.

$\therefore \quad \Delta z = \Delta x$ and $\Delta y = 0$

∴ $f'(z) = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$

Let $\Delta z \to 0$ along the imaginary axis.

∴ $\Delta z = i\Delta y$ and $\Delta x = 0$

∴ $f'(z) = \lim_{\Delta y \to 0} = \frac{i\Delta y}{i\Delta y} = -1$

∴ Two limits obtained are different.

i.e. limit is not unique.

∴ Given function $f(z) = \bar{z}$ is not differentiable in $\mathbb{C}$ to check $\Delta z \to 0$. $x = y$, we get $\frac{1-i}{1+i}$.

2) Show that the function $f(z) = |z|^2$ is differentiable only at the origin. (2006)

Solution : Let $f(z) = x^2 + y^2$.

Since, $x^2 + y^2$ is continuous everywhere, *f(z)* is continuous everywhere.

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z_0 + \delta z) - f(z_0)}{\delta z} = \lim_{\delta z \to 0} \frac{|z_0 + \delta z|^2 - |z_0|^2}{\delta z}$$

$$= \lim_{\delta z \to 0} \frac{(z_0 + \delta z)(\overline{z_0} + \overline{\delta z}) - z_0 \overline{z_0}}{\delta z} = \lim_{\delta z \to 0} \frac{(z_0 + \delta z)(\overline{z_0} + \overline{\delta z}) - z_0 \overline{z_0}}{\delta z}$$

$$= \lim_{\delta z \to 0} \frac{z_0 \delta \bar{z} + \overline{z_0} \delta z + \delta z \delta \bar{z}}{\delta z} = \lim_{\delta z \to 0} \frac{z_0 \delta \bar{z} + \overline{z_0} \delta z + \delta z \delta \bar{z}}{\delta z}$$

$$= \lim_{\delta z \to 0} z_0 \frac{\delta \bar{z}}{\delta z} + \overline{z_0} + \delta \bar{z}$$

(i) When δz is real: Then $\delta y = 0$ and $\delta \bar{z} = \delta z = \delta x$. As $\delta z \to 0, \delta x \to 0$

∴ $f'(z_0) = \lim_{\delta x \to 0} z_0 \frac{\delta \bar{z}}{\delta z} + \overline{z_0} + \delta \bar{z} = \lim_{\delta x \to 0} z_0 + \overline{z_0} + \delta x = z_0 + \overline{z_0}$

(ii) When δz is imaginary: Then $\delta x = 0$ and $\delta z = i\delta y, \delta \bar{z} = -i\delta y$. As $\delta z \to 0, \delta y \to 0$

∴ $f'(z_0) = \lim_{\delta z \to 0} z_0 \frac{\delta \bar{z}}{\delta z} + \overline{z_0} + \delta \bar{z} = \lim_{\delta x \to 0} z_0 (\frac{-i\delta \bar{z}}{\delta z}) + \overline{z_0} + i\delta y = -z_0 + \overline{z_0}$

Since the two limits are different along two different paths except at *z=0*, $f'(z_0)$ does not exist anywhere except at *z=0*

Hence, *f(z)* is not differentiable anywhere except at *z=0*

3) Let $f(z)=\begin{cases}\dfrac{x^3-y^3+i\left(x^3+y^3\right)}{x^2+y^2} & x\neq 0,\ y\neq 0 \text{ i.e. } z\neq 0\\ 0 & x=0,\ y=0 \text{ i.e. } z=0\end{cases}$

Prove that C.R. equations are satisfied at the origin but $f'(0)$ does not exist i.e. $f(z)$ is not differentiable there.

Solution : Let $z=x+iy$ and $f(x)=u(z)+iv(z)$

$\therefore \quad f(z)=u(z)+iv(z)$

$$=\begin{cases}\dfrac{x^3-y^3+i\left(x^3+y^3\right)}{x^2+y^2} & z=(x,y)\neq 0\\ 0 & z=0\end{cases}$$

$$\therefore \quad u(x,y)=\begin{cases}\dfrac{x^3-y^3+i\left(x^3+y^3\right)}{x^2+y^2} & z\neq 0\\ 0 & z=0\end{cases}$$

$$\therefore \quad v(x,y)=\begin{cases}\dfrac{x^3+y^3+i\left(x^3+y^3\right)}{x^2+y^2} & z\neq 0\\ 0 & z=0\end{cases}$$

$$\therefore \quad \frac{\partial u}{\partial x}(x,y)=\lim_{h\to 0}\frac{u(x+hy)-u(x,y)}{h}$$

$$\frac{\partial u}{\partial x}(0,0)=\lim_{h\to 0}\frac{u(h,0)-u(0,0)}{h}$$

$$=\lim_{h\to 0}\frac{\dfrac{h^3-0}{h^2-0}-0}{h}=\lim_{h\to 0}\frac{h}{h}=1$$

Similarly,

$$\frac{\partial u}{\partial y}(0,0)=\lim_{h\to 0}\frac{u(0,u)-u(0,0)}{h}=\lim_{h\to 0}\frac{\dfrac{0-h^3}{0-h^2}-0}{h}=\lim_{h\to 0}\frac{-h}{h}=-1$$

$$\therefore \quad \frac{\partial v}{\partial x}(0,0)=\lim_{h\to 0}\frac{\dfrac{h^3+0}{h^2}-0}{h}=\lim_{h\to 0}\frac{h}{h}=1$$

$$\therefore \quad \frac{\partial v}{\partial y}(x, y) = \lim_{h \to 0} \frac{v(x, y+h) - v(x, y)}{h} = \lim_{h \to 0} \frac{\frac{0+h^3}{0+h^2}}{h} = \lim_{h \to 0} \frac{h^3}{h^2} = 1$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$$

$\therefore$ Cauchy-Riemann equations are satisfied at the origin.

$$\therefore \quad f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{z}$$

$$\therefore \quad f'(0) = \lim_{\Delta z \to 0} \frac{f(z) - f(0)}{z - 0}$$

where $z \to 0$, in any manner in Complex Plane.
Along the x-axis, $y = 0$ and $z = x$

$$\therefore \; f'(0) = x \lim_{h \to 0} \frac{f(x) - f(0)}{x} = \lim_{x \to 0} \frac{\frac{x^3 + ix^3}{x^2} - 0}{x} = \lim_{x \to 0} \frac{\cancel{x^3}(1+i)}{\cancel{x^3}}$$

$= 1 + i$

Along the y-axis, $x = 0$ and $z = iy$

$$\therefore \; f'(0) = \lim_{iy \to 0} \frac{f(y) - f(0)}{iy} = \lim_{iy \to 0} \frac{\frac{-y^3 + iy^3}{y^2}}{iy}$$

$$= \frac{i^2 + i}{i} = \frac{i(i+1)}{i} = 1 + i$$

Let $z \to 0$ along the line $y = x$, $x = y$, and $z = x + ix = x(1+i)$.

$$f'(0) = \lim_{x(1+i) \to 0} \frac{f(x) - f(0)}{x(1+i)} = \lim_{x \to 0} \frac{i \,.\, \cancel{2x^3}}{\cancel{2x^2}(1+i)} = \lim_{x \to 0} \frac{ix}{1+ix} = \frac{i}{1+i}$$

Hence, $f'(0)$ is not unique.

$\therefore$ The given $f(z)$ is not differentiable at origin.

4) If $f : G \to \mathbb{C}$ defined by $f(z) = u(z) + iv(z)$ is analytic and suppose that $\psi = \psi(u, v)$ be any continuously differentiable function.

Prove that $\left(\frac{\partial \psi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial y}\right)^2 = \left[\left(\frac{\partial \psi}{\partial u}\right)^2 + \left(\frac{\partial \psi}{\partial v}\right)^2\right] | f'(z) |^2$.

Solution : Let $z = x + iy$ and $f(x, y) = u(x, y) + iv(x, y)$

Given, $f(z)$ is analytic.

$\therefore$ Cauchy Riemann equations are satisfied.

$\therefore \qquad \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = \dfrac{-\partial v}{\partial x} \qquad (1)$

Now, $\psi = \psi(u, v)$ where $u = u(x, y)$ and $v = v(x, y)$

$\therefore \qquad \dfrac{\partial \psi}{\partial x}$ and $\dfrac{\partial \psi}{\partial y}$ exist.

$$\frac{\partial \psi}{\partial x} = \frac{\partial}{\partial x}\left[\psi(u, v)\right]$$

$$= \frac{\partial \psi}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial \psi}{\partial v} \cdot \frac{\partial v}{\partial x} \qquad (2)$$

Similarly,

$$\frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial \psi}{\partial v} \cdot \frac{\partial v}{\partial y}$$

By C.R. equations

$$\frac{\partial \psi}{\partial y} = \frac{-\partial \psi}{\partial u} \cdot \frac{\partial v}{\partial x} + \frac{\partial \psi}{\partial v} \cdot \frac{\partial u}{\partial x} \qquad (3)$$

Squaring and adding equations (2) and (3)

$$\left(\frac{\partial \psi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial u} \cdot \frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial v} \cdot \frac{\partial v}{\partial x}\right)^2 + 2\frac{\partial \psi}{\partial u} \cdot \frac{\partial u}{\partial x} \cdot \frac{\partial \psi}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$+\left(\frac{\partial \psi}{\partial u} \cdot \frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial v} \cdot \frac{\partial u}{\partial x}\right)^2 - 2\frac{\partial \psi}{\partial u} \cdot \frac{\partial u}{\partial x} \cdot \frac{\partial \psi}{\partial v} \cdot \frac{\partial u}{\partial x}$$

$$= \left[\left(\frac{\partial \psi}{\partial u}\right)^2 + \left(\frac{\partial \psi}{\partial v}\right)^2\right]\left(\frac{\partial u}{\partial x}\right)^2 + \left[\left(\frac{\partial \psi}{\partial u}\right)^2 + \left(\frac{\partial \psi}{\partial v}\right)\right]^2 \left(\frac{\partial v}{\partial x}\right)$$

$$= \left[\left(\frac{\partial \psi}{\partial u}\right)^2 + \left(\frac{\partial \psi}{\partial v}\right)^2\right]\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] \qquad (*)$$

$\because \qquad f$ is analytic at z.

$\Rightarrow \qquad f$ is differentiable at z.

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \text{ and } f'(z) = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial x}$$

$$(*) \Rightarrow \left[\left(\frac{\partial \psi}{\partial u}\right)^2 + \left(\frac{\partial \psi}{\partial v}\right)^2\right]\left[\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\left(\frac{\partial u}{\partial x} - i\frac{\partial v}{\partial x}\right)\right]$$

$$\left(\frac{\partial \psi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial y}\right)^2 = \left[\left(\frac{\partial \psi}{\partial u}\right)^2 + \left(\frac{\partial \psi}{\partial v}\right)^2\right] |\, f(z)|^2$$

5) Prove that the function *Re z* is differentiable anywhere in $\mathbb{C}$.

Solution : $\dfrac{f(z + \Delta z)}{\Delta z} \to f'(z) \qquad$ (a unique limit)

as $\Delta z \to 0$ in any manner in $\mathbb{C}$ plane.

Here $f(z) = \operatorname{Re} Z$

$\therefore \quad f(\Delta + \Delta z) = \operatorname{Re}\left(\overline{z + \Delta z}\right)$

6) Prove that Cauchy Riemann equation can be written polar Co-ordinates as $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{dv}{d\theta}$ and $\frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}$.

Solution : Let $f(r,\theta) = u(r,\theta) + iv(r,\theta)$

Let $x = r\cos\theta$ and $y = r\sin\theta$

$\Rightarrow r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\left(y/x\right)$

Now, $u = u(r,\theta)$ where r and θ are functions of x and y.

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}.\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}.\frac{\partial \theta}{\partial x}$$

$$\because \quad r = \sqrt{x^2 + y^2} \text{ and } \theta = \tan^{-1}\left(y/x\right)$$

$$\therefore \quad \frac{\partial r}{\partial x} = \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2 + y^2}} = \frac{\cancel{r}\cos\theta}{\cancel{r}} = \cos\theta$$

and

$$\frac{\partial \theta}{\partial x} = \frac{1}{1 + \left(y/x\right)^2} \times \left[-x^{-2}y\right] = \frac{-y}{\cancel{x^2}\left(\frac{x^2 + y^2}{\cancel{x^2}}\right)} = \frac{-r\sin\theta}{r^2} = \frac{-\sin\theta}{r}$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}.\cos\theta + \frac{\partial u}{\partial \theta}\left(\frac{-\sin\theta}{r}\right)$$

(1)

Similarly, $\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}.\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}.\frac{\partial \theta}{\partial y} = \frac{\partial u}{\partial r}.\sin\theta + \frac{1}{r}\cos\theta.\frac{\partial u}{\partial \theta}$

(2)

Similarly, $\frac{\partial v}{\partial x} = \cos\theta.\frac{\partial v}{\partial r} - \frac{\sin\theta}{r}.\frac{\partial v}{\partial \theta}$

(3)

and $\quad \frac{\partial v}{\partial y} = \sin\theta.\frac{\partial v}{\partial r} + \frac{\cos\theta}{r}.\frac{\partial v}{\partial \theta}$

(4)

By Cauchy – Riemann Equations

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x} \Rightarrow \quad \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0$

Consider,

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = \frac{\partial u}{\partial r}(\cos\theta) - \frac{1}{r}\frac{\partial u}{\partial \theta}\sin\theta - \frac{\partial v}{\partial r}\sin\theta - \frac{1}{r}\frac{\partial v}{\partial \theta}\cos\theta$$

$$\cos\theta\left(\frac{\partial u}{\partial r}-\frac{1}{r}\frac{\partial v}{\partial \theta}\right)-\sin\theta\left(\frac{\partial v}{\partial r}+\frac{1}{r}\frac{\partial u}{\partial \theta}\right)=0$$

(5)

and $\dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x}=0$

$$\frac{\partial u}{\partial r}\sin\theta+\frac{1}{r}\cos\theta.\frac{\partial u}{\partial \theta}+\cos\theta.\frac{\partial v}{\partial r}-\frac{\sin\theta}{r}\frac{\partial v}{\partial \theta}=0$$

$$\sin\theta\left(\frac{\partial u}{\partial r}-\frac{1}{r}\frac{\partial v}{\partial \theta}\right)+\cos\theta\left(\frac{1}{r}.\frac{\partial u}{\partial \theta}+\frac{\partial v}{\partial r}\right)-=0$$

(6)

Multiplying equation (3) $\times\cos\theta$ and equation (6) $\times\sin\theta$ and adding.

$$\cos^2\theta\left(\frac{\partial u}{\partial r}-\frac{1}{r}\frac{\partial v}{\partial \theta}\right) - \sin\theta.\cos\theta\left(\frac{\partial v}{\partial r}+\frac{1}{r}\frac{\partial u}{\partial \theta}\right) = 0$$

$$+ \quad \sin^2\theta\left(\frac{\partial u}{\partial r}-\frac{1}{r}\frac{\partial v}{\partial \theta}\right) - \sin\theta.\cos\theta\left(\frac{\partial v}{\partial r}+\frac{1}{r}\frac{\partial u}{\partial \theta}\right) = 0$$

$$\therefore \left(\sin^2\theta+\cos^2\theta\right)\left(\frac{\partial u}{\partial r}-\frac{1}{r}\frac{\partial v}{\partial \theta}\right)=0$$

$$\therefore \frac{\partial u}{\partial r}-\frac{1}{r}\frac{\partial v}{\partial \theta}=0$$

$$\therefore \boxed{\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta}}$$

Multiplying equation (3) $\times\sin\theta$ and equation (6) $\times\cos\theta$ and subtracting.

$$\sin\theta.\cos\theta\left(\frac{\partial u}{\partial r}-\frac{1}{r}\frac{\partial v}{\partial \theta}\right) - \sin^2\theta\left(\frac{\partial v}{\partial r}+\frac{1}{r}\frac{\partial u}{\partial \theta}\right) = 0$$

$$- \quad \sin\theta.\cos\theta\left(\frac{\partial u}{\partial r}-\frac{1}{r}\frac{\partial v}{\partial \theta}\right) + \cos^2\theta\left(\frac{\partial v}{\partial r}+\frac{1}{r}\frac{\partial u}{\partial \theta}\right) = 0$$

$$- \qquad\qquad -$$

$$-\left(\sin^2\theta+\cos^2\theta\right)\left(\frac{\partial v}{\partial r}+\frac{1}{r}\frac{\partial u}{\partial \theta}\right) = 0$$

$$\therefore -\left(\sin^2\theta+\cos^2\theta\right)\left(\frac{\partial v}{\partial \theta}+\frac{1}{r}\frac{\partial u}{\partial r}\right)=0$$

$$\therefore \quad -\frac{\partial v}{\partial r}-\frac{1}{r}\frac{\partial u}{\partial \theta}=0$$

$$\therefore \boxed{\frac{\partial v}{\partial r}=\frac{1}{r}\frac{\partial u}{\partial \theta}}$$

6.5 THE FUNCTIONS $e^z, \sin(z), \cos(z)$ etc

Exponential Function : The exponential function in Complex Plane $\mathbb{C}$ denoted by e^z, is defined by as

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots$$

1) The exponential function $f(z) = e^z$ is analytic in the whole Complex Plane and $f'(z) = f(z)$ with $f(0) = 1$.

Solution : We have, $f'(z) = e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$.

$\therefore \quad a_n = \frac{1}{n!} \quad \Rightarrow \quad a_{n+1} = \frac{1}{(n+1)!}$

$\therefore \quad R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{1}{n!} \times \frac{(n+1)!}{1} \right|$

$= \lim_{n\to\infty} |(n+1)|$

$\because$ The radius of convergence of the given power series is $R = \infty$.

$\therefore$ The power series converges for all z and convergence is uniform for each compact subset of $\mathbb{C}$.

$\therefore$ By using corollary.

[If $f(z) = \sum a_n z^n$ has a radius of convergence $R > 0$ then f is analytic in $B(0, R)$]

$f(z) = e^z$ is analytic in whole Complex Plane.

Note: Similarly, $\sin z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}, \cos z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$ are analytic in whole Complex Plane.

Definition :

Entire Function : If the function f is analytic everywhere in whole Complex Plane $\mathbb{C}$ (except at ∞) is called an <u>Entire function</u> or <u>integral function</u>. e.g. $e^z, \cos z, \sin z$.

6.6 SUMMARY

1) A function f is said to be <u>analytic</u> (or <u>holomorphic</u> or <u>regular</u>) at a point $z = z_0$, if f is differentiable at every point of some nbd of z_0.

2) If the power series $\sum_{n=0}^{\infty} a_n z^n$ has radius of cgs. $R>0$, then $f(z)=\sum a_n z^n$ is analytic on $B(0;R)$.

3) Let u and v be real valued function defined on the domain $G \subset \mathbb{C}$ and suppose that u and v have continuous partial derivatives then $f: G \to \mathbb{C}$ defined by $f(z)=u(z)+iv(z)$ is analytic iff u and v satisfy Cauchy Riemann equation. i.e. $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$.

4) $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+...$

6.7 UNIT END EXERCISES

1) Give an example of a function which is not differentiable at the origin but the partial derivatives exist and satisfy the Cauchy-Riemann equations there .

Solution: Consider $f:\mathbb{C} \to \mathbb{C}$ defined by

$$f(x,y)=\frac{xy(x+iy)}{x^2+y^2} (x,y) \neq (0,0)$$

$$=0 \qquad (x,y)=(0,0).$$

$f_x(0,0)=0$ similarly $f_y(0,0)=0$

But $\lim_{z\to 0} \frac{f(z)-f(0)}{z} = \lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ does not exist. For, on the line $y=ax \because \frac{f(z)-f(0)}{z} = \frac{\alpha}{1+a^2}$ for $z \neq 0$. ∴ The limit depends on real number a.

2) Check at what points does the function $f(z)=|z|^2$ is differentiable.

Solution : $f(z)=|z|^2$. Let $z=x+iy$.

$\Rightarrow f(z)=x^2+y^2$. $\therefore f_x = 2x, \quad f_y = 2y$

$\Rightarrow f$ has continuous partial derivatives for all z.

$\therefore f$ is differentiable at $z \in \mathbb{C}$, provided that $f_y = if_x$.

$\Rightarrow 2y = 2ix \Rightarrow 2y - i2x = 0 \Rightarrow (x,y)=(0,0)$.

$\therefore f$ is differentiable only at the origin $(0,0)$

3) Show that $f(z) = x^2 + iy^2$ is differentiable at all points on the line $y = x$.

Solution: $\because f(z) = x^2 + iy^2 . \therefore f_x = 2x,\ f_y = 2iy$

$\Rightarrow f_x, f_y$ exist and are continuous functions of $z = x + iy$

$\Rightarrow f_y = 2iy = if_x = 2xi$ iff $x - y = 0$.

This is possible iff $x = y$.

$\Rightarrow$ By proposition f is differentiable at all points on the line $y = x$.

4) Suppose f is analytic in a region and at every point, either $f = 0$ or $f^2 = 0$. Show that f is a constant function.

(Hint: Consider The derivative of $f^2(z)$)

5) Find all analytic functions $f = u + iv$ with $u(x, y) = x^2 - y^2$

Solution: $\because f = u + iv$ is analytic.

$\because f$ satisfies Cauchy- Riemann equations.

$\Rightarrow u_x = v_y,\ v_x = -u_{y.\ \Rightarrow 2x=} v_y,\ -2y = -v_x$

$\Rightarrow v_y = 2x,\ v_x = 2y.$

$\therefore v(x, y) = 2xy + c,$ where c is any real constant.

$\Rightarrow f = (x^2 - y^2) + 2ixy + ic$

6) If f is a analytic in a region and if $|f|$ is constant there, then show that f is constant.

Solution: If $|f| = 0$, the proof is immediate, otherwise assume that $|f| \neq 0$. Let $f = u + iv \Rightarrow u^2 + v^2 \neq 0$.

$\therefore$ Taking the partial derivatives w.r.t. x and y, we see that $\therefore uu_x + vv_x = 0,\ uu_y + vv_y = 0$.

Making use of the Cauchy- Riemann equations, we get,

$\therefore uu_x - vu_y = 0,\ vu_x + uu_y = 0$

$\Rightarrow (u^2 + v^2)u_x = 0 \Rightarrow u_x = v_y = 0$, similarly $u_y = v_x = 0$.

$\Rightarrow f$ is constant.

7) Show that $\frac{d}{dz}(\sin(z)) = \cos(z)$. (Hint: Use that $\sin(z) = \frac{1}{2i}(e^{iz} - e^{-iz})$)

8) Find a power series representation for $\cos(z)$.

Solution: $\because \cos(z) = \frac{1}{2}(e^{iz} + e^{-iz}), e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

$e^{iz} = \sum_{n=0}^{\infty} \frac{(iz)^n}{n!}, e^{iz} = \sum_{n=0}^{\infty} \frac{(-iz)^n}{n!}$

$\therefore \frac{1}{2}(e^{iz} + e^{-iz}) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$

$\therefore \cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + ...$

9) Show that $\log(x^2 + y^2)$ is harmonic in $\mathbb{C} - 0$.

Solution: Let $u(x, y) = \log(x^2 + y^2)$.

$\Rightarrow u_x = \frac{2x}{x^2 + y^2}, u_y = \frac{2y}{x^2 + y^2}$ $\mathbb{C} - 0$

$\Rightarrow u_{xx} = \frac{2y^2 - 2x^2}{(x^2 + y^2)^2}, u_{yy} = \frac{2x^2 - 2y^2}{(x^2 + y^2)^2}$

$\Rightarrow u_{xx}, u_{yy}$ exist and are continuous functions of z on $\mathbb{C} - 0$, also $\because u_{xx} + u_{yy} = 0$.

$\Rightarrow u(x, y)$ is harmonic in $\mathbb{C} - 0$.

10) For the function *f(z)* defined by

$$f(z) = \frac{(\overline{z})^2}{z} \quad if\ z \neq 0$$

$$= 0 \quad if\ z = 0$$

Prove that C-R eq. are not satisfied at the origin , but the function *f(z)* is not differentiable at the origin(2009)

11) Find the holomorphic function *f(z)* whose real part is *2xy+2x* (2008)

12) Find the analytic function $f(z) = u(r, \theta) + i\, v(r, \theta)$ whose real part is $r^2 \cos 2\theta$.

13) For the function, $f(z)$ defined by $f(z)=\begin{cases} \dfrac{(\bar{z})^2}{z}, & z \neq 0 \\ 0\ , & z = 0 \end{cases}$.

Prove that the Cauchy Riemann equations are satisfied at $(0,0)$ but the function is not differentiable at $(0,0)$.

14) If $f : G \to \mathbb{C}$ is analytic, prove that $\left(\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}\right)\left|\operatorname{Re} f(z)\right|^2 = 2\left| f'(z)\right|^2$

15) Construct an analytic function $f(z)=u(z)+iv(z)$, whose real part is *cos x cosh y*. express the result as a function of *z*.

16) Construct an analytic function $f(z)=u(z)+iv(z)$, whose real part is $e^x(x\cos y - y\sin y)$.

Express the result as a function of *z*.

*****

7

COMPLEX INTEGRATION

Unit Structure

7.0 Objectives
7.1 Introduction
7.2 Complex Line integrals
7.3 Integration along piecewise smooth path, The Closed Curve Theorem
7.4 Summary
7.5 Unit End Exercises

7.0 OBJECTIVES

Through this unit we shall study the concept of complex integration, an integration of the form $\int_{\gamma} f(z)dz$ taken over a piecewise smooth path γ further we shall derive certain properties of this integral. We would like to know further that what can be the integral of an entire function along a boundary of a rectangle in a complex plane, the answer is given in form of a closed curve theorem.

7.1 INTRODUCTION

We have to recall theorem on differentiability of a power series that states that a power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ converges for $|z|CR$. Then $f'(z)$ exists and $f'(z) = \sum_{n=1}^{\infty} na_n z^{n-1}$ on the open disc $|z|CR$. Therefore an everywhere convergent power series represents an entire function. Our main goal in this unit is to study the converse of this result namely that every entire function can be expanded as an everywhere convergent power series. This result has a consequence that every entire function is infinitely differentiable. We shall also arrive at these results by discussing integrals. Let us start by defining a Line integral.

7.2 COMPLEX INTEGRATION

Definition : Trace of a curve :

If $x:[a,b]\to\mathbb{C}$ is a curve, then the set $\{x(t)\}$ is called the trace x and is denoted by $\{x\}=\{x(t): a\le t\le b\}$

The trace of x is always a compact set.

Definition : Contour : A contour is a piecewise smooth curve.

Definition : A complex valued function f is said to be continuous on a smooth curve $x:[a,b]\to\mathbb{C}$ if, $f(z)=f(x(t))=u(t)+iv(t)$ is continuous.

7.3 INTEGRATION ALONG A PIECEWISE SMOOTH PATH, THE CLOSED CURVE THEOREM

Definition : Complex Line Integral :

Suppose f is complex valued, continuous and defined on open set $G\subset\mathbb{C}$ and that $x:[a,b]\to\mathbb{C}$ is a piecewise smooth curve with $\{x\}\subset G$.

Then, the expression

$$\int_x f(z)\,dz=\int_a^b f[x(t)]\,x'(t)\,dt=\sum_{j=1}^{n-1}\int_i^{j+1} f[x(t)]\,x'(t)\,dt$$

where $a=t_0<t_1<t<...<t_{n-1}=b$ is called the complex line integral of f over x.

This curve x is called path of integration of this integral.

Connection between Real and Complex line integral :

If $f(z)=u(z)+iv(z)$ then the complex line integral $\int_x f(z)\,dz$ can be expressed as

$$\int_x f(z)\,dz=\int_x u\,dx-v\,dy+i\int_x u\,dy+v\,dx$$

Theorem : *Let x be such that $x(t)=x(t)+i\,y(t)$, $a\le t\le b$ is a smooth curve and suppose that f and g are continuous function on open set $G\subset\mathbb{C}$ containing $\{x\}$ then ;*

i) $\int\limits_{x} \alpha\, f(z)\, dz = \alpha \int\limits_{x} f(z)\, dz$ *where* α *is a complex constant.*

ii) $\int\limits_{x} \left[f(z) + g(z) \right] dz = \int\limits_{x} f(z)\, dz + \int\limits_{x} g(z)\, dz$

Example 1 : Evaluate $\int\limits_{x} \frac{1}{z}\, dz$, where $x(t) = e^{it},\ t \in [0, 2\pi]$.

Solution : By the definition of complex line integral.

$$\int\limits_{x} f(z)\, dz = \int\limits_{a}^{b} f\left[x(t)\right] x'(t)\, dt$$

Here, $f(z) = \frac{1}{z}, \quad x(t) = e^{it}, \quad a = 0, \quad b = 2\pi$

$\therefore \quad f\left[x(t)\right] = \frac{1}{e^{it}}, \quad x'(t) = i\, e^{it}$

$$\int\limits_{x} \frac{1}{z}\, dz = \int\limits_{0}^{2\pi} \frac{1}{\cancel{e^{it}}} . i\, \cancel{e^{it}}\, dt = \int\limits_{0}^{2\pi} i\, dt = i\,[\ t\ ]_{0}^{2\pi} = 2\pi\, i$$

Definition : Rectifiable Curve :A curve is rectifiable if it has finite length.

Note : Every piecewise smooth curve is Rectifiable.

Definition : If x is s.t. $x(t).\ a \le t \le b$ is rectifiable, then its length $L(x)$ us defined by $L(x) = \int\limits_{a}^{b} \left| x'(t) \right| dt$

Example 1 : Find the length of the curve $x(t) = 4\, e^{it},\ t \in [0,\ 2\pi]$.

Solution : Length of $x = L(x) = \int\limits_{0}^{2\pi} \left| 4i\, e^{it} \right| dt \quad = \int\limits_{0}^{2\pi} |\ 4\ ||\ i\ |\left| e^{it} \right| dt$

$$= \int\limits_{0}^{2\pi} 4dt = 4 \int\limits_{0} \frac{2\pi}{dt} = 8\pi$$

Example 2 : Find the length of the curve $x(t) = (1+i)\, t,\ t \in [0, 4]$.

Solution :

$\because \quad x(t) = (1+i)\, t$

$\therefore \quad x'(t) = (1+i)$

$\therefore$ Length of

$$x = L(x) = \int_0^4 |(1+i)|\,dt = \int_0^4 \sqrt{1^2+1^2}\,dt = \int_0^4 \sqrt{2}\,dt = \sqrt{2}\,[\,t\,]_0^4 = 4\sqrt{2}$$

Definition : Opposite Curve :
If $x:[a,b]\to\mathbb{C}$ is a given curve then, the opposite curve $-x$ to x is defined as $-x(t) = x(a+b-t); \quad t\in[a,b]$

Example : Let $x(t) = e^{it}; \quad t\in[0,\pi]$.

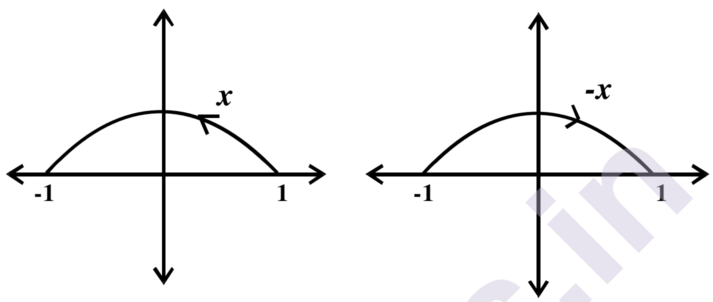


Fig. 7.1

Definition : Let $x_1:[a,b]\to\mathbb{C}$ and $x_2:[a,b]\to\mathbb{C}$ be two smooth curves such that $x_1(b_1) = x_2(a_2)$. Then we define the path $x_1+x_2:[a_1, b_1+b_2-a_2]\to\mathbb{C}$ as follows.

$$(x_1+x_2)(t) \quad \textbf{OR} \quad (x_1\cup x_2)(t) = \begin{cases} x_1(t) & \text{if } t\in[a_1,b_1] \\ x_2(t-b_1+a_2) & \text{if } t\in[b_1, b_1+b_2-a_2] \end{cases}$$

The path x_1+x_2 is called the sum of two curves x_1 or x_2 or the union of two curves x_1 and x_2.

Example 1 : Let $x(t) = t, \quad t\in[0,3]$ and $x_2(t) = 3e^{it}, \quad t\in\left[0, 3\pi/2\right]$.

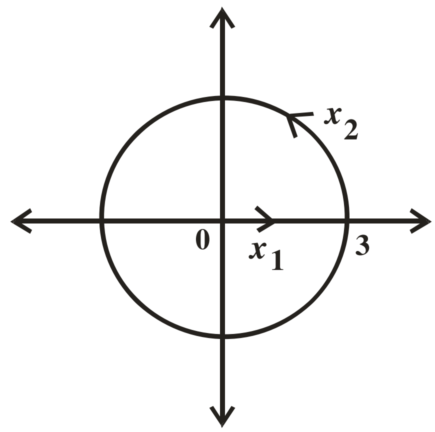


Fig 7.2

Theorem : *Let x be s.t.* $x(t) = x_1(t) + iy_1(t)$ *be a smooth curve and suppose that f is a continuous function on an open set G containing* $\{x\}$. *Then,*

i) $\int_{x} f(z)\,dz = -\int_{x} f(z)\,dz$

ii) $\left|\int_{x} f(z)\,dz\right| \le \int_{x} |s(z)|\,|dz|$

iii) *If* $M = \underset{t\in[a,b]}{\text{Max}} \left| f(x(t)) \right|$ *and* $L = L(x)$ *(Length of x)*

then $\left|\int_{x} f(z)\,dz\right| \le ML$.

(This property is called standard estimate for the integral.)

iv) *If* x_1 *and* x_2 *are smooth curves in G then,* $\int_{x_1+x_2} f(z)\,dz = \int_{x_1} f(z)\,dz + \int_{x_2} f(z)\,dz$, *where* $x_1 + x_2$ *are sum of 2 curves.*

Proof : i) By definition of opposite curve

$$-x(t) = x(a+b-t)$$

By definition of complex line integral.

$$\int_{-x} f(z)\,dz = \int_{a}^{b} f[-x(t)][-x'(a+b-t)]\,dt$$

Put $a+b-t=u \quad \Rightarrow \quad dt=-du$, When $t=a, \quad u \Rightarrow b$
$t=b, \quad u=a$

$$\int_{-x} f(z)\,dz = -\int_b^a f[x(u)].[x'(u)](-du) = \int_a^b f[x(u)]x'(u)\,du$$

$$= -\int_x f(z).dz$$

ii) If $\int_x f(z)\,dz = 0$, then there is nothing to prove.

Let $\int_x f(z)\,dz \neq 0$

Put $u = e^{-i\theta}$, where $\theta = \arg\left[\int_x f(z)\,dz\right]$

$\therefore \quad |u| = 1$

$$\left|\int_x f(z)\,dz\right| = e^{i\theta}\int_x f(z)\,dz = u\int_x f(z)\,dz \qquad (1)$$

$\because \quad |z| = \text{Re}|z|$

$$\left|\int_x f(z)\,dz\right| = \text{Re}\left|\int_x f(z)\,dz\right| = \text{Re}\left[u\int_x f(z)\,dz\right] \text{ from (1)}$$

$$= \text{Re}\int_x u\,f(z)\,dz = \int_x re[u\,f(z)\,dz]$$

$$\leq \int_x |u\,f(z)\,dz| \qquad (\because \text{Re}\, z \leq |z|)$$

$$= \int_x |u|.|f(z)||dz|$$

$$\left|\int_x f(z)\,dz\right| \leq \int_x |f(z)||dz| \qquad (\because |u|=1)$$

iii) Given that, $M = \underset{t\in[a,b]}{\text{Max}}|f(x(t))|$ and $L = L(x)$

By using part (ii),

$$\left|\int_x f(z)\,dz\right| \leq \int_x |f(z)||dz|$$

$$\left| \int_x f(z)\,dz \right| = \int_a^b \left| f(x(t)) \right| \left| x'(t) \right| dt \le M \int_a^b \left| x'(t) \right| dt = ML(x)$$

$$\left| \int_x f(z)\,dz \right| \le ML$$

iv) Let $x_1 : [a_1, b_1] \to G$ and $x_2 : [a_2, b_2] \to G$ with $x_1(b_1) = x_2(a_2)$.

$\therefore$ We define $x_1 + x_2 : [a_1\, b_1 + b_2 - a_2] \to G$

$$(x_1 + x_2)(t) = \begin{cases} x_1(t) & \text{if } t \in [a_1, b_1] \\ x_2(t - b_1 + a_2) & \text{if } t \in [b_1, b_1 + b_2 - a_2] \end{cases}$$

$\therefore$ $x_1 + x_2$ has derivative $x_1'(t)$ in $[a_1, b_1]$ and $(t\, b_1 + a_2)$ in $[b_1, b_1 + b_2 - a_2]$

$$\int_{x_1+x_2} f(z)\,dz = \int_{a_1}^{b_1+b_2-a_2} f\left[(x_1 + x_2)(t)\right](x_1 + x_2)'(t)\,dt$$

$$= \int_{a_1}^{b_1} f\left[x_1(t)\right] x_1'(t)\,dt + \int_{b_1}^{b_1+b_2+b_3} \left[x_2(t - b_1 + a_2)\, x_2'(t - b_1 + a_2)\,dt\right]$$

$$= \int_{x_1} f(z)\,dz + \int_{b_1}^{b_1+b_2+b_3} f\left[x_2(t - b_1 + a_2)\, x_2'(t - b_1 + a_2)\,dt\right]$$

Put $t - b_1 + a_2 = u \;\Rightarrow\; dt = du$, then $t = b_1$, $u = a_2$, $t = b_1 + b_2 - a_2$, $u = b_2$.

$$= \int_{x_1} f(z)\,dz + \int_{a_2}^{b_2} f\left[x_2(u)\right] x_2'(u)\,du$$

$$= \int_{x_1} f(z)\,dz + \int_{x_2} f(z)\,dz$$

Note : i) If x is piecewise smooth then, there is a partition $P \div a = t_0 < t_1 < \ldots < t_n = b$ of $[a, b]$ s.t. the restriction x_k of curve x to $[t_{k-1}, t_k]$ is smooth for $1 \le k \le n$.

$$x = x_1 + x_2 + \ldots + x_n$$

$$\therefore \int_x f(z)\,dz = \int_{x_1+x_2+x_n} f(z)\,dz = \int_{x_1} f(z)\,dz + \int_{x_2} f(z)\,dz + \ldots + \int_{x_n} f(z)\,dz$$

ii) $\text{Re}\left[\int_x f(z)\,dz\right] \ne \int_x \text{Re}\left[f(z)\right] dz$

Example 1 : Let $f(z)=1$ and $x(t)=it$, $t\in[0,1]$.

$$\int_x f(z)\,dt=\int_a^b f[x(t)]\,x'(t)\,dt=\int_0^1 1.i.dt=i$$

$$\text{Re}\left[\int_x f(z)\,dt\right]=\text{Re}\, i=0 \qquad (1)$$

and $$\int_x \text{Re}[f(z)]\,dz=\int_0^1 1.i\,dt=1 \qquad (2)$$

From equation (1) and (2)

$$\text{Re}\left[\int_x f(z)\,dz\right]\neq\int_x \text{Re}[f(z)]\,dz$$

Change of Parameter :

Let $x:[a,b]\to\mathbb{C}$ and $\sigma:[c,d]\to\mathbb{C}$ be two smooth curve. Then the curve σ is equivalent to curve x if, there is a function $\varnothing:[c,d]\to[a,b]$ which is contain non-decreasing and with $\varnothing(c)=a$ and $\varnothing(d)=b$ s.t. $\sigma=x\circ\varnothing$.

Here, we call the function $\varnothing$ a change of parameter. This new curve $x\circ\varnothing$ is called the Reparametrization of the curve x.

Theorem : *Let $x:[a,b]\to\mathbb{C}$ be a smooth curve and suppose that $\varnothing:[c,d]\to[a,b]$ is a continuous non-decreasing function with $\varnothing(c)=a$ and $\varnothing(d)=b$. If f is continuous on $\{x\}$ then*

$$\int_x f(z)\,dz=\int_{x\circ\varnothing} f(z)\,dz$$

Proof : Given that $x:[a,b]\to\mathbb{C}$ is a smooth curve and $\varnothing:[c,d]\to[a,b]$ is continuous non-decreasing function with $\varnothing(c)=a$ and $\varnothing(d)=b$.

By hypothesis, there is a change of parameter $\varnothing[c,d]\to[a,b]$ s.t.

$$(x\circ\varnothing)(s)=x[\varnothing(s)]$$

and $$(x\circ\varnothing)'(s)=x'[\varnothing(s)]\varnothing'(s) \qquad \text{for } s\in[c,d]$$

$$\therefore \quad \int_{x\circ\varnothing} f(z)\,dz=\int_c^d f[x(\varnothing(s))](x\circ\varnothing)'(s)\,ds$$

$$= \int_c^d f\left[x(\varnothing(s))\right] x'(\varnothing(s))\, \varnothing'(s)\, ds$$

Put $t = \varnothing(s) \Rightarrow \quad dt = \varnothing'(s)$

when $s = c$, $t = \varnothing(c) = a$ and when $s = d$, $t = \varnothing(d) = b$

$$\int_{x\circ\varnothing} f(z)\, dz = \int_a^b f(x(t))\, x'(t)\, dt$$

$$\therefore \int_{x\circ\varnothing} f(z)\, dz = \int_x f(z)\, dz$$

Fundamental theorem of calculus :

If f is continuous on $[a,b]$ and $F'(x) = f(x)$ in $[a,b]$ then $\int_s^b f(x)dx = F(b) - F(a)$.

Primitive or Antiderivative of a function :

A function $f : G \to \mathbb{C}$ is said to be primitive or antiderivative of f in G if, F is analytic in G and $F'(z) = f(z)$ in G.

Theorem : *Let G be an open set in $\mathbb{C}$ and suppose that $f : G \to \mathbb{C}$ is a continuous function with primitive $F : G \to \mathbb{C}$. If $x : [a,b] \to G$ is a smooth curve, then $\int_x f(z)\, dz = F[x(b)] - F[x(a)]$.*

In particular, if x is closed then $\int_x f(z)\, dz = 0$.

Proof : Given that, $f : G \to \mathbb{C}$ is a continuous function with primitive $f : G \to \mathbb{C}$.

$\therefore \quad F'(z) = f(z)$

$$\therefore \quad \int_x f(z)\, dz = \int_a^b f(x(t))\, x'(t)\, dt$$

$$= \int_a^b (F \circ x)'(t)\, dt \qquad (\because \; F'(x(t)) = f(x(t)))$$

$$= \left[F(x(t))\right]_a^b \qquad \text{(by using fundamental)}$$

$$= F[x(b)] - F[x(a)]$$

If x is closed, then $x(a) = x(b)$.

$$\int_x f(z)\, dz = F[x(a)] - F[x(b)] = 0$$

Example : Evaluate $\int_x z^2\,dz$ where $x(t) = t + \dfrac{it^2}{\pi}, \quad t \in [0, \pi]$.

Solution : Given integral is $\int_x z^2\,dz$.

Here, $f(z) = z^2$

$F(z) = \dfrac{z^3}{3}$ is primitive of *f*.

$a = 0, \quad b = \pi$.

By previous theorem,

$$\int_x f(z)\,dz = \left[F(x(t))\right]_a^b$$

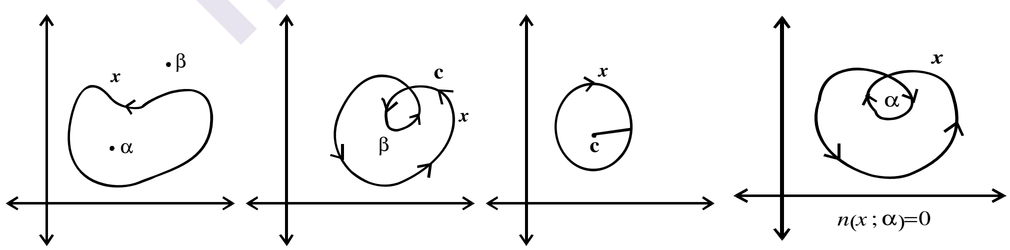
$$\int_x z^2\,dz = \left[f\left(t + \frac{it^2}{\pi}\right)\right]_0^\pi = \left[\left(t + \frac{it^2}{\pi}\right)^3 \cdot \frac{1}{3}\right]_0^\pi$$

$$= \frac{1}{3}\left[\left(\pi + i\,\frac{\pi^2}{\pi}\right)^3 - 0\right] = \frac{1}{3}\left[(\pi + i\,\pi)^3\right]$$

$$= \frac{\pi^3}{3}(1+i)^3$$

Definition : The index of curve or winding number : If x is a closed rectifiable curve in $\mathbb{C}$ and if $\alpha \notin \{x\}$, then, $\eta(x, \alpha) = \dfrac{1}{2\pi i}\int_x \dfrac{dz}{z - \alpha}$ is called the index of x w.r.t the point α.

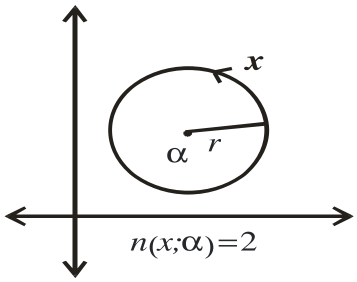
It is also called the winding number about α.



$\eta(x;\alpha) = 1$ $\quad$ $\eta(x;\alpha) = 2$ $\quad$ $\eta(x;c) = -1$

$\eta(x;\beta) = 0$ $\quad$ $\eta(x;\beta) = 1$

Fig. 7.3



$$x : x(t) = \left\{ z : z - \alpha = r\,e^{it},\ t \in [0, 4\pi] \right\}$$

$$| z - \alpha | = | r\,e^{it} | = | r || e^{it} | = | r |$$

Fig 7.4

Theorem : *If* $x : [0,1] \to \mathbb{C}$ *is a closed rectifiable curve and* $\alpha \notin \{x\}$ *then* $\eta(x;\alpha)$ *is an integer.*

Proof : Define $g : [0,1] \to \mathbb{C}$ by,

$$g(t) = \int_0^t \frac{x'(s)\,ds}{x(s) - \alpha} \quad t \in [0,1]$$

$\therefore \quad g(0) = 0$, $\quad g$ is continuous on $[0,1]$ and

$$g(1) = \int_x \frac{dz}{z - \alpha} \qquad (*)$$

(put $x(s) = z \Rightarrow x'(s)\,ds = dz$)

To prove that $g(1) = 2\pi in$ for some integer η.

$\therefore$ from equation (1),

$$g'(t) = \frac{x'(t)}{x(t) - \alpha}, \quad t \in [0,1]$$

Now, $\frac{d}{dt}\left[e^{-g(t)} \left(x(t) - \alpha \right) \right] = e^{-g(t)} x'(t) + \left(x(t) - \alpha \right) e^{-g(t)} \left(-g'(t) \right)$

$$= e^{-g(t)} \left[x'(t) - \cancel{\left(x(t) - \alpha \right)} \left(\frac{x(t)}{\cancel{x(t) - \alpha}} \right) \right] = e^{-g(t)} \left[x'(t) - x'(t) \right]$$

$$= e^{-g(t)}.0 = 0$$

$\therefore \quad e^{-g(t)}.\left(x(t) - \alpha \right) =$ constant $= K$ (say)

OR $\quad x(t) - \alpha = K.e^{g(t)} \qquad$ (2)

To find K, put $t = 0$

$$x(0) - \alpha = K\,e^{g(0)} = K\,e^0 = K \qquad \{\because e^0 = 1\}$$

$\therefore \quad K = x(0) - \alpha$

Putting the value of K in (2), we get

$$x(t)-\alpha=(x(0)-\alpha).e^{g(t)}$$

Put $t=1$,

$$e^{g(1)}=\frac{x(1)-\alpha}{x(0)-\alpha}=1 \qquad (\because x \text{ is closed } x(1)=x(0))$$

$$\therefore \quad e^{g(1)}=e^{2\pi in} \qquad (\because e^{2\pi in}=1)$$

$$g(1)=2\pi in$$

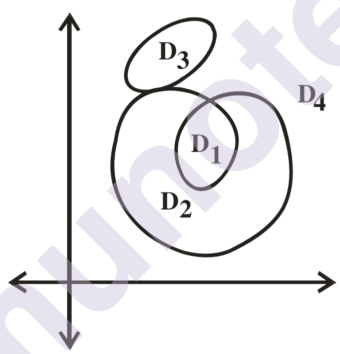
For some integer n substitute the above value in (*), we get

$$2\pi in=\int_x \frac{dz}{z-\alpha}$$

$$\therefore \quad n=n(x;\alpha)=\frac{1}{2\pi i}\int_x \frac{dz}{z-\alpha}$$

Component of a Metric space:

A subset D of a metric space X is a component of X if D is a maximal connected subset of X i.e. D is connected and there is no connected subset of X that properly contains D.



$G=\mathbb{C}/\{x\}$

Fig 7.5

Note: If G is open then component of G also open.

Simply and multiply connected domains :

Definition : A domain D is said to be simply connected if any simple closed curve which lies in D can be shrunk to a point without leaving domain D.

Definition : A domain which is not simply connected is said to be a multiply connected domain.

Green's Theorem :

Let $M(x, y)$ and $N(x, y)$ be continuous and have continuous partial derivatives in a domain Ω and on its boundary *x*, Green's theorem states that,

$$\int_x M\ dx + N\ dy = \iint_\Omega \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy$$

Cauchy Theorem :

Let G be an open set in $\mathbb{C}$ and suppose that $x:[a,b] \to G$ is a smooth curve. If f is analytic with f' continuous inside and on a simple closed curve x then, $\int_x f(z)\, dz = 0$.

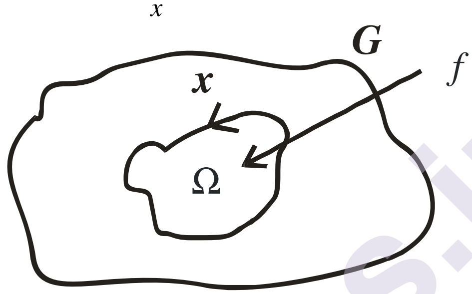


Fig 7.6

Proof : Let $z = x + iy$,

$\therefore \quad f(z) = u(z) + iv(z)$

and $\quad \Omega = \text{Int } x$

$\therefore \quad \int_x f(z)\, dz = \int_x [u(z) + iv(z)](dx + idy)$

$= \int_x (udx - vdy) + i\int_x vdx + udy \qquad (1)$

Given that, f is analytic in Ω and on its boundary *x*.

$\therefore$ f is continuous in Ω and hence *u* and *v* are continuous.

Also, given that, f' is continuous inside and on a simple closed curve *x*.

$\therefore$ Partial derivatives of *u* and *v* are also continuous in Ω and on its boundary *x*.

$\therefore$ By Green's theorem

$$\int_x u\, dx - v\, dy = \iint_\Omega \left(\frac{-\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx\, dy$$

and $\int_{x} v\,dx + u\,dy = \iint_{\Omega} \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dx, dy$

Substituting above values in equations (1), we get

$$\int_{x} f(z)\,dz = \iint_{\Omega} \left(\frac{-\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx\,dy + i \iint_{\Omega} \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dx\,dy$$

Given that, f is analytic.

∴ Cauchy Riemann equations are satisfied.

$$\frac{\partial u}{\partial x} = \frac{-\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial y} = \frac{-\partial v}{\partial x}$$

$$\therefore \quad \int_{x} f(z)\,dz = \iint_{\Omega} \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y} \right) dx\,dy + i \iint_{\Omega} \left(\frac{\partial v}{\partial y} - \frac{\partial v}{\partial y} \right) dx\,dy$$

$$\int_{x} f(z)\,dz = 0$$

Note : In Cauchy's theorem, Cauchy assumed the continuity of derived function $f'(z)$. It was Goursat who first proved that this condition can removed from the hypothesis in the theorem. The revised form of the theorem is known as Cauchy- Goursat theorem which we shall study in the next chapter.

8.4 SUMMARY

1) $\int_{x} f(z)dz = \int_{a}^{b} f\left[x(t)\right]x'(t)\,dt = \sum_{j=1}^{n-1} \int_{i}^{j+1} f\left[x(t)\right]x'(t)\,dt$

$\int_{x} f(z)\,dz = \int_{x} u\,dx - v\,dy + i \int_{x} u\,dy + v\,dx$

2) If $x:[a,b] \to \mathbb{C}$ is a given curve then, the opposite curve $-x$ to x is defined as

$$-x(t) = x(a+b-t); \quad t \in [a,b]$$

3) If x is piecewise smooth then, there is a partition $P \div a = t_0 < t_1 < ... < t_n = b$ of $[a,b]$ s.t. the restriction x_k of curve x to $[t_{k-1}, t_k]$ is smooth for $\therefore \int_{x_1} f(z)dz = -\int_{-x_2} f(z)dz = \int_{x_2} f(z)dz\varepsilon\delta|h| < \delta$

$$F(z+h) - F'(z) = f(z)\int_{z_0}^{z+h} f(w)dw - \int_{z_0}^{z} f(w)dw = \int_{z_0}^{z+h} f(w)dw + \int_{z}^{z_0} f(w)dw = \int_{z}^{z+h} f(w)dw$$

$$\lim_{h\to 0}\frac{F(z+h)-F(z)}{h}=f(z)=\frac{1}{h}\int_{z}^{z+h} f(w)dw-\frac{f(z)}{h}h$$

$$=\frac{1}{h}\left[\int_{z}^{z+h} f(w)dw-f(z)\int_{z}^{z+h} dw\right]=\frac{1}{h}\int_{z}^{z+h}(f(w)-f(z))dw$$

$$=0\le\frac{1}{|h|}\int_{z}^{z+h}|f(w)-f(z)|.|dw|<\frac{1}{|h|}\varepsilon\int_{z}^{z+h}|dw|=\frac{1}{|h|}\varepsilon|h|=\varepsilon$$

$1\le k\le n$.

$$\int_{x} f(z)\,dz=\int_{x_1+x_2+x_n} f(z)\,dz=\int_{x_1} f(z)\,dz+\int_{x_2} f(z)\,dz+...+\int_{x_n} f(z)\,dz$$

$$x=x_1+x_2+...+x_n$$

4) Let $x:[a,b]\to\mathbb{C}$ and $\sigma:[c,d]\to\mathbb{C}$ be two smooth curve. Then the curve σ is equivalent to curve x if, there is a function $\varnothing:[c,d]\to[a,b]$ which is contain non-decreasing and with $\varnothing(c)=a$ and $\varnothing(d)=b$ s.t. $\sigma=x\circ\varnothing$.

5) Fundamental theorem of calculus :

If f is continuous on $[a,b]$ and $F'(x)=f(x)$ in $[a,b]$ then $\int_a^b f(x)dx=F(b)-F(a)$.

6) Primitive or Antiderivative of a function :

A function $f:\text{K}\mathbb{C}$ is said to be Primitive or Antiderivative of f in G if, F is analytic in G and $F'(z)=f(z)$ in G.

7) The index of curve or winding number :

If x is a closed rectifiable curve in $\mathbb{C}$ and if $\alpha\notin\{x\}$, then, $\eta(x,\alpha)=\frac{1}{2\pi i}\int_{x}\frac{dz}{z-\alpha}$ is called the index of x w.r.t. the point α.

It is also called the winding number about α.

8) A domain D is said to be simply connected if any simple closed curve which lies in D can be shrunk to a point without leaving domain D.

8.5 UNIT END EXERCISES

1) Suppose $f(z)=x^2+iy^2$ where $z=x+iy$. Then evaluate $\int_e f(z)dz$, where

$c: z(t)=t+it, 0\le t\le 1$.

Solution: Consider $c: z(t)=t+it, 0\le t\le 1$. Then $z(t)=1+i$, and $\int_c f(z)dz=\int_0^1(t^2+it^2)(1+i)dt=\frac{2i}{3}$.

2) Find the integral of the function $f(z)=\frac{1}{z}$, taken over a circle of radius R.

Solution: $\because f(z)=\frac{1}{z}=\frac{x}{x^2+y^2}-\frac{iy}{x^2+y^2}$

$\because c: z)t)=R\cos(t)+iR\sin(t), 0\le t\le 2\pi, R>0$. Then

$\int_c f(z)dz=\int_0^{2\pi}\left[\left(\frac{\cos(t)}{R}\right)-\left(\frac{i\sin(t)}{R}\right)\right]\left(-R\sin(t)+iR\cos(t)\right)=2\pi i$

3) Let c be any smooth curve in. Let $f(z)=1$. Then find $\int_c f(z)dz$.

4) Let c be the Unit circle and suppose that $f\ll 1$ on c. Then prove that $\int_c f(z)dz\ll 2\pi$.

(Hint: $M=1, L=2\pi$. Apply ML formula.)

5) Let c be any closed curve not passing through the origin, then show that

$\int_c\frac{1}{z^2}dz=0:\int_c z^k dz=0, k\ne -1, k$ is any integer.

(Hint: $\because g(z)=\frac{1}{z^2}=F'(z)$ where $F(z)=-\frac{1}{z}$ and $F(z)$ is analytic everywhere except at the origin .)

6) Evaluate $\int_c(z-i)dz$, where c is the parabolic segment $c: z(t)=t+it^2, -1\le t\le 1$

Solution : Let $f(z)=z-i$. Then f is the derivative of an analytic function $F(z)=\frac{z^2}{2}-iz$.

$\therefore$ By proposition , $\int_c f(z)dz = F(z(b)) - F(z(a))$

$\therefore \int_c (z-i)dz = \left[\frac{z^2}{2} - iz\right]_{-1+i} - \left[\frac{z^2}{2} - iz\right]_{1+i} = 0 \because \int_c (z-i)dz = 0$.

7) Find $\int_y z^{\frac{1}{z}} dz$. Where (a) γ is the upper half of the Unit circle from +1 to -1.

(b) γ is the lower half of the Unit circle from +1 to -1.

(Hint: (a) Let $\gamma(t) = \cos(t) + i\sin(t), 0 \le t \le \pi$

(b) Let $\gamma(t) = -\cos(t) - i\sin(t), \pi \le t \le 2\pi$

8) Let $\gamma(t) = 2e^{it}$, for $-\pi \le t \le \pi$. Find $\int_\gamma (z^2 - 1)^{-1} dz$.

9) Prove the following integration by parts formula . Let f and g be analytic in G , let γ be a rectifiable curve from a to b in G . Then show that $\int_\gamma fg' = f(b)g(b) - f(a)g(a) - \int_\gamma f'g$.

10) Evaluate the integral $\int_\gamma \left(|z| - e^z \sin z + \bar{z}\right) dz$ where γ is the circle $|z| = 2$.

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8

CAUCHY THEOREM

Unit Structure

8.0. Objectives
8.1. Introduction
8.2. Cauchy Theorem for an Open Star shaped Domain
8.3. Cauchy Integral Formula
8.4. Summary
8.5. Unit End Exercises

8.0. OBJECTIVES

Our main goal in this unit is to show that a function analytic in a disc can be represented as a power series. We shall prove Cauchy's theorem for an open star-shaped domain and Cauchy integral formula for an analytic function f in a disc.

8.1. INTRODUCTION

We have seen that a function is analytic on a closed curve c but $\int_c f \neq 0$. For example consider the function $f : \mathbb{C} - 0 \to \mathbb{C}$ defined as $f(z) = \frac{1}{z}$. In this example $\int_{|z|=1} f(z)dz = \int_{|z|=1} \frac{1}{z} dz = 2\pi i$. Whereas, the closed curve theorem states that if f is analytic throughout a disc, the integral around any closed curve is 0. We shall try to find the most general type of domain in which the closed curve theorem is valid. We should note that $f(z) = \frac{1}{z}$ is analytic in the punctured plane $\mathbb{C} - 0$. We shall see that the existence of a hole at $z = 0$ allows us to construct an example above, for which the integral is non-vanishing. The property of a domain, which assures that it has no holes is called simple connectedness. The formal definition is as follows.

Definition : A domain D is said to be simply connected if any simple closed curve which lies in D can be shrunk to a point without leaving domain D.

Definition :<u>Singular point</u> **:**
A point at which the function f is not analytic is said to be a singular point or singularity of the function f.

e.g. $f(x) = \dfrac{z^2}{z-3}$

Here, f is not defined at $z = 3$ and hence not analytic at $z = 3$, therefore $z = 3$ is singular point.

8.2 CAUCHY THEOREM FOR AN OPEN STAR SHAPED DOMAIN

Theorem : *Let G be star like w.r.t. point z_0 and suppose that f is analytic in G. Then there exists an analytic function F in G s.t. $F'(z) = f(z)$ in G.*

In particular, $\int_x f(z)\,dz = 0$*, for every closed, piecewise smooth curve x in G.* *(2008)*

Proof : Given that, G is a starlike w.r.t point z_0 and f analytic in G.

$\therefore$ By definition, $[z_0, z] \subset G \quad \forall z \in G$

Fix a point z in G and define

$$F(z) = \int_{[z_0, z]} f(\xi)\,d\xi = \int_{z_0}^{z} f(\xi)\,d\xi$$

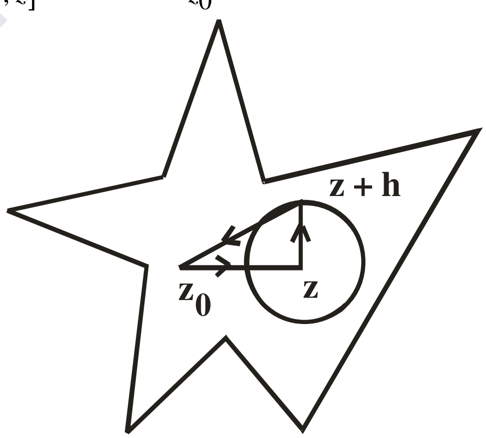


Fig 8.1

Choose $h \in \mathbb{C}$ with $|h| > 0$ s.t.

$\overline{B}(z;(h)) \subset G$ and $[z, z+h] \subset G$

Since *G* is starlike w.r.t. point z_0.

$\therefore$ The triangle $\Delta = [z_0, z_1, z+h]$ is contained in *G*.

$\therefore$ f is analytic inside and on the boundary of the triangle Δ.

$\therefore$ By Cauchy Goursat theorem,

$$\int_{\delta_\Delta} f(z)dz \left(= \int_{\delta_\Delta} (\xi)d\xi \right)$$

$$\int_{z_0}^{z} f(\xi)d\xi + \int_{z}^{z+h} f(\xi)d\xi + \int_{z+h}^{z_0} f(\xi)d\xi = 0$$

$$F(z) + \int_{z}^{z+h} f(\xi)d\xi - F(z+h) = 0$$

$$\therefore F(z+h) - F(z) = \int_{z}^{z+h} f(\xi)d\xi$$

$$\therefore F(z+h) - F(z) - h\,f(z) = \int_{z}^{z+h} f(\xi)d\xi - \int_{z}^{z+h} f(z)d\xi$$

$$\frac{F(z+h) - F(z)}{h} - f(z) = \frac{1}{h}\int_{z}^{z+h} \left[f(\xi) - f(z)\right]d\xi$$

$$\left| \frac{F(z+h) - E(z)}{h} - f(z) \right| \le \frac{1}{|h|}\int_{z}^{z+h} |f(\xi) - f(z)||d\xi| \qquad (1)$$

Given that, F is analytic in *G*.

$\therefore f$ is differentiable in *G*.

$\therefore f$ is continuous at a point $z \in G$.

For a given $\varepsilon > 0$, $\exists\ \delta > 0$ s.t. $|f(\xi) - f(z)| < \varepsilon$ whenever $|\xi - z| < \delta$

Choose $|\xi - z| = |h| < \varepsilon$

$\therefore$ From equation (1),

$$\left| \frac{F(z+h) - F(z)}{h} \right| < \frac{\varepsilon}{|h|}\int_{z}^{z+h} |d\xi| = \frac{\varepsilon}{|h|}.|h|$$

By definition $\lim_{h \to 0} \dfrac{F(z+h) - F(z)}{h} = f(z)$

$$F'(z) = f(z) \text{ in } B(z;|h|)$$

Since z is fixed but arbitrary.

$\therefore \quad F'(z) = f(z)$ in G. (2)

$\Rightarrow$ The derivative F' exists and is continuous at every point z in G.

$\therefore \quad F$ is analytic in G.

From equation (2),

F is primitive of f.

$\therefore$ By using the theorem

$\int_x f(z)\,dz = 0$ for every closed, piecewise smooth curve x in G.

In Cauchy's theorem, Cauchy assumed the continuity of derived function $f'(z)$. It was Goursat who first prove that this condition can removed from the hypothesis in the theorem. The revised form of the theorem is known as Cauchy-Goursat theorem.

Cauchy-Goursat Theorem : (Cauchy Triangular Theorem) :

Let f be analytic in an open set $G \subset D$. Let z_1, z_2, z_3 be points in G. Assume that the triangle Δ with vertices z_1, z_2, z_3 is continuous in G then $\int_x f(z)\,dz = 0$ where $\partial\Delta$ is the boundary of a triangle Δ.

(2007,2008, 2009)

Proof : Given that, the triangle Δ with vertices z_1, z_2, z_3 is continued in G. Let m_1, m_2, m_3 be mid points of line segment $[z_1, z_2], [z_2, z_3], [z_3, z_1]$ respectively.

Then we get 4 smallest triangles $\Delta^1, \Delta^2, \Delta^3, \Delta^4$.

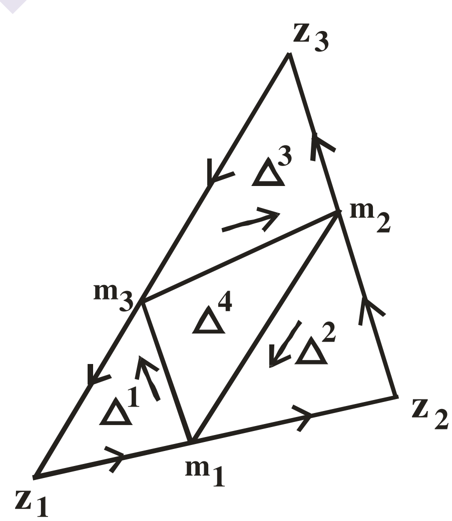


Fig 8.2

$$\therefore \int_{\partial\Delta} f(z)\,dz = \int_{z_1 m_1 z_2 m_2 z_3 m_3 z_1} f(z)\,dz$$

$$= \int_{m_3 z_1 m_1 m_3} f(z)\,dz + \int_{m_1 z_2 m_2 m_1} f(z)\,dz$$

$$+ \int_{m_2 z_3 m_3 m_2} f(z)\,dz + \int_{m_3 m_1} f(z)\,dz$$

$$+ \int_{m_1 m_2} f(z)\,dz + \int_{m_2 m_3} f(z)\,dz$$

$$= \int_{\partial\Delta^1} f(z)\,dz + \int_{\partial\Delta^2} f(z)\,dz + \int_{\partial\Delta^3} f(z)\,dz + \int_{\partial\Delta^4} f(z)\,dz$$

$$= \sum_{K=1}^{4} \int_{\partial\Delta^K} f(z)\,dz$$

Among this 4 triangles, there is one triangle, call it Δ, s.t.

$$\left| \int_{\partial\Delta_1} f(z)\,dz \right| \geq \left| \int_{\partial\Delta^K} f(z)\,dz \right| \qquad K = 1, 2, 3, 4$$

$$\therefore \quad \left| \int_{\partial\Delta} f(z)\,dz \right| \leq 4 \left| \int_{\partial\Delta_1} f(z)\,dz \right|$$

Let $L(\partial\Delta)$ be the perimeter of a triangle Δ, then

$L(\partial\Delta_1) = \frac{1}{2} L(\partial\Delta)$ and diam. $\Delta_1 = \frac{1}{2}$ diam. Δ (diameter of triangle means the length of its largest side)

Now, perform the same process on the triangle Δ_1 getting a triangle Δ_2 with analogus properties.

(i) $\Delta = \Delta_0 \supset \Delta_1 \supset \Delta_2$

(ii) $\left| \int_{\partial\Delta} f(z)\,dz \right| \leq 4^2 \left| \int_{\partial\Delta_2} f(z)\,dz \right|$

(iii) $L(\partial\Delta_2) = \frac{1}{2^2} L(\partial\Delta)$

(iv) $diam\,\Delta_2 = \frac{1}{2^2}\, diam\,\Delta$

Continue the process and at the n^{th} stage, we get

(i) $\Delta = \Delta_0 \supset \Delta_1 \supset \Delta_2 \supset \ldots \supset \Delta_n$

(ii) $\left| \int_{\partial\Delta} f(z)\,dz \right| \leq 4^n \left| \int_{\partial\Delta_n} f(z)\,dz \right|$

(iii) $L(\partial\Delta_n) = \frac{1}{2^n} L(\partial\Delta)$

(iv) $diam\,\Delta_n = \frac{1}{2^n} diam\,\Delta$

A metric space $\times$ is complete iff for any sequence $\{F_n\}$ of non-empty closed sets with $F_1 \supset F_2 \supset F_3 \supset ...$ and $diam\,F_n \to 0$, $\bigcap_{n=1}^{\infty} F_n$ consists of a single point.

Since $\mathbb{C}$ is complete and $\{\Delta_n\}$ is a sequence of non-empty closed sets with $\Delta_0 = \Delta \supset \Delta_1 \supset ...$

and $\lim_{n\to\infty} diam\,\Delta_n = \lim_{n\to\infty}\left[\left(\frac{1}{2}\right)^n diam\,\Delta_n\right]$

$= 0 \qquad \left\{\because \lim_{n\to\infty} x^n = 0 \text{ if } |x| < 1\right\}$

By using cantor's theorem,

$\bigcap_{n=0}^{\infty} \Delta_n$ consists of a single point say z_0.

In particular $z_0 \in \Delta G$.

Given that, f is analytic on G.

$\therefore f$ is differentiable at a point $z_0 \in G$ for every $\varepsilon > 0$, $\exists\ \delta > 0$ s.t.

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \varepsilon \text{ where } 0 < |z - z_0| < \varepsilon$$

By increment theorem,

$f(z) = f(z_0) + (z - z_0) f'(z_0) + (z - z_0)\eta(z)$ with $\eta(z)$ is continuous and $|\eta(z)| < \varepsilon$ for $|z - z_0| < \delta$.

$$\int_{\partial\Delta_n} f(z)\,dz = \int_{\partial\Delta_n} f(z_0)\,dz + \int_{\partial\Delta_n} (z - z_0) f'(z_0)\,dz + \int_{\partial\Delta_n} (z - z_0)\eta(z)\,dz$$

$= 0 + 0 + \int_{\partial\Delta_n} (z - z_0)\eta(z)\,dz \qquad (\because \int_x f(z) = 0$ if x is closed cure and here $\partial\Delta_n$ is closed.)

$$\left| \int_{\partial\Delta_n} f(z)\,dz \right| \leq \int_{\partial\Delta_n} |z - z_0|\,|\eta(z)|\,|dz|$$

Choose η s.t. $diam \Delta_n = \frac{1}{2^n} diam\, \Delta < \delta$.

$\because \quad z_0 \in \Delta_n \Rightarrow \Delta_n \subset B(z_0; \delta)$

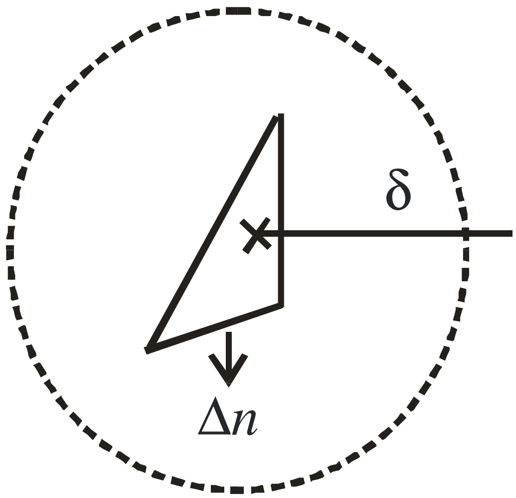


Fig 8.3

$\therefore \quad \left| \int_{\partial \Delta_n} f(z)\, dz \right| < \varepsilon\, diam\, \Delta_n \int_{\partial \Delta_n} |\, dz\, |$

($\because \; |\, \eta(2) \,| < \varepsilon$ and $|\, z - z_0 \,| < diam\, \Delta_n$)

$= \varepsilon\, diam\, \Delta_n\, L(\partial \Delta_n)$

($\because \; L(\partial \Delta_n) = \int_{\partial \Delta_n} |\, d_2 \,|$)

$= \varepsilon . \frac{1}{2^n} diam\, \Delta . \frac{1}{2} L(\partial \Delta)$

($\because \; diam\, \Delta_n = \frac{1}{2^n} diam\, \Delta$ and $L(\partial \Delta_n) = \frac{1}{2^n} L(\partial \Delta)$

$\therefore \quad 4^n \left| \int_{\partial \Delta n} f(z)\, dz \right| < \varepsilon\, diam\, \Delta . L(\partial \Delta)$

$\therefore \quad \left| \int_{\partial \Delta_n} f(z)\, dz \right| < \varepsilon\, diam\, \Delta . L(\partial \Delta)$ (by (ii),

$\because \left| \int_{\partial \Delta} f(z) dz \right| \le 4^n \left| \int_{\partial \Delta_n} f(z)\, dz \right|$)

Since $\varepsilon > 0$ is arbitrary.

$\therefore \int_{\partial \Delta} f(z)\, dz = 0$

Check your progress:

1) State and prove Cauchy Goursat Theorem for a closed quadrilateral.
2) State and prove Cauchy Goursat Theorem for a closed Rectangle.

Theorem : *Let G be an open set in $\mathbb{C}$. Let f be analytic in G except possibly at a point $z_0 \in G$. Assume that f is continuous in G and that the triangle Δ with vertex at z_0 is contained in G. Then $\int_{\partial\Delta} f(z)dz = 0$, where $\partial\Delta$ is a boundary of the triangle Δ.*

Proof : Let $\Delta = [z_0, z_1, z_2]$.

Let ξ_1 be a point on the line segment $[z_0, z_1]$ and ξ_2 be a point on the line segment $[z_0, z_2]$.

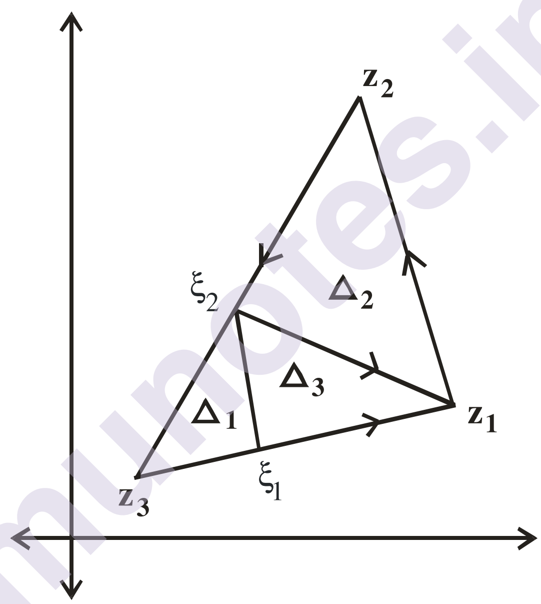


Fig 8.4

Consider the subtriangles,

$$\Delta_1 = [\xi_2, z_0, \xi_1],\ \Delta_2 = [z_1, z_2, \xi_2],\ \Delta_3 = [\xi_1, z_1, \xi_2]$$

$$\therefore \int_{\partial\Delta} f(z)\,dz = \int_{\xi_2 z_0 \xi_1 z_1 z_2 \xi_2} f(z)\,dz$$

$$= \int_{\xi_2 z_0 \xi_1 \xi_2} f(z)\,dz + \int_{\xi_1 z_1 \xi_2 \xi_1} f(z)\,dz + \int_{z_1 z_2 \xi_2 \xi_1} f(z)\,dz$$

$$= \int_{\partial\Delta_1} f(z)\,dz + \int_{\partial\Delta_2} f(z)\,dz + \int_{\partial\Delta_3} f(z)\,dz = \int_{\partial\Delta_1} f(z)\,dz + 0 + \ldots$$

(by Cauchy Goursat theorem)

$$\therefore \quad \left| \int_{\partial\Delta} f(z)\,dz \right| \le \int_{\partial\Delta_1} |f(z)||dz|$$

Put $M = \max_{z \in \partial\Delta} |f(z)|$

$$\therefore \quad \left| \int_{\partial\Delta} f(z)\,dz \right| \le M \int_{\partial\Delta_1} |dz| = ML(\partial\Delta_1)$$

As ξ_1 and ξ_2 tend to z_0, perimeter of the triangle Δ_1 tends to zero i.e. $L(\partial\Delta_1) \to 0$.

$$\therefore \quad \int_{\partial\Delta} f(z)\,dz = 0$$

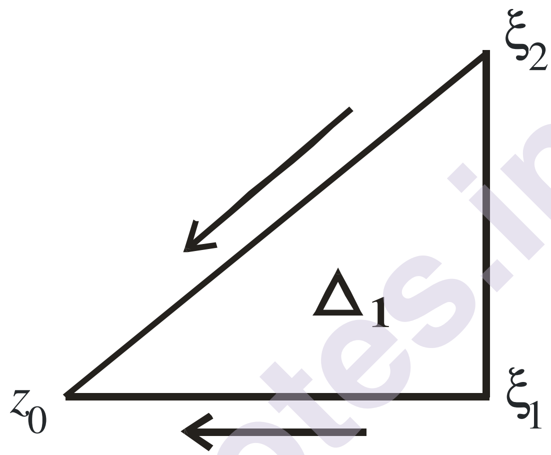


Fig 8.5

Theorem : *Let G be an open set in* $\mathbb{C}$. *Let* f *be analytic* $G/\{\alpha\}$ *for some* $\alpha \in G$. *If* f *is continuous on G, then* $\int_{\partial\Delta} f(z)\,dz = 0$, *where* $\partial\Delta$ *is a boundary of the triangle* Δ *contained in G.*

Star Shaped Domains :

Definition : A set G in $\mathbb{C}$ is said to be convex if, given any two points z and w in G, the line segment $[z, w]$ lies entirely in G.

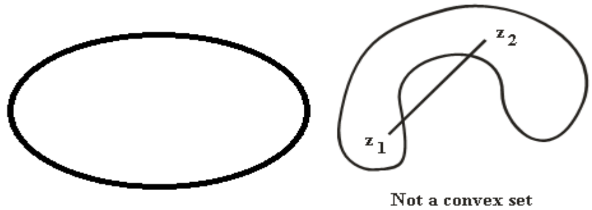
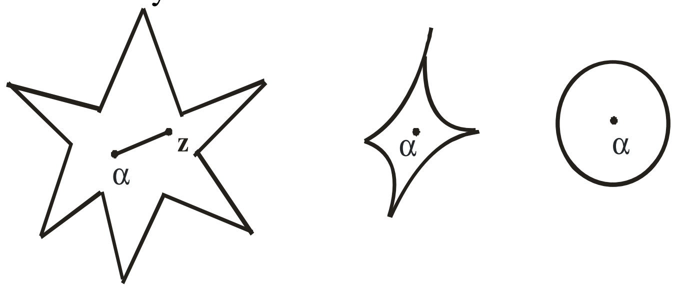


Fig 8.6

Definition : A set G is said to be starlike (or star shaped) w.r.t. points $\alpha \in G$ if for any point $z(\neq \alpha)$ in G, the line segment $[\alpha, z]$ lies entirely in G.



The above set is starlike w.r.t. α

Open sets, closed, half planes if $\mathrm{Re}\, z > 0, \mathrm{Im}\, z > 0$ etc. are

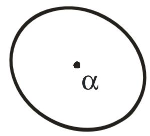


Fig 8.7

Punctured disk is not starlike w.r.t . α

Note : Every starlike set is not convex but every convex set is starlike.

Question : If f is analytic in a simply connected domain D, then $\int_{z_1}^{z_2} f(z)\, dz$ is independent of path in D, joining any two points z_1 and z_2 in D.

Solution : Let x_1 and $x_2 : [a, b] \to G$ be two smooth paths in G such that $x_1(a) = x_2(a) = z_1$ and $x_1(b) = x_2(b) = z_2$.

$x_1(t_1) \neq x_2(t_2), \quad t_1, t_2(-a, b)$

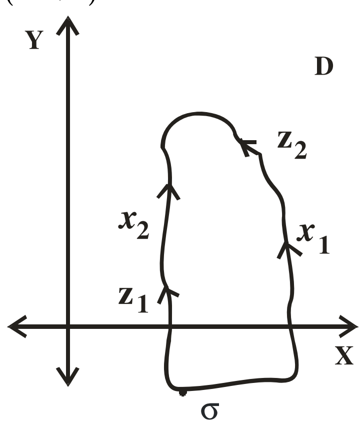


Fig 8.8

Form a simple closed curve σ which moves from z_1 to z_2 via x_1 and z_2 to z_1 via $-x_2$.

$\therefore$ f is analytic inside and on a simple closed curve σ.

By Cauchy Goursat theorem,

$$\int_{\sigma} f(z)\,dz = 0$$

$$\therefore \quad \int_{x_1} f(z)\,dz + \int_{-x_2} f(z)\,dz = 0 \quad \therefore$$

$$\int_{x_1} f(z)\,dz = -\int_{-x_2} f(z)\,dz$$

$$\therefore \quad \int_{x_1} f(z)\,dz = \int_{x_2} f(z)\,dz = \int_{z_1}^{z_2} f(z)\,dz$$

$\int_{z_1}^{z_2} f(z)\,dz$ is independent of path.

Cauchy Deformation Theorem :

Statement : *If f is analytic in c domain bounded by two simple closed curves x_1 and x_2 (where x_2 is inside x_1) and on these curves, then $\int_{x_1} f(z)\,dz = \int_{x_2} f(z)\,dz$ where x_1 and x_2 are both traversed in anticlockwise direction.*

Proof : Join two curves x_1 and x_2 by lines AB and CD. Denote, $x_1\ell$ = lower section of x_1 from A to D.

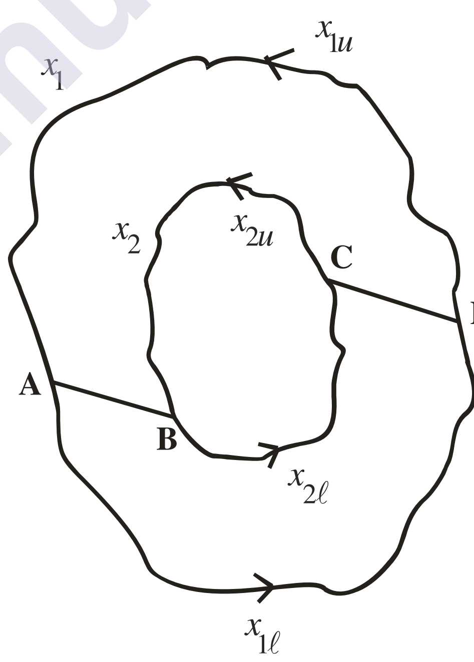


Fig 8.9

$x_1 u$ = the upper section of x_1 from *D* to *A*.
$x_2 \ell$ = the lower section of x_2 from *B* to *C*.
$x_2 u$ = the upper section of x_2 from *C* to *B*.

Form a simple closed curve σ_1 by transversing from *A* to *B* then from *B* to *C* by $x_2 u$, then from *C* to *D* and finally from *C* to *D* and finally from *D* back to *A* by $x_1 u$.

$\therefore$ f is analytic inside and on the simple closed curve σ_1.

$\therefore$ By Cauchy- Goursat theorem.

$$\int_{\sigma_1} f(z)\,dz = 0$$

$$\therefore \int_{AB} f(z)\,dz + \int_{-x_2 u} f(z)\,dz + \int_{CD} f(z)\,dz + \int_{x_1 u} f(z)\,dz = 0 \qquad (1)$$

From a simple closed curve σ_2 by transversing from *A* to *D* by $x_1 \ell$ then from *D* to *C*, then from *C* to *B* by $-x_2 \ell$ and finally from *B* back to *A*.

$\therefore$ f is analytic inside and on the simple closed curve σ_2.

$\therefore$ By Cauchy - Goursat theorem,

$$\int_{\sigma_2} f(z)\,dz = 0$$

$$\therefore \int_{x_1 \ell} f(z)\,dz + \int_{DC} f(z)\,dz + \int_{-x_2 \ell} f(z)\,dz + \int_{BA} f(z)\,dz = 0 \qquad (2)$$

Adding equations (1) and (2), we get

$$\int_{x_1 u} f(z)\,dz + \int_{x_1 \ell} f(z)\,dz - \left[\int_{x_2 \ell} f(z)\,dz + \int_{x_2 u} f(z)\,dz\right] = 0$$

$$\left\{\because \int_{x} f(z)\,dz = -\int_{-x} f(z)\,dz\right\}$$

$$\int_{x_1} f(z)\,dz - \int_{x_2} f(z)\,dz = 0$$

$$\therefore \int_{x_1} f(z)\,dz = \int_{x_2} f(z)\,dz$$

Generalization of Cauchy Deformation Theorem :

Statement : *If* f *is analytic in a domain bounded by non-intersecting simple closed curves* $x, x_1, ..., x_n$ *where* $x, x_1, ..., x_n$ *are inside* x *and on this curves, then*

$$\int_{x} f(z)\,dz = \int_{x_1} f(z)\,dz + \int_{x_2} f(z)\,dz + \ldots + \int_{x_n} f(z)\,dz$$ *where,* $x, x_1, x_2, \ldots, x_n$ *are traversed in anticlockwise direction.*

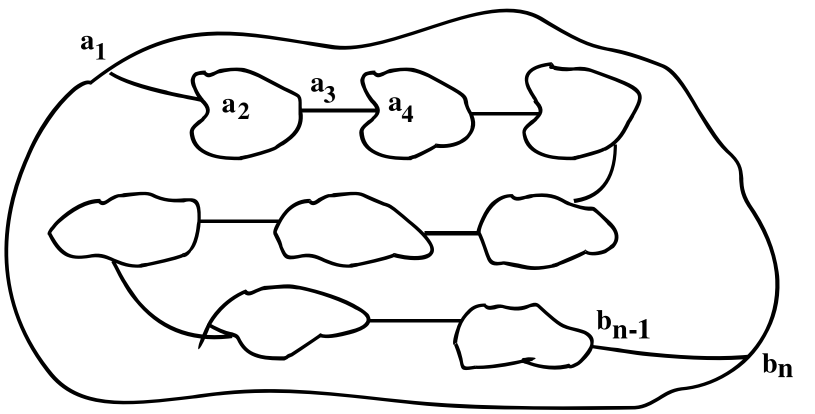


Fig 8.10

8.3 CAUCHY INTEGRAL FORMULA

Statement : Let f be analytic in a simply connected domain *G*, $G \subset D$. If x is a simple closed curve in *G* and be any point inside *x* then,

$$\int_{x} \frac{f(z)}{z-\alpha}\,dz = 2\pi i\left[f(z)\right]_{z=\alpha} = 2\pi i\, f(\alpha)$$ where, x is traversed in anticlockwise direction.

Proof : Given that, f is analytic in a simply connected domain *G*. construct a circle Γ with centre at α and radius $r > 0$ so that Γ lies entirely inside x.

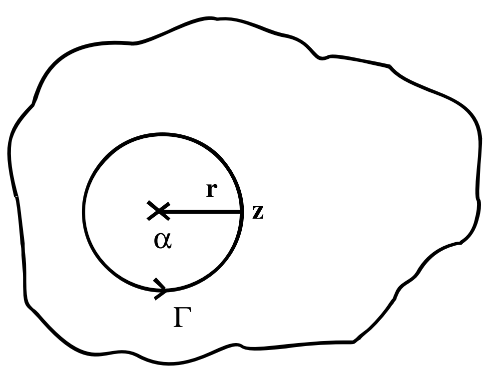


Fig 8.11

$\therefore$ The function $\dfrac{f(z)}{z-\alpha}$ is analytic in a domain which is bounded by two simple closed curves x and Γ and on these curves.

$\therefore$ By Cauchy deformation theorem,

$$\int_x \frac{f(z)}{z-\alpha}dz = \int_\Gamma \frac{f(z)}{z-\alpha}dz = \int_\Gamma \frac{f(z)-f(\alpha)+f(\alpha)}{z-\alpha}dz$$

$$\int_x \frac{f(z)}{z-\alpha}dz = \int_\Gamma \frac{f(z)-f(\alpha)}{z-\alpha}dz + f(\alpha)\int_\Gamma \frac{dz}{z-\alpha} \qquad (1)$$

Consider, $\displaystyle\int_\Gamma \frac{dz}{z-\alpha}$

$\therefore$ Equation of the circle Γ is, $|z-\alpha| = r$

or $z = \alpha + er^{it}$, $t \in [0, 2\pi]$

$dz = i\,re^{it}dt$

$|z-\alpha| = r$

$z = r + re^{i\theta}$, $\theta \in [0, 2\pi]$

$= \alpha + re^{it}$, $t \in [0, 2\pi]$

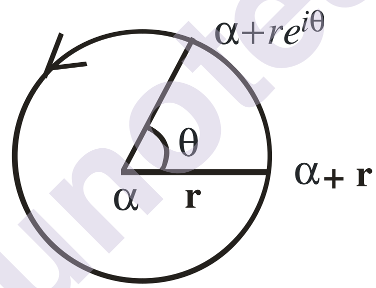


Fig 8.12

$$\therefore \quad \int_\Gamma \frac{dz}{z-\alpha} = \int_0^{2\pi} \frac{ire^{it}}{re^{it}}dt = i\int_0^{2\pi} dt = 2\pi i$$

$\therefore$ Equation (1), becomes,

$$\int_x \frac{f(z)}{z-\alpha}dz = \int_\Gamma \frac{f(z)-f(\alpha)}{z-\alpha}dz + 2\pi i\, f(\alpha)$$

$$\frac{1}{2\pi i}\int_x \frac{f(z)}{z-\alpha}dz - f(\alpha) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)-f(\alpha)}{z-\alpha}dz$$

$$\therefore \quad \left|\frac{1}{2\pi i}\int_x \frac{f(z)}{z-\alpha}dz - f(\alpha)\right| \le \frac{1}{2\pi}\int_\Gamma \frac{|f(z)-f(\alpha)|}{|z-\alpha|}|dz| \qquad (2)$$

$(\because |2\pi i| = |2\pi||i| = 2\pi \quad \because |i| = 1)$

Given that, f is analytic in *G*.

$\therefore$ f is differentiable in *G*.

$\therefore$ f is continuous in *G*.

$\therefore$ f is continuous at a point $\alpha \in G$.

$\therefore$ For a given $\varepsilon > 0$, $\exists\ \delta > 0$ such that,

$$\sqrt{|z-\alpha|} < \delta \quad \Rightarrow \quad |f(z) - f(\alpha)| < \varepsilon$$

Choose, r s.t. $|r| < \delta$

$\therefore$ From equation (2)

$$\left| \frac{1}{2\pi i} \int_x \frac{f(z)}{z-\alpha}\, dz - f(\alpha) \right| < \frac{\varepsilon}{2\pi} \int_\Gamma \frac{|dz|}{r} \qquad \{\because z - \alpha = \gamma\}$$

$$= \frac{\varepsilon}{2\pi r} \int_\Gamma |dz| = \frac{\varepsilon}{2\pi r} . L(T) = \frac{\varepsilon}{2\pi r} . 2\pi r$$

$\therefore$ $$\left| \frac{1}{2\pi i} \int_x \frac{f(z)}{z-\alpha} dz - f(\alpha) \right| < \varepsilon$$

$\because$ ε is arbitrary.

$\because$ $$\int_x \frac{f(z)}{z-\alpha} = 2\pi i\, f(\alpha)$$

Theorem : *Let f be analytic in a simply connected domain G. $G \subset \mathbb{C}$. If x is closed rectifiable curve in G and $\alpha \in G / \{x\}$, then* $\frac{1}{2\pi i} \int_x \frac{f(z)}{z-\alpha} dz = f(\alpha)\, \eta(x; \alpha)$.

Proof : Given that, f is analytic in a simply connected domain G. Define $F(z)$ as follows,

$$F(z) = \begin{cases} \dfrac{f(z) - f(\alpha)}{z-\alpha}, & z \neq \alpha \\ f'(x), & z = \alpha \end{cases} \qquad (1)$$

$F(z) = \dfrac{f(z) - f(\alpha)}{z-\alpha}$ is analytic in $G / \{\alpha\}$.

($\because$ $f(z) - f(\alpha)$ and $\dfrac{1}{z-\alpha}$ are analytic in $G / \{\alpha\}$)

$\therefore$ $$\lim_{z\to\alpha} f(z) = \lim_{z\to\alpha} \frac{f(z) - f(\alpha)}{z-\alpha} = f^{'}(\alpha) = F(\alpha) \qquad \text{by (1)}$$

$\therefore$ *F* is continuous at a point $\alpha \in G$ and hence *F* is continuous in *G*.

$\therefore$ *F* is analytic in *G*.

$\therefore$ By Cauchy- Goursat theorem $\int_x F(z)\,dz = 0$

$$\int_x \frac{f(z)-f(\alpha)}{z-\alpha}\,dz = 0$$

$$\int_x \frac{f(z)}{z-\alpha}\,dz = f(\alpha)\int_x \frac{dz}{z-\alpha}$$

$$\int_x \frac{f(z)}{z-\alpha}\,dz = f(\alpha).2\pi\, in(x,\alpha)\ldots\left(\eta(x;\alpha)\right) = \frac{1}{2\pi i}$$

$$\frac{1}{2\pi i}\int_x \frac{f(z)}{z-\alpha}\,dz = f(\alpha).\eta(x;\alpha)$$

Example 1 : Use Cauchy integral formula, to evaluate $\int_x \frac{z^2+2}{z-1}\,dz$ where x is a circle $|z| = 2$.

Solution : Given that, $\int_x \frac{z^2+2}{z-1}\,dz$

$$F(z) = \frac{f(z)}{z-1} = \frac{z^2+2}{z-1}$$

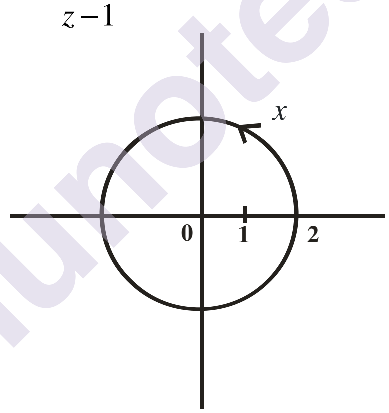


Fig 8.13

$\therefore$ F has singular point at $z = 1$.

Given equation of circle is $|z| = 2$.

$\therefore$ Centre is origin and radius is 2.

$\therefore$ The singular point $z = 1$, lies inside the circle.

$\therefore$ We use Cauchy integral formula

$$\int_x \frac{f(z)}{z-\alpha}\,dz = 2\pi i\left[f(z)\right]_{z=\alpha}$$

$$\int_x \frac{z^2+2}{z-1}\,dz = 2\pi i\left[z^2+2\right]_{z=1} = 2\pi i[1+2] = 2\pi i(3) = 6\pi i\ .$$

2) Evaluate $\int_x \cot z\, dz$ where x is a circle $|z| = \frac{1}{2}$.

Solution: $\int_x \cot z\, dz = \int_x \frac{\cos z}{\sin z} dz$

For finding singular point put $\sin z = 0$

$\Rightarrow \quad z = \sin^{-1}(0) = n\pi$

Singular point $z=0$, lies inside the circle $|z| = \frac{1}{2}$.

Hence by Cauchy's integral formula $\int_x \frac{\cos z}{\sin z} dz = 2\pi i(\cos 0) = 2\pi i$

3) Evaluate $\int_c \frac{\sin(z\pi)}{z - \frac{1}{2}}$ where C is the unit circle oriented clockwise.

Solution: Let I= $\int_c \frac{\sin(z\pi)}{z - \frac{1}{2}}$

$z = \frac{1}{2}$ is a point of singularity and lies inside $|z| = 1$

$\therefore z_0 = \frac{1}{2}, \quad f(z) = \sin(z\pi)$

$\therefore I = 2\pi i f(z_0) = 2\pi i \sin\left(\frac{\pi}{2}\right)$

Theorem : *Let f be analytic in a simply connected domain G, $G \subset \mathbb{C}$ and suppose x is a simple closed curve in G. If α is any point inside x then,*

$$f'(\alpha) = \frac{1}{2\pi i}\int_x \frac{f(z)}{(z-\alpha)^2} dz$$, *where x is traversed in anticlockwise direction.*

(**Note:** To prove the theorem we will need the following theorem Boundedness Theorem :Let f be continuous on a compact set S. then, f is bounded on S i.e. there exists a number M s.t. $|f(z)| \le M \quad \forall\, z \in S$)

Proof : Given that, x is a simple closed curve in G and α is any point inside x.
By Cauchy-integral formula,

$$f(\alpha) = \frac{1}{2\pi i}\int \frac{f(z)}{z-\alpha} dz \text{ and } f(\alpha+h) = \frac{1}{2\pi i}\int_x \frac{f(z)}{z-(\alpha+h)} dz$$

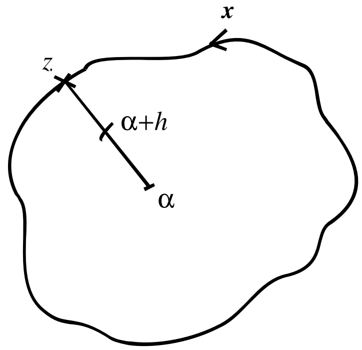


Fig 8.14

$$\therefore \quad f(\alpha+h)-f(\alpha)=\frac{1}{2\pi i}\left[\int_x \frac{f(z)}{z-\alpha}\,dz-\int_x \frac{f(z)}{z-(\alpha+h)}\,dz\right]$$

$$=\frac{1}{2\pi i}\left[\int_x \frac{\left[z-(\alpha+h)\right]f(z)-(z-\alpha)f(z)}{(z-\alpha)\left[z-(\alpha+h)\right]}\right]$$

$$=\frac{1}{2\pi i}\left[\int_x \frac{hf(z)}{\left[z-(\alpha+h)\right](z-\alpha)}\,dz\right]$$

$$\frac{f(\alpha+h)-f(\alpha)}{h}=\frac{1}{2\pi i}\int_x \frac{f(z)}{\left[z-(\alpha+h)\right](z-\alpha)}\,dz$$

$$=\frac{1}{2\pi i}\left[\int_x \frac{(z-\alpha)f(z)-\left[z-(\alpha+h)\right]f(z)}{\left[z-(\alpha+h)\right](z-\alpha)^2}\,dz\right]$$

$$=\frac{1}{2\pi i}\int_x \frac{h\,f(z)}{\left[z-(\alpha+h)\right](z-\alpha)^2}\,dz$$

$$=\frac{h}{2\pi i}\int_x \frac{f(z)}{\left[z-(\alpha+h)\right](z-\alpha)^2}\,dz$$

$$\left|\frac{f(\alpha+h)-f(\alpha)}{h}-\frac{1}{2\pi i}\int_x \frac{f(z)}{(z-\alpha)^2}\,dz\right|=\frac{|h|}{2\pi}\int_x \frac{|f(z)||dz|}{|z-(\alpha+h)|(z-\alpha)^2} \quad (1)$$

Theorem will be proved if L.H.S. of equation (1) tends to zero as $h \to 0$. Choose $r=\inf\{|z-\alpha|: z \in x\} \Rightarrow r \le |z-\alpha|$.

Choose $|\alpha-(\alpha+h)|=|h|<\frac{r}{2}$

$$\therefore \quad |z-(\alpha+h)| \ge |z-\alpha|-|h|$$

$\{\because |a-b| \le |a|-|b|\}$

$$> r-\frac{r}{2}=\frac{r}{2}$$

$\because$ f is analytic in G.

$\therefore$ f is continuous on $\{x\}$ $\left\{\because x^2 \text{ is compact set}\right\}$

$\therefore$ By boundedness theorem, $\exists\ M > 0$ s.t.
$|f(z)| \le M.$ $\forall z \in \{x\}$

$\therefore$ From equation (1),

$$\left|\frac{f(\alpha+h)-f(\alpha)}{h} - \frac{1}{2\pi i}\int_x \frac{f(z)}{(z-\alpha)^2}\,dz\right| \le \frac{|h|}{2\pi}\int_x \frac{M.|dz|}{r/2 . r^2}$$

$$= \frac{|h|.M}{\pi r^3}\int_x |dz| = \frac{|h|.M}{\pi r^3} L(x)$$

L.H.S. tends to zero as $h \to 0$.

$$\therefore \lim_{h\to 0}\frac{f(\alpha+h)-f(\alpha)}{h} = \frac{1}{2\pi i}\int_x \frac{f(z)}{(z-\alpha)^2}\,dz$$

$$\therefore f'(x) = \frac{1}{2\pi i}\int_x \frac{f(z)}{(z-\alpha)^2}\,dz$$

Generalization of the above theorem :

Theorem : *Let f be analytic in a simply connected domain G, $G \subset \mathbb{C}$ and suppose x is a simple closed curve in G. If α is any point inside x, then*

$$f^n(\alpha) = \frac{n!}{2\pi i}\int_x \frac{f(z)}{(z-\alpha)^{n+1}}\,dz$$

where $n = 0, 1, 2, \ldots$ x is traversed in anticlockwise direction.

Prove this theorem by induction on *n*.

Note : If a function f is analytic at a point $\alpha \in G$, then its derivatives of all orders are also analytic at a point α.

Example 1 :

Use Cauchy integral formula or theorem, to evaluate $\int_x \frac{\sin \pi z + \cos \pi z}{(z-1)(z-2)}\,dz$ where *x* is a circle $|z| = \frac{3}{2}$.

Solution : Given integral $\int_x \frac{\sin \pi z + \cos \pi z}{(z-1)(z-2)}\,dz$

By partial fraction,

$$\int_x \frac{\sin \pi z + \cos \pi z}{(z-1)(z-2)} dz = \int_x \frac{\sin \pi z + \cos \pi z}{z-1} dz + \int_x \frac{\sin \pi z + \cos \pi z}{z-2} .$$

Here, $F(z) = F(z) = \dfrac{f(z)}{(z-1)(z-2)}$ has singular point at $z=1$ and $z=2$.

(**Note :** If the singular points lies inside the circle then we use Cauchy integral formula and it lies outside the circle then we use Cauchy integral theorem.)

Given equation of circle, $|z| = 3/2$ i.e. $x^2 + y^2 = \dfrac{9}{4}$

∴ Singular point $z=1$ lies inside the circle and $z=2$ lies outside the circle.

∴ For $z=1$, we use Cauchy integral formula

$$\int_x \frac{f(z)}{z-\alpha} dz = 2\pi i \left[f(z) \right]_{z=\alpha}$$

$$\int_x \frac{\sin \pi z + \cos \pi z}{z-1} dz = 2\pi i \left[\sin \pi z + \cos \pi z \right]_{z=1}$$

$= 2\pi i [\sin \pi + \cos \pi] = 2\pi i [0-1]$

$= -2\pi i$ ....(a)

∴ For $z=2$, we use Cauchy integral formula theorem

∴ $\int_x F(z)\, dz = 0$

∴ $\displaystyle\int_x \frac{\sin \pi z + \cos \pi z}{z-2} = 0$ (b)

∴ Substituting (a) and (b) in equation (1), we get

$$\int_x \frac{\sin \pi z + \cos \pi z}{(z-1)(z-2)} dz = 0 - 2\pi i = -2\pi i$$

2) $\displaystyle\int_x \frac{e^{2z}}{\left(z - \frac{1}{2}\right)^3} dz$ where x is the rectangle with vertices at $\pm i$ and $1 \pm i$.

Solution : Here, $F(z) = \dfrac{f(z)}{\left(z-\frac{1}{2}\right)^3}$ has singular point at $z = 1/2$ of order 3.

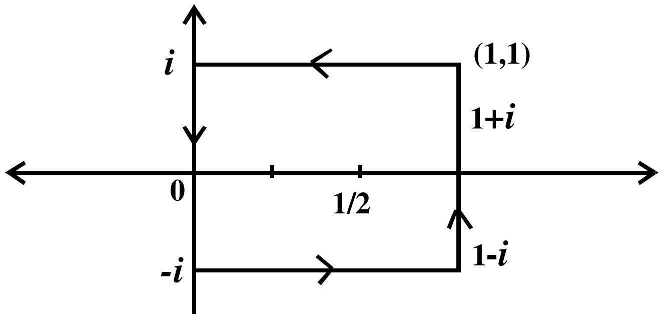


Fig 8.15

By Generalization of derivative of an analytic function.

$$f^{n}(\alpha)=\frac{n!}{2\pi i}\int_{x}\frac{f(z)}{(z-\alpha)^{n+1}}\,dz$$

$$\therefore \int_{x}\frac{f(z)}{(z-\alpha)^{n+1}}\,dz=\frac{2\pi i}{n!}f^{n}(\alpha)$$

$$\therefore \int_{x}\frac{e^{z}}{\left(z-\frac{1}{z}\right)^{3}}\,dz=\frac{\not{2}\pi i}{\not{2}!}f^{2}\left(\frac{1}{2}\right)=\pi i\times\left[\frac{d^{2}}{dz^{2}}\left(e^{2}z\right)\right]_{z=\frac{1}{2}}$$

$$=\pi i\left[4e^{2z}\right]_{z=1/2}=\pi i\left[4e^{2\times\frac{1}{2}}\right]\quad=4\pi ie$$

Exercise : Use Cauchy integral formula to evaluate,

i) $\int_{x}\frac{\cos\pi z}{z^{2}-1}$ where x is a rectangle with vertices at $2\pm i$ and $-2\pm i$.

ii) $\int_{x}\tan z\,dz$ where x is a circle $\left|z-\frac{\pi}{2}\right|=\frac{\pi}{2}$.

iii) $\int_{x}\frac{e^{z}-z}{(z-2)^{3}}\,dz$ where x is a circle $|z|=3$.

iv) $\int_{x}\frac{z^{2}+1}{z-\frac{1}{2}-i}\,dz$ where x is a circle $|z-3|=1$.

Singular point is $P=\frac{1}{2}+i$ and here centre $c=(3,0)$

$$d(c,p)=d\left(3,\frac{1}{2}+i\right)=\sqrt{\left(3-\frac{1}{2}\right)^{2}+(0,-1)^{2}}=\sqrt{\frac{2p}{4}>1}.$$

$\therefore$ Singular point p lies outside the circle.

v) $\int\limits_{x} \frac{z^2+2z+3}{z-i}$ where x is a circle $|z-i|=2$.

Cauchy estimate or Cauchy inequality :

Theorem: *If f is analytic in an open disk $B(\alpha; R)$ and*

$|f(z)| \leq M \quad \forall\ z \in B(\alpha, R)$

then $\left|f^n(\alpha)\right| \leq \frac{n!M}{R^n} \qquad n = 0, 1, 2, \ldots$

Proof : For $0 < r < R$,

Construct a circle x with centre at α and radius γ.

By generalization of the theorem on Derivative of an analytic function.

$$f^n(\alpha) = \frac{n!}{2\pi i} \int\limits_{x} \frac{f(z)}{(z-\alpha)^{n+1}} dz \qquad n = 0, 1, 2, \ldots$$

$$\therefore \quad \left|f^n(\alpha)\right| \leq \frac{n!}{2\pi} \int\limits_{x} \frac{|f(z)|}{|z-\alpha|^{n+1}} |dz| \qquad (1)$$

Given that,

$$|f(z)| \leq M \qquad \forall\ z \in x$$

For any point z on x, we have $|z-\alpha| = r$

$\therefore$ From equation (1),

$$\left|f^{(n)}(\alpha)\right| \leq \frac{n!}{2\pi} \int\limits_{x} \frac{M}{r^{n+1}} |dz| = \frac{n!}{2\pi} \frac{M}{r^{n+1}} \int\limits_{x} |dz| = \frac{n!}{2\pi} \cdot \frac{M}{r^{n+1}} \times 2\pi r$$

$\therefore \left|f^{(n)}(\alpha)\right| \leq \frac{n!M}{r^n}$

$\because$ $r < R$ is arbitrary.

$\therefore$ As $r \to R$, we have $\left|f^{(n)}(\alpha)\right| \leq \frac{n!M}{R^n} \qquad n = 0, 1, 2, \ldots$

Cauchy Integral formula for Multiply connected domains:

Theorem: *It f is analytic in a domain which is bounded by two simple closed curves x_1 and x_2 (where x_2 lies inside x_1) and on these curves and if z_0 is any point in G. Then*

$$f(z_0) = \frac{1}{2\pi i} \int\limits_{x_1} \frac{f(z)}{z-z_0} dz - \frac{1}{2\pi i} \int\limits_{x_2} \frac{f(z)}{z-z_0} dz$$

Where x_1 and x_2 are transverse in anticlockwise direction

Proof: Construct a circle x with centre at z_0 and radius r, so that x lies inside x_1

$\therefore$ The function $\frac{f(z)}{z-z_0}$ is analytic in a domain which is bounded by non-intersecting simple closed curves x_1, x, x_2 where (x_1 *and* x_2 lies inside x_1) and on these curves

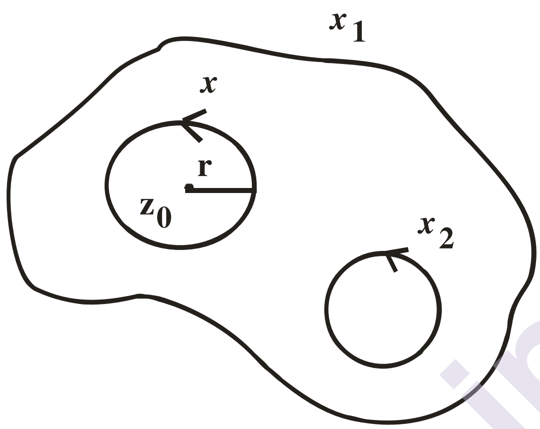


Fig 8.16

By using Cauchy determination theorem,

$$\int_{x_1} \frac{f(z)}{z-z_0} dz = \int_{x_1} \frac{f(z)}{z-z_0} dz + \int_{x_2} \frac{f(z)}{z-z_0} dz \qquad \text{(I)}$$

By using Cauchy integer formula, $\int_{x} \frac{f(z)}{z-z_0} dz = 2\pi i\, f(z_0)$

Put this value in equation (I),

$$\int_{x_1} \frac{f(z)}{z-z_0} dz = 2\pi i\, f(z_0) + \int_{x_2} \frac{f(z)}{z-z_0} dz$$

$$\therefore f(z_0) = \frac{1}{2\pi i} \int_{x_1} \frac{f(z)}{z-z_0} dz - \frac{1}{2\pi i} \int_{x_2} \frac{f(z)}{z-z_0} dz$$

8.4 SUMMARY

1) Let G be starlike w.r.t. point z_0 and suppose that f is analytic in G. Then there exists an analytic function F in G s.t. $F'(z) = f(z)$ in G.

In particular, $\int_{x} f(z)\, dz = 0$, for every closed, piecewise smooth curve x in G.

2) Cauchy-Goursat Theorem : (Cauchy Triangular Theorem) :

Let f be analytic in an open set $G \subset D$. Let z_1, z_2, z_3 be points in G. Assume that the triangle Δ with vertices z_1, z_2, z_3 is continuous in G then $\int_x f(z)\,dz = 0$, where $\partial\Delta$ is the boundary of a triangle Δ.

3) Cauchy Deformation Theorem :

Statement : If f is analytic in c domain bounded by two simple closed curves x_1 and x_2 (where x_2 is inside x_1) and on these curves, then $\int_{x_1} f(z)\,dz = \int_{x_2} f(z)\,dz$ where x_1 and x_2 are both transverse in anticlockwise direction.

4) Statement : Let f be analytic in a simply connected domain G, $G \subset D$. If x is a simple closed curve in G and be any point inside x then, $\int_x \frac{f(z)}{z-\alpha}\,dz = 2\pi i\left[f(z)\right]_{z=\alpha} = 2\pi i\, f(\alpha)$

where, x is traversed in anticlockwise direction.

5) Theorem : Let f be analytic in a simply connected domain G, $G \subset \mathbb{C}$ and suppose x is a simple closed curve in G. If α is any point inside x then, $f'(\alpha) = \frac{1}{2\pi i}\int_x \frac{f(z)}{(z-\alpha)^2}\,dz$, where x is traversed in anticlockwise direction.

6) Generalization of the above theorem :

Theorem : Let f be analytic in a simply connected domain G, $G \subset \mathbb{C}$ and suppose x is a simple closed curve in G. If α is any point inside x, then

$$f^n(\alpha) = \frac{n!}{2\pi i}\int_x \frac{f(z)}{(z-\alpha)^{n+1}}\,dz$$,where $n = 0, 1, 2, \ldots$ x is traversed in anticlockwise direction.

8.5 UNIT END EXCERCISES

1) Suppose $f : G \to \mathbb{C}$ be an analytic function, define $\varnothing : G \times G \to \mathbb{C}$ by

$$\varnothing(z, w) = \frac{f(z) - f(w)}{z - w} \text{ if } z \neq w$$
$$= f'(z) \quad \text{if } z = w.$$

Prove that $\varnothing$ is a continuous and for each fixed *w,* $z \to \varnothing(z,w)$ is analytic function of z.

(Hint: Take $z = z_0 + h, w = w_0 + k$ and $z_0 + h \neq w_0 + k$ for any h,k. Consider $\lim_{(h,k)\to(0,0)} \varnothing(z_0 + h, w_0 + k) = \varnothing(z_0, w_0)$, similarly $\lim_{(h,k)\to(0,0)} \varnothing(z_0 + h, w_0 + k) = \varnothing(z_0, w_0)$ for $z_0 = w_0$ and $h = k$ or $h \neq k$.

2) Let δ be a closed rectifiable curve in $\mathbb{C}$ and $a \notin \{\gamma\}$. Then show that for $n \geq 2$,
$\int_\gamma (z-a)^{-n}\, dz = 0$.

Solution: Use the lemma that, if γ is a rectifiable curve and ϕ is a function defined and continuous on $\{\gamma\}$.

For each $m \geq 1$, $F_m(z) = \int_\gamma \varnothing(w)(w-z)^{-m}\, dw$ for $z \notin \{\gamma\}$. Then each F_m is analytic on
$\mathbb{C} - \{\gamma\}$ and $F'_m = mF_{m+1}$. Take $m = 1, \varnothing = 1$ on $\{\gamma\}$. Then $F_1(z) = \int_\gamma (w-z)^{-1} dw$.

$\therefore F_1'(z) = F_2(z)$ on $\mathbb{C} - \{\gamma\}$. Here $a \in \mathbb{C} - \{\gamma\}$.

$\therefore F_1'(a) = F_2(a)$ $\because$ $F_2(a) = 0$, since $F_1(a)$ is constant number independent of a.

$\therefore \int_\gamma (z-a)^{-2} dz = 0$ Inductively $\int_\gamma (z-a)^{-n} dz = 0$ for $n \geq 2$.

3) Let f be analytic on $D = B(0,;1)$. Suppose $|f(z)| \leq 1$ for $|z| \leq 1$. Then show that
$|f'(0)| \leq 1$

(**Hint:** f is analytic in a simply connected set $B(0,;1)$.

Let $\gamma = boundary\left(B\left(0;\frac{1}{2}\right)\right)$, then γ is a simple closed curve and α=0 is a point inside γ.

Consider $f'(0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z^1} dz$

$$\therefore \ |f'(0)| = \left| \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z^2} dz \right| \le \frac{1}{2\pi} \int_\gamma \left| \frac{f(z)}{z^2} \right| \le \frac{1}{2\pi} .2\pi = 1 .$$

4) Let $\gamma(t) = 1 + e^{it}$ for $0 \le t \le 2\pi$ .Find $\int_\gamma \left(\frac{z}{z-1} \right)^n dz$ for all positive integers n .

(**Hint :** Put $\frac{f(z)}{z^2} f(z) = z^n$. Then $\int_\gamma \left(\frac{z}{z-1} \right)^n dz = \int_\gamma \left(f(z)/(z-1)^n \right) dz$. Where $\gamma(t) = (1+\cos(t), \sin(t))$ for $0 \le t \le 2\pi$. Apply the following Cauchy- Integral formula

$$f^n(a)\eta(\gamma;a) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^{n+1}} dz , \quad \text{for } n \ge 1 .$$

5) Use Cauchy- Integral formula to evaluate

(i) $\int_\gamma \frac{\cos(nz)}{z^2 - 1} dz$, γ is a rectangle with vertices at $2 \mp i, -2 \mp i$

(ii) $\int_\gamma \frac{e^z - z}{(z-2)^3} dz$, γ is a circle $|z| = 3$.

Solution (ii) Let $f(z) = e^z - z$.

By generalisation of Cauchy formula for derivative of analytic function

$$f^n(a)\eta(\gamma;a) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^{n+1}} dz , \quad \text{for } n \ge 1 .$$

Here $a = 2$ and we have $|z| = 3$ and $n = 2$.

$$f^2(2)\eta(\gamma;2) = \frac{2i}{2\pi i} \int_\gamma \frac{e^z - z}{(z-2)^3} dz \quad .$$

$\int_\gamma \frac{e^z - z}{(z-2)^3} dz = \pi i f^2(2)$, since $\eta(\gamma;2) = 1$.

$f^{(2)}(z) = e^z$. $\Rightarrow$ $f^2(2) = e^z$. $\quad \int_\gamma \frac{e^z - z}{(z-2)^3} dz = i\pi e^2$.

(iii) $\int_\gamma \frac{\sin z}{(z-\pi)\left(z - \pi/2\right)} dz$ where γ is the cicle $|z| = 2$

6) Evaluate the integral $\int_{\gamma}\frac{dz}{z^2+1}, \gamma(\theta)=2|\cos(2\theta)|e^{i\theta}$ for $0 \le \theta \le 2\pi$.

7) Use Cauchy-integral theorem or formula to evaluate $\int_{r}\frac{\cos(\pi z)+\sin(\pi z)}{z^2+1}dz$ where γ is the circle $|z|=2$, taken in positive sense.

8) Evaluate $\int_{\gamma}\frac{\cos(e^z)}{z(z+2)}dz$, where γ is a unit circle.

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9

THEOREMS IN COMPLEX ANALYSIS

Unit Structure

9.0 Objectives
9.1 Introduction
9.2 Morera's Theorem
9.3 Liouville's Theorem
9.4 Taylor's Theorem
9.5 Fundamental Theorem of Algebra
9.6 Summary
9.7 Unit End Exercises.

9.0 OBJECTIVES

In this unit we shall prove the important theorems in complex analysis.

1) Morera's theorem
2) Liouville's theorem
3) Taylor's theorem
4) Fundamental theorem of Algebra

9.1. INTRODUCTION

Given an entire function f, we saw that f has a power series representation as $\sum_{k=0}^{\infty} a_k$, where each $a_k = \frac{f^k(0)}{k!}$. In fact being an entire function the $k\,th$ order derivative $f^k(z)$ exists $\forall k \geq 0$.

In this unit we propose to prove some important theorems in complex analysis .Let us start with the Taylor's theorem.

9.2 MORERA'S THEOREM

Note: this is a sort of converse of Cauchy Goursat thm.

Theorem:

If f(z) is continuous in a simply connected domain D and $\int_x f(z)dz = 0$ *where x is rectifiable curve in D , then f(z) is analytic in D*

Proof: Suppose z is any variable point and z_0 is a fixed in the region D. Also suppose x_1 and x_2 are any two continuous rectifiable curve in D joining z_0 to z and x is the closed continuous rectifiable curve consisting of x_1 and $-x_2$. Then we have

$$\int_x f(z)dz = \int_{x_1} f(z)dz + \int_{-x_2} f(z)dz \text{ and } \int_x f(z)dz = 0 \quad \text{(given)}$$

$$\therefore \int_{x_1} f(z)dz = -\int_{-x_2} f(z)dz = \int_{x_2} f(z)dz$$

i.e. the integral along every rectifiable curve in D joining z_0 to z is the same

Now, consider a function $F(z)$ defined by $F(z) = \int_{z_0}^{z} f(w)dw$ ……..(1)

As discussed above (1) depends only on the end points z_0 and z

If $z+h$ is a point in the neighbourhood of z, then we have $F(z+h) = \int_{z_0}^{z+h} f(w)dw$ …..(2)

From(1)and (2), we have

$$F(z+h) - F(z) = \int_{z_0}^{z+h} f(w)dw - \int_{z_0}^{z} f(w)dw = \int_{z_0}^{z+h} f(w)dw + \int_{z}^{z_0} f(w)dw$$

$$= \int_{z}^{z+h} f(w)dw \quad \text{…….(3)}$$

Since the integral on the RHS of (3) is path independent therefore it may be taken along the straight line joining z to $z+h$, so that

$$\frac{F(z+h) - F(z)}{h} - f(z) = \frac{1}{h}\int_{z}^{z+h} f(w)dw - \frac{f(z)}{h}h$$

$$= \frac{1}{h}\left[\int_{z}^{z+h} f(w)dw - f(z)\int_{z}^{z+h} dw\right] = \frac{1}{h}\int_{z}^{z+h} (f(w) - f(z))dw \quad \text{…….(4)}$$

The function f(w)is given to be continuous at x therefore for a given $\varepsilon > 0$ there exist $\delta > 0$ s.t. $|f(w) - f(z)| < \varepsilon$ s.t. $|w - z| < \delta$

Since h is any arbitrary therefore choosing $|h|<\delta$ so that every point w lying on the line
joining z to $z+h$ satisfies ….. (5)
From (4)and(5), we have

$$\left|\frac{F(z+h)-F(z)}{h}-f(z)\right|\leq\frac{1}{|h|}\int_{z}^{z+h}|f(w)-f(z)|.|dw|$$

$$<\frac{1}{|h|}\varepsilon\int_{z}^{z+h}|dw|=\frac{1}{|h|}\varepsilon|h|=\varepsilon$$

Since ε is small and positive, therefore we have $\left|\frac{F(z+h)-F(z)}{h}-f(z)\right|=0$ or $\lim_{h\to 0}\frac{F(z+h)-F(z)}{h}=f(z)$

Hence $F'(z)=f(z)$

i.e. *F(z)* is differentiable for all values of z inD. Therefore *F(z)* is analytic in D. Since the deriviative of an analytic function therefore *f(z)* is analytic in D

9.3 LIOUVILLE'S THEOREM

Statement : *If f is an entire and bounded function, then f is constant. (2006, 2008)*

Proof : Given that, f is an entire and bounded function.

$\therefore$ $\exists\ M>0$ s.t. $|f(z)|\leq M \quad \forall\ z\in\mathbb{C}$

$\because$ f is an entire function and hence f is analytic everywhere in Complex Plane $\mathbb{C}$ and $\mathbb{C}=B(\alpha;R)$ (say)

$$\therefore \quad \left|f^{(n)}(\alpha)\right|\leq\frac{n!M}{R^n} \qquad n=0,1,2,...$$

Put $z=\alpha$

$$\left|f^{(n)}(z)\right|\leq\frac{n!M}{R^n} \qquad n=0,1,2,...$$

Put $n=1$

$$|f'(z)|\leq\frac{1!M}{R'}$$

$$\therefore \quad |f'(z)|\leq\frac{M}{R}\to 0 \quad \text{as} \quad R\to\infty.$$

$$\therefore \quad |f'(z)|=0 \quad \forall\ z\in\mathbb{C}.$$

$\Rightarrow$ f is constant.

Aliter :

Given that f is an entire and bounded function. Let z_1 and z_2 be any two points in $\mathbb{C}$.

Construct a circle x with centre at z_1 and radius R so that point z_2 lies inside x.

By Cauchy's integral formula

$$f(z_1)=\frac{1}{2\pi i}\int_x \frac{f(\xi)}{\xi - z_1}\,d\xi \text{ and } f(z_2)=\frac{1}{2\pi i}\int_x \frac{f(\xi)}{\xi - z_2}\,d\xi$$

$$f'(z_1)-f(z_2)=\frac{1}{2\pi i}\left[\int_x \frac{f(\xi)}{\xi - z_2}\,d\xi - \int_x \frac{f(\xi)}{\xi - z_1}\,d\xi\right]$$

$$\leq \frac{1}{2\pi}|z_2 - z_1|\left|\int_x \frac{|f(\xi)|}{|\xi - z_2||\xi - z_1|}\right|d\xi| \qquad (1)$$

Choose R, so large that $|z_2 - z_1| < \frac{R}{2}$, since ξ is any point on the circle x.

$\therefore \quad |\xi - z_1| = R$

Now, $|\xi - z_2| = |\xi - z_1 + z_1 - z_2| \geq |\xi - z_1| - |z_2 - z_1| > R - \frac{R}{2} = \frac{R}{2}$

Given that f is bounded function.

$\therefore \quad |f(\xi)| \leq M \quad \forall\ \xi \in x$

$\therefore$ From equation (1)

$$|f(z_2)-f(z_1)| \leq \frac{1}{2\pi}|z_2 - z_1|\left|\int_x \frac{M}{\frac{R}{2}\cdot R}\right|d\xi|$$

$$=\frac{|z_2 - z_1|}{\pi R^2}\cdot M\int_x |d\xi|$$

$$=\frac{|z_2 - z_1| M \times 2\pi R}{\pi R^2} = \frac{2|z_2 - z_1| M}{R}$$

$\therefore \quad |f(z_2)-f(z_1)| \leq \frac{2}{R}|z_2 - z_1| M \to 0$ as $R \to \infty$.

$\therefore \quad |f(z_1)-f(z_2)| = 0$

$\therefore \quad f(z_1)=f(z_2)$ for any two points z_1 and z_2 in $\mathbb{C}$.

$\therefore$ f is constant.

Note : If f is a non-constant entire function the f is unbounded.

Example : Let $f(z)=u(z)+iv(z)$ be an entire function and suppose $|u(z)| \leq M \quad \forall\ z \in \mathbb{C}$. Prove that f is constant.

Proof : Given that, $f(z)=u(z)+iv(z)$ is an entire function.

Define $g(z)=e^{f(z)}$

$\Rightarrow$ g is an entire function.

$\therefore$ $|g(z)|=\left|e^{f(z)}\right|=\left|e^{u(z)+iv(z)}\right|=\left|e^{u(z)}\right|$

$\left(\because \left|e^{i\theta}\right|=1\right)$

$|g(z)|<e^{M}$ constant.

$\Rightarrow$ g is bounded.

$\left(\because |u(z)|\le M \ \forall\ z\in\mathbb{C}\right)$

Thus, g is entire and bounded function.
By Liouville's theorem,
g is constant.

$\therefore$ $e^{f(z)}$ is constant.

$\therefore$ $f(z)=u(z)+iv(z)$ is constant.

$\Rightarrow$ u and v are constant.

Note : If $f(z)=\sum_{n=0}^{\infty} a_n(z-\alpha)^n$ has radius of convergence.
$R>0$, then f is analytic in $B(\alpha;R)$.

9.4 TAYLOR'S THEOREM

If f is analytic in a domain G, then for any point $z\in B(\alpha,R)\subset G$, f has Taylor series expansion, $f(z)=\sum_{n=0}^{\infty} a_n(z-\alpha)^n$ where $a_n=\dfrac{f^{(n)}(\alpha)}{n!}$.

Proof : Given that f is analytic in G.
For $0<r<R$, construct a circle x with centre at α and radius r so that the point z lies inside x.

By Cauchy integral formula, $f(z)=\dfrac{1}{2\pi i}\displaystyle\int_x \frac{f(\xi)}{\xi-z}\,d\xi$ (1)

Now, $\dfrac{1}{\xi-z}=\dfrac{1}{\xi-\alpha+\alpha-z}=\dfrac{1}{(\xi-\alpha)\left[1-\dfrac{(z-\alpha)}{(\xi-\alpha)}\right]}$

$$=\frac{1}{(\xi-\alpha)}\left[1-\frac{(z-\alpha)}{(\xi-\alpha)}\right]^{-1}$$

$$=\frac{1}{(\xi-\alpha)}\left[1+\frac{(z-\alpha)}{(\xi-r)}+\ldots+\frac{(z-\alpha)^{n-1}}{(\xi-\alpha)^{n-1}}+\frac{(z-\alpha)^{n}}{(\xi-\alpha)^{n}}+\ldots\right]$$

$\left\{\because (x)^{-1}=1+x+x^2+\ldots\right\}$

$$=\frac{1}{(\xi-\alpha)}\left[1+\frac{(z-\alpha)}{(z-\alpha)}+\ldots+\frac{(z-\alpha)^{n-1}}{(\xi-\alpha)^{n-1}}+\frac{(z-\alpha)^{n}}{(\xi-\alpha)^{n}}\cdot\frac{1}{1-\frac{(z-\alpha)}{(\xi-\alpha)}}\right]$$

$\left[\because (1-x)^{-1}=1+x+x^2+\ldots+x^{n-1}+x^n+x^{n+1}+\ldots\right.$

$$=1+x+x^2+\ldots+x^{n-1}+x^n\left(1+x+x^2+\ldots\right)$$

$$\left.=1+x+x^2+\ldots+x^{n-1}+x^n(1-x)^{-1}\right]$$

$$=\left[\frac{1}{\xi-\alpha}+\frac{(z-\alpha)}{(\xi-\alpha)^2}+\ldots+\frac{(z-\alpha)^{n-1}}{(\xi-\alpha)^{n}}+\frac{(z-\alpha)^{n}}{(\xi-\alpha)^{n}}-\frac{1}{\xi-z}\right] \quad (2)$$

Multiplying equation (2) by $\frac{f(\xi)}{2\pi i}$ and integrating w.r.t. ξ over.

$$\therefore\ f(z)=\frac{1}{2\pi i}\int_x\frac{f(\xi)}{\xi-z}\,d\xi$$

$$=\frac{1}{2\pi i}\int_x\frac{f(\xi)}{\xi-\alpha}\,d\xi+\frac{(z-\alpha)}{2\pi i}\int_x\frac{f(\xi)}{(\xi-\alpha)^2}\,d\xi+\ldots$$

$$+(z-\alpha)^{n-1}\int_x\frac{f(\xi)}{(\xi-\alpha)^n}\,d\xi+\frac{(z-\alpha)^n}{2\pi i}\int_x\frac{f(\xi)}{(\xi-\alpha)(\xi-z)}\,d\xi \quad (3)$$

By generalization of the theorem on derivative of analytic function.

$$f^{(n)}(\alpha)=\frac{n!}{2\pi i}\int_x\frac{f(\xi)}{(\xi-\alpha)^{n+1}}\,d\xi \qquad n=0,1,2,\ldots$$

OR $$\frac{f^{(n)}(\alpha)}{n!}=\frac{1}{2\pi i}\int_x\frac{f(\xi)}{(\xi-\alpha)^{n+1}}\,d\xi$$

Substituting all these values in equation (3), we get

$$f(z)=f(\alpha)+(z-\alpha)f'(x)+\frac{(2-\alpha)^2}{2!}f''(\alpha)+\ldots+\frac{(z-\alpha)^{n-1}f^{(n-1)}}{(n-1)!}+A_n$$

$$= a_0 + a_1(z-\alpha) + a_2(z-\alpha)^2 + \ldots + a_{n-1}(z-\alpha)^{n-1} + A_n$$

$$\left\{\because a_n = \frac{f^{(n)}(\alpha)}{n!} \text{ and } A_n = \frac{(z-\alpha)}{2\pi i}\int_x \frac{f(\xi)}{(\xi-\alpha)^n(\xi-z)}\, d\xi\right\}$$

The theorem will be proved if $\lim\limits_{n\to\infty} A_n = 0$

$$A_n = \frac{(z-\alpha)^n}{2\pi i}\int_x \frac{f(\xi)}{(\xi-\alpha)^n(\xi-z)}\, d\xi$$

$$|A_n| \le \frac{|z-\alpha|^n}{2\pi}\int_x \frac{|f(\xi)|}{|\xi-\alpha|^n\,|\xi-z|}\,|d\xi| \qquad (4)$$

Choose $|\varsigma| > 0$ s.t. $|z-\alpha| = \varsigma \qquad (0 < \varsigma < r)$

Since ξ is any point on the circle x.

Now, $|\xi - z| = |\xi - \alpha + \alpha - z| \ge |\xi-\alpha| - |z-\alpha| = r - \varsigma$

Given that, f is analytic in G.

$\therefore \quad f$ is continuous on $\{x\}$ $\qquad (\because \{x\}$ is compact set$)$

By boundness theorem, $\exists\ M > 0$ s.t. $|f(\xi)| \le M \quad \forall\ \xi \in x$

$\therefore \quad$ From equation (4),

$$|A_n| \le \frac{\varsigma^n}{2\pi}\int_x \frac{M}{r^n(r-\varsigma)}|d\xi| = \frac{M}{2\pi(r-\varsigma)}\left(\frac{\varsigma}{r}\right)^n \int_x |d\xi|$$

$$= \frac{M}{\cancel{2\pi}(r-\varsigma)}\left(\frac{\varsigma}{r}\right)^n . \cancel{2\pi} r$$

$$\therefore \quad |A_n| \le \frac{Mr}{(r-\varsigma)}\left(\frac{\varsigma}{r}\right)^n \to 0 \text{ as } n \to \infty$$

$\left(\because \lim\limits_{n\to\infty} x^n = 0,\ 0 \le x < 1 \text{ and hence } 0 < \varsigma < 1 \Rightarrow 0 < \varsigma < r\right)$

$\therefore \quad \lim\limits_{n\to\infty} A_n = 0$

$\therefore \quad$ Given series is convergent and we write

$$f(z) = f(\alpha) + (z-\alpha) f'(x) + (z-\alpha)^2 f''(\alpha) + \ldots$$

$$\boxed{f(z) = \sum_{n=0}^{\infty} a_n (z-\alpha)^n} \quad \text{where } a_n = \frac{f^{(n)}(\alpha)}{n!}$$

Put $\alpha = 0$ in equation (1), we get,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{where} \quad a_n = \frac{f^n(0)}{n!}$$

Example 1 : Expand $\sin z$ in a Taylor series about $z = \dfrac{\pi}{4}$.

Solution : By Taylor series

$$f(z) = f(\alpha) + (z-\alpha) f'(\alpha) + \frac{(z-\alpha)^2}{2!} f''(\alpha) + \frac{(z-\alpha)^3}{3!} f'''(\alpha) + \ldots$$

Here, $f(z) = \sin z$ and $\alpha = \dfrac{\pi}{4}$

$$f(\alpha) = \sin\alpha = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$f'(z) = \cos z \Rightarrow f'(\alpha) = \cos\alpha = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$f''(z) = -\sin z \Rightarrow f''(\alpha) = -\sin\alpha = -\sin\frac{\pi}{4} = \frac{-1}{\sqrt{2}}$$

$$f'''(z) = -\cos z \Rightarrow f'''(\alpha) = -\cos\alpha = -\cos\frac{\pi}{4} = \frac{-1}{\sqrt{2}}$$

$\therefore$ Substituting above values in equation (1), we get

$$\sin z = \frac{1}{\sqrt{2}} + \left(z - \frac{\pi}{4}\right)\frac{1}{\sqrt{2}} + \frac{\left(z-\frac{\pi}{4}\right)^2}{2}\left(\frac{-1}{\sqrt{2}}\right) + \frac{\left(z-\frac{\pi}{4}\right)^3}{6}\left(\frac{-1}{\sqrt{2}}\right) + \ldots$$

$$= \frac{1}{\sqrt{2}}\left[1 + \left(z - \pi/4\right) - \frac{\left(z-\pi/4\right)^2}{2} - \frac{\left(2-\pi/4\right)^3}{6} + \ldots\right]$$

Zeros of an Analytic Function :

Definition : A complex polynomial $p(z)$ of degree n is an expression of the form $a_0 + a_1 z + a_2 z + \ldots + a_n z^n$, where $a_0, a_1, \ldots, a_n$ are complex constants and $a_n \neq 0$.

Definition : Let G be an open set and suppose $f : G \to \mathbb{C}$ is a given function. A point $z_0 \in G$ is said to be zero (or root) of f if $f(z_0) = 0$.

e.g. $f(z) = z^2 - 5z + 6 = z^2 - 3z - 2z + 6 = (z-3)(z-2)$. Here roots or zero of, f are $z = 2$ and $z = 3$.

Definition : If $f : G \to \mathbb{C}$ is analytic and α in G satisfies $f(\alpha) = 0$, then α is a zero of f of order (multiplicity) $m \geq 1$, if $\exists$ an analytic function $g : G \to \mathbb{C}$ s.t. $f(z) = (z-\alpha)^m g(z)$ where $g(\alpha) \neq 0$.

Example : Let $f:G\to\mathbb{C}$ be an analytic function then f has a zero of order $m\geq 1$ at $z=\alpha$ if $f(z)=(z-\alpha)^m g(z)$ where g is analytic on G and $g(\alpha)\neq 0$. (2006)

Solution : Let $f(z)=(z-\alpha)^m g(z)$ where g is analytic on G and $g(\alpha)\neq 0$ (1)

$\therefore$ By Taylor series,

For any $z\in B(\alpha;\gamma)\subset G$, g has Taylor series expansion

$$g(z)=g(\alpha)+(z-\alpha)\,g'(\alpha)+\frac{(z-\alpha)^2}{2!}\,g''(\alpha)+\ldots$$

$$f(z)=(z-\alpha)^m\left[g(\alpha)+(z-\alpha)\,g'(\alpha)+\frac{(z-\alpha)^{m+2}}{2!}\,g''(\alpha)+\ldots\right]$$

$$=(z-\alpha)^m\,g(\alpha)+(z-\alpha)^{m+1}\,g'(\alpha)+\frac{(z-\alpha)^{m+2}}{2!}\,g'(\alpha)+\ldots$$

$$f(z)=\frac{(z-\alpha)^m}{m!}\,f^{(m)}(\alpha)+\frac{(z-\alpha)^{m+1}}{(m+1)!}\,f^{(m+1)}(\alpha)+\frac{(z-\alpha)^{m+2}}{(m+2)!}$$

$$f^{(m+2)}(\alpha)+\ldots$$

Clearly, this is a Taylor series expansion about $z=\alpha$ and

$f(\alpha)=f'(\alpha)=f'(\alpha)=\ldots=f^{(m-1)}(\alpha)=0$ and $f^{(m)}(\alpha)\neq 0$.

$\Rightarrow$ f has a zero of order $m\geq 1$ at $z=\alpha$.

Conversely, assume that f has a zero of order $m\geq 1$ at $z=\alpha$.

$\therefore$ $f(\alpha)=f'(\alpha)=\ldots=f^{(m-1)}(\alpha)=0$

$\therefore$ $f(z)=\sum\limits_{n=0}^{\infty} a_n\,(z-\alpha)^n$

$\therefore$ $f(z)=\sum\limits_{n=m}^{\infty} a_n\,(z-\alpha)$

$=a_m\,(z-\alpha)^m+a_{m+1}\,(z-\alpha)^{m+1}+a_{m+2}\,(z-\alpha)^{m+2}+\ldots$

$=(z-\alpha)^m\left[a_m+a_{m+1}\,(z-\alpha)+a_{m+2}\,(z-\alpha)^2+\ldots\right]$

$=(z-\alpha)^m\sum\limits_{n=0}^{\infty} a_{n+m}\,(z+\alpha)^n$ $\quad f(z)=(z-\alpha)^m\,g(z)$ $\qquad g(\alpha)\neq 0$

Note : $f'(z) = m(z-\alpha)^{m-1}\, g(z) + (z-\alpha)^m\, g'(z)$

$$f^{(m)}(z) = \frac{m(m-1)(m-2)\ldots\times 2\times \left| g(z) \right| + \ldots + \ldots}{m!}$$

Each of these terms have $(z-\alpha)$ as one form.

$$f^{(m)}(\alpha) = m!\, g(\alpha) + 0 + 0 + \ldots$$

$$f^{(m)}(\alpha) = m!\, g(\alpha)$$

Definition : A zero of an analytic function f is said to be isolated if it has a neighbourhood in which there is no other zero of *f*.

Theorem : *Any zero of an analytic function is isolated in the set of its zeros.* *(2009)*

Proof : Let $f : G \to \mathbb{C}$ be an analytic function.

Suppose f has a zero of order *m* at $z = \alpha$.

$\therefore \quad f(z) = (z-\alpha)^m\, g(z)$ (1)

where *g* is analytic on *G* and $g(\alpha) \neq 0$.

Let $\varepsilon > 0$ be given.

Put $\left| g(\alpha) \right| = 2\varepsilon > 0$

$\because \quad g$ is analytic on G.

$\Rightarrow \quad g$ is continuous at $\alpha \in G$.

$\Rightarrow$ for the above $\varepsilon > 0,\ \exists\ \delta > 0$

s.t. $\left| z-\alpha \right| < \delta \ \Rightarrow\ \left| g(z) - g(\alpha) \right| < \varepsilon$

When $\left| z-\alpha \right| < \delta$ i.e. $z \in B(\alpha, \delta)$

$\therefore \quad |g(z)| \geq |g(\alpha)| - |g(\alpha) - g(z)| \geq \left| g(\alpha) \right| - \left| g(\ ) + g(\alpha) \right| > 2\varepsilon - \varepsilon = \varepsilon$

$\therefore \quad \left| g(z) \right| > \varepsilon > 0$

$\therefore \quad g(z) \neq 0 \quad \forall\ z \in B(\alpha, \delta)$

$\therefore \quad g(z) \neq 0 \quad \forall\ z \in B(\alpha, \delta) - \{\alpha\}$

and $\quad \left| z-\alpha \right| \neq 0 \quad \forall\ z \in B(\alpha, \delta) - \{\alpha\}$

or $\quad \left| z-\alpha \right| \neq 0 \quad \forall\ z \in B(\alpha, \delta) - \{\alpha\}$

From equation (1)

$$f(z) = (z-\alpha)^m\, g(z) \neq 0 \qquad \forall\ z \in B(\alpha, \delta) - \{\alpha\}$$

$\because \quad \alpha$ is arbitrary.

$\therefore$ Any zeros of an analytic function is isolated in the set of its zeros.

9.4 FUNDAMENTAL THEOREM OF ALGEBRA

Statement: *Every non-constant complex polynomial has a root.*
(2009)

OR

If $p(z)$ is a non-constant complex polynomial then, there is a complex number and α with $p(\alpha)=0$

Proof : Given that $p(z)$ is a non-constant complex polynomial T.P.T. $p(\alpha)=0$.

Assume that this is not true.

Suppose $p(z) \neq 0 \quad \forall\ z$.

$\therefore \quad p(z)$ is an entire function.

Let $f(z)=\dfrac{1}{p(z)}$ (1)

$\Rightarrow \quad f$ is an entire function.

$\because \quad p(z)$ is non-constant entire function.

$\Rightarrow \quad p(z)$ is unbounded (by contra positive statement of Lioville's theorem)

$\therefore \quad \lim\limits_{z\to\infty} p(z)=\infty$

$\therefore \quad$ From equation (1), $\lim\limits_{z\to\infty} f(z)= \lim\limits_{z\to\infty}\left[\dfrac{1}{p(z)}\right]=\dfrac{1}{\infty}=0$

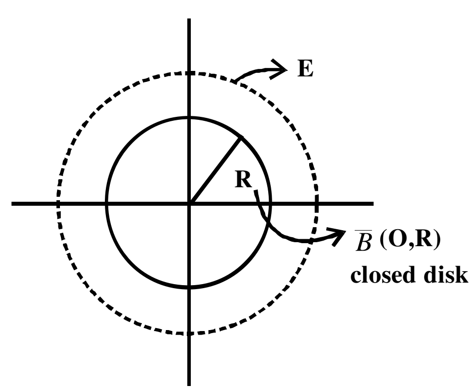
Let f be defined on an bounded set E.

If for a given $\varepsilon>0$, $\exists\ R>0$ s.t.

$\left|f(z)-\ell\right|<\varepsilon \qquad$ whenever $\qquad |z|>R$ and $z\in E$

Then, we say that $f(z)\to\ell$ as $z\to\infty$.

$\therefore \quad \lim\limits_{z\to\infty} f(z)=\ell$



(closed disk $\overline{B}$ (O,R) is compact set)

Fig 9.1

$\therefore$ For given $\varepsilon > 0$, $\exists\ R > 0$ s.t. $|f(z)| < \varepsilon$ whenever $|z| > R$

$\because$ f is an entire function.

$\therefore$ f is continuous on $\overline{B}(0; R)$.

$\therefore$ By boundedness theorem.

f is bounded on $\overline{B}(0; R)$.

$\therefore$ $\exists\ M > 0$ s.t. $|f(z)| \le M \quad \forall\ z \in \overline{B}(0; R)$

$\therefore$ f is entire and bounded on $\overline{B}(0; R)$

$\therefore$ By Liouville's theorem, f is constant.

$\therefore$ From equation (1),

$$p(z) = \frac{1}{f(z)} = \frac{1}{\text{constant}} = \text{constant}$$

which contradicts the hypothesis that $p(z)$ is non-constant.

$\therefore$ Our assumption is wrong.

$\therefore$ There is a Complex Number α with $p(\alpha) = 0$.

Exercise : Prove that a complex polynomial $p(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$ has exactly n roots where $a_0, a_1, \ldots, a_n$ are complex constant and $a_n \neq 0$. (Use fundamental theorem of Algebra)

Theorem : *Suppose that f is analytic in domain G. If z_f, the set of zeros of f in G, has limit point in G, then $f(z) = 0$ in G.*

Proof : Given that f is analytic in a domain G.

$z_f = \{z \in : f(z) = 0\}$ and α is a limit point of z_f.

Let $\{z_n\}$ be a sequence of zeros of f in G, such that $\lim_{n \to \infty} z_n = \alpha$

$\because$ f is analytic on G, $\quad f$ is continuous on G.

$\therefore$ $f(\alpha) - f\left(\lim_{n \to \infty} z_n\right) = \lim_{n \to \infty} f(z_n) = 0$

$\{\because z_n$ is a zero of $\Rightarrow f(z_n) = 0\}$

$\therefore$ $f(\alpha) = 0$

$\because$ zeros of an analytic function are isolated.

$\therefore$ either $f(z) = 0 \quad \forall\ z \in B(\alpha; \delta)$

OR $f(z) \neq 0 \quad \forall\ B(\alpha; \delta) - \{\alpha\}$

x is connected if the only sets of x which are both open and closed are $\varnothing$ and x.

Assume that $f(z) \neq 0$ in $B(\alpha; \delta) - \{\alpha\}$

$\because$ α is a limit point of z_f $\Rightarrow$ every nbd of α continuously infinitely many points of z_f.

For sufficiently large *n*, there is a point z_n such that $f(z_n)=0$ in $B(\alpha;\delta)-\{\alpha\}$

$\therefore$ Our assumption is wrong.

Hence, we must have $f(z)=0 \quad \forall\ z\in B(\alpha;\delta)$

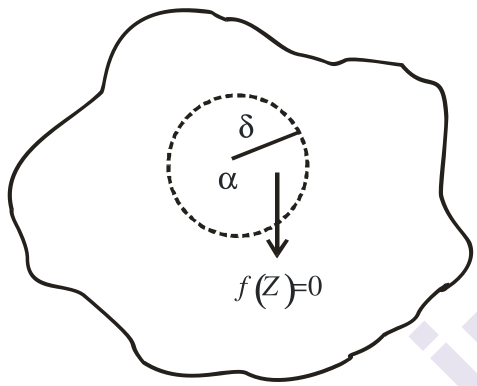


Fig 10.2

Given that, *G* is a domain.

$\therefore$ G is open and connected.

We split the set *G* into two sets.

$A=\{\xi\in G:\xi \text{ is a point of } z_f\}$

$B=\{\xi\in G:\xi\notin A\}$, where z_f is the set of zeros of f in *G*.

$\therefore$ $A\cap B=\varnothing$ and $A\cup B=G$

$\because$ $\xi\in A$ is a limit point of z_f in *G*.

$\therefore$ $f(z)=0 \quad \forall\ z\in B(\xi;\delta)$

$\Rightarrow$ $z\in A$

$\therefore$ $z\in B(\xi;\delta)\Rightarrow z\in A$

$\therefore$ $B(\xi,\delta)\subset A$

$\Rightarrow$ A is an open set and $A\neq\varnothing$ $\quad(\because \alpha\in A)$

Let $\xi'\in B$, then ξ' is not a limit point of z_f.

$\therefore$ By continuity of f at ξ', $\exists\ \delta>0$ s.t.

$f(z)\neq 0 \quad \forall\ z\in B(\xi';\delta)$

$\Rightarrow$ $z\in B$

Thus, $z\in B(\xi',\delta)\Rightarrow z\in B$

$\therefore$ $B(\xi';\delta)\subset B$

$\Rightarrow$ B is an open set.

$\because$ G is connected.

It can not be written as a union of two non-empty disjoint open sets.

$\Rightarrow \quad A=\varnothing$ or $B=\varnothing$

But $\quad A\neq\varnothing \qquad (\because \alpha\in A)$

$\therefore \quad B=\varnothing$

$\therefore \quad A=G \qquad (\because A\cup B=G$ and $B=\varnothing)$

$\therefore$ Each point of G is a limit point of z_f.

$\therefore \quad f(z)=0 \qquad \forall\, z\in G$

Theorem: *Let f and g be analytic in a domain G. If T is a subset of G having limit point α in G and if $f(z)=g(z)\ \forall\, z\in T$, then $f(z)=g(z)\ \forall\, z\in G$.*

$F(z)=f(z).g(z)$, $T=\{z\in G: F(z)=0\}$ and use of previous theorem.

Theorem : *Let f be analytic in a domain G such that for some α in G and $f^{(n)}(\alpha)=0$, $n=0,1,2,\ldots$ then $f(z)=0\ \forall\, z\in G$.* (Use Taylor's theorem)

Exercise : Prove that the function $f(z)=ze^z-z$ has a zero of order 2 at origin.

$f(z)=ze^z-z$

$\therefore$ by Maclaurin expansion

$$f(z)=ze^z-z$$

$$=z\left(1+\frac{z}{1!}+\frac{z^2}{2!}+\frac{z^3}{3!}+\frac{z^4}{4!}+\frac{z^5}{5!}+\frac{z^6}{6!}\right)-z$$

$$=\left(z+\frac{z^2}{1!}+\frac{z^3}{2!}+\frac{z^4}{3!}+\frac{z^5}{4!}+\frac{z^6}{5!}+\frac{z^7}{6!}\right)-z$$

$$=\frac{z^2}{1!}+\frac{z^3}{2!}+\frac{z^4}{3!}+\frac{z^5}{4!}+\frac{z^6}{5!}+\frac{z^7}{6!}$$

$$=z^2\left(\frac{1}{1!}+\frac{z}{2!}+\frac{z^2}{3!}+\frac{z^3}{4!}+\frac{z^4}{5!}+\frac{z^5}{6!}\right)$$

$\therefore$ since lowest power of z is 2

9.5 SUMMARY

1) Morera's Theorem :

Statement : Let G be a domain in $\mathbb{C}$ and let $f: G\to\mathbb{C}$ be a continuous function s.t. $\int_{\partial\Delta} f(z)\,dz=0$ for any triangle Δ in G then

f is analytic in G. (This is a partial converse of Cauchy – Goursat theorem.)

2) Liouville's Theorem :

Statement : If f is an entire and bounded function, then f is constant.

1) **Taylor's Theorem:** If f is analytic in a domain G, then for any point $z \in B(\alpha, R) \subset G$, f has taylor series expansion, $f(z) = \sum_{n=0}^{\infty} a_n (z-\alpha)^n$ where $a_n = \frac{f^{(n)}(\alpha)}{n!}$.

3) Fundamental Theorem of Algebra :

Statement : Every non-constant complex polynomial has a root.

OR

If $p(z)$ is a non-constant complex polynomial then, there is a complex number and α with $p(\alpha) = 0$.

9.6 UNIT END EXERCISES

1) Show that an entire function is infinitely differentiable .

Solution: If f is entire , by Taylors expansion of f, f has a power series representation. In fact $f^k(0)$ exist $\forall k \geq 1$.

$$f(z) = \sum_{k=0}^{\infty} \frac{f^k(0)}{k!} z^k, \qquad \forall z \in \mathbb{C}$$

We can see that $f(z)$ has an infinite radius of convergence . $f(z)$ converges for all $z \in \mathbb{C}$.

By the result that Power series are infinitely differentiable within their domain of convergence, $f(z)$ is infinitely differentiable.

$$f^1(z) = \sum_{k=1}^{\infty} \frac{kf^k(0)}{k!} z^{k-1} = \sum_{k=1}^{\infty} \frac{f^k(0)}{(k-1)!} z^{k-1}$$

$$f^2(z) = \sum_{k=2}^{\infty} \frac{k)k-1)f^k(0)}{k!} z^{k-2} = \sum_{k=2}^{\infty} \frac{f^k(0)}{(k-2)!} z^{k-2}$$ and so on.

2) Find the power series expansion of $f(z)=z^2$ around $z=2$.

Solution: $\therefore\ f(z)=f(2)+f'(2)(z-2)+\frac{f''(2)}{2!}(z-2)^2+\ldots$

$f(z)=4+4(z-2)+(z-2)^2$

3) Find the power series expansion for e^z about any point a.

4) Suppose an entire function f is bounded by M, along $|z|=R$. Show that the coefficients in it's power series expansion about $z=0$ satisfy $|c_k|\le\frac{M}{R^k}$. (Hint: $c_k=\frac{f^k(0)}{k!}$,

by Cauchy formula $\therefore\ f^k(0)=\frac{k!}{2\pi i}\int_{|z|=R_z}\frac{f(z)}{k+1}dz \quad \forall k\ge 1$)

5) Let f be an entire function, if for some integer $k\ge 0$, there exist positive constants A and B such that $|f(z)|\le A+B|z|^k$, then f is a polynomial of degree atmost k.
(Hint: Use Liouville's Theorem)

6) Using Morera's theorem show that the function f defined by $f(z)=\int_0^\infty\frac{|e^{zt}|}{t+1}dt$ is analytic in the left half plane $D:\mathrm{Re}(z)<0$.

Solution: $\therefore\ \int_0^\infty\frac{|e^{zt}|}{t+1}dt<\int_0^\infty e^{xt}dt=-\frac{1}{x}$, for $\mathrm{Re}(z)=x<0$.

This integral is absolutely convergent and $|f(z)|\le\frac{1}{|x|}$,

Consider $\int_\Gamma f(z)dz=\int_\Gamma\left(\int_0^\infty\frac{e^{zt}}{t+1}dt\right)dz$. Here Γ= The boundary of some closed rectangle in D.

Since, $\int_\Gamma\int_0^\infty\frac{|e^{zt}|}{t+1}dtdz$ converges hence we can interchange the order of integration .

$\therefore\int_\Gamma f(z)dz=\int_0^\infty\int_\Gamma\frac{e^{zt}}{t+1}dtdz=\int_0^\infty 0dt=0$, since $\frac{e^{zt}}{t+1}$ is analytic inside and on a closed curve Γ.

$\therefore$ By Morera's Theorem f is analytic in D.

7) Show that $\int_0^1 \frac{\sin(2t)}{t} dt$ is an entire function.(Use Morera's Theorem.)

8) Show that α is a zero of multiplicity k if and only if $P(a) = P'(a) = ... = P^{k-1}(a) = 0$ but $P^k(a) \neq 0$.
(Hint : Use the Fundamental theorem of Algebra .)

9) Find the order of zero at $z = 0$ of the function $f(z) = z(ze^z - z)$.

10) Find the Maclaurin series expansion of $f(z) = \sin^2 z$.

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10

MAXIMUM AND MINIMUM MODULUS PRINCIPLE

Unit Structure

10.0 Objectives

10.1 Introduction

10.2 Maximum Modulus Principle, Schwarz Lemma, Open Mapping Theorem

10.3 Automorphisms of the Unit Disc

10.4 Summary

10.5 Unit End Exercises

10.0 OBJECTIVES

After going through this unit, we shall understand 1)Maximum modulus principle and open mapping theorem for analytic functions . 2) Corollaries on open mapping theorem and maximum modulus principle. 3) Possible Automorphisms of the Unit disc $B(0;1)$. 4)Harmonic functions and their properties.

10.1 INTRODUCTION

In previous sections, we have studied the connections between everywhere convergent power series and entire functions. We shall now turn our attention to the general relationship between power series and analytic functions. According to a theorem, every power series represents an analytic function inside it's circle of convergence.

Our first goal is the converse of this theorem. We then turn to the questions of analytic functions in arbitrary open sets and local behaviour of such functions.

10.2 THE MAXIMUM MODULUS PRINCIPLE

Definition : Let G be any subset of $\mathbb{C}$. A complex function f defined on G is said to have local maximum modulus at a point α in G if, there exists $\delta > 0$ s.t. $B(\alpha; \delta) \subset G$ and

$|f(z)| \le |f(\alpha)| \quad \forall z \in B(\alpha, \delta)$.

Similarly, f has local minimum modulus at a point α in G, if $\exists \; \delta > 0$ s.t. $B(\alpha; \delta) \subset G$ and

$|f(z)| \ge |f(\alpha)| \quad \forall \; z \in B(\alpha, \delta)$.

Theorem : *Suppose f is analytic in a domain G and there is a point α in G s.t. $|f(z)| = |f(\alpha)| \; \forall \; z \in G$. Then f is constant i.e. if $|f|$ attains to maximum modulus in G then f is constant.*

Proof : Given that, f is analytic in domain G (open and connected set)

$\because$ $\quad \alpha \in G$ and G is open.

$\therefore$ $\quad \exists \; r > 0$ s.t. $\overline{B}(\alpha; r) \subset G$

where $x = \partial \overline{B}(\alpha; r) =$ boundary of closed disk $\overline{B}(\alpha; r)$

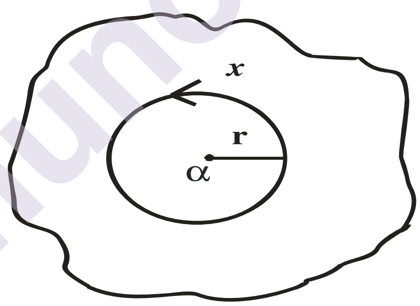


Fig 10.1

$\therefore$ $\quad$ By Cauchy integral formla,

$$f(\alpha) = \frac{1}{2\pi i} \int_x \frac{f(z)}{z - \alpha} dz$$

Here x is the circle $|z - \alpha| = r$

$z = \alpha + re^{it} \qquad t \in [0, 2\pi]$, $\qquad dz = i\, re^{it} dt$

$$\therefore \quad f(\alpha) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f\left(\alpha + re^{it}\right)}{re^{it}} \cdot i\, re^{it}\, dt = \frac{1}{2\pi} \int_0^{2\pi} f\left(\alpha + re^{it}\right) dt \tag{1}$$

Given that, $\left| f(z) \right| \leq \left| f(\alpha) \right| \quad \forall\ z \in G$

$$\therefore \quad \left| f\left(\alpha + re^{it}\right) \right| \leq \left| f(\alpha) \right|$$

$\therefore$ From equation (1), we get

$$\left| f(\alpha) \right| \leq \frac{1}{2\pi} \int_0^{2\pi} \left| f\left(\alpha + re^{it}\right) \right| dt \leq \frac{1}{2\pi} \int_0^{2\pi} \left| f(\alpha) \right| dt$$

$$= \frac{\left| f(\alpha) \right|}{2\pi} \int_0^{2\pi} dt = \frac{\left| f(\alpha) \right|}{2\pi} [t]_0^{2\pi}$$

$$= \frac{\left| f(\alpha) \right|}{2\pi} \times 2\pi = \left| f(\alpha) \right|$$

$$\therefore \left| f(\alpha) \right| \leq \frac{1}{2\pi} \int_0^{2\pi} \left| f\left(\alpha + re^{it}\right) \right| dt \leq \left| f(\alpha) \right|$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left| f\left(\alpha + re^{it}\right) \right| dt$$

$$\therefore \quad 2\pi \left| f(\alpha) \right| = \int_0^{2\pi} \left| f\left(\alpha + re^{it}\right) \right| dt$$

$$\int_0^{2\pi} \left| f(\alpha) \right| dt = \int_0^{2\pi} \left| f\left(\alpha + re^{it}\right) \right| dt$$

$$\left(\because 2\pi \left| f(\alpha) \right| = \int_0^{2\pi} \left| f(\alpha) \right| dt \right)$$

$$\int_0^{2\pi} \left[\left| f(\alpha) \right| - \left| f\left(\alpha + re^{it}\right) \right| \right] dt = 0$$

Here, the integral $\left| f(\alpha) \right| - \left| f\left(\alpha + re^{it}\right) \right|$ is continuous and non-negative.

$$\therefore \quad \left| f(\alpha) \right| - \left| f\left(\alpha + re^{it}\right) \right| = 0 \quad \forall\ t \in [0, 2\pi]$$

$$\therefore \quad \left| f\left(\alpha + re^{it}\right) \right| = \left| f(\alpha) \right| \quad \forall\ t \in [0, 2\pi]$$

$$\therefore \quad \left| f(z) \right| = \left| f(\alpha) \right| \quad \forall\ z \in x$$

This equation holds on all circles $|z-\alpha|=\delta \quad 0 \le s \le r$

$\therefore$ $f(z)$ is constant in $B(\alpha; r)$

$\therefore$ $f(z)$ is constant in $B(\alpha; r)$

($\because$ If $f: G \to \mathbb{C}$ is analytic and $|f(z)| =$ constant if $z \in G$ then, f is constant on G.

By using theorem [Let f and g be analytic in a domain G. If T is a subset of G having limit point α in G and if $f(z) = g(z) \quad \forall\ z \in T$ then $f(z) = g(z) \quad \forall\ z \in G$]

f is constant in G.

Maximum Modulus Principle :

Suppose f is analytic in a bounded domain D and continuous on $\overline{D}$ (Closure of D). Then , $|f|$ attains its maximum on the boundary ∂D of D. *(2006, 2012)*

Proof : If f is constant, then there is nothing to prove. Let f be a non-constant function.

Given that, f is continuous on $\overline{D}$ ($\overline{D}$ is a compact set).

$\therefore$ $|f|$ attains its maximum value at same point in $\overline{D}$.

Maximum modulus principal,

$\therefore$ $|f|$ does not attain its maximum in D.

$\therefore$ $|f|$ attains its maximum on the boundary ∂D of D.

$\left(\overline{D} = D \cup \partial D\right)$

Minimum Modulus Principle :

Suppose f is a non-constant and analytic function in a domain G. If $|f|$ attains its local minimum G at α, then $f(\alpha) = 0$ or f is constant *(2006,2007)*

Proof : Given that, f is a non-constant analytic function in a domain G and $|f|$ attains its local minimum at a point α in G.

$\therefore \quad |f(\alpha)| \le |f(z)| \qquad \forall\ z \in B(\alpha; \delta) \subset G \qquad (1)$

To prove that $f(\alpha) = 0$

Assume that this is not a true.

i.e. $\quad f(\alpha) \ne 0$ in some open disk $B(\alpha, r) \subset G$.

Let $\quad g(z)=\frac{1}{f(z)} \quad (2)$

$\therefore \quad g$ is analytic in $B(\alpha; r)\subset G$.

$\therefore \quad$ From equation (1),

$$\frac{1}{|f(\alpha)|}\geq\frac{1}{|f(z)|}$$

OR

$$|g(z)|\leq|g(\alpha)| \qquad \forall\ z\in B(\alpha, r)$$

$\therefore \quad g$ has a local maximum modulus at a point α in *G*.

$\therefore \quad$ By maximum modulus principle,
G is constant in *G*.

$\therefore \quad$ From equation (2),

$$f(z)=\frac{g}{g(z)}=\frac{1}{\text{constant}}=\text{constant}$$

$\Rightarrow \quad f$ is constant in *G*.

which is contradicts that f is a non-constant function.

$\therefore \quad$ Our assumption is wrong.

$\therefore \quad f(\alpha)=0$

Theorem : *If f is a non-constant analytic function in a bounded domain G and $f(z)\neq 0$ for any $z\in G$, then $|f|$ can not attain its minimum in G.*

Example : Let $f(z)=e^z$ and $T=\overline{B}(2+3i,1)$. Find a point in T at which $|f|$ attains its maximum value.

Solution : Given function, $f(z)=e^z\ T=\overline{B}(2+3i,1)$

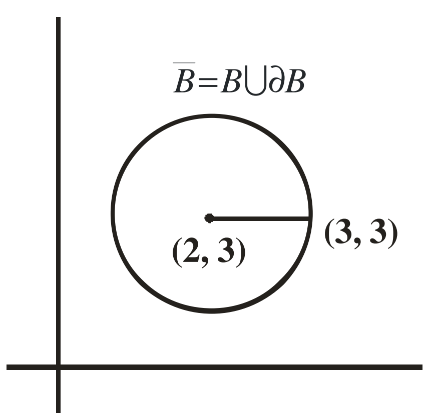


Fig 10.2

The boundary of T is the circle.

$$\left| z-(2+3i) \right| = 1$$

$\therefore$ By maximum modulus theorem, $|f|$ attains its maximum value on ∂B.

$$\therefore \left| f(z) \right| = \left| e^z \right| = \left| e^{(2+3i)+(1)e^{i\theta}} \right| = \left| e^{2+\cos\theta} . e^{i(3+\sin\theta)} \right| = \left| e^2 + \cos\theta \right|$$

$$\left(\because \left| e^{i\theta} \right| = 1\right)$$

$$= e^2 + \cos\theta$$

We know that, the value of $\cos\theta$ is maximum when $\theta = 0$.

$$\therefore \quad \left| e^z \right| = e^{2+1} = e^3$$

$\left| e^z \right| = e^3$ is the maximum value of f at a point $(3,3)$ or $3+3i$ in *T*.

Exercise : Let $f(z) = z$ and $T = \overline{B}(0;1)$.

Prove that the function $|f|$ cannot attain its minimum value on the boundary of *T*.

Schwarz's Lemma

Let $D = \{z \in \mathbb{C} : |z| < 1\}$ *be the unit disk and suppose f is analytic in D with, (i) f (0) = 0 and (ii)* $|f(z)| \le 1$ *for* $z \in D$. *Then,* $|f(z)| \le |z| \quad \forall z \in D$ *and* $|f'(0)| \le 1$. *Moreover, if* $|f(z)| = |z|$ *for some* $z \ne 0$, *then there is a constant C with* $|c| = 1$ *s.t.* $f(\omega) = c.\omega \ \forall \omega \in D$.*(2004, 2005, 2008)*

Proof: Given that, f is analytic in D, with (i) $f(0) = 0$ *and* (ii) $|f(z)| \le 1$ for $z \in$ *D.* Define $g : D \to \mathbb{C}$ by

$$g(z) = \begin{cases} f(z)/z & , \ z \ne 0 \\ f'(0) & \ \ z = 0 \end{cases}$$

$\Rightarrow g$ is analytic in D

Choose r s.t. $0 < r < 1$

$\therefore$ on the circle $|z| = r$

$$|g(z)| = \left| \frac{f(z)}{z} \right|$$

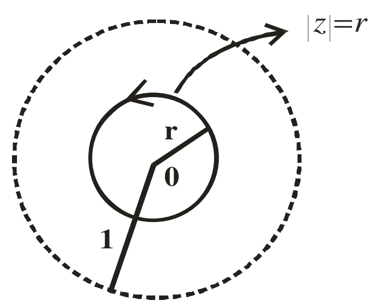


Fig 10.3

$\left|g(z)\right| \le -\dfrac{1}{r}$ on the circle $|z| = r$

$\ldots\ldots\ldots\left(\because \left|f(z)\right| \le 1\, and\, |z| = r\right)$

$\therefore$ by Maximum Modulus theorem,

$\left|g(z)\right| \le \dfrac{1}{r} \qquad \forall z \in \overline{B}(0;r)$

As $r \to 1$, $\left|g(z)\right| \le 1 \qquad \forall z \in D = B(0,1)$ ------------- (2)

$\therefore \left|f(z)\right| \le 1$ --------- (by (1))

$\Rightarrow \left|f(z)\right| \le |z| \qquad \forall z \in D$

Again, from equation (1)

$\left|f'(0)\right| = \left|g(0)\right| \le 1$ --------- (2)

$\therefore \left|f'(0)\right| \le 1$

If $\left|f(z)\right| = |z|$ for some $z \ne 0$ then,

$\left|g(z)\right| = \dfrac{\left|f(z)\right|}{|z|} = \dfrac{|z|}{|z|} = 1$

$\therefore |g|$ attains its maximum value of some point z inside D.

$\therefore$ by maximum Modules principle,

g is constant in D

i.e. $g(z) = c$ where, c is constant and $|c| = 1$

$\therefore \dfrac{f(z)}{z} = c$

or $f(z) = cz \qquad \forall\, z \in D$

or $f(\omega) = c\omega \qquad \forall\, \omega \in D$

Example: $\alpha; z$ in D, define the M.T. $\phi_\alpha(z) = \dfrac{z-\alpha}{1-\overline{\alpha}z}$

Let f be analytic in D and let $\left|f(z)\right| < 1, then$

(i) $\left|\dfrac{f(z)-f(\alpha)}{1-f(\alpha)f(z)}\right| \le \left|\dfrac{z-\alpha}{1-\overline{\alpha}z}\right|$ **OR** $\left|\phi_{f(x)}\, f(z)\right| \le \left|\phi_\alpha(z)\right| \;\forall\, \alpha, z \in D$

(ii) $\dfrac{\left|f'(z)\right|}{1-\left|f(z)\right|^2} \le \dfrac{1}{1-|z|^2} \quad \forall\, z \in D$

Solution:

Fix a pint α in D arbitrarily

Put $\omega = \dfrac{z-\alpha}{1-\overline{\alpha}z} = \phi_\alpha(z) \Rightarrow z = \dfrac{\omega+\alpha}{1+\overline{\alpha}\omega}$

Define $\phi_{f(\alpha)} f(z) = g(\omega) = \dfrac{f(z) - f(\alpha)}{1 - f(\alpha) f(z)}$ -------------------- (1)

$$\phi_{f(\alpha)} f(z) = \frac{f\left(\dfrac{\omega+\alpha}{1+\bar{\alpha}\omega}\right) - f(\alpha)}{1 - f(\alpha) f\left(\dfrac{\omega+\alpha}{1+\bar{\alpha}\omega}\right)}$$

$\therefore$ g is analytic in D, g(o) = o and $|g(\omega)| < 1$

By using Schwarz's Lemma,

$|g'(o)| < 1 \; and \; |g(\omega)| \le |\omega| \qquad \forall\, \omega \in D$

$$\Rightarrow \left|\frac{f(z) - f(\alpha)}{1 - f(\alpha) f(z)}\right| \le \left|\frac{z-\alpha}{1-\bar{\alpha}z}\right|$$

OR

$\left|\phi_{f(x)} f(z)\right| \le |\phi_\alpha(z)| \;\; \forall\, z \in D$

From equation (1)

$$g'(\omega)\frac{d\omega}{dz} = \frac{d}{dz}\left[\frac{f(z) - f(\alpha)}{1 - f(\alpha) f(z)}\right]$$

$$g'(\omega)\left[\frac{(1-\bar{\alpha}z)(1) + (z-\alpha)\bar{\alpha}}{(1-\bar{\alpha}z)^2}\right] =$$

$$= \frac{\left(1-\overline{f(\alpha)} f(z) \cdot f'(z) + \left(f(z) - f(\alpha)\right)\overline{f(\alpha)} f'(z) f(\alpha)\right)\overline{f(\alpha)} f'(z)}{\left[1 - f(\alpha) f(z)\right]^2}$$

$$g'\left(\frac{z-\alpha}{1-\bar{\alpha}z}\right)\left[\frac{\left(1-|\alpha|^2\right)}{\left(1-\bar{\alpha}z\right)^2}\right] = \frac{\left[1-|f(\alpha)|^2\right] f'(z)}{\left[1-\overline{f(\alpha)} f(z)\right]^2}$$

Put $z = \alpha$

$$g'(o)\left[\frac{1-|\alpha|^2}{\left(1-|\alpha|^2\right)^2}\right] = \frac{\left[1-|f(\alpha)|^2\right] f'(\alpha)}{\left[1-|f(\alpha)|^2\right]^2}$$

$$\therefore |g'(o)| = \left|\frac{\left(1-|\alpha|^2\right) f'(\alpha)}{1-|f(\alpha)|^2}\right| \le 1 \qquad \left(\because |g'(o)| \le 1\right)$$

$$\therefore \ \frac{|f'(\alpha)|}{1-|f(\alpha)|^2} \le \frac{1}{1-|\alpha|^2}$$

$$\therefore \ \frac{|f'(\alpha)|}{1-|f(\alpha)|^2} \le \frac{1}{1-|\alpha|^2}$$

Put $\alpha = z, \dfrac{f'(z)}{1-|f(z)|^2} \le \dfrac{1}{1-|z|^2} \quad \forall\, z \in D$

iii) $|f(z)| \le \dfrac{|f(o)+|z||}{1+|f(o)||z|}$

$\because |g(\omega)| \le |\omega| \qquad \forall\, \omega \in D$

$$\left|\frac{f(z)-f(o)}{1-\overline{f(o)}f(z)}\right| \le 1\left|\frac{z-\alpha}{1-\overline{\alpha}z}\right|$$

Put $\alpha = o$

$$\left|\frac{f(z)-f(o)}{1-f(o)f(z)}\right| \le |z|$$

If $|a|<1 \ and \ |b|<1$

$$\frac{|a|-|b|}{1-|a||b|} \le \left|\frac{|a-b|}{1-\overline{a}b}\right| \le \frac{|a|+|b|}{1+|a||b|}$$

$$\frac{|f(z)|-|f(o)|}{1-|f(z)||f(o)|} \le \left|\frac{f(z)-f(o)}{1-f(o)\ and\ f(z)}\right| \le (z)$$

$$\frac{|f(z)|-|f(o)|}{1-|f(z)||f(o)|} \le |z|$$

$$\therefore \ |f(z)-f(o)| \le |z|\left(1-|f(z)||f(o)|\right)$$

$$\therefore |f(z)-f(o)| \le |z|-|z|-|f(z)||f(o)|$$

$$\therefore |f(z)|+|z||f(z)||f(o)| \le |f(o)|+|z|$$

$$f(z) \le \frac{|f(o)|+|z|}{1+|f(o)||z|}$$

<u>Counting zero</u>

Definition : A zero of order one is said to be a simple zero.

e.g. Let $f(z) = z^2 - 3z + 2$

Here, f has simple zeros at $z = 1$ and $z = 2$.

$f'(z) = 2z - 3$

$f'(1) = -1 \neq 0$

$f'(2) = -1 \neq 0$

Prove that all zeros of the function $\sin z$ are simple.

To find zeros at $\sin z$ put $\sin z = 0$

$\therefore \quad \sin z = 0 \Rightarrow z = \sin^{-1}(0) = 2^{\pi}, \quad n = 0, \pm 1, \pm 2, ... \quad n \in \mathbb{Z}$

$f(z) = \sin z$

$f'(z) = \cos z$

$f'(z) = \cos n\pi = (-1)^n \neq 0, \quad \forall\ n \in \mathbb{Z}$

$\therefore$ All zeros of $\sin z$ are simple.

Theorem : *Let f be analytic in a domain G with zeros $\alpha_1, \alpha_2, ..., \alpha_m$ (repeated according to order)*

If x is a smooth closed curve in G which does not pass through any $\alpha_k{}'s$ then

$$\frac{1}{2\pi i} \int_x \frac{f'(z)}{f(z)} dz = \sum_{k=1}^{m} n(x; \alpha_k)$$

Proof : Given that, f is analytic in a domain G with zeros $\alpha_1, \alpha_2, ..., \alpha_n$ (repeated according to order or multiplicities.)

$\therefore \quad f(z) = (z - \alpha_1)(z - \alpha_2)...(z - \alpha_m)$, where g is analytic and $g(\alpha_k) \neq 0, \quad k = 1, 2, ..., m$

Taking log on both sides and differentiating w.r.t. z, we get

$$\frac{f'(z)}{f(z)} = \frac{1}{z - \alpha_1} + \frac{1}{z - \alpha_2} + ... + \frac{1}{z - \alpha_m} + \frac{g'(z)}{g(z)}$$

Multiply this equation by $\frac{1}{2\pi i}$ and integrate w.r.t z over $\{x\}$ on both side.

$$\frac{1}{2\pi i} \int_x \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi i} \int_x \frac{dz}{z - \alpha_1} + ... + \frac{1}{2\pi i} \int_x \frac{dz}{z - \alpha_m} + \frac{1}{2\pi i} \int_x \frac{g'(z)}{g(z)} dz$$

$\because \quad g$ and g' are analytic in G.

$\therefore \quad \frac{g'}{g}$ is analytic in G and x is a smooth closed curve.

$\therefore \quad$ By Cauchy theorem, $\int_x \frac{g'(z)}{g(z)} dz = 0$

$\therefore \qquad \frac{1}{2\pi i}\int_x \frac{f'(z)}{f(z)}dz = n(x;\alpha_1)+n(x;\alpha_2)+...+n(x;\alpha_m)$

$= \sum_{k=1}^{m} n(x;\alpha_k)$

Note :

$\frac{1}{2\pi i}\int_x \frac{f'(z)}{f(z)}dz = \sum_{k=1}^{m} n(x,\alpha_k) =$ Number of zeros of f inside x, where each zero is counted according to its order.

e.g. for $f(z)-(z-\alpha_1)(z-\alpha_2)^2 g(z)$

$$\frac{1}{2\pi i}\int_x \frac{f'(z)}{f(z)}dz = \eta(x;\alpha_1)+\eta(x,\alpha_2)+3\eta(x,\alpha_3) = 1+0+3=4$$

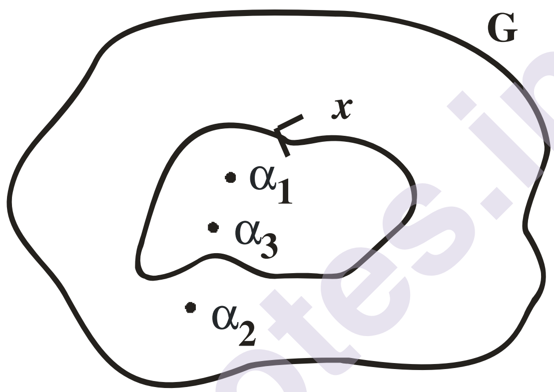


Fig 10.4

Corollary : *Let f, G and x be as in the preceding theorem except that $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n$ are the points in G that satisfy the equation $f(z)=\alpha$ then,*

$$\frac{1}{2\pi i}\int_x \frac{f'(x)}{f(z)-\alpha}dz = \sum_{k=1}^{m} \eta(x;\alpha_k) \qquad \textit{Number of zeros of}$$

$f(z)=\alpha$ inside x.

Example : Evaluate $\int_x \frac{f'(z)}{f(z)}dz$ where $f(z)=\frac{z(z-1)^2}{z^3+5}$ and x is the circle $|z|=1.2$.

Solution : Given function, $f(z)=\frac{z(z-1)^2}{z^3+5}$.

Here, $f(z)$ has simple zero at $z=0$ and $z=1$ is a zero of order 2.

Given, equation of circle, $|z|=1.2$.

$\therefore$ Zero $z=0$ and $z=1$ lies inside x.

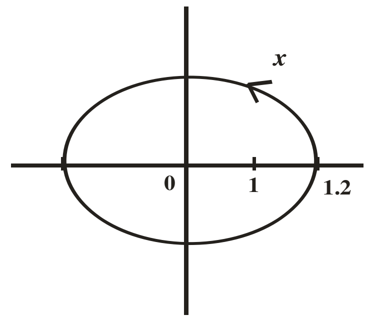


Fig 10.5

By theorem, $\frac{1}{2\pi i}\int_x \frac{f'(z)}{f(z)}dz$ = number of zeros of inside x where each zero is counted according to its order.

$$\therefore \quad \frac{1}{2\pi i}\int_x \frac{f'(z)}{f(z)}dz = 1+2=3 \qquad \Rightarrow \int_x \frac{f'(z)}{f(z)}dz = 6\pi i$$

Exercise: Evaluate $\int_x \frac{f'(z)}{f(z)}dz$ where $f(z)=\frac{z^2+(z-1)(z+3)}{z^3+2}$ and x is the circle $|z|=1.5$.

Note : Let $x:[0,1]\to G$ be a closed (Smooth) curve in $\mathbb{C}$ and suppose $f:G\to\mathbb{C}$ is an analytic function. Then $\sigma = f\circ x$ is also a closed curve in w-plane. If α is a Complex Number $\alpha\notin\{\sigma\}=f(\{x\})$, we write,

$$\eta(\sigma;\alpha)=\frac{1}{2\pi i}\int_\sigma \frac{dw}{w-\alpha}=\frac{1}{2\pi i}\int_0 \frac{\sigma'(t)}{\sigma(t)-\alpha}dt$$

$$=\frac{1}{2\pi i}\int_0 \frac{f'(x(t))x'(t)\,dt}{f(x(t))-\alpha}$$

$$=\frac{1}{2\pi i}\int_x \frac{f'(z)}{f(z)}dz = 1+2=3$$

$$\int_x \frac{f'(z)}{f(z)}dz = 6\pi i$$

Exercise : Evaluate $\int_x \frac{f'(z)}{f(z)}dz$ where $f(z)=\frac{z^2(z-1)(z+3)}{z^3+2}$ and x is the circle $|z|=1.5$.

Note : Let $x:[0,1]\to G$ be a closed (smooth) curve in $\mathbb{C}$ and suppose $f:G\to\mathbb{C}$ is an analytic function. Then $\sigma = f\circ x$ is also a closed curve in w-plane. If α is a Complex Number $\alpha \notin \{\sigma\} = f(\{x\})$, we write,

$$\eta(\sigma;\alpha) = \frac{1}{2\pi i}\int_x \frac{dw}{w-\alpha} = \frac{1}{2\pi i}\int_x \frac{\sigma'(t)}{\sigma(t)-\alpha}\,dt$$

$$= \frac{1}{2\pi i}\int_0 \frac{f'(x(t))x'(t)\,dt}{f(x(t))-\alpha} = \frac{1}{2\pi i}\int_x \frac{f'(z)}{f(z)}\,dz = 1+2=3$$

$= \sum_{k=1}^{m} \eta(x,\alpha_k) =$ numbers at zeros of $f(z)-\alpha$ inside x where each zero is counted according to its order.

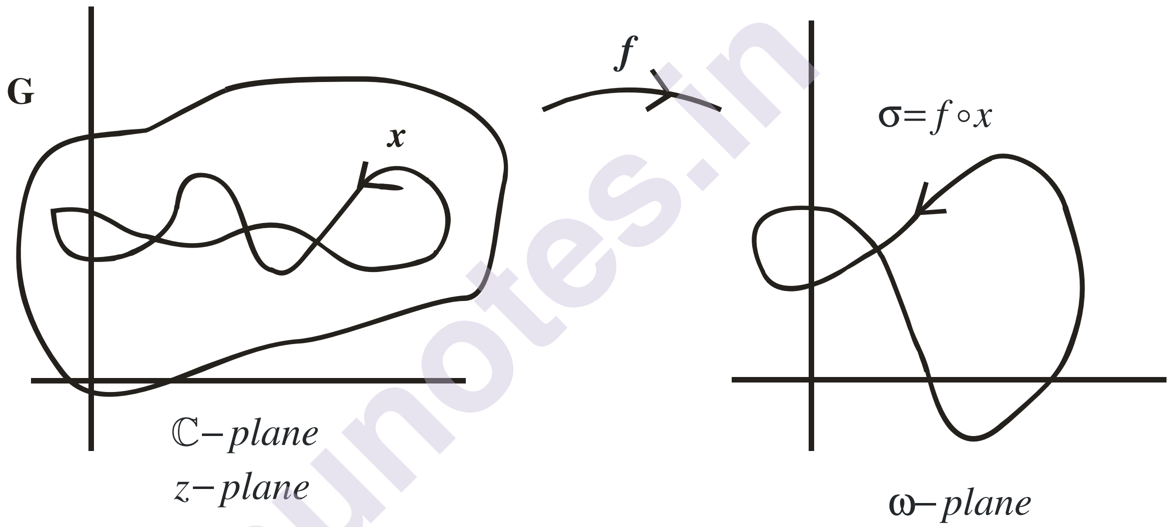


Fig 10.6

where $\alpha_1,\alpha_2,...,\alpha_n$ are points in G with $f(\alpha_k)=\alpha$.

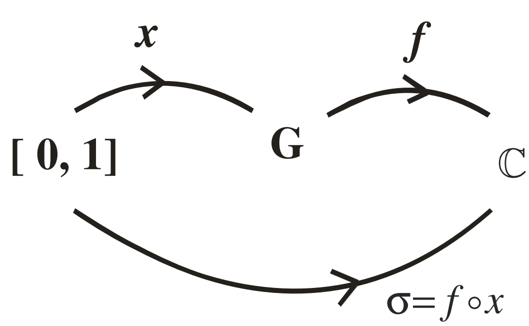


Fig 10.7

Theorem : *Suppose that* f *is analytic in* $B(a;R)$ *and let* $f(a)-\alpha$ *has a zero of order m at* $z=a$ *then there is an* $\varepsilon>0$ *and* $\delta>0$ *s.t.* $|\xi-\alpha|<\delta$ *and the equation* $f(z)-\xi$ *has exactly m simple roots in* $B(a;\varepsilon)$.

Proof : Given that, $f(\alpha)-\alpha$ has a zero of order *m* at $z=a$.

$\because$ Zeros of an analytic function are isolated.

$\therefore$ We can choose $\varepsilon>0$ s.t. $\varepsilon<R/2$.

$f(z)-\alpha$ has no solution with $0<|z-a|<2\varepsilon$ and $f'(z)\neq 0$ if $0<|z-a|<2\varepsilon$.

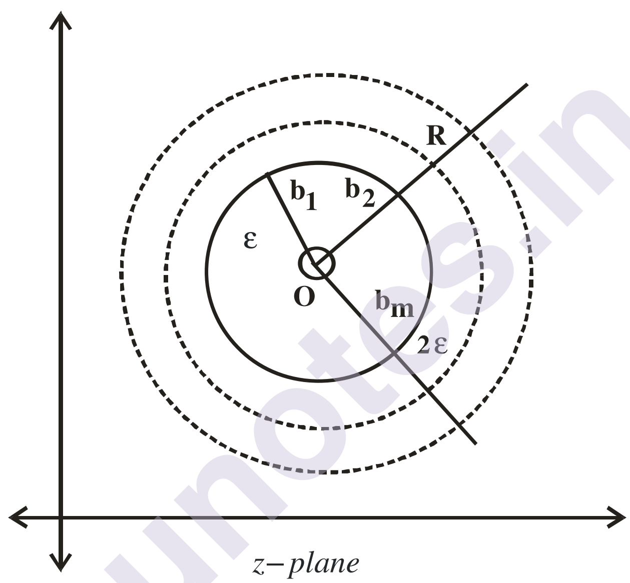


Fig 10.8

Let *x* be the circle, $|z-a|=\varepsilon$

i.e. $x(t)=a+\varepsilon e^{2\pi it},\quad t\in[0,1]$.

$x:[0,1]\to B(a;R)$

$\because$ f is an analytic on $B(a;R)$

$\therefore$ $\sigma=f\circ x$ is also a closed curve in *w*-plane.

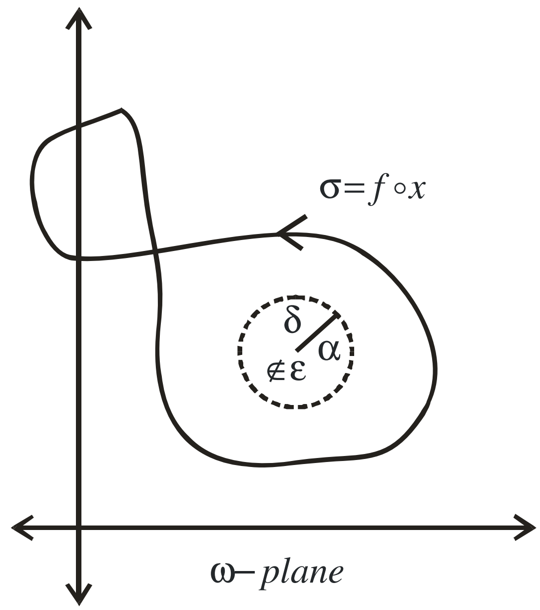


Fig 10.9

Now, $\alpha \notin \{\sigma\}$

So, $\exists\ \delta > 0$ s.t. $B(\alpha;\delta) \cap \{\sigma\} = \varnothing$

(It means open disk $B(\alpha;\delta)$ does not touch trace of σ.

$\therefore$ $B(\alpha;\delta)$ is contained in the same component of $w/\{\sigma\}$.

For $\xi \in B(\alpha;\delta)$ i.e. $|\xi - \alpha| < \delta$

$$\Rightarrow \quad \eta(\sigma;\alpha) = \eta(\sigma;\xi) \qquad (1)$$

Now, $\eta(\sigma;\alpha) = \dfrac{1}{2\pi i} \displaystyle\int_{\sigma} \dfrac{dw}{w-\alpha}$ $\quad \because \omega = f(z) \Rightarrow dw = f'(z)\,dz.$

$$\Rightarrow \eta(\sigma;\alpha) = \frac{1}{2\pi i} \int_{x} \frac{f'(z)}{f(z)-\alpha}\, dz$$

= Number of zeroes of $f(z) - \alpha$ inside x, where each zero is counted according to its order.

= Thm $f(z) - \alpha$ has a zero of order m at $z = a$)

Again, $\eta(\sigma;\xi) = \dfrac{1}{2\pi i} \displaystyle\int_{\sigma} \dfrac{dw}{w-\xi}$

$$\eta(\sigma;\xi) = \frac{1}{2\pi i} \int_{x} \frac{f'(z)}{f(z)-\xi}\, dz$$

$\therefore \quad \frac{1}{2\pi i}\int\limits_{x}\frac{f'(z)}{f(z)-\xi}dz=\eta(\sigma;\xi)=n(\sigma;\alpha)=m$ By (1) and $\eta(\sigma;\alpha)=m$

$\Rightarrow \quad f(z)=\xi$ has exactly m-roots in $B(a;\xi)$

$\because \quad f'(z)\neq 0$ for $0<|z-a|<\varepsilon$

$\Rightarrow$ The equation $f(z)=\xi$ has exactly m-simple roots in $B(a,\xi)$.

Definition : If x and α are Metric spaces and $f:X\to Y$ has the property that $f(0)$ is open in α whenever *U* is open in *X*, then *f* is called an open map.

Open Mapping Theorem :

Statement : *Suppose G is a domain in* $\mathbb{C}$, f *is a non-constant analytic function on G. Then for any open set U in G,* $f(U)$ *is an open.* *(2007, 2009)*

Proof : Given that, f is a non-constant analytic function on G.

Let $a\in U$ and $f(a)=\alpha$

$\because$ U is open $\Rightarrow \exists\ \varepsilon>0$ s.t. $B(a;\varepsilon)\subset U$.

$\because$ f is non-constant analytic function on *G*.

$\therefore$ by fundamental theorem of algebra, $\exists$ an integer $m\geq 1$.

$f(z)-\alpha$ has a zero of order *m* at $z=a$.

$\therefore$ by using previous theorem, for the above $\varepsilon>0$, $\exists\ \delta>0$ s.t. for the above $\varepsilon>0$, $\exists\ \delta>0$ s.t. $|\xi-\alpha|<\delta$ and the equation $f(z)=\xi$ has exactly m simple roots in $B(\alpha;\varepsilon)$.

Thus $\forall\ \xi\in B(\alpha;\delta)$, we an find m points in $B(\alpha;\varepsilon)$ which are mapped to ξ by *f*.

$\therefore \quad B(\alpha;\delta)\subset f(B(\alpha;\varepsilon))$

$\therefore \quad B(\alpha;\delta)\subset f(U)$ $\quad (\because B(\alpha;\varepsilon)\subset U)$

$\therefore \quad \alpha$ is interior point of $f(0)$.

But α is arbitrary.

$\therefore f(U)$ is open.

10.3 AUTOMORPHISMS OF THE UNIT DISC

A function $f: D \to D$ is said to be an Analytic automorphism or Automorphism of the Unit disc D if f is bijective and if both f, f^{-1} are analytic in D.

Note: Let $0 \neq \alpha \in D = \{ z \in \mathbb{C} : |z| < 1 \}$

For $\alpha, z \text{ in } D$, define the Mobius transformation $\phi_\alpha(z) = \dfrac{z-\alpha}{1-\bar{\alpha} z}$

This Mobius transformation is analytic in D and also in $\bar{D} = DU\partial D$, $\phi_\alpha(z)$ is not analytic at a point $z = \dfrac{1}{\alpha}$.

Which lies outside the disk $\bar{D}$. $\quad (\because \alpha \in D \Rightarrow |\alpha| < 1)$

$$\Rightarrow \frac{1}{|\alpha|} = \frac{1}{|\bar{\alpha}|} > 1 \text{ and } |z| = |\bar{z}|$$

Note: $\phi_\alpha : 0 \to D$ is an analytic automorphism.

1. $\phi_\alpha(0) = -\alpha$ *and* $\phi_\alpha(\alpha) = 0$
2. for any point $z \in D$, $\phi_\alpha(z) \in D$

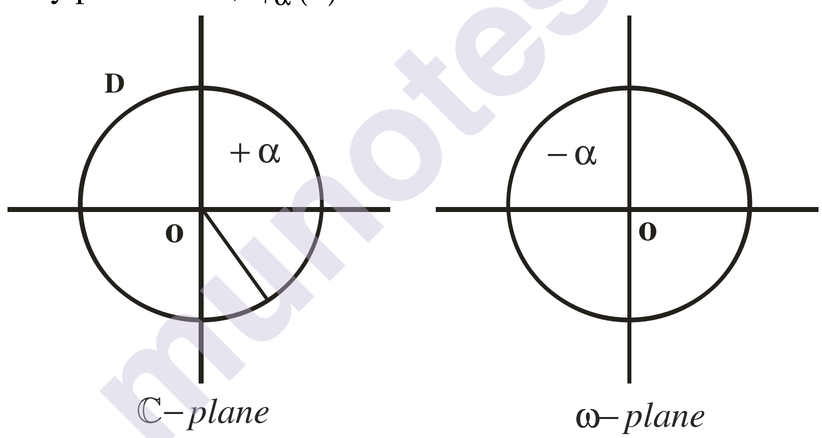


Fig 10.10

i.e. $|\phi_\alpha(z)| < 1 \quad \forall z \in D$

Since, $\quad |\phi_\alpha(z)|^2 = \dfrac{(z-\alpha)(\bar{z}-\bar{\alpha})}{(1-\bar{\alpha} z)(1-\alpha \bar{z})}$

$$\left(\because \; |z|^2 = z \cdot \bar{z}\right)$$

$$= \frac{z \cdot \bar{z} - z\bar{\alpha} - \alpha\bar{z} + \alpha\bar{\alpha}}{1 - \alpha\bar{z} - \bar{\alpha}z + \alpha\bar{\alpha}\, z\bar{z}} = \frac{|z|^2 + |\alpha|^2 - (z\bar{\alpha} + \alpha\bar{z})}{1 + |\alpha|^2 |z|^2 - (\alpha\bar{z} + \bar{\alpha}z)} < \frac{2 - (z - \bar{\alpha} + \alpha\bar{z})}{2 - (z\bar{\alpha} + \alpha\bar{z})}$$

$\ldots.(\because z\alpha \in D \Rightarrow |z| < 1, |\alpha| < 1)$

$\therefore \left|\phi_{\alpha}(z)\right|^{2} < 1$

$\therefore \left|\phi_{\alpha}(z)\right| < 1$

$\Rightarrow \left|\phi_{\alpha}(z)\right| \in D$

$\therefore \phi_{\alpha}$ maps D onto itself i.e. $\phi_{\alpha}(0) = D$

(3) If $\alpha \in D$, so is $-\alpha$ and for any $z \in D$,

$[\phi_{\alpha} \circ \phi_{\alpha}](z) = z = [\phi_{-\alpha} \circ \phi_{\alpha}](z)$ i.e. $\phi_{\alpha}^{-1} = \phi_{\alpha}$

Now $\left[\phi_{-\alpha} \circ \phi_{\alpha}\right](z) = \phi_{\alpha}\left[\phi_{\alpha}(z)\right] = \phi_{-\alpha}\left[\dfrac{z-\alpha}{1-\overline{\alpha}z}\right] = \dfrac{\left(\dfrac{z-\alpha}{1-\overline{\alpha}z}\right)+\alpha}{1+\overline{\alpha}\left(\dfrac{z-\alpha}{\alpha}\right)}$

$$[\phi_{-\alpha} \circ \phi_{\alpha}](z) = \frac{z-\alpha+\alpha-|\alpha|^{2} z}{1-\overline{\alpha}z+\overline{\alpha}z-|\alpha|^{2}} = \frac{z\left(1-|\alpha|^{2}\right)}{1-|\alpha|^{2}} = z$$

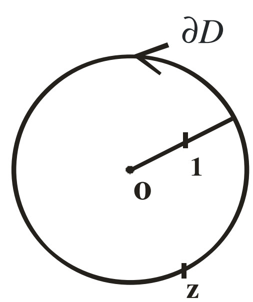
Similarly, $[\phi_{\alpha} \circ \phi_{-\alpha}](z) = z$

$\therefore \phi_{\alpha}^{-1} = \phi_{-\alpha}$

$\Rightarrow \phi_{\alpha}$ maps D onto D in a one-one manner Hence, ϕ_{α} *and* ϕ_{α}^{-1} are automorphisms of the Unit disc.

4. For $z \in \partial D, \phi_{z}(z) \in \partial D$ i.e. $\left|\phi_{\alpha}(z)\right| = 1 \ \forall z \in \partial D$

Since, $\left|\phi_{\alpha}(z)\right| = \left|\dfrac{z-\alpha}{1-\overline{\alpha}z}\right|$



$\because z \in \partial D$

$\therefore$ For any point $z \in \partial D$

$z = e^{i\theta}$, $\theta \in [0, 2\pi]$

$$\therefore \left|\phi_{\alpha}(z)\right| = \left|\frac{e^{i\theta}-\alpha}{1-\overline{\alpha}\, e^{i\theta}}\right| = \frac{\left|e^{i\theta}-\alpha 1\right|}{\left|e^{i\theta}\right| \left|e^{-i\theta}-\overline{\alpha}\right|} = \frac{\left|e^{i\theta}\right.}{\left|e^{i\theta}\right.}$$

11. …

11

$\left(\because \left|e^{i\theta}\right| = 1 \textit{ and } e^{i\theta} = e^{-i\theta}\right)$

$\left|\phi_{\alpha}(z)\right| = 1$ $\left(\because |z| = |\overline{z}|\right)$

$\Rightarrow \phi_{\alpha}(z) \in \partial D$

$\therefore \phi_{\alpha}$ *maps* ∂D *onto* ∂D

i.e. $\phi_{\alpha}(\partial D) = \partial D$

combining results (2) and (4), we get

ϕ_{α} *maps* $\overline{D}$ *onto* $\overline{D}$.

5. Mobius transformation $\phi_{\alpha}(z)=\dfrac{z-\alpha}{1-\bar{\alpha}z}$ is analytic in D,

$$\phi'_{\alpha}(z)=\frac{(1-\bar{\alpha}z)(1)+(z-\alpha)\bar{\alpha}}{(1-\bar{\alpha}z)^2}=\frac{1-|\alpha|^2}{(1-\bar{\alpha}2)^2}$$

In particular,

$\phi'_{\alpha}(o)=1-|\alpha|^2 \neq 0$ $\qquad\qquad (\because\ \alpha\in D\Rightarrow|\alpha|<1$

$\phi'_{\alpha}(\alpha)=\dfrac{1}{1-|\alpha|^2}\neq 0$

$\Rightarrow|\alpha|^2<1\ \therefore\ 1-|\alpha|^2+0)$

Proposition: *If* $|z|<1$ *then,* ϕ_{α} *is one map of D onto itself. The inverse of* ϕ_{α} *is* $\phi_{-\alpha}$. *Furthermore,* ϕ_{α} *maps* ∂D *onto* ∂D,

$\phi_{\alpha}(\alpha)=0,\Rightarrow\phi'_{\alpha}(0)=1-|z|^2$ *and* $\phi'_{\alpha}(\alpha)=\dfrac{1}{1-|\alpha|^2}$.

Theorem: *Let* $f:D\to D$ *be a one-one analytic map of D onto itself, with* $f(\alpha)=0$ *and suppose that* ϕ_{α} *is a one-one analytic map of D onto itself with analytic inverse* ϕ_{α}. *Then, there is a complex no. C with* $|C|=1$ *s.t.* $f=C\phi_{\alpha}$.

Proof: Given that, $f:D\to D$ is an one-one analytic map of D onto itself with $f(\alpha)=o$.

Put $\omega=\dfrac{z-\alpha}{1-\bar{\alpha}\omega}=\phi_{-\alpha}(\omega)$ $\quad , \quad |\alpha|<1$

Define,

$g=f\circ\phi_{-\alpha}$

$\Rightarrow g$ is a one-one analytic map of D onto itself and $g(o)=f[\phi_{-\alpha}(o)]=f(\alpha)=o$

$\ldots..\left(\because f(\alpha)=o \text{ and } \phi_{-\alpha}(\omega)=\dfrac{\omega+\alpha}{1+\bar{\alpha}\omega}\right)$

and $|g(\omega)|=|f[\phi_{-\alpha}(\omega)]|=|f(z)|<1$ $\quad\ldots\ldots\quad$ $(\because\ z=\phi_{-\alpha}(\omega)$ and $f:D\to D$ *and* D is unit Disk

$\therefore\ z\in D\Rightarrow f(z)\in D\Rightarrow|f(z)|<1)$

$\therefore$ by Schwarz's Lemma,

$|g(\omega)|\leq|\omega|$ $\qquad\qquad \forall\,\omega\in D$

and $|g'(o)|\leq 1$

let $|g(\omega)| = |\omega|$ for some $\omega \neq o$ in D

$\therefore$ by second part of Schwarz's Lemma,

There is a complex no. C with $|C| = 1$ s.t.

$g(\omega) = c.\omega \quad \forall\, \omega \in D$

$\Rightarrow f\left[\phi_{-\alpha}(\omega)\right] = c\omega$

$\Rightarrow f(z) = c\phi_{\alpha}(z)$

$\Rightarrow f = c\phi_{\alpha}$

10.4 SUMMARY

1) The Maximum Modulus Principle:
Let G be any subset of $\mathbb{C}$. A complex function f defined on G is said to have local maximum modulus at a point α in G if, there exists $\delta > 0$ s.t. $B(\alpha;\delta) \subset G$ and $|f(z)| \leq |f(\alpha)| \quad \forall\, z \in B(\alpha;\delta)$.

Similarly, f has local minimum modulus at a point α in G, if $\exists\ \delta > 0$ s.t. $B(\alpha;\delta) \subset G$ and $|f(z)| \geq |f(\alpha)| \quad \forall\, z \in B(\alpha,\delta)$.

2) Minimum Modulus Principle : Suppose f is a non-constant and analytic function in a domain G. If $|f|$ attains its local minimum G at α, then $f(\alpha) = 0$.

3) Schwarz's Lemma: *Let* $D = \{z \in \mathbb{C} : |z| < 1\}$ *be the unit disk and suppose f is analytic in D with, (i) f (0) = 0 and (ii)* $|f(z)| \leq 1$ *for* $z \in D$. *Then,* $|f(z)| \leq |z| \quad \forall z \in D$ *and* $|f'(0)| \leq 1$. *Moreover, if* $|f(z)| = |z|$ *for some* $z \neq 0$, *then there is a constant C with* $|c| = 1$ *s.t.* $f(\omega) = c.\omega\ \forall \omega \in D$.

4) Theorem : If f is a non-constant analytic function in a bounded domain G and $f(z) \neq 0$ for any $z \in G$, then $|f|$ cannot attain its minimum in G.

5) Theorem : Let f be analytic in a domain G with zeros $\alpha_1, \alpha_2, ..., \alpha_m$ (repeated according to order)
If x is a smooth closed curve in G which does not pass through any $\alpha_k's$ then

$$\frac{1}{2\pi i} \int_x \frac{f'(z)}{f(z)}\, dz = \sum_{k=1}^{m} n(x;\alpha_k)$$

6) Open Mapping Theorem :

Statement : Suppose G is a domain in $\mathbb{C}$, f is a non-constant analytic function on G. Then for any open set U in G, $f(U)$ is an open.

7) A function $f: D \to D$ is said to be an Analytic automorphism or Automorphism of the unit disc D, if f is bijective and if both f, f^{-1} are analytic in D.

10.5 UNIT END EXERCISES

1) Find the maximum modulus of $z^2 - z$ in the disc $|z| \le 1$.

Solution: $\therefore \quad z^2 - z = z(z-1)$

$\therefore$ The maximum modulus is assumed at the boundary of the disc $|z| \le 1$ That is at $z = -1$.

$\therefore \max_{|z|\le 1} z^2 - z = 2$

2) Show that the maximum modulus of e^z is always assumed on the boundary of the compact domain .

Solution: Since $\left|e^z\right| = e^x$ where $z = x + iy$

$\therefore \left|e^z\right|$ is maximum at a point in the domain with maximal x.

(At a point farthest to the right.)

3) Suppose *f,g* both are analytic in a compact domain *D*. Show that $|f(z)| + |g(z)|$ takes it's maximum on the boundary.

(Hint: Take $f(z) = Ae^{-i\alpha}, g(z) = Be^{i\beta}$ then put

$h(z) = |f(z)| + |g(z)|$

$\Rightarrow |h(z)| = |f(z)| + |g(z)| \le |f(z)| + |g(z)|$

Let z_0 be an interior point of a compact domain D. Assume that $|f| + |g|$ takes maximum values inside D, say $|f(z_0)| + |g(z_0)|$

$\therefore |h(z)| = |f(z)| + |g(z)|$

$\le |f(z)| + |g(z)| \le |f(z_0)| + |g(z_0)| = |h(z_0)|$

$\therefore |h(z)| \le |h(z_0)|$

$\therefore$ The analytic function $h(z)$ assumes it's maximum at the interior point z_0 (not on the boundary), which is not possible.

$\therefore |f(z)| + |g(z)|$ takes it's maximum on the boundary.

4) Let f be analytic and bounded by 1 in the Unit disc and $f\left(\frac{1}{2}\right)=0$ Estimate $\left|f\left(\frac{3}{4}\right)\right|$.

Solution: Since $f\left(\frac{1}{2}\right)=0$, define $g:\mathbb{C}\to\mathbb{C}$ as follows:

$$g(z)=\frac{f(z)}{\left(z-\frac{1}{2}\right)/\left(1-\frac{z}{2}\right)} \quad \text{for } z\neq\frac{1}{2}$$

$$=\frac{3}{4}f'\left(\frac{1}{2}\right) \quad \text{for } z=\frac{1}{2}$$

Then g is analytic in $|z|\le 1$ Letting $|z|\to 1$ we find that $|g|\le 1$ on the disc .

$$\therefore |f(z)|\le\left|\frac{z-\frac{1}{2}}{1-\frac{2}{2}}\right| \quad \therefore \text{ for } z=\frac{3}{4}, \quad \therefore \left|f\left(\frac{3}{4}\right)\right|\le\frac{2}{5}$$

5) Show that among all functions , which are analytic and bounded by 1, in the Unit disc, Max $\left|f'\left(\frac{1}{3}\right)\right|$ is assumed, when $f\left(\frac{1}{3}\right)=0$.

Solution: Suppose $f\left(\frac{1}{3}\right)\neq 0$, consider $g(z)=\dfrac{f(z)-f\left(\frac{1}{3}\right)}{1-f\left(\frac{1}{3}\right)f(z)}$

$$\therefore \left|\frac{w-f\left(\frac{1}{3}\right)}{1-f\left(\frac{1}{3}\right)w}\right|=1 \text{ when } |w|=1 \text{ and } |f|<1 \text{ in } |z|<1,$$

$\therefore$ By Maximum-Modulus Theorem $|g|<1$ in $|z|<1$.

By direct calculations $g'\left(\frac{1}{3}\right)=\dfrac{f'\left(\frac{1}{3}\right)}{\left(1-\left|f\left(\frac{1}{3}\right)\right|^2\right)}$ $\therefore$ $\left|g'\left(\frac{1}{3}\right)\right|>\left|f'\left(\frac{1}{3}\right)\right|$

This is a contradiction.

6) Show that the automorphisms of the Unit Disc are of the form $g(z)=e^{i\theta}\left(\frac{z-\alpha}{a-z\tilde{\alpha}}\right), |\alpha|<1$.

Solution: Let $g(z)=\left(\frac{z-\alpha}{1-z\tilde{\alpha}}\right)$. Then $|g(z)|=1$ for $|z|=1$.

Since $g(\alpha)=0 \Rightarrow g$ is an automorphism of the Unit disc.

Assume that f is an automorphism of the Unit Disc with $f(\alpha)=0$.

$\therefore\ h = fog^{-1}$ is an automorphism with $h(0)=0$.

$\therefore$ By the lemma that describes automorphisms of the Unit disc ,

$\therefore\ h(z)=e^{i\theta}z$ or $f(z)=e^{i\theta}\left(\frac{z-\alpha}{1-z\tilde{\alpha}}\right)$.

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11

SINGULARITIES

Unit Structure

11.0 Objectives
11.1 Introduction
11.2 Three Types of isolated Singularities
11.3 Laurent's Theorem
11.4 Classification of Singularities by the Principal Part of Laurent's Expansion
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11.0 OBJECTIVES

After going through this unit , you will understand the concept of continuing an analytic function to another region. We shall also study three types of singularities of a function $f(z)$ and the theorems like Casorati-Weirstrass theorem and the Laurent's theorem.

Given a singularity z_0 of a function $f(z)$, we shall try to classify the singularities by finding the principal part of Laurent series expansion of a function $f(z)$.

11.1 INTRODUCTION

We shall recall the uniqueness theorem that states that if f is analytic in a region D and $\{z_n\}$ is a sequence of distinct points such that $f(z_n)=0\ \forall n$ and $\{z_n\}$ converges to some $z_0 \in \mathbb{C}$, then f is identically zero in a region D. Suppose we are given a function f, which is analytic in region D. The question is that of continuing f analytically to a region D_1 such that $g=f$ on $D_1 \cap D$. By the uniqueness theorem such continuation of f is uniquely determined. The Schwarz reflection principle is an example of how, in some cases, an analytic function can be continued beyond it's original domain of analyticity. In this unit, we shall examine the possibilities of such extensions for functions given by power series.

11.2 THREE TYPES OF ISOLATED SINGULARITIES

Definition :
A point at which the function f is not analytic is said to be a singular point or singularity of the function f.

e.g. $f(x) = \dfrac{z^2}{z-3}$

Here, f is not defined at $z=3$ and hence not analytic at $z=3$, therefore $z=3$ is singular point.

Definition : A point at which the function f is analytic is said to be a Regular point.

Definition : A function f has isolated singular point at $z = z_0$ if $\exists$ an $R>0$ s.t. f is defined and analytic in $0<|z-z_0|<R$ but not $B(z_0, R)$.

e.g. 1) $f(z) = \dfrac{z^2}{(z-1)(z-3)}$

$z=1$ and $z=3$ are points of singularity

2) $f(z) = \cot z = \dfrac{\sin z}{\cos z}$

Put $\cos z = 0 \Rightarrow z = n\pi$

$\therefore$ Singular points are $n\pi,\ n \in \mathbb{Z}$.

Definition : Let f be analytic $0<|z-z_0|<R$. Let z_0 be an isolated singular point of f. A point $z=z_0$ is said to be a Removable singularity of f, if $\exists$ an analytic function $g : B(z_0, R) \to \mathbb{C}$ g s.t. $f(z) = g(z)$ for $0<|z-z_0|<R$.(2007)

Or

Definition : If a single valued function $f(z)$ is not defined at a point $z = z_0$ but $\lim\limits_{z \to z_0} f(z)$ exists. Then $z = z_0$ is said to be a removable singularity of f.

e.g. $f(z) = \dfrac{\sin z}{z},\ z \neq 0$

In this case, f is not defined at a point $z=0$ but $\lim_{z\to 0}\frac{\sin z}{z}=1$ exists.

$\therefore$ $z=0$ is a Removable singularity of f.

OR

Define $g:B(0;R)\to\mathbb{C}$ s.t.

$$g(z)=\begin{cases}\frac{\sin z}{z} & z\neq 0\\ 1 & z=0\end{cases}$$

$\therefore$ $f(z)=g(z)$ for $0<|z-0|<R$

$$f(z)=\frac{z^2-9}{z-3}, z\neq 3$$

Here, f is not defined at $z=3$.

But, $\lim_{z\to 3} f(z)=\lim_{z\to 3}\frac{z^2-9}{z-3}=\lim_{z\to 3}\frac{(z-3)(z+3)}{z-3}=6$ exist.

$\therefore$ $z=3$ is a removable singularity of f.

Definition: A singular point which is not isolated is said to be Non-isolated singular point.

e.g. $f(z)=cosec\left(\frac{1}{z}\right)=1/\sin(z)$

In the delta nbd at zeros, there are other singular point of f $\therefore$ z = 0 is a non isolated singular point of f.

For, Singular points, Put $sin\left(\frac{1}{z}\right)=0$

$$\therefore \frac{1}{z}=n\pi \quad n\in\mathbb{Z}$$

$z=\frac{1}{n\pi}\to 0 n\to\infty$, $n=0,\pm 1,\pm 2,....$ .Since $z=\frac{1}{n\pi}\to 0$ as $n\to\infty$

Here, z = *0* is a non-isolated singular point, whereas other singular points are isolated.

Theorem: *If f has an isolated singular point at* z_0*, then* $z=z_0$ *is a removable singularity of f iff* $\lim_{z\to z_0}(z-z_0)f(z)=0$

Proof: Let $\lim_{z\to z_0}(z-z_0)f(z)=0$

T.P.T. $z=z_0$ is a removable singularity of f.

Given, f has an isolated singular point at $z=z_0$

$\therefore$ there exists $R > 0$ s.t. f is defined and analytic in $B(z_0;R)\setminus\{z_0\}$ but not in $B(z_0;R)$

Define

$$h(z)=\begin{cases}(z-z_0)\,f(z) & , \; z\neq z_0\\ 0 & , \; z=z_0\end{cases} \qquad \text{(I)}$$

$\therefore$ h is analytic in $B(z_0;R)\setminus\{z_0\}$ and

$$\lim_{z\to z_0} h(z)=\lim_{z\to z_0}(z-z_0)f(z)=0=h(z_0) \qquad \text{from (I)}$$

$\therefore$ h is continuous in $B(z_0;R)$

T.P.T. h is analytic in $B(z_0;R)$

i.e. T.P.T. $\int_{\partial\Delta} h(z)dz=0$ for every triangle $\Delta=\operatorname{int}\Delta+\partial\Delta$ in $B(z_0;R)$

There are four cases :

$\therefore$ By Morera's Theorem,

h is analytic in $B(z_0,R)$

from equation (I), $h(z_0)=0$

$\therefore$ z_0 is a zero (root) of h

$\therefore$ f an analytic function $g:B(z\,;R)\to\mathbb{C}$ $s.t.$ $h(z)=(z-z_0)\,g(z)$

where $g(z_0)\neq 0$

$\therefore$ for $0<|z-z_0<R|$

$$h(z)=(z-z_0)\,g(z)=(z-z_0)\,f(z) \qquad \text{by (I)}$$

$\therefore f(z)=g(z)$ $\qquad$ for $0<|z-z_0|<R$

$\Rightarrow$ $z=z_0$ is a removable singularity of *f*.

Conversely,

Suppose $z=z_0$ is a removable singularity of *f*.

T.P.T. $\lim_{z\to z_0}(z-z_0)\,f(z)=0$

By definition, f an analytic function $g:B(z_0,R)\to\mathbb{C}$

s.t. $f(z)=g(z)$ $\qquad$ for $0<|z-z_0|<R$

$$\lim_{z\to z_0} f(z)=\lim_{z\to z_0} g(z)=g(z_0)\neq 0$$

$$\therefore \lim_{z\to z_0}(z-z_0)\,f(z)=0\times g(z_0)=0$$

$$\therefore \lim_{z\to z_0}(z-z_0)\,f(z)=0$$

Definition:

If f has an isolated singular point at z_0, then $z = z_0$ is a pole of f if $\lim_{z \to z_0} f(z) = \infty$ i.e. for any M >0,

$\exists\ \delta > 0$ s.t. $|f(z)| \geq M$ where $0 < |z - z_0| < \delta$

Definition:

If f has a pole at $z = z_0$ and $m \geq 1$ is the smallest positive integer s.t. $(z - z_0)^m f(z)$ has a removable singularity at $z = z_0$ then, f has a pole of order m at $z = z_0$.

Definition:

A pole of order one is said to be a simple pole.

e.g. $f(z) = \dfrac{z^3}{z - 4}$

Here, z = 4 is an isolated singular point at f.

$\lim_{z \to 4} f(z) = \lim_{z \to 4} \dfrac{z^3}{z - 4} = \infty$

$\therefore z = 4$ is a simple pole of *f*.

$f(z) = \dfrac{z^2}{(z - z)(z - 1)^4}$

Here *f* has simple pole at $z = 2$ and $z = 1$ is a pole of order 4.

Essential Singularity:

An isolated Singular point which is neither a pole nor a removable singularity is said to be Essential singularity. e.g. $f(z) = e^{1/z}$

Here, f has essential singularity at $z = 0$. T.P.T. $z = 0$ neither a pole nor a removable singularity.

Theorem: *If a function f(z) of analytic for all finite values of z and as* $|z| \to \infty, |f(z)| = a|z|^k$ *then f(z) is a polynomial of degree* $\leq k$.

Proof: Since f(z) is analytic for all finite values of z therefore it can be expanded by Taylor's theorem in the form $f(z) = \sum_{n=0}^{\infty} a_n z^n$, for $|z| < R$, where R is large.

Let max $|f(z)| = M$ on the circle $|z| = r\,(r < R)$. Then by Cauchy's inequality, we have $|a_n| \leq \dfrac{M}{r^n}$ for all values of n

$= \frac{Ar^k}{r^n} = Ar^{n-k} r \to \infty \forall n > k \leq k$, since $|f(z)| = A|z|^k$ when $|z| \to \infty$

$= \frac{Ar^k}{r^n} = Ar^{n-k}$, which tends to zero when $r \to \infty$ since $n>k$.

Thus $a_n=0,\ n > k$.

Hence, we have $f(z)=a_0+ a_1z+ a_2 z^2+\dots\dots\dots +a_k z^k$, which is a polynomial of degree $\leq k$.

11.3 LAURENT'S THEOREM

Theorem: *If f is analytic in $G = a_{nn}(\alpha; R_2, R_1)$, $R_2 > 0$ then for any point z in G, f has unique representation $f(z) = \sum_{n=0}^{\infty} a_n (z-\alpha)^n + \sum_{n=1}^{\infty} \frac{bn}{(z-\alpha)^n}$ where, $a_n = \frac{1}{2\pi i}\int_{x_1} \frac{f(\xi)}{(\xi-\alpha)^{n+1}} d\xi$, $n = 0, 1, 2$ .. and $b_n = \frac{1}{2\pi i}\int_{x_2} \frac{f(\xi)}{(\xi-\alpha)^{-n+1}} d\xi$, $n = 1, 2, \dots\dots$*

and x_1, x_2 are circles $|\xi-\alpha| = r_1$, $|\xi-\alpha| = r_2$ respectively with $R_2 < r_2 < r_1 < R_1$

Proof: for a given $z \in G$, choose r_1 *and* r_2 s.t. $R_2 < r_2 < |z-\alpha| < r_1 < R_1$ by using Cauchy integral formula for multiply connected domain

$$f(z) = \frac{1}{2\pi i}\int_{x_1} \frac{f(\xi)}{\xi - z} d\xi - \frac{1}{2\pi i}\int_{x_2} \frac{f(\xi)}{\xi-2} d\xi \qquad (1)$$

Consider, $\frac{1}{2\pi i}\int_{x_1} \frac{f(\xi)}{\xi-2} d\xi$

For any point ξ *on* x_1

Consider $\frac{1}{\xi - z} = \frac{1}{\xi-\alpha+\alpha-z} = \frac{1}{(\xi-\alpha)\left[1-\frac{(z-\xi)}{(\xi-\alpha)}\right]}$

$$= \frac{1}{(\xi-\alpha)}\left[1+\frac{(z-\alpha)}{(\xi-\alpha)}+\dots\dots+\frac{(z-\alpha)^{n-1}}{(\xi-\alpha)^{n-1}}+\frac{(z-\alpha)^n}{(\xi-\alpha)^n}-\frac{1}{1-\frac{(z-\alpha)}{(\xi-\alpha)}}\right]$$

$$\dots\dots\left(\because \frac{1}{1-x} = 1+x+x^2+x^3+1\dots.+x^{n-1}+x^n\frac{1}{1-x}\right)$$

$$= \frac{1}{(\xi-\alpha)} + \frac{(z-\alpha)}{(\xi-\alpha)^2} + \ldots.. + \frac{(z-\alpha)^{n-1}}{(\xi-\alpha)^n} + \frac{(z-\alpha)^n}{(\xi-\alpha)^n(\xi-\alpha)}$$

Multiply the above equation by $\frac{f(\xi)}{2\pi i}$ // and then integrating w.r.t. ξ *over* x_1

$$\therefore \frac{1}{2\pi i}\int_{x_1}\frac{f(\xi)}{\xi-z}d\xi = \frac{1}{2\pi i}\int_{x_1}\frac{f(\xi)}{\xi-\alpha}d\xi + \frac{(z-\alpha)}{z\pi i}\int_{x_1}\frac{f(\xi)}{(\xi-\alpha)^2}d\xi + \ldots.$$

$$\ldots\ldots + \frac{(z-\alpha)^{n-1}}{2\pi i}\int_{x_1}\frac{f(\xi)}{(\xi-\alpha)^n}d\xi + \frac{(z-\alpha)^n}{2\pi i}\int_{x_1}\frac{f(\xi)}{(\xi-\alpha)^n(\xi-\alpha)}$$

Given that, $an = \frac{1}{2\pi i}\int_{x_1}\frac{f(\xi)}{(\xi-\alpha)^{n+1}}d\xi$ n = 0, 1, 2, 3 ……..

$$\therefore \frac{1}{2\pi i}\int_{x_1}\frac{f(\xi)}{\xi-z}d\xi = a_0 + a_1(z-\alpha) + 0\ldots. + a_{n-1}(z-\alpha)^{n-1} + R_n \qquad (2)$$

Where, $R_n = \frac{(z-\alpha)^n}{2\pi i}\int_{x_1}\frac{f(\xi)}{(\xi-\alpha)^n(\xi-z)}d\xi$

T.P.T. $\lim_{n\to\infty} R_n = 0$

$$|R_n| \le \frac{|z-\alpha|^n}{2\pi}\int_{x_1}\frac{|f(\xi)|}{|\xi-\alpha|^n}\frac{|d\xi|}{|\xi-z|} \qquad (3)$$

Choose $\varsigma > 0$ s.t. $|z-\alpha| = \varsigma$ equation of the circle x_1 is, $|\xi-\alpha| = r$,

Now, $|\xi-z| = |\xi-\alpha+\alpha-z| \ge |\xi-\alpha| - |z-\alpha| = r_1 - \varsigma$

Given that, f is analytic in G

$\therefore$ f is continuous on x_1 (Compact set)

By boundedness theorem

$\exists\, M_1 > 0$ s.t. $|f(\xi)| \le M$ $\forall\, \xi \in x_1$

Put all the above values in equation (3) we get

$$|R_n| \le \frac{\varsigma^n}{2\pi}\int_{x_1}\frac{M_1}{r_1^{\,n}(r_1-\varsigma)}|d\xi| = \frac{\varsigma^n M_1}{2\pi\, r_1^{\,n}(r_1-\varsigma)}\int_{x_1}|d\xi|$$

$$= \left(\frac{\varsigma}{r_1}\right)^n \frac{M_1}{2\pi(r_1-\varsigma)} \cdot 2\pi r_1 = \frac{M_1 r_1}{r_1-\varsigma}\left(\frac{\varsigma}{r_1}\right)^n \to 0 \; as \;\; n\to\infty$$

$$\therefore \lim_{n\to\infty} R_n = 0$$

$\therefore$ from equation (2)

$$\frac{1}{2\pi i}\int_{x_1}\frac{f(\xi)}{(\xi-z)}d\xi=\sum_{n=0}^{\infty}a_n(z-\alpha)^n \qquad (4)$$

Consider,

$$\frac{-1}{2\pi i}\int_{x_2}\frac{f(\xi)}{(\xi-z)}d\xi$$

For any point ξ on x_2,

Consider $\frac{-1}{\xi-z}=\frac{1}{z-\xi}=\frac{1}{z-\alpha+\alpha-\xi}=\frac{1}{z-\alpha\left[1-\frac{(\xi-\alpha)}{(z-\alpha)}\right]}$

$$=\frac{1}{(z-\alpha)}\left[1+\frac{(\xi-\alpha)}{(z-\alpha)}+......+\frac{(\xi-\alpha)^{n-1}}{(z-\alpha)^{n-1}}+\frac{(\xi-\alpha)^n}{(z-\alpha)^n}-\frac{1}{1-\frac{\xi\alpha}{z-\alpha}}\right]$$

$$=\frac{1}{z-\alpha}+\frac{(\xi-\alpha)}{(z-\alpha)^2}+.....+\frac{(\xi-\alpha)^{n-1}}{(z-\alpha)^n}+\frac{(\xi-\alpha)^n}{(z-\alpha)^n(z-\alpha)}$$

Solving in the same manner as above, we get,

$$\frac{-1}{2\pi i}\int_{x_2}\frac{f(\xi)}{(\xi-z)}d\xi=\sum_{n=1}^{\infty}\frac{b_n}{(z-\alpha)^n} \quad \_\_\_\_\_\_\_\_\_\_ (*)$$

From equation (1), (4) and $(*)$, we get

$$f(z)=\sum_{n=0}^{\infty}a_n(z-\alpha)^n+\sum_{n=1}^{\infty}\frac{b_n}{(z-\alpha)^n} \quad \_\_\_\_\_\_\_\_ \qquad (**)$$

Note: (i) equation $(**)$ can also be written as $f(z)=\sum_{n=-\infty}^{\infty}a_n(z-\alpha)^n$

for $R_2<|z-\alpha|<R_1$

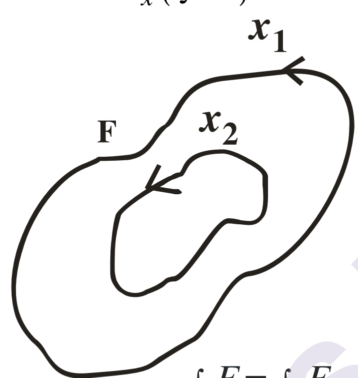
1) Where x is the circle $|\xi-\alpha|=r$ with $R_2<r<R_1$

$$a_n=\frac{1}{2\pi i}\int_{x}\frac{f(\xi)}{(\xi-\alpha)^{n+1}}d\xi \qquad n=0,\pm 1,\pm 2$$

Proof: If x is the circle $|\xi-\alpha|=r$ s.t. $R_2<r<R_1$, then both functions $\frac{f(\xi)}{(\xi-\alpha)^{n+1}}$ *and* $\frac{f(\xi)}{(\xi-\alpha)^{-n+1}}$ are analytic in $R_2<|\xi-\alpha|<R_1$ by using Cauchy De-formation Theorem,

$$a_n=\frac{1}{2\pi i}\int_{x_1}\frac{f(\xi)}{(\xi-\alpha)^{n+1}}\,d\xi=\frac{1}{2\pi i}\int_{x}\frac{f(\xi)}{(\xi-\alpha)^{n+1}}\,d\xi$$

$$b_n=\frac{1}{2\pi i}\int_{x_2}\frac{f(\xi)}{(\xi-\alpha)^{-n+1}}\,d\xi=\frac{1}{2\pi i}\int_{x}\frac{f(\xi)}{(\xi-\alpha)^{+n+1}}\,d\xi$$



$$\int_{x_1}F=\int_{x_2}F$$

Fig 11.1

We observe that, $b_n=a_{-n}$

From $(**)$

$$f(z)=\sum_{n=0}^{\infty}a_n(z-\alpha)^n+\sum_{n=1}^{\infty}a_{-n}(z-\alpha)^{-n}$$

$$=\sum_{n=0}^{\infty}a_n(z-\alpha)^n+\sum_{n=-1}^{-\infty}a_n(z-\alpha)^n$$

$$f(z)=\sum_{n=-\infty}^{\infty}a_n(z-\alpha)^n$$

Where $a_n=\frac{1}{2\pi i}\int_{x}\frac{f(\xi)}{(\xi-\alpha)^{n+1}}\,d\xi \qquad n=0,\pm1,\pm2.......$

2. The Laurent Series expansion is Unique.

Proof: Suppose that we have another Laurent series expansion

$$f(z)=\sum_{m=-\infty}^{\infty}A_m(z-\alpha)^m \quad \text{for } R_2<|z-\alpha|<R_1 \qquad (1)$$

We prove that equation (1) is identical with

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-\alpha)^n \qquad \text{for } R_2 < |z-\alpha| < R_1$$

Where $$a_n = \frac{1}{2\pi i} \int_x \frac{f(\xi)}{(\xi-\alpha)^{n+1}} d\xi, \qquad n = 0, \pm 1, \pm 2 ....$$

To prove that $a_n = A_m$

Let x be the circle $|\xi - \alpha| = r$ with $R_2 < r < R_1$

$$a_n = \frac{1}{2\pi i} \int_x \frac{f(\xi)}{(\xi-\alpha)^{n+1}} d\xi = \frac{1}{2\pi i} \int_x \sum_{m=-\infty}^{\infty} A_m (\xi-\alpha)^m \cdot \frac{1}{(\xi-\alpha)^{n+1}} d\xi$$

$$= \frac{1}{2\pi i} \int_x \sum_{m=-\infty}^{\infty} A_m (\xi-\alpha)^{m-n-1} d\xi = \frac{1}{2\pi i} \sum_{m=-\infty}^{\infty} A_m \int_x (\xi-\alpha)^{m-n-1} d\xi$$

$$= \frac{1}{2\pi i} \sum_{m=-\infty}^{\infty} A_m \int_0^{2\pi} \left(re^{i\theta}\right)^{m-n-1} i\, re^{i\theta}$$

$\because |\xi - \alpha| = r$

$\Rightarrow \xi = \alpha + re^{i\theta} \quad \theta \in [0, 2\pi]$

$$= \frac{1}{2\pi i} \sum_{m=-\infty}^{\infty} A_m r^{m-n} \int_0^{2\pi} e^{i(m-n)\theta} d\theta$$

. This integral $= 2\pi \quad , m = n$

$= 0 \qquad m \neq n$

$$= \frac{1}{2\pi i} A_m r^0 \, 2\pi$$

$a_n = A_m$

$\therefore$ Laurent series is unique in G.

Note:

1. The Laurent series for given function

is $$f(z) = \sum_{n=-1}^{-\infty} a_n (z-\alpha)^n + \sum_{n=0}^{-\infty} a_n (z-\alpha)^n \qquad R_2 < |z-\alpha| < R_1$$

1) The part $\sum_{n=-1}^{\infty} a_n (z-\alpha)^n$ of Laurent series is called the principal part of f(z) at $z = \alpha$.

2) The part $\sum_{n=0}^{\infty} a_n (z-\alpha)^n$ of Laurent series is called the Analytic part of f(z) at $z = \alpha$.

2. If f has a pole of order m at $z=\alpha$, then $f(z)=\dfrac{g(z)}{(z-\alpha)^m}$ where, g is analytic at a point α and $g(\alpha)\neq 0$

e.g. $f(z)=\dfrac{z^2}{(z-3)^3}$

$g(z)=z^2$ is analytic at point 3 and $g(3)=3^2=9\neq 0$.

11.4 CLASSIFICATION OF SINGULARITIES BY THE PRINCIPAL PART OF LAURRENT'S EXPANSION

Corollary: Let $z=\alpha$ be an isolated singularity of $f(z)$ and let $f(z)=\sum_{n=-\infty}^{\infty} a_n (z-\alpha)^n$ be its Laurent expansion in $a_{nn}(\alpha;0,R)$ $0<|z-\alpha|<R$ (Punctured disk or deleted nbd of α).

Then,

i) $z=\alpha$ is a removable singularity of f iff $a_n=0$ *for* $n\leq -1$ i.e. (Principal part is zero) (2008)

ii) $z=\alpha$ is a pole of order m iff $a_{-m}\neq 0$ *and* $a_n\neq 0$ for $n\leq -(m+1)$. i.e (Principal part is finite)

iii) $z=\alpha$ is an essential singularity of f iff $a_n\neq 0$ for infinitely many negative integers n. i.e. (Principal part is infinite)

Proof: Given, $f(z)=\sum_{n=-\infty}^{\infty} a_n (z-\alpha)^n$

$$f(z)=\sum_{n=-1}^{-\infty} a_n (z-\alpha)^n+\sum_{n=0}^{\infty} a_n (z-\alpha)^n \qquad (1)$$

i) Let $a_n=0$ for $n\leq -1$

T.P.T. $z=\alpha$ is a removable singularity of f.

From equation (1)

$$f(z)=\sum_{n=0}^{\infty} a_n (z-\alpha)^n \qquad (\because\ a_n=0\ \ for\ \ n\leq 1)$$

$$f(z)=a_0+\sum_{n=1}^{\infty} a_n (z-\alpha)^n$$

$$\lim_{z\to\alpha} f(z)=\lim_{z\to\alpha}\left[a_0+\sum_{n=1}^{\infty} a_n (z-\alpha)^n\right]=a_0+0=a_0\neq 0$$

$$\lim_{z\to\alpha}(z-\alpha)f(z)=\lim_{z\to\alpha}(z-\alpha)\bullet\lim_{z\to\alpha}f(z)=0\bullet a_0=0$$

$\therefore z=\alpha$ is a removable singularity of f

Conversely, suppose, $z=\alpha$ is a removable singularity of f.

$\therefore \exists$ an analytic function $g:B(\alpha;R)\to\mathbb{C}$ s.t.

$$f(z)=g(z) \qquad \text{in } 0<|z-\alpha|<R$$

$\because g$ is analytic in $B(\alpha;R)$

$\therefore$ for any point $z\in B(\alpha;R)$

g has Taylor series expansion

$$\therefore g(z)=\sum_{n=0}^{\infty}a_n(z-\alpha)^n$$

$\therefore$ the Laurent series expansion for f(z) must coincide with the Taylor series expansion for g(z) about $z=\alpha$

$$\therefore f(z)=\sum_{n=0}^{\infty}a_n(z-\alpha)^n$$

$\Rightarrow a_n=0 \;\; for \;\; n\le -1$ (Compare equation (1) and above equation)

E.g. $f(z)=\dfrac{z-\sin z}{z^3} \qquad , \; z\ne 0$

$$f(z)=\frac{z-\left[z-\frac{z^3}{3!}+\frac{z^5}{5!}+\frac{z^7}{7!}+\ldots\ldots\right]}{z^3}=\frac{1}{3!}-\frac{z^2}{5!}+\frac{z^4}{7!}-\ldots\ldots\ldots\ldots$$

Principal part is zero i.e. $a_n=0 \quad for \;\; n\le -1$.

This is a Laurent series expansion for f(z) but principal part contain no negative power of z.

i.e. $a_n=0 \quad for \;\; n\le -1$

$\therefore z=0$ is a removable singularity of *f*.

ii) Given $a_{-m}\ne 0 \;\; and \;\; a_n = \;\; for \;\; n \le -(m+1)$

$$f(z)=\sum_{n=-\infty}^{\infty}a_n(z-\alpha)^n=\sum_{n=-1}^{\infty}a_n(z-\alpha)^n+\sum_{n=0}^{\infty}a_n(z-\alpha)^n$$

$$f(z)=\frac{a_{-m}}{(z-\alpha)^m}+\ldots\ldots+\frac{a_{-1}}{(z-\alpha)}+\sum_{n=0}^{\infty}a_n(z-\alpha)^n$$

Multiplying above equation by $(z-\alpha)^m$,

$$(z-\alpha)^m f(z) = a_{-m} + \ldots\ldots + a_{-1}(z-\alpha)^{m-1} + (z-\alpha)^m \sum_{n=0}^{\infty} a_n (z-\alpha)^n$$

$$\lim_{z\to\alpha} (z-\alpha)^m f(z) = \lim_{z\to\alpha} \left[a_{-m} + \ldots\ldots + a_{-1}(z-\alpha)^{m-1} + (z-\alpha)^m \sum_{n=0}^{\infty} a_n (z-\alpha)^n \right]$$

$$= a_{-m} + 0 + 0 + \ldots. + 0 + 0 \;\; = a_{-m} \neq 0$$

$$\therefore \lim_{z\to\alpha} (z-\alpha)^{m+1} f(z) = \lim_{z\to\alpha} (z-\alpha) a_{-m} = 0$$

$\therefore$ the function $(z-\alpha)^m f(z)$ has a removable singularity at $z=\alpha$

$\therefore$ $z=\alpha$ is a pole of order m

Converse, (Exercise)

E.g. $f(z) = \dfrac{e^z}{z^3}$, Here f has pole of order 3 at z = 0

$$f(z) = \frac{\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots.\right)}{z^3} = \frac{1}{z^3}\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots.\right)$$

$$= \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{2!.z} + \frac{1}{3!} + \frac{z}{4!} + \ldots\ldots.$$

This is a Laurent series expansion for *f (z)* but principal part of Laurent series is finite.

$\therefore$ f has a pole of order 3 at $z = 0$

iii) Combine part (i) and (ii) and by definition essential singularity, we see that $z-\alpha$ is an essential singularity of f iff $a_n \neq 0$ for infinitely many negative integers n.

e.g. $f(z) = e^{1/z}$

$$= \underbrace{1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \frac{1}{4!z^4} + \ldots\ldots.}_{\textit{Prinicpal part}}$$

This is a Laurent series expansion for *f(z)* and principal part of Laurent series is infinite.

$\therefore z=0$ is an essential singularity of *f*.

Theorem: *Let f be analytic in* $0 < |z-\alpha| < R \;\; (R>0)$. *Then, f has a pole of order m at* $z=\alpha$ *iff there exists an analytic function,* $g : B(\alpha; R) \to \mathbb{C}$ *s.t.* $f(z) = \dfrac{g(z)}{(z-\alpha)^m}$ *where* $g(\alpha) \neq 0$

(**Note:** We know that the geometric series $\sum_{n=0}^{\infty} z^n$ converges for $|z|<1$ and we write $\frac{1}{1-z}=\sum_{n=0}^{\infty} z^n = 1+z+z^2+z^3+.......$

Similarly, the geometric series $\sum_{n=0}^{\infty} (-1)^n z^n$ converges for $|z|<1$ and we write $\frac{1}{1+z}=\sum_{n=0}^{\infty} (-1)^n z^n = 1-z+z^2-z^3+z^4......$)

Example:1

Expand $f(z)=\frac{5}{z^2+z-6}$ in a Laurent series valid for (i) $2<|z|<3$ (ii) $|z|<2$ (iii) $|z|>3$ (iv) $0<|z-2|<4$

Solution: Give function

$$f(z)=\frac{5}{z^2+z-6}=\frac{5}{(z-2)(z+3)}=\frac{1}{z-2}-\frac{1}{z+3} \qquad (1)$$

1) For $2<|z|<3$, if $2<|z|$ then $\frac{2}{|z|}<1$

$$\frac{1}{z-2} = \frac{1}{z\left(1-\frac{2}{z}\right)} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n = \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}}$$

if $|z|<3$ *then* $\frac{|z|}{3}<1$

$$\frac{1}{z+3}=\frac{1}{3\left(1+\frac{z}{3}\right)}=\frac{1}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n = \sum_{n=0}^{\infty}(-1)^n \frac{z^n}{3^{n+1}}$$

From equation (1), we get

$$f(z)=\sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}}-\sum_{n=0}^{\infty}(-1)^n\frac{z^n}{3^{n+1}}$$

$$=.........+\frac{2^3}{z^4}+\frac{2^2}{z^3}+\frac{2}{z^2}+\frac{1}{2}-\left[\frac{1}{3}-\frac{z}{3^2}+\frac{z^2}{3^3}-.....\right]$$

$$=........+\frac{2^3}{z^4}+\frac{2^2}{z^3}+\frac{2}{z^2}+\frac{1}{z}-\frac{1}{3}+\frac{z}{3^2}+\frac{z^2}{3^3}+.......$$ (In this case, z = 0 is an essential singularity)

This is the required Laurent series for $2<|z|<3$

ii) For $|z| < 2$

$\because \ |z| < 2 < 3 \Rightarrow \frac{|z|}{2} < 1 \ \ and \ \ \frac{|z|}{3} < 1$

$$\frac{1}{z-2} = \frac{-1}{2\left(1-z/2\right)} = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n = -\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}$$

$$\frac{1}{z+3} = \frac{1}{3\left(1+\frac{z}{3}\right)} = \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n = \sum_{n=0}^{\infty}(-1)^n\frac{z^n}{3^{n+1}}$$

$\therefore$ from equation (1)

$$f(z) = -\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}} - \sum_{n=0}^{\infty}(-1)^n\frac{z^n}{3^{n+1}}$$

This is the required Laurent series for $|z| < 2$ (In this case, z = 0 is removable singularity)

iii) For $|z| > 3$

$\because |z| > 3 > 2 \Rightarrow \frac{2}{|z|} < 1 \ and \ \frac{3}{|z|} < 1$

$$\frac{1}{z-2} = \frac{1}{z\left(1-2/z\right)} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n = \sum_{n=0}^{\infty}\frac{2^n}{2^{n+1}}$$

$$\frac{1}{z+3} = \frac{1}{z\left(1+3/z\right)} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{3}{z}\right)^n = \sum_{n=0}^{\infty}\frac{3^n}{z^{n+1}}$$

from equation (1),

$$f(z) = \sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}} - \sum_{n=0}^{\infty}\frac{3^n}{z^{n+1}}$$

This is the required Laurent series (In this case z = 0 is essential singularity)

iv) For $0 < |z-2| < 4$

put z – 2 = u $\Rightarrow$ z = u + 2

$\therefore$ from equation (1)

$$f(z) = \frac{5}{(z-2)(z+3)} = \frac{5}{(z-2)(z-2+5)} = \frac{5}{u(u+5)}$$

$$0 < |u| < 4 < 5 \ \Rightarrow \ \frac{|u|}{5} < 1$$

$$\therefore f(z) = \frac{5}{4\cdot 5\left(1+u/5\right)} = \frac{1}{u}\sum_{n=0}^{\infty}(-1)^n\left(\frac{u}{5}\right)^n$$

Example: 2

Expand $f(z)=\dfrac{4}{z^2+2z-3}$ in a Laurent series valid for

(i) $1<|z|<3$ (ii) $|z|>3$ (iii) $|z|<1$

(iv) $0<|z-1|<4$

Example 3:

Expand $f(z)=\dfrac{\cos(z)-1}{z^4}$ in a Laurent series about z = 0 and name the singularity .

Solution: $f(z)=\dfrac{Cosz-1}{z^4}$

$$=\frac{\left[1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+........\right]-1}{z^4}=\frac{-1}{2!z^2}+\frac{1}{4!}-\frac{z^2}{6!}+..........$$

Z = 0 is a pole of order 2

Example 4: Find the Laurent series of *f(z)*= $\dfrac{1}{z(z-1)(z-2)}$ in the annular region $1<|z|<2$. (2012)

Solution: *f(z)*= $\dfrac{1}{z(z-1)(z-2)}$

By partial fraction $for\quad 1<|z|<2,\quad \frac{1}{z}<1 \quad \& \quad \frac{z}{2}<1$

$$\therefore f(z)=\frac{1}{2z}-\frac{1}{z\left(1-1/z\right)}-\frac{1}{4\left(1-z/2\right)}$$

$$=\frac{1}{2z}-\frac{\left(1-1/z\right)^{-1}}{z}-\frac{\left(1-z/2\right)^{-1}}{4}$$

$$==\frac{1}{2z}-\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+.....\right)-\frac{1}{4}\left(1+\frac{z}{2}+\frac{z^2}{4}+........\right)$$

Definition:

A set D in a Metric space X is said to be Dense iff $\overline{D}=X$, where $\overline{D}$, is closure of D.

OR

Let D be a subset of $\mathbb{C}(D\subseteq\mathbb{C})$, we say that D is dense in $\mathbb{C}$ if, for any $\omega_0\in\mathbb{C}\ and \in>0\ \ B(\omega_0,\in)\,n\,D\neq\phi$

11.5 CASORATI -WEIERSTRASS THEOREM

If f has an essential singularity at $z=\alpha$, *then for every* $\delta>0$, $f\left[a_{nn}(\alpha;0,\delta)\right]$ *is dense in* $\mathbb{C}$. *(2005, 2008)* ***OR*** $\overline{f\left(a_{nn}(\alpha;0,\delta)\right)}=\mathbb{C}$

Proof: Let $G=a_{nn}(\alpha;0,\delta)=0<|z-\alpha|<\delta=B(\alpha;\delta)\setminus\{\alpha\}$
Given that, f has an essential singularity at $z=\alpha$
$\therefore$ f is analytic in G.
T.P.T. f(G) is dense in $\mathbb{C}$ i.e. $\{f(G)\}=\mathbb{C}$
i.e. T.P.T. for given $\omega_0\in\mathbb{C}$,$\in>0$, $\delta>0,\exists z$
s.t. $|z-\alpha|<\delta$ *and* $|f(z)-\omega_0|<\in$

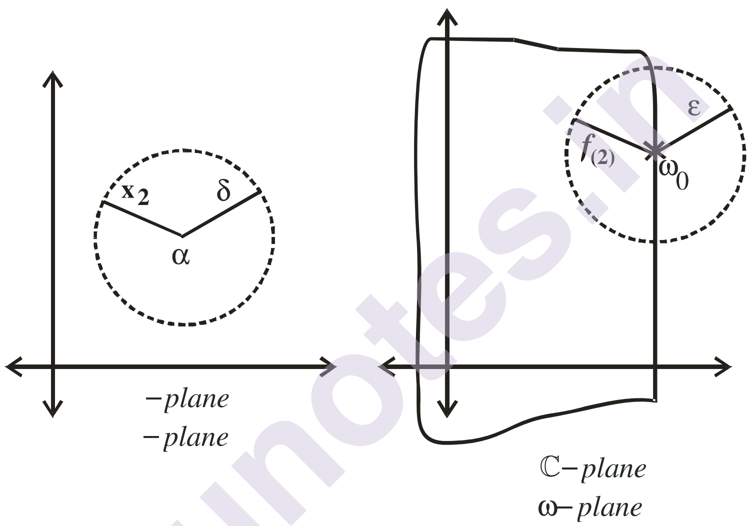


Fig 11.1

Assume this is not true
i.e. Assume there is $\omega_0\in\mathbb{C}$ *and* $\in>0$ s.t.

$|f(z)-\omega_0|\geq\in \qquad \forall 2\in G=a_{nn}(\alpha;0,\delta)$

$\therefore \lim_{z\to\alpha}\frac{|f(z)-\omega_0|}{|z-\alpha|}=\infty$

The function $\frac{f(z)-\omega_0}{z-\alpha}$ has a pole at $z=\alpha$
$z=z_0$ is a pole of *f* if $\lim_{z\to z_0} f(z)=\infty$

Note: If $\lim_{x\to\alpha}|x-\alpha|=0$ then $\lim_{x\to\alpha}|x-\alpha|^2=0$

If $m \geq 1$ is the order of this pole then, $(z-\alpha)^m \left[\frac{f(z)-\omega_0}{(z-\alpha)}\right]$ has a removable singularity at $z=\alpha$.

$\therefore \lim_{z\to\alpha} |z-\alpha|^{m+1} \; |f(z)-\omega_0| = 0$ (1)

Now, $|z-\alpha|^{m+1}|f(z)| = |z-\alpha|^{m+1}|f(z)-c+c|$

$\therefore \lim_{z\to\alpha}\left[|z-\alpha|^{m+1}|f(z)|\right] \leq \lim_{z\to\alpha}\left[|z-\alpha|^{m+1}|f(z)-c| + |z-\alpha|^{m+1}|c|\right]$

= 0 + 0 --------- by (1)

$\therefore \lim_{z\to\alpha} |z-\alpha|^{m+1} |f(z)| = 0$

$\therefore$ the function $(z-\alpha)^m f(z)$ has a removable singularity at $z=\alpha$

$\therefore f$ has a pole of order m at $z=\alpha$ which contradicts the hypothesis that $z=\alpha$ is an essential singularity of *f*.

$\therefore$ Our assumption was wrong

Hence, $|f(z)-\omega_0| < \in$ $\forall z \in G$

$\Rightarrow f(G)$ is dense in $\mathbb{C}$ i.e. $\{f(G)\} = \mathbb{C}$

$* \; z = z_0$ is a removable singularity $\Leftrightarrow \lim_{z\to z_0} (z-z_0) f(z) = 0$

$**$ If $z=\alpha$ is an essential singularity of *f* then f has Laurent series expansion about $z=\alpha$.

$***$ If f is analytic in $B(\alpha;\gamma)$ than for any $z \in B(\alpha;\gamma)$ and has a Taylor series expansion about $z = \alpha$.

$\because \; g(z) = (z-\alpha)^m f(z) \Rightarrow g^{(m-1)}(z) = \frac{d^{m-1}}{dz^{m-1}}(z-\alpha)^m f(z)$

$\Rightarrow g^{(m-1)}(\alpha) = \lim_{z\to\alpha} g^{(m-1)}(z) = \lim_{z\to\alpha} \frac{d^{m-1}}{dz}(z-\alpha)^m f(z)$

11.6 SUMMARY

1) A point at which the function f is not analytic is said to be a singular point or singularity of the function *f*.

2) A function f has isolated singular point at $z = z_0$ if $\exists$ an $R > 0$ s.t. f is defined and analytic in $0 < |z - z_0| < R$ but not $B(z_0, R)$.

3) If f has an isolated singular point at z_0, then $z = z_0$ is a removable singularity of f iff $\lim_{z \to z_0} (z - z_0) f(z) = 0$

4) If f has a pole at $z = z_0$ and $m \geq 1$ is the smallest positive integer s.t. $(z - z_0)^m f(z)$ has a removable singularity at $z = z_0$ then, f has a pole of order m at $z = z_0$

5) Laurent Theorem: If f is analytic in $G = a_{nn}(\alpha; R_2, R_1)$, $R_2 > 0$ then for any point z in G, f has unique representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - \alpha)^n + \sum_{n=1}^{\infty} \frac{bn}{(z - \alpha)^n}$$

Where, $a_n = \frac{1}{2\pi i} \int_{x_1} \frac{f(\xi)}{(\xi - \alpha)^{n+1}} d\xi$ n = 0, 1, 2, … , $b_n = \frac{1}{2\pi i} \int_{x_2} \frac{f(\xi)}{(\xi - \alpha)^{-n+1}} d\xi$ n = 1, 2, 3, ……………

and x_1, x_2 are circles $|\xi - \alpha| = r_1$, $|\xi - \alpha| = r_2$ respectively with $R_2 < r_2 < r_1 < R_1$.

6) Let f be analytic in $0 < |z - \alpha| < R$ $(R > 0)$ Then, f has a pole of order m at $z = \alpha$ iff there exists an analytic function,

$g : B(\alpha; R) \to \mathbb{C}$ s.t.

$f(z) = \frac{g(z)}{(z - \alpha)^m}$ where $g(\alpha) \neq 0$

7) Casorati Weierstrass Theorem:

If f has an essential singularity at $z = \alpha$, then for every $\delta > 0$, $f\left[a_{nn}(\alpha; 0, \delta)\right]$ is dense in $\mathbb{C}$.

11.7 UNIT END EXERCISES

1) Each of the following functions f has an isolated singularity at $z = 0$. Determine it's nature , if it is removable singularity , define $f(0)$, so that f is analytic at $z = 0$ if it is a pole, find the singular part; if it is an essential singularity determine $f(\{z : 0 < |z| < \delta\})$ for arbitrarily small values.

(a) $f(z)=\frac{\sin(z)}{z}$ (b) $f(z)=\frac{\cos(z)-1}{z}$

(c) $f(z)=\frac{z^2+1}{z(z-1)}$ (d) $f(z)=z\sin\left(\frac{1}{z}\right)$.

Solution: (a) $\therefore\ \lim_{z\to 0} f(z)=\lim_{z\to 0}\frac{\sin(z)}{z}=1$ exists.

$\therefore\ z=0$ is a removable singularity of f.

Define $g:B(0;r)\to\mathbb{C}$ as $g(z)=\frac{\sin(z)}{z}, z\neq 0 = 1, z=0$.

Then $f(z)=g(z)$ for $0<|z|<R$ and g is analytic on $B(0;r)$.

(b) $\therefore\ \lim_{z\to 0}(z-0)f(z)=\lim_{z\to 0}\frac{z(\cos(z)-1)}{z}=0$

$\therefore$ By theorem on removable singularity, $z=0$ is a removable singularity of f. $\therefore$ Define $f(0)=0$.

Define $g:B(0;r)\to\mathbb{C}$ as $f(z)=\frac{\cos(s)-1}{z}$ $\quad z\neq 0$

$\quad =0 \quad z=0$

Then $f(z)=g(z)$ for $0<|z|<R$ and g is analytic on $B(0;r)$.

(c) f has a pole at $z=0$.

2) Classify the singularities of (a) cot(z) (b) $\frac{\exp(1/z^2)}{z-1}$

(c) $f(z)=\frac{\sin z}{z(z-1)(z-2)^2}$ (2005)

Solution: (b) Let $f(z)=\frac{\exp\left(\frac{1}{z^2}\right)}{z-1}$.

$\because \lim_{z\to 1}|f(z)|=\lim_{z\to 1}\left|\frac{\exp\left(\frac{1}{z^2}\right)}{z-1}\right|=\infty \Rightarrow f(z)$ has a pole at $z=1$.

Since $f(z)=\frac{A(z)}{B(z)}$, where $A(z)=\exp\left(\frac{1}{z^2}\right), B(z)=z-1$.

$A(1)\neq 0, B(1)=0 \Rightarrow B$ has a zero of order 1 at $z=1\Rightarrow$
$f(z)$ has a pole of order 1 at $z=1$.

We know that f is analytic in a deleted neighbourhood $B(0;R)-0$ of 0 and f is not analytic at $z=0\Rightarrow z=0$ is a singularity of f. Since we know that $\lim_{z-1}\frac{\exp\left(\frac{1}{z^2}\right)}{z-1}$ does not exist.

$\therefore\ z=0$ is not a removable singularity of *f*.

$f(z)$ cant be written as $f(z)=\dfrac{A(z)}{B(z)}=\dfrac{A(z)}{z^2}$, where A and B are analytic at $z=0$ $\therefore$ $z=0$ is not a pole of *f*.

$\therefore\ z=0$ is an essential singularity of *f*.

(c) $f(z)=\dfrac{\sin z}{z(z-1)(z-2)^2}$

Solution: *f(z)* has pole at *z=0, z=1, z=2* of order *1,1,2* respectively.

3) Find the Laurent series expansion of (a) $\dfrac{1}{z^2+z^2}$ about $z=0$.

(b) $\dfrac{1}{z^2-4}$ about $z=0$.

Solution:

(a)

Let $f(z)=\dfrac{1}{z^4+z^2}=\dfrac{1}{z^2}-\dfrac{1}{1+z^2}=z^{-2}-\dfrac{1}{1-(-z^2)}=z^{-2}-\sum_{k=0}^{\infty}(-1)^k z^{2k}$

$=\sum_{k-1}^{\infty}(-1)^{k+1}z^{2k}$

4) Check whether $z=0$ is a removable singularity of $f(z)=\dfrac{\sin(z)}{z}$ or not.

Solution: $f(z)=\dfrac{\sin(z)}{z}$

$\therefore\ f(z)=\dfrac{\sin(z)}{z}=1-\dfrac{z^2}{3!}+\dfrac{z^4}{5!}-\ldots$

Here all the coefficients $c_{-k}=0$ for $k>0$.

$\therefore$ *f(z)* has removable singularity at *z=0*

5) Show that the image of $B(0;1)-0$ under the function $f(z)=\operatorname{cosec}\left(\dfrac{1}{z}\right)$ is dense in the Complex plane. (Hint : $z=0$ is essential singularity of $\sin\left(\dfrac{1}{z}\right)$. Make use of the following theorem: If *f* is analytic in a deleted neighbourhood *D* of z_0 except for poles at all points of a sequence $\{z_n\}\to z_0$. Then $f(D)$ is dense in the Complex plane.

6)Expand $f(z) = ze^{1/z^2}$ in a Laurent series about z = 0 and name the singularity.

7) Expand $f(z) = \frac{e^{2z}}{(z-1)^3}$ in a Laurent series about z = 1 and name the singularity. (Hint: $f(z) = \frac{e^{2z}}{(z-1)^3} = \frac{e^{2z} \cdot e^{-2} e^2}{(z-1)^3} = \frac{e^2 \left(e^{2(z-1)}\right)}{(z-1)^3}$)

8) Determine the number of zeroes, counting multiplicities, of the polynomial $z^4-2z^3+9z^2+z-1$ inside the circle $|z| = 2$

9) Expand $f(z) = \frac{z}{(z-1)(2-z)}$ in a Laurent series valid for (i) $0 < |z| < 2$ (ii) $|z| < 1$ (2008)

10) Expand $f(z) = \frac{e^{2z}}{(z-1)^3}$ in a Laurent series about $z=1$ and name the singularity. (2007)

11) Expand $f(z) = \frac{z}{(z-1)(z-2)}$ in a Laurent series valid for (i) $1 < |z| < 2$ (ii) $|z| > 2$ (2007)

12) Expand $f(z) = \frac{-1}{(z-1)(z-2)}$ in a Laurent series valid for

(i) $1 < |z| < 2$, (ii) $2 < |z| < \infty$.

13) Expand $f(z) = (z-3)\sin\left(\frac{1}{z+2}\right)$ in a Laurent series about $z = -2$ and name the singularity.

(14) Expand $f(z) = \frac{1}{z^2+4z+3}$ in a Laurent series valid for

(i) $1 < |z| < 3$, (ii) $|z| < 3$.

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12

RESIDUE CALCULUS AND MERMORPHIC FUNCTIONS

Unit Structure

12.0 Objectives
12.1 Introduction
12.2 The Residue Theorem and it's Application
12.3 Evaluation of Standard Types of Integrals by the Residue Calculus Method
12.4 Argument Principle
12.5 Rouche's Theorem
12.6 Summary
12.7 Unit End Exercises

12.0 OBJECTIVES

In this unit we shall study the generalisation of the Cauchy closed curve theorem to functions having isolated singularities. We shall prove the Residue theorem and further we shall use it to evaluate the standard types of integrals like $\int_{-\infty}^{\infty} \frac{\sin(x)}{x} dx, \int_0^{\infty} \frac{dx}{1+x^2}, \int_{|z|=1} f(z)dz$ etc. . We shall also prove the Argument Principle and Rouche's theorem for Meromorphic functions in the complex plane $\mathbb{C}$.

12.1 INTRODUCTION

In this unit, we now seek to generalize the Cauchy closed curve theorem to functions, which have isolated singularities. If is a circle surrounding a single isolated singularity z_0 and $f(z) = \sum_{k=-\infty}^{\infty} c_k (z - z_0)^k$ in a deleted neighbourhood of z_0 that contains trace of a circle γ, then $f_{\gamma} f = 2\pi i c_{-1}$. Thus the coefficient c_{-1} is of special significance in this context. We shall see some of the applications of the Residue theorem. Let us start defining the Meromorphic functions.

Residues: Let f has an isolated singularity at $z=\alpha$ and

Let $f(z)=\sum_{n=-\infty}^{\infty} a_n(z-\alpha)^n$ be its Laurent expansion about $z=\alpha$ Res $(f;\alpha)=$ Coefficient of $(z-\alpha)^{-1}$ in Laurent series $=a_{-1}$.

12.2 THE RESIDUE THEOREM AND IT'S APPLICATIONS

Proposition: *If f has a pole of order m at $z=\alpha$ and $g(z)=(z-\alpha)f(z)$ then, Res $(f;\alpha)=\frac{g^{(m-1)}(\alpha)}{(m-1)!}$*

Proof: Given that, f has a pole of order m at $z=\alpha$

$\therefore z=\alpha$ is an isolated singularity of f

$\therefore$ by definition, $\exists R>0$ s.t.

$\therefore f$ is analytic in $0<|z-\alpha|<R$ *or* $B(\alpha;R)\setminus\{\alpha\}$

$\because$ f has a pole of order m at $z=\alpha$

$\therefore f(z)=\frac{g(z)}{(z-\alpha)^m}$, *where* $g(\alpha)\neq 0$ and g is analytic in $B(\alpha;R)$.

$\therefore$ for any $z\in B(\alpha;R)$, g has Taylor expansion about $z=\alpha$

$$g(z)=\sum_{n=0}^{\infty} a_n(z-\alpha)^n \text{ where } a_n=\frac{g^{(n)}(\alpha)}{n!} \quad \text{---------} \quad (2)$$

$$g(z)=a_0+a_1(z-\alpha)+........+a_{m-1}(z-\alpha)^{m-1}+a_m(z-\alpha)^m+a_{m+1}(z-\alpha)^{m+1}$$

$$\therefore f(z)=\frac{g(z)}{(z-\alpha)^m}=\frac{a_0}{(z-\alpha)^m}+\frac{a_1}{(z-\alpha)^{m-1}}+.......+\frac{a_{m-1}}{(z-\alpha)}$$

$$+a_m+a_{m+1}(z-\alpha)+a_{m+2}(z-\alpha)^2+.........$$

This is a Laurent expansion for f(z) about $z=\alpha$

$\therefore$ by definition of residue,

Res $(f;\alpha)=$ Coefficient of $(z-\alpha)^{-1}$ in a Laurent series $=a_{m-1}$

$$\text{Res } (f;\alpha)=\frac{g^{(m-1)}}{(m-1)!} \quad \text{-----------------------} \quad \left(\because a_n=\frac{g^{(n)}(\alpha)}{n!}\right)$$

Calculation of Residues:

1) If f has a pole of order mat $z=\alpha$ then,

Res $(f;\alpha)=\frac{1}{(m-1)!}\lim_{z\to\alpha}\left[\frac{d^{m-1}}{dz^{m-1}}(z-\alpha)^m f(z)\right]$ Where m = order

2) If f has a simple pole at $z=\alpha$, then,

Res $(f;\alpha)=\lim_{z\to\alpha}\left[(z-\alpha)f(z)\right]$

Example: Determine the residue of $f(z)=\frac{e^2}{(z-2)(z-3)^2}$ at its poles

Solution: Given function, $f(z)=\frac{e^z}{(z-2)(z-3)^2}$

Here f has simple pole at z = 2 and z = 3 is a pole of order 2.

For z= 2:

Res $(f;\alpha)=\lim_{z\to\alpha}\left[(z-\alpha)f(z)\right]$

Res$(f;2)=\lim_{z\to 2}\left[(2-2)\cdot\frac{e^2}{(z-2)(2-3)^2}\right]=\lim_{z\to 2}\frac{e^2}{(2-3)^2}$

$=\frac{e^2}{(2+3)^2}=\frac{e^2}{(-1)^2}\;=e^2$

For z=3

$Res(f;\alpha)=\frac{1}{(m-1)!}\lim_{z\to\alpha}\left[\frac{d^{m-1}}{dz^{m-1}}(z-\alpha)^m f(z)\right]$

$\alpha=3,\quad m=2$

$\therefore \operatorname{Re}s(f;3)=\frac{1}{(2-1)!}\lim_{z\to 3}\left[\frac{d}{d^2}\cancel{(z-3)^2}\frac{e}{(z-2)\cancel{(z-3)^2}}\right]$

$=\lim_{z\to 3}\left[\frac{d}{d^2}\left(\frac{c^2}{z-2}\right)\right]$

$=\lim_{z\to 3}\left[e^z\frac{-1}{(z-2)^2}+\frac{1}{(z-2)}e^2\right]=-e^3\times\frac{1}{(3-2)^2}+\frac{e^3}{(3-2)}\;=-e^3+e^3\quad=0$

Example: Determine the residues of $f(z)=\cot(z)$ at its poles

Solution: Here

$$\int_{x} f(z)dz = \int_{x_1} f(z)dz + \int_{x_2} f(z)dz + ... + \int_{x_m} f(z)dz$$

$$\therefore \int_{x_k} f(z)dz = \sum_{n=-\infty}^{\infty} a_n \int_0^{2\pi} (r_k e^{i\theta})^n i r_k e^{i\theta} d\theta$$

$$0 < |z - z_0| < \delta 0 < |z| < 1$$

$$f(z) = \frac{e^{2z}}{(z-1)^3}$$

$$f(z) = \cot z = \frac{\cos z}{\sin z}$$

$\therefore f$ has simple pole at $n\pi$, where n is an integer

$$\operatorname{Re}s(f;\alpha) = \lim_{z\to\alpha}\left[(z-\alpha)f(z)\right]$$

$$\operatorname{Re}s(f,n\pi) = \lim_{z\to n\pi}\left[(z-n\pi).\frac{\cos z}{\sin z}\right] \qquad \frac{0}{0} \text{ form}$$

By using L'hospital rule, $\displaystyle \lim_{z\to n\pi}\frac{-(z-n\pi)\sin z + \cos z}{\cos z} = 1$

Example: Compute the residue of $f(z) = \dfrac{\sin z}{z^4}$ at the pole $z=0$. (2012)

Solution:

$$f(z) = \frac{\sin z}{z^4} = \frac{1}{z^4}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + ...\right)$$

$$= \frac{1}{z^3} - \frac{1}{6z} + \frac{z}{5!} - \frac{z^3}{7!}.....$$

$$\operatorname{Re}s(f,0) = \textit{coefficient of } 1/z = \frac{-1}{6}$$

Residue Theorem or Cauchy Residue Theorem

Let f be analytic in a domain G except for the isolated singular points $z_1, z_2, ...z_m$. If x is a simple closed curve which does not pass through an of the points z_k then

$$\int_x f(z)d\,z = 2\pi i \sum_{k=1}^{m} \operatorname{Re}s(f;z_k)$$

$= 2\pi i \times$*[sum of residue of f at its pole inside x], where x is traversed in anticlockwise direction. (2008)*

Proof: Given that, $z_1, z_2, z_3 ....z_m$ are isolated singular points in G. Assume these points lie inside x

Choose positive numbers $r_1, r_2, ...r_m$ so small that no two circles $x_k : |z - z_k| = r_k$ intersect $k = 1, 2........m$ and every circle $x_k \ (k-1, 2, ...m)$ is inside x.

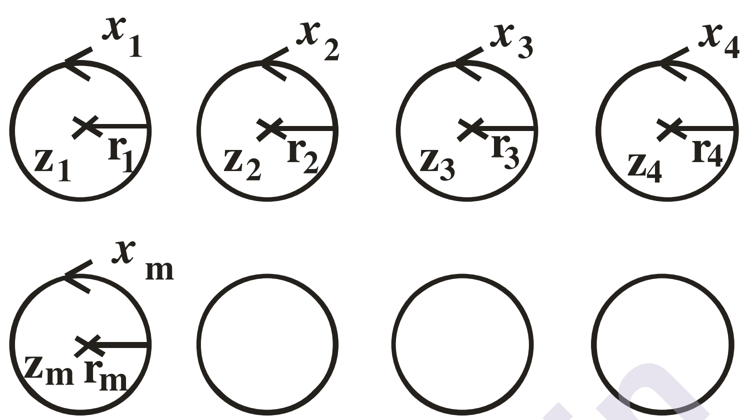

Fig 12.1

$\therefore$ the function f is analytic in a domain which is bounded by non-intersecting closed curves $x_1, x_2, ...x_m$ and on the curves.

By using Cauchy Deformation Theorem

$$\int_x f(z)dz = \int_{x_1} f(z)dz + \int_{x_2} f(z)dz + ... + \int_{x_m} f(z)dz$$

$$\therefore \int_x f(z)dz = \sum_{k=1}^{m} \int_{x_k} f(z)dz \qquad \text{..............}(1)$$

$\therefore f$ has an isolated singular point at $z = z_k$

$\therefore f$ has Laurent expansion about $z = z_k$

$$\therefore f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_k)^n \qquad 0 < |z - z_k| < r_k$$

Any point on the circle = centre + radius$\times e^{i\theta}$

Consider,

$$x_k \int f(z)\, dz = x_k^{\int \sum_{n=-\infty}^{\infty} a_n (z - z_k)^n} = \sum_{n=-\infty}^{\infty} x_k^{\int (z-z_k)^n dz}$$

Any point on the circle x_k is,

$$z = z_k + r_k e^{i\theta} \qquad \theta \in [0, 2\pi]$$

$$dz = i r_k e^{i\theta} d\theta$$

$$\therefore \int_{x_k} f(z)dz = \sum_{n=-\infty}^{\infty} a_n \int_0^{2\pi} (r_k e^{i\theta})^n i r_k e^{i\theta} d\theta = \sum_{n=-\infty}^{\infty} a_n . i r_k^{n+1} \int_0^{2\pi} e^{i(n+1)\theta} d\theta$$

> Note: $\int_0^{2\pi} e^{i(n+1)\theta} d\theta = 2\pi \quad , n=-1$
>
> $0 \quad , n \neq -1$

$$\therefore \int_{x_k} f(z)\,dz = a_{-1}\, i\, r_k^{-1+1}(2\pi) = 2\pi i\, a_{-1} = 2\pi i . \operatorname{Re}s(f\; i\; z_k)$$ by definition of residue.

Put this value in equation (1) to get.

$$\int_x f(z)\,dz = \sum_{k=1}^{m} 2\pi i \quad \operatorname{Re}s(f; z_k) = 2\pi i \sum_{k=1}^{m} \quad \operatorname{Re}s(f, z_k)$$

Example 1: Use Residue Theorem to evaluate

$\int_x \frac{z^2+1}{z(z-2)(z+4)^2}$ where x is the circle $|z|=3$

Solution:- By using Residue Theorem

$\int_x f(z)\,dz = 2\pi i$ sum of residues of f at its poles inside x]……..(1)

Here $f(z) = \frac{z^2+1}{z(z-2)(z+4)^2}$

$\therefore f$ has a simple pole at $z=0$ and $z=2 \quad i$ and $z=-4$ is a pole of order 2.

But, simple poles $z=0,\; z=2$ lies inside x

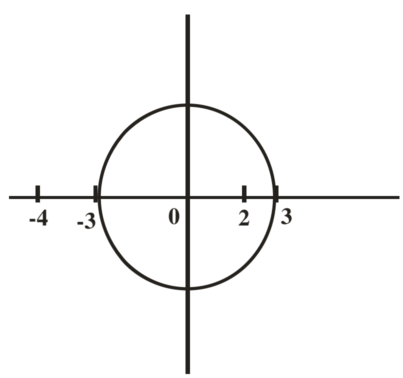

Fig 12.2

For **simple pole** at z=0

$$\operatorname{Re}s(f;\alpha)=\lim_{z\to\alpha}(z-\alpha)f(z)$$

$$\operatorname{Re}s(f;0)=\lim_{z\to 0}(z-0)\times\frac{z^2+1}{z(z-2)(z+4)^2}$$

$$=\lim_{z\to 0}\frac{\not{z}\left(2^2+1\right)}{\not{z}(z-2)(z+4)^2}\quad=\lim_{z\to 0}\frac{z^2+1}{(z-2)(z+4)^2}=\frac{0+1}{(0-2)(0+4)^2}=$$

$$-\frac{1}{32}$$

Similarly,

$$\therefore\frac{e^z+e^{-z}}{2}=0\Rightarrow e^z+e^{-z}=0\Rightarrow e^z=-e^{-z}$$

$$\Rightarrow e^{2z}=-1\Rightarrow e^{(2n+1)\pi i}\Rightarrow 2z=(2n+1)\pi i$$

$$\Rightarrow z=(2n+1)\frac{\pi}{2}i\quad |z|=3$$

$z\in x_1$

$$\boxed{\operatorname{Re}s(f;2)=\frac{5}{72}}$$

$\therefore$ from equation (1)

$$\int_x\frac{z^2+1}{z(z-2)(z+4)^2}=2\pi i\left[\operatorname{Re}s(f;0)+\operatorname{Re}s(f;2)\right]$$

$$=2\pi i\left[\frac{-1}{32}+\frac{5}{72}\right]=2\pi i\left[\frac{-1}{8\times4}+\frac{5}{8\times9}\right]=2\pi i\left[\frac{-9+20}{72\times4}\right]=\not{2}\pi i\times\frac{11}{72\times\not{4}_2}$$

$$\boxed{\int_x\frac{z^2+1}{z(z-2)(z+4)^2}=\frac{11}{144}\pi i}$$

(2) Use Residue Theorem to evaluate

$\int_x\frac{\sin z}{\left(z-\pi/4\right)^3}dz$ where z is a closed square bounded by $x=\pm2,$ and $y=\pm2$.

Solution : By using residue Theorem,

$\int_x f(z)\,dz=2\pi i$ [sum of residue of f at its poles inside x]…….(1)

Here $f(z)=\dfrac{\sin z}{\left(z-\pi/4\right)^3}$

$\therefore f$ has a pole of order 3 at $z\cdot=\pi/4$

But, the pole $z=\dfrac{\pi}{4}$ lies inside x

For pole of order $m=3$,

$$\operatorname{Re}s(f;\alpha=\frac{1}{(3-1)!}\lim_{z\to\alpha}\frac{d^{m-1}}{dz^{3-1}}(z-\alpha)^m f(z)$$

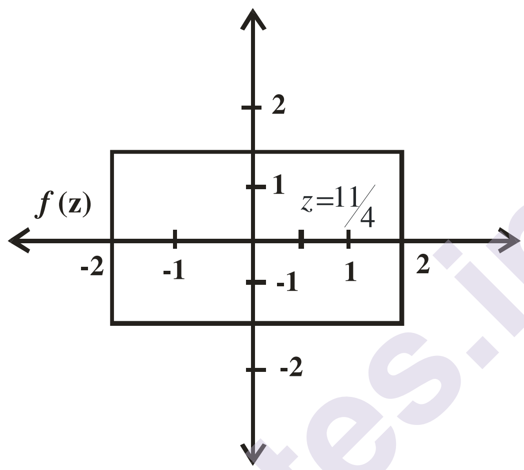

Fig 12.3

$$\operatorname{Re}s\left(f;\frac{\pi}{4}\right)=\frac{1}{(3-1)!}\lim_{z\to\alpha}\frac{d^{3-1}}{dz^{3-1}}\left(z-\frac{\pi}{4}\right)^3\cdot\frac{\sin 2}{\left(z-\pi/4\right)^3}$$

$$=\frac{1}{2}\lim_{z\to\frac{\pi}{4}}\frac{d^2}{dz^2}(\sin z)=\frac{1}{2}\lim_{z\to\frac{\pi}{4}}(-\sin z)$$

$$=\frac{1}{2}\left(-\sin\frac{\pi}{4}\right)=\frac{1}{2}\times\frac{-1}{2\sqrt{2}}$$

$$\therefore \boxed{\operatorname{Re}s\left(f;\frac{\pi}{4}\right)=\frac{-1}{2\sqrt{2}}}$$

$$\therefore \int_x f(z)\,dz=\int_x\frac{\sin z}{\left(z-\pi/4\right)^3}dz=2\pi i\left[\frac{-1}{2\sqrt{2}}\right]=\frac{-1}{\sqrt{2}}\pi i$$

(3) $\displaystyle\int\frac{e^z}{\cosh z}dz$ where x is circle $|z|=3$

Solution :- by using residue Theorem

$\int_x f(z)\,dz = 2\pi i$ [sum of residue of f at its poles

Here $f(z) = \dfrac{e^z}{\cosh z}$

Let *coshz=0*

$\therefore \dfrac{e^z + e^{-z}}{2} = 0 \Rightarrow e^z + e^{-z} = 0 \Rightarrow e^z = -e^{-z}$

$\Rightarrow e^{2z} = -1 \Rightarrow e^{(2n+1)\pi i} \Rightarrow 2z = (2n+1)\pi i$

$\Rightarrow z = (2n+1)\dfrac{\pi}{2} i$

$\therefore f$ has simple poles at $z = i\left(\dfrac{2n+1}{2}\right)\pi$

But , the simple poles $z = \pm i\dfrac{\pi}{2}$ lies inside $|z| = 3$

For simple pole

$\operatorname{Res}(f;\alpha) = \lim_{z\to\alpha}(z-\alpha)f(z)$

$\operatorname{Res}\left(f; i\dfrac{\pi}{2}\right) = \lim_{z\to i\frac{\pi}{2}}\left(z - i\dfrac{\pi}{2}\right)\dfrac{e^2}{\cosh z}$ $\left(\dfrac{0}{0}\right)$form

$= \lim_{z\to i\frac{\pi}{2}}\left[\dfrac{\left(z-i\frac{\pi}{2}\right)e^z + e^2(1)}{\sinh z}\right] = \dfrac{\cos{\pi/2} + i\sin{\pi/2}}{i\sin\frac{\pi}{2}} \qquad = \dfrac{0+i.1}{i.1} \qquad = 1$

$\operatorname{Res}\left(f; -i\dfrac{\pi}{2}\right) = \lim_{z\to -i\frac{\pi}{2}}\left[\left(z + i\dfrac{\pi}{2}\right)\dfrac{e^3}{\cosh z}\right]$

$= \lim_{z\to i\frac{\pi}{2}}\left[\dfrac{\left(z+i\frac{\pi}{2}\right)e^2 + e^2(1)}{\sinh z}\right] = \dfrac{0+e^{-i\pi/2}}{\sinh\left(i\frac{\pi}{2}\right)} \quad = \dfrac{\cos\left(-\pi/2\right) + i\sin\left(\pi/2\right)}{i\sin\left(\pi/2\right)} = 1$

$$\boxed{\int_x \frac{e^z}{\cosh z} = 2\pi i[1+1] = 4\pi i}$$

(4) $\displaystyle\int_x \frac{z^2+1}{(z-1)(z-2)^2(2+5)}$ where x is the circle $|z| = 4$

12.3 EVALUATION OF STANDARD TYPES OF INTEGRALS BY THE RESIDUE CALCULUS METHOD

* Application of Residue Theorem to evaluate real Integrals.

Type-I

Integral of the type $\int_0^{2\pi} F(\cos, \sin\theta)\, d\theta$, Where $\int_0^{2\pi} F(\cos\theta, \sin\theta)$ is rational function of $\cos\theta$ and $\sin\theta$

Consider, $\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta$

Put $z = e^{i\theta}$, $\theta \in [0 , 2\pi]$, $dz = i - e^{i\theta} d\theta \Rightarrow d\theta = \frac{dz}{i.e^{i\theta}} = \frac{dz}{iz}$

$$\boxed{\begin{aligned}\cos\theta &= \frac{e^{i\theta}+e^{-i\theta}}{2}\\ \sin\theta &= \frac{e^{i\theta}-e^{-i\theta}}{2i}\end{aligned}}\begin{aligned}&= \frac{z+z^{-1}}{2} = \frac{z+1/z}{2} = \frac{z^2+1}{2z}\\ &= \frac{z+z^{-1}}{2i} = \frac{z-1/z}{2i} = \frac{z^2-1}{2iz}\end{aligned}$$

$$\therefore \int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta = \int_x F\left(\frac{z^2+1}{2z}, \frac{z^2+1}{2z}\right)\frac{dz}{iz} \text{ ,Where}$$

$$f(z) = F\left(\frac{z^2+1}{2z}, \frac{z^2+1}{2z}\right)\frac{i}{iz}$$

$= 2\pi i$ [sum of residue of f at its pole inside x] (by residue Theorem)

$ax^2 + bx + c$ is polynomial than the root are $x = \frac{(-b) \pm \sqrt{b^2 - 4ac}}{2a}$

(1) Use Residue Theorem to evaluate $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$

Solution :- the given integral is $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$

Put $z = e^{i\theta}$, $\theta \in [0 , 2\pi]$, $dz = i - e^{i\theta} d\theta$

$$d\theta = \frac{dz}{i.e^{i\theta}} \Rightarrow d\theta = \frac{dz}{iz}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + z^{-1}}{2} = \frac{z^2+1}{2z}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \int_x \left(\frac{1}{2+\frac{z^2+1}{2z}} \right) \frac{dz}{iz} \quad \text{where } x \text{ is the unit circle } |z|=1$$

$$= \frac{1}{i} \int_x \frac{1}{\not z} \left(\frac{1}{\frac{4z+z^2+1}{2\not z}} \right) dz = \frac{2}{i} \int_x \frac{1}{z^2+4z+1}\, dz$$

………(1)

Put $f(z) = \dfrac{1}{z^2+4z+1}$

f has simple pole at $z = \dfrac{(-4) \pm \sqrt{16-4}}{2} = \dfrac{\not 2\left(-2 \pm \sqrt{3}\right)}{\not 2}$ i.e. $z = -2 \pm \sqrt{3}$

but, the simple pole $= -2+\sqrt{3} \approx -2+1.73$ lies inside x

For simple pole, $\operatorname{Res}(f;\alpha) = \lim\limits_{z\to\alpha} (z-\alpha) f(z)$

$$\operatorname{Res}\left(f; -2+\sqrt{3}\right) = \lim_{z\to -2+\sqrt{3}} \left[z - \left(-2+\sqrt{3}\right) \right] \times \frac{1}{z^2+4z+1} \qquad \left(\frac{0}{0}\right) \text{form}$$

$$= \lim_{z\to -2+\sqrt{3}} \frac{1}{2z+4} = \frac{1}{2\left(-2+\sqrt{3}\right)+4} = \frac{1}{2\sqrt{3}}$$

$\therefore$ by using Residue Theorem

$\int_x f(z)\, dz = 2\pi i$ [sum of residue of f at its poles inside x]

$$\int_x \frac{dz}{z^2+4z+1} = \not 2\pi i \left[\frac{1}{\not 2\sqrt{3}} \right] = \frac{i\pi}{\sqrt{3}}$$

Put this value in equation (1)

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \frac{2}{i} \times \frac{i\pi}{\sqrt{3}}$$

$$\therefore \boxed{\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \frac{2\pi}{\sqrt{3}}}$$

(2) use residue Theorem to evaluate $\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} \quad a > |b|$

Given , $\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta}$

Put $z = e^{i\theta} \qquad \theta \in [0, 2\pi] \quad \Rightarrow dz = ie^{i\theta} d\theta \qquad \Rightarrow d\theta = \frac{d^2}{ie^{i\theta}} = \frac{dz}{iz}$

$$\sin\theta = \frac{e^{-i\theta}}{2i} = \frac{z - z^{-1}}{2i} = \frac{z^2 - 1}{2iz}$$

$\therefore \int_x^{2\pi} \frac{d\theta}{a+b\cos\theta} = \int_x \frac{dz/iz}{a + b\left(\frac{z^2-1}{2iz}\right)}$ where x is the unit circle $|z| = 1$

$$= \int_x \frac{dz}{iz} \times \frac{1}{\frac{2aiz + bz^2 - b}{2aiz}} = 2 \int_x \frac{dz}{bz^2 2aiz - b} \quad \ldots\ldots\ldots(1)$$

Put $f(z) = \frac{1}{bz^2 + 2aiz - b}$

$\therefore f$ has simple pole at $z = -2ai \pm \sqrt{\frac{4a^2(i)^2 + 4b^2}{2b}}$

$$= \frac{-2ai \pm 2\sqrt{b^2 - a^2}}{2b}$$

i.e. f has simple pole at $z = -ai \pm \sqrt{\frac{b^2 - a^2}{b}} = -i\left[\frac{a \pm \sqrt{a^2 - b^2}}{b}\right]$

Let $\alpha_1 = -i\left[\frac{a - \sqrt{a^2 - b^2}}{b}\right]$ and $\beta = -i\left[\frac{a + \sqrt{a^{2-}b^2}}{b}\right]$

$\because \alpha\beta = 1 \Rightarrow |\alpha||\beta| = 1 \Rightarrow |\alpha| = \frac{1}{|\beta|} < 1 \ldots\ldots\ldots\ldots(\because |b| < a)$

But the pole α lies inside x

For simple pole,

$$\operatorname{Re}s(f;\alpha) = \lim_{z\to\alpha} (z - \alpha) f(z)$$

$$\operatorname{Re}s(f;\alpha_1) = \lim_{z\to\alpha_1} (z - \alpha_1) . \frac{1}{bz^2 + 2aiz - b} \left(\frac{0}{0}\right) \text{form}$$

$$= \lim_{z \to \alpha_1} \frac{1}{2bz + 2ia}$$

$$= \frac{1}{2\cancel{b}(-i)\left[\frac{a-\sqrt{a^2-b^2}}{\cancel{b}}\right]+2ia} = \frac{1}{-\cancel{2ai}+2i\sqrt{a^2-b^2}+\cancel{2ai}}$$

$$\boxed{\operatorname{Res}(f;\alpha_1) = \frac{1}{2i\sqrt{a^2-b^2}}}$$

By using Residue Theorem,

$$\int_x f(z)\,dz = 2\pi i \quad [\text{sum of residue of } f \text{ at its poles inside } x]$$

$$= 2\pi i \times \frac{1}{2i\sqrt{a^2-b^2}}$$

$$\boxed{\int_x f(z)\,dz = \frac{\pi}{\sqrt{a^2-b^2}}}$$

Put this value in equation (1) we get

$$\boxed{\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = 2\times\frac{\pi}{\sqrt{a^2-b^2}} = \frac{2\pi}{\sqrt{a^2-b^2}}}$$

3. Use residue theorem to evaluate $\int_0^{\pi} \frac{d\theta}{a+b\cos\theta}$ a>b!

Hint: First evaluate $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}$

Then use the property $\boxed{\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = 2\int_0^{\pi} \frac{d\theta}{a+b\cos\theta}}$

4. Evaluate $\int_0^{\pi} \frac{1+2\cos\theta}{5+3\cos\theta}\,d\theta$ &

5. Evaluate $\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}\,d\theta$ &

6. Evaluate $\int_0^{\pi} \frac{d\theta}{a+\cos\theta} \quad a>1$

Type-II

Integral of the type $\int_{-\infty}^{\infty} f(x)dx$

Improper Integral where $f(x)=\frac{P(x)}{Q(x)}, where\ p(x), Q(x)$ are polynomial in x.

Example: If $|f(z)|\le \frac{M}{R^K}$ for $z=\mathrm{Re}^{i\theta}$ where K>1 and M are constants.

Then $\lim_{R\to\infty}\int_{x_R} f(z)dz=0$ where, x_R is the semi circle are of radius R as shown in figure.

Proof: Given $|f(z)|\le \frac{M}{R^K}$ for $z=\mathrm{Re}^{i\theta}$

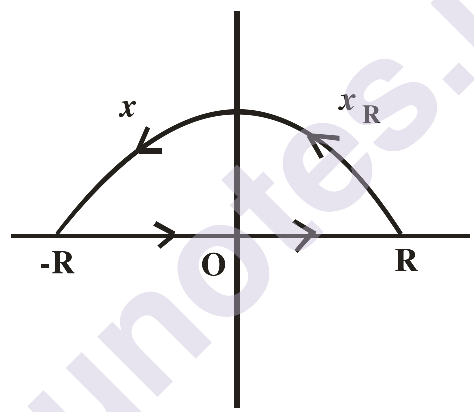

Fig 12.4

$$\therefore \left|\int_{x_R} f(z)dz\right| \le \int_{x_R} |f(z)||dz| \le \frac{M}{R^K}\int_{x_R}|dz|$$

Put $z=\mathrm{Re}^{i\theta}\Rightarrow dz=Rie^{i\theta}d\theta$

$$\therefore \left|\int_{x_R} f(z)dz\right| \le \frac{M}{R^K}\int_0^{\pi}\left|Rie^{i\theta}d\theta\right| = \frac{M}{R^K}\times \pi R$$

$$\therefore \left|\int_{x_R} f(z)dz\right| \le \frac{\pi M}{R^{K-1}} \to 0 \quad as \quad R\to\infty$$

$$\therefore \lim_{R\to\infty}\int_{x_R} f(z)dz=0$$

Example: Evaluate $\int_{-\infty}^{\infty} f(x)dx$, where $f(x) = \frac{P(x)}{Q(x)}, P(x)\ and\ Q(x)$ are polynomials in x .

Solution:

Consider $\int_{x} f(z)dz$ where x is a closed curve consisting of large semicircle x_R of radius R and the real axis from –R to R traversed in the anticlockwise direction.

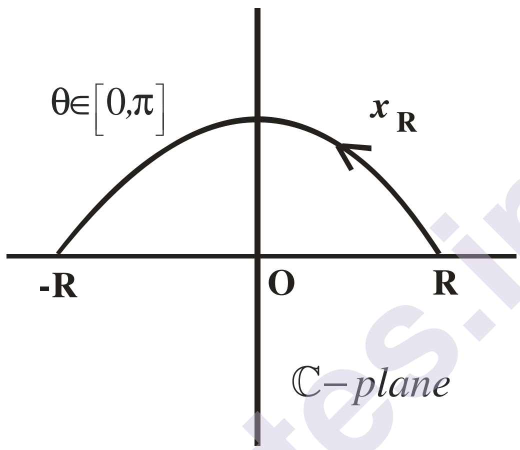

Fig 12.5

We choose only those poles of f which lie in the upper half of the complex plane.

$\therefore$ by residue theorem,

$$\int_{x} f(z)dz = 2\pi i \text{ [sum of residues of f at its pole inside } x]$$

$$\int_{x} f(z)dz + \int_{-R}^{R} f(x)dx = 2\pi i \text{ [sum of residues of f at its pole inside } x]$$

on real axis $z = x$

$$\therefore \lim_{R\to\infty} \int_{x_R} f(z)dz + \lim_{R\to\infty} \int_{-R}^{R} f(x)dz = 2\pi i \text{ [sum of residues of f at its poles inside } x]$$

$$O + \int_{\infty}^{\infty} f(x)dx = 2\pi i \text{ [sum of residues of f at its poles inside } x]$$

……….. (by previous example $\lim_{R\to\infty} \int_{x=0} f(z)dz$

Note: If $f(z)=\dfrac{P(z)}{Q(z)}$ where, P(z) and Q(z) are polynomial in z such that,
(i) Q(z) = 0 has no real roots.
(ii) The degree of Q(z) is greater than that of P(z) by atleast 2, then,
$\int_{x} f(z)dz = 2\pi i$ [Sum of residue of f at its poles inside x]

Example:

(1) Evaluate $\int_{-\infty}^{\infty} \dfrac{x^2+3}{\left(x^2+1\right)\left(x^2+4\right)}dx$

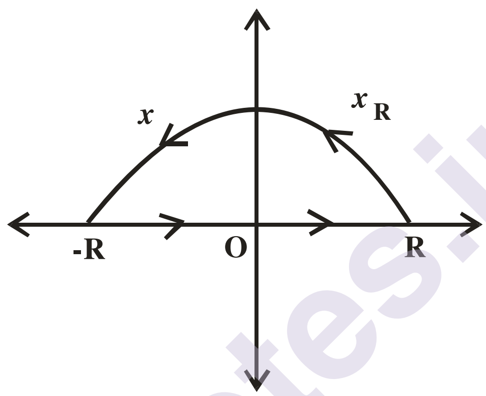

Fig 12.6

Where x is a closed curve consisting of large semicircle x_R of radius R and real axis form –R to R traversed in the anticlockwise direction.

degQ(z)≥degP(z)+2

$P(z)=z^2+3$ & $Q(z)=\left(z^2+1\right)\left(z^2+4\right)$

$\therefore$ this general method is applicable

Consider,

$$\int \frac{z^2+3}{\left(z^2+1\right)\left(z^2+4\right)}$$

Here, $f(z)=\dfrac{z^3+3}{\left(z^2+1\right)\left(z^2+4\right)}$

Here f has simple poles at $z=\pm i$ and $z=\pm 2i$ but the poles $z=i$ and $z=2i$ lies inside x.

For simple pole

$$\text{Res }(f;\alpha)=\lim_{z\to\alpha}(z-\alpha)f(z)$$

Res

$$(f;i)=\lim_{z\to i}(z-i)\times\frac{z^2+3}{\left(z^2+1\right)\left(z^2+4\right)}$$

$$=\lim_{z\to i}\frac{(z-i)\left(z^2+3\right)}{(z-i)(z+i)\left(z^2+4\right)}=\lim_{z\to i}\frac{\left(z^2+3\right)}{(z+i)\left(z^2+4\right)}$$

$$=\frac{i^2+3}{(i+i)\left(i^2+4\right)}=\frac{(-1)+3}{2i(-1+4)}=\frac{3-1}{2i(4-1)}=\frac{2}{2i(3)}$$

$$\boxed{\text{Re}\,s(f;i)=\frac{1}{3i}}$$

$$\text{Res}(f;2i)=\lim_{z\to 2i}(z-2i)\frac{z^2+3}{\left(z^2+1\right)\left(z^2+4\right)}$$

$$=\lim_{z\to 2i}\frac{(z-2i)\left(z^2+3\right)}{\left(z^2+1\right)(z_r-2i)(z_r+2i)}=\lim_{z\to 2i}\frac{z^2+3}{\left(z^2+1\right)(z+2i)}$$

$$=\frac{(2i)^2+3}{\left[(2i)^2+1\right][2i+2i]}=\frac{4i^2 3}{\left(4i^2+1\right)(4i)}=\frac{-4+3}{4i(-4+1)}$$

$$\boxed{\text{Re}\,s(f;2i)=\frac{1}{P2i}}$$

By using Residue Theorem

$\int_x \frac{z^2+3}{\left(z^2+1\right)\left(z^2+4\right)}dz=2\pi i$ [Sum of Residue of f at its pole lies inside x]

$$=2\pi i\left[\frac{1}{3i}+\frac{1}{|2|}\right]=2\pi i\left[\frac{4+1}{12i}\right]=\frac{5}{6}\pi$$

$$\therefore \int_{x_R}\frac{z^2+3}{\left(z^2+1\right)\left(z^2+4\right)}dz+\int_{-R}^{R}\frac{x^2+3}{\left(x^2+1\right)\left(x^2+4\right)}dx=\frac{5}{6}\pi$$

$$\therefore \lim_{R\to\infty}\int_{x_R}\frac{z^2+3}{\left(z^2+1\right)\left(z^2+4\right)}dz\ \lim_{R\to\infty}\int_{-R}^{R}\frac{x^2+3}{\left(x^2+1\right)\left(x^2+4\right)}dx=\frac{5}{6}\pi \quad \text{----(1)}$$

T.P.T. $\lim\limits_{R\to\infty} \int\limits_{x_R} \frac{z^2+3}{\left(z^2+1\right)\left(z^2+4\right)} = 0$

$$\left|\int\limits_{x_R} \frac{z^2+3}{\left(z^2+1\right)\left(z^2+4\right)} dz\right| \le \int\limits_0^{\pi} \frac{\left|R^2e^{2i\theta}+3\right|\left|\mathrm{Re}^{i\theta}\, d\theta\right|}{\left|R^2e^{2i\theta}+1\right|\left|R^2e^{2i\theta}+4\right|}$$

$$\left\{\begin{array}{r} \because put\ z = \mathrm{Re}^{i\theta}, \theta \in [0, 2\pi] \\ \Rightarrow dz = i\,\mathrm{Re}^{i\theta}\, d\theta \end{array}\right\} \le \int\limits_0^{2\pi} \frac{\left(R^2+3\right)R\, d\theta}{\left(R^2-1\right)\left(R^2-4\right)}$$

$$\left(\because |z_1+z_2| \ge |z_1|-|z_2| \Rightarrow \left|\mathbb{R}^2e^{2i\theta}+1\right| \ge \left|R^2e^{2i\theta}\right|-|1| \Rightarrow \frac{1}{\left|R^2e^{2i\theta}+|1|\right|} \le \frac{1}{R^2-1}\right)$$

$$\left|\int\limits_{x_R} \frac{z^3+3}{\left(z^2+1\right)\left(z^2+4\right)} dz\right| \le \frac{R^3\left(1+{}^{3}\!/_{R^2}\right)}{R^2\left(1-{}^{1}\!/_{R^2}\right)\left(1-{}^{4}\!/_{R^2}\right)} \int\limits_0^{\pi} d\theta$$

$$= \frac{1}{R} \frac{\left(1+{}^{3}\!/_{R^2}\right)\cdot\pi}{\left(1-{}^{1}\!/_{R^2}\right)\left(1-{}^{4}\!/_{R^2}\right)} \to 0 \ \ as \ \ R\to\infty$$

$$\therefore \lim_{R\to\infty} \int\limits_{x_R} \frac{z^2+3}{\left(z^2+1\right)\left(z^2+4\right)} dz = 0$$

Put this value in equation (1)

$$0 + \int\limits_{-\infty}^{\infty} \frac{x^2+3}{\left(x^2+1\right)\left(x^2+4\right)} dx = \frac{5\pi}{6}$$

$$\boxed{\int\limits_{-\infty}^{\infty} \frac{x^2+3}{\left(x^2+1\right)\left(x^2+4\right)} dx = \frac{5}{6}\pi}$$

(2) Use Residue theorem to evaluate

$$\int\limits_0^{\infty} \frac{x^2}{\left(1+x^2\right)^2} dx$$

Hint: First calculate $\int\limits_{-\infty}^{\infty} \frac{x^2}{\left(1+x^2\right)^2} dx$ and then use the property

$$\int_{-\infty}^{\infty} \frac{x^2}{\left(1+x^2\right)^2} dx = 2\int_{0}^{\infty} \frac{x^2}{\left(1+x^2\right)^2} dx$$

Solution: Consider $\int_x \frac{z^2}{\left(1+z^2\right)^2} dz$ where x is a closed curve consisting of large semicircle x_R of Radius R and real axis from –R to R traversed in the anticlockwise direction. Here $f(z) = \frac{z^2}{\left(1+z^2\right)^2}$,

Here, f has pole of order z at $z = \pm i$ but, the pole $z = i$ lies inside x.
For pole of order 2

Res $(f;i) = \frac{1}{(2-i)!} \lim_{z\to i} \frac{d}{dz}(z-i)^2 f(z)$

$$= \lim_{z\to i} \frac{d}{dz}(z-i)^2 \frac{z^2}{\left(1+z^2\right)^2} = \lim_{z\to i} \frac{(z-i)^2 \ z^2}{(z-i)(z+i)^2} = \lim_{z\to i} \frac{d}{dz} \frac{z^2}{(z+i)^2}$$

(3) $\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} dx$

Type-III: Integral of the type $\int_{-\infty}^{\infty} sim\, mx\, f(x) dx$ or $\int_{-\infty}^{\infty} \cos(mx) f(x)\, dx$

m > 0 where $f(x) = \frac{P(x)}{Q(x)}$

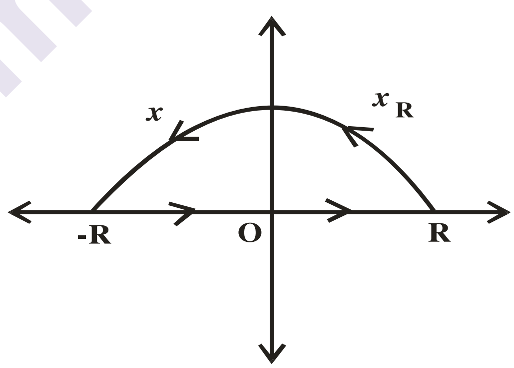

Fig 12.7

Consider $\int_x e^{imz} f(z)\, dz$ where x is a closed curve consisting of large semicircle x_R of radius R and the real axis from –R to R traversed in the anticlockwise direction.

$\therefore$ by residue theorem

$\int_{x} e^{imz} f(z) dz = 2\pi i$ [Sum of residue of f at its pole lies inside x]

$\therefore \int_{R} e^{imz} f(z)\, dz + \int_{-R}^{R} e^{imz} f(x)\, dx = 2\pi i$ [Sum of residue of f at its pole lies inside x]

$\therefore \lim_{R\to\infty} \int_{x} e^{imz} f(z) dz + \lim_{R\to\infty} \int_{-R}^{R} e^{imx} f(x)\, dx = 2\pi i$ [Sum of residue of f at its pole inside x] --------- (1)

$\therefore \lim_{R\to\infty} \int_{x_R} e^{imz} f(z) dz = 0$ --------(by next example)

$\therefore$ from equation (1),

$0 + \int_{-\infty}^{\infty} (Cosmx + i\sin mx) f(x) dx = 2\pi i$ [Sum of residue of its poles inside x]

Example: 1) Use Residue Theorem, to evaluate $\int_{-\infty}^{\infty} \frac{\cos 3x}{x^2+4} dx$.

Solution: Consider $\int_{x} \frac{e^{3iz}}{z^2+4} dz$

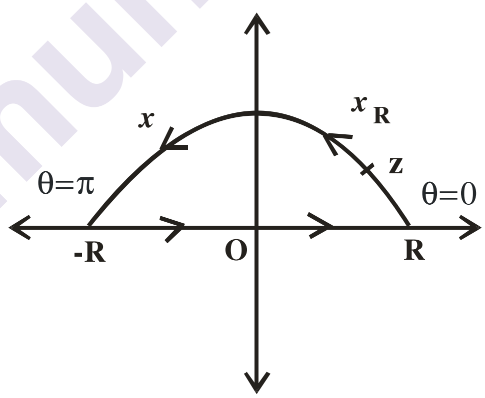

Fig 12.8

Where x is a closed curve consisting of large semicircle x_R of radius R and the real axis from –R to R traversed in the anticlockwise direction.

Put $f(z) = \frac{e^{3iz}}{z^2+4}$

Here, f is a simple poles at $z = \pm 2i$ but, the pole $z = zi$ lies inside x.
For simple pole,

$$\text{Res } (f;\alpha) = \lim_{z\to 2i} (z-\alpha) f(z)$$

$$\text{Res}(f;2i) = \lim_{z\to 2i} (z-2i)\times\frac{e^{3iz}}{z^2+4} = \lim_{z\to 2i}\frac{(z-2i)e^{3iz}}{(z-2i)(z+2i)}$$

$$= \lim_{z\to 2i}\frac{e^{3iz}}{z+2i} = \frac{e^{3i(2i)}}{(2i+2i)} = \frac{e^{6i^2}}{4i}$$

$$\boxed{\text{Re}\, s(f;2i) = \frac{e^{-6}}{4!}}$$

- by residue theorem,
- $\int_x f(z)dz = 2\pi i$ [Sum of residue of f at its all poles inside x]

$$\therefore \int_x \frac{e^{3iz}}{z^2+4}dz = 2\pi i\left[\frac{e^{-6}}{4i}\right]$$

$$\therefore \boxed{\int_x \frac{e^{3iz}}{z^2+4}dz = \frac{\pi . e^{-6}}{2}}$$

$$\therefore \int_{x_R} \frac{e^{3iz}}{z^2+4}dz + \int_{-R}^{R}\frac{e^{3iz}}{x^2+4}dx = \frac{\pi}{2}e^{-6}$$

$$\lim_{R\to\infty}\int_{x_R}\frac{e^{3iz}}{z^2+4}dz + \lim_{R\to\infty}\int_{-R}^{R}\frac{e^{3ix}}{x^2+4}dx = \frac{\pi}{2}e^{-6} \quad \text{-------------} \ (1)$$

T.P.T. $\lim_{R\to\infty}\int_{x_R}\frac{e^{3iz}}{z^2+4}dz = 0$

Put $z = \text{Re}^{i\theta} \qquad \theta\in[0,2\pi]$

$$dz = i.R.e^{i\theta}\, d\theta$$

$$\left|R^2e^{2i\theta}+4\right| \geq R^2-4 \Rightarrow \frac{1}{\left|R^2e^{2i\theta}+4\right|} \leq \frac{1}{R^2-4}$$

$$\left|\int_{x_R}\frac{e^{3iz}}{z^2+4}dz\right| \leq \int_0^{2\pi}\frac{\left|e^{3i.\text{Re}^{i\theta}}\right|\left|i\,\text{Re}^{i\theta}\,d\theta\right|}{\left|R^2e^{2i\theta}+4\right|}$$

$$\leq \frac{R}{(R^2-4)}\int_0^{2\pi}\left|e^{3iR}(Cos\,\theta + i\sin\theta)\right|d\theta = \frac{R}{(R^2-4)}\int_0^{\pi}e^{-3R\,Sin\theta}$$

$$\ldots\left(\because \left|e^{3iR\,Cos\theta}\right| = 1\right)$$

$$\because Sin\theta \geq \frac{20}{\pi} \quad , \quad \theta \in \left[0, \pi/2\right]$$

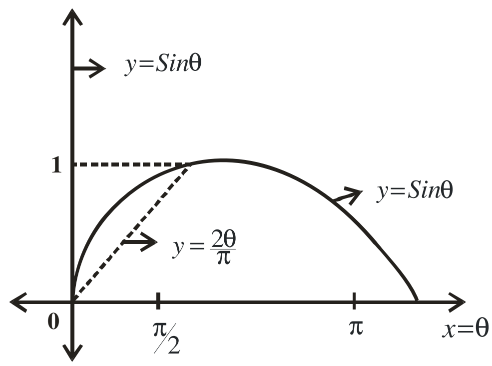

Fig 12.9

$$\therefore \left|\int_{x_R} \frac{e^{3iz}}{z^2+4}\,dz\right| \leq \frac{R}{\left(R^2-4\right)} \cdot 2\int_0^{\pi/2} e^{\frac{-6R\theta}{\pi}}\,d\theta \qquad \ldots\ldots\left(\because \; Sin\theta \geq \frac{2\theta}{\pi}\right.$$

$$\Rightarrow - Sin\,\theta \leq \frac{-2\theta}{\pi} \Rightarrow -3R\,Sin\theta \leq \frac{-6R\theta}{\pi}$$

$$= \frac{2R}{\left(R^2-4\right)}\left[\frac{e^{\frac{-6R\theta}{\pi}}}{\frac{-6R}{\pi}}\right]_0^{\pi/2} = \frac{2R}{\left(R^2-4\right)} \times \frac{\pi}{-6R}\left[e^{\frac{-6R}{\pi}\times\frac{\pi}{2}} - e^{\frac{-6R}{\pi}\times 0}\right]$$

$$= \frac{-\pi}{3\left(R^2-4\right)}\left[e^{-3R}-1\right] = \frac{\pi}{3R^2\left(1-4/R^2\right)}\left[1-e^{-3R}\right] \rightarrow 0 \quad as \;\; R \rightarrow 0$$

$$\therefore \lim_{R\to\infty} \int_{x_R} \frac{e^{3iz}}{z^2+4}\,dz = 0$$

Put this value in equation (1)

$$0 + \int_{-\infty}^{\infty} \frac{Cos3x + i\,Sin3x}{x^2+4} = \frac{\pi}{4}e^{-6}$$

Equating Real part from both sides

$$\boxed{\int_{-\infty}^{\infty} \frac{Cos3x}{x^2+4}\,dx = \frac{\pi}{2}e^{-6}}$$

(2) $\int_{0}^{\infty} \frac{Sinmx}{x\left(x^{2}+a^{2}\right)} dx \qquad m>0 \qquad a \in R$

(3) $\int_{0}^{\infty} \frac{Cosmx}{\left(1+x^{2}\right)^{2}} dx \qquad$ [$(z=\pm i)$ is a pole of order 2]

(4) $\int_{-\infty}^{\infty} \frac{Sinx}{x^{2}+4x+5} dx$

Type-IV: Poles on the Real axis

In this case, we cannot use residue theorem because the pole z = 0 lie on real axis and hence on x.

Example: Prove that $\int_{0}^{\infty} \frac{\sin x}{x} dx=\frac{\pi}{2}$

Solution: Consider $\int_{0}^{\infty} \frac{e^{iz}}{z} dz$

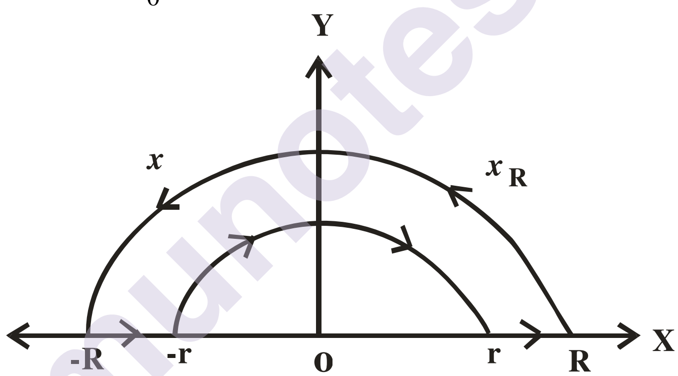

Fig 12.10

Where x is a closed curve consisting of

(i) a real axis from r to R

(ii) a large semicircle x_R of radius R.

(iii) a real axis from –R to –r

(iv) a small semicircle x_r of radius r.

$\because$ the pole z = 0 lie outside x,

$\therefore$ by Cauchy- Goursat theorem,

$$\int_{x} \frac{e^{iz}}{z} dz=0$$

$$\therefore \int_{r}^{R} \frac{e^{ix}}{x} dx + \int_{x_R} \frac{e^{iz}}{z} dz + \int_{-R}^{-r} \frac{e^{ix}}{x} dx + \int_{x_r} \frac{e^{iz}}{z} dz = 0$$

In 3rd integral x is negative $\Rightarrow dx$ will be $-dx$, using this negative sign make limits r to R.

$$\int_{r}^{R} \frac{e^{ix}}{x} dx - \int_{r}^{R} \frac{e^{-ix}}{x} dx + \int_{x_R} \frac{e^{iz}}{2} dz + \int_{x_r} \frac{e^{iz}}{z} dz = 0$$

$$2i \int_{r}^{R} \frac{e^{ix} - e^{-ix}}{2i - x} dx + \int_{x_R} \frac{e^{iz}}{z} dz + \int_{x_r} \frac{e^{iz}}{z} dz = 0$$

$$2i \int_{r}^{R} \frac{Sinx}{x} dx + \int_{x_R} \frac{e^{iz}}{z} dz + \int_{x_r} \frac{e^{iz}}{z} dz = 0 \qquad \left(\because \ Sinz = \frac{e^{iz} - e^{iz}}{2} \right)$$

Taking limit $R \to \infty$ *and* $r \to 0$

$$\lim_{\substack{R\to\infty \\ r\to 0}} 2i \int_{r}^{R} \frac{Sinx}{x} dx + \lim_{R\to\infty} \int_{x_R} \frac{e^{iz}}{z} dz + \lim_{r\to\infty} \int \frac{e^{iz}}{z} dz = 0 \quad \text{------------(1)}$$

T.P.T. $\lim_{R\to\infty} \int_{x_R} \frac{e^{iz}}{z} dz = 0$ (Proof as similar as Type III)

Consider $\int_{x_R} \frac{e^{iz}}{z} dz$

Put $z = r\,e^{i\theta} \quad \Rightarrow dz = ir\,e^{i\theta} d\theta$

$$\int_{x_r} \frac{e^{iz}}{z} dz = -\int_{0}^{\pi} \frac{e^{i\,r\,e^{i\theta}}}{r\,e^{i\theta}} \times ire^{i\theta} d\theta \qquad \text{--------} \left(\because \int_x f = \int_x f \right)$$

Taking limit as $r \to 0$

$$\lim_{x\to 0} \int_{x_r} \frac{e^{iz}}{z} dz = -i \int_{0}^{\pi} e^{0} d\theta = -i\pi$$

$\therefore$ from equation (1),

$$2i \int_{0}^{\infty} \frac{Sinx}{x} dx + 0 - i\pi = 0$$

$$\therefore \int_{0}^{\infty} \frac{Sinx}{x} dx + 0 - i\pi = 0$$

$$\therefore \int_0^\infty \frac{Sinx}{x} = \frac{i\pi}{2i} \qquad \therefore \boxed{\int_0^\infty \frac{Sinx}{x} = \frac{\pi}{2}}$$

Meromorphic Function (M.F):

If f is defined and analytic in an open set G $\subset \mathbb{C}$ except for poles. Then f is a Meromorphic function on G.

$$e.g.\ f(z) = \frac{z^2}{z(z-5)}$$

Theorem *(1) If f has a zero of order m at* $z = \alpha$, *then Res* $\left(\frac{f'(z)}{f(z)};\alpha\right) = m$

Proof: Given that, f has a zero of order m at $z = \alpha$

$\therefore \exists$ an analytic function $g : B(\alpha; R) \to \mathbb{C}$

s.t. $f(z) = (z-\alpha)^m g(z)$ where $g(\alpha) \neq 0$

Diff. w.r.t. z

$$f'(z) = m(z-\alpha)^{m-1} g(z) + (z-\alpha)^m g'(z)$$

$$\frac{f'(z)}{f(z)} \frac{(z-\alpha)^{m-1}[m\,g(z) + g'(z)]}{(z-\alpha)^m g(z)} = \frac{m\,g(z) + g'(z)}{(z-\alpha)g(z)}$$

Laurent expansion for $\frac{f'(z)}{f(z)}$ about $z = \alpha$

$$\frac{f'(z)}{f(z)} = \frac{m}{(m-\alpha)} + \frac{g(z)}{g(z)}, 0 < |z-\alpha| < R$$

$\because \frac{g'(z)}{g(z)}$ is analytic in $B(\alpha, R)$ and hence it has Taylor series expansion about $z = \alpha$.

$\therefore$ By the definition of residue,

Res $\left(\left(\frac{f'(z)}{f(z)}\right);\alpha\right) = m =$ Coefficient of $(z-\alpha)$ in the Laurent series expansion.

***Theorem* (2)** *If f has a pole of order n at* $z = \beta$ *then Res* $\left(\frac{f'(z)}{f(z)};\beta\right) = -n$

Proof: Given that, f h as a pole of order n at $z=\beta$

$\therefore \exists$ an analytic function $g : B(\beta; R) \to \mathbb{C}$

s.t. $f(z) = \dfrac{g(z)}{(z-\beta)^n}$ where $g(\beta) \pm 0$

Diff. w.r.t. z

$$f'(z) = \frac{(z-\beta)^n g'(z) - g(z) n (z-\beta)^{n-1}}{\left[(z-\beta)^n\right]^2}$$

$$\frac{f'(z)}{f(z)} = \frac{(z-\beta)^{n-1}\left[(z-\beta) g'(z) - ng(z)\right]}{(z-\beta)^n \times (z-\beta)^n} \times \frac{(z-\beta)^n}{g(z)}$$

$$= \frac{(z-\beta) g'(z) - n\, g(z)}{(z-\beta) g(z)} = \frac{g'(z)}{g(z)} - \frac{n}{(z-\beta)}$$

$$\frac{f'(z)}{f(z)} = \frac{-n}{(z-\beta)} + \frac{g'(z)}{g(z)} \qquad 0 < |z-\beta| < R$$

$\because \dfrac{g'(z)}{g(z)}$ is analytic in $\beta(\beta; R)$ and hence it has Taylor series expansion about $z=\beta$

$\therefore$ by definition

$$\text{Res}\left(\left(\frac{f'(z)}{f(z)}\right); \beta\right) = -n$$

12.4 THE ARGUMENT PRINCIPLE

Let f be a Meromorphic function in a domain G and have only finitely many zeros and poles. If x is a simple closed curve in G s.t. no zeroes and poles of f lie, on x, then

$$2\pi i \int_x \frac{f'(z)}{f(z)} dz = Z_f - P_f$$

Where, Z_f, P_f, denote respectively the number of zeros and poles of f inside x, each counted according to their order or multiplicity. *(2004, 2007)*

Proof: Given that, f is a Meromorphic function in domain G.

Put $F(z) = \dfrac{f'(z)}{f(z)}$

$\Rightarrow$ the singular points of F inside x are the zeros and poles of f.

$\therefore$ by Residue Theorem,

$\int_x F(z)\,dz = \int_x \frac{f'(z)}{f(z)}\,dz = 2\pi i$ [Sum of residues of F at its singular points inside x. -----------(1)

If α_j is a zero of f of order m_j, then Res $\left(\frac{f'(z)}{f(z)};\alpha_j\right)=m_j$

If B_k is a pole of f of order n_k, then Res $\left(\frac{f'(z)}{f(z)};\mathrm{B}_\kappa\right)=-n_k$

From equation (1)

$$\int_x \frac{f'(z)}{f(z)}\,dz = 2\pi i\left[\operatorname{Res}\left(\frac{f'(z)}{f(z)}\ ;\alpha_j\right)+\operatorname{Res}\left(\frac{f'(z)}{f(z)}\ ;B_K\right)\right]$$

$$\frac{1}{2\pi i}\int_x \frac{f'(z)}{f(z)}\,dz = \sum_i m_j - \sum_K n_K$$

$$\boxed{\frac{1}{2\pi i}\int_x \frac{f'(z)}{f(z)}\,dz = Z_f - P_f}$$

Example:

(1) Use Argument Principal to evaluate

$\int_x \frac{f'(z)}{f(z)}\,dz$ where $f(z)=\frac{(z-2)}{z(z-1)(z-4)}$ and x is the circle $|z|=3$.

Solution: by Argument Principal

$\frac{1}{2\pi i}\int_x \frac{f'(z)}{f(z)}\,dz = Z_f - P_f$ where, Z_f *and* Z_f are the no. of zeros and poles of f inside x each --(1)

Given function,

$$f(z)=\frac{(z-2)}{z(z-1)(z-4)}$$

Here f has simple zeros at z = 2 and z = 0, Z = 1 and z = 4 are simple poles

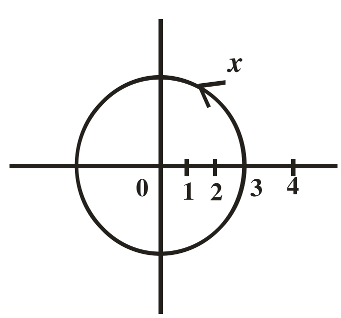

Fig 12.11

Given equation of circle $|z| = 3$

But, simple zero z = 2 and simple poles z = 0, 1, lies inside x.

$\therefore$ from equation (1)

$$\frac{1}{2\pi i}\int_x \frac{f'(z)}{f(z)}\,dz = 1-2 = -1 \Rightarrow \int_x \frac{f'(z)}{f(z)} = -2\pi i$$

(2) Use the argument principal to evaluate $\int_x \frac{f'(z)}{f(z)}\,dz$ where $f(z) = \frac{(z-1)}{z^2(z-2)(z-3)}$ and x is the circle $|z| = \pi$

Solution:

By argument principle

$$\frac{1}{2\pi i}\int_x \frac{f'(z)}{f(z)}\,dz = Z_f - P_f$$

Given function $f(z) = \frac{(z-1)}{z^2(z-2)(z-3)}$

Here f has simple zero at $z = 1$ *and* $z = 0$ is a poles of order 2 and z = 2, 3 are simple poles.

Given equation at circle, $|z| = \pi$

But simple zero z = 1 and simple poles z = 2,3 and z = 0 is pole of order 2 lie inside x.

From equation (1),

$$\int_x \frac{f'(z)}{f(z)} = 2\pi i\,[1-4] = 2\pi i[-3] = -6\pi i\ .$$

(3) Evaluate $\int_{|z|=\pi} \cot \pi z$ by using argument principle

Solution: $\int_{|z|=\pi} \cot \pi z = \int_{|z|=\pi} \frac{\cos \pi z}{Sin \pi z}dz = \frac{1}{\pi}\int_{|z|=\pi} \frac{\pi.\cos \pi z}{Sin \pi z}dz$

$\therefore$ by argument principal

$$\int_{|z|=\pi} \cot z dz = \frac{1}{\pi}2\pi i\left[Z_p - P_f\right] = 2i\left[Z_f - P_f\right] \qquad \text{............. (1)}$$

Here $f(z) = Sin\pi\, z$ has simple zeros at

$\pi z = n\pi \qquad n \in \mathbb{Z}$

$\therefore z = n \qquad n \in \mathbb{Z}$

$\therefore z = 0, \pm 1, \pm, 2, \pm 3, ...$ are simple zeros of f and f has no poles but zeros $z = 0, 1, \pm 2, \pm 3$ lies inside x i.e. $|z| = \pi$

$$\int_{|z|=\pi} \cot(\pi z) = 2i[7-0] = 14i$$

(4) Evaluate $\int_x \tan \pi z \, dz$ where x is the circle $|z| = \pi$

Take as an Exercise.

12.5 ROUCHE'S THEOREM

Suppose f and g are Meromorphic functions in a nbd. of $\overline{B}(\alpha; R)$ *with no zeros and poles on the circle* $x = \{z; |z - \alpha| = R\}$. *If* $z_f, z_g\ (P_f, P_g)$ *are the no. of zeros (poles) of f and g counted according to their order and if* $|f(z) - g(z)| < |g(z)|\ \forall 2 \in x$ *then,* $Z_f - P_f = Z_g - P_g$. *(2005, 2008,2009)*

Proof: Given that, f and g are Meromorphic functions in a nbd. of $\overline{B}(\alpha; R)$

Put $F(z) = \dfrac{f(z)}{g(z)}$

$\Rightarrow F$ is a Meromorphic function in a neighbourhood of $\overline{B}(\alpha; R)$

Let $Z = x(t)$ be any point on the circle x.

Then, $z = x(t) = \alpha + \mathrm{Re}^{it} \qquad t \in [0, 2\pi]$

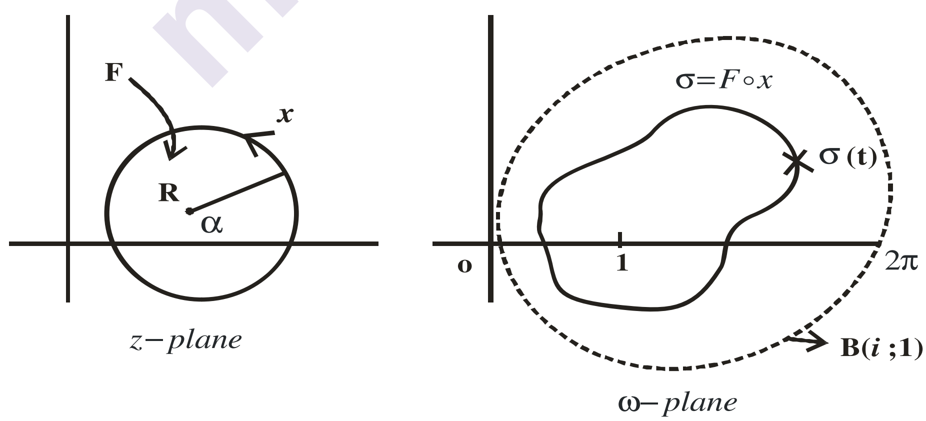

Fig 12.12

$\because x$ is a simple closed curve in Z-plane and F is analytic.

$\therefore \sigma = F.x$ is also a closed curve in w-plane

$\therefore$ for any $t \in [0, 2\pi]$

$$|\sigma(t)-1| = |F(x(t)-1)| = \left|\frac{f(x(t))}{g(x(t))}-1\right| \qquad \text{-----------------} (1)$$

Given that

$$|f(z)-g(z)| < |g(z)| \qquad \forall z \in x$$

$$\left|\frac{f(z)}{g(z)}-1\right| < 1$$

Put $z = x(t)$

$$\left|\frac{f(x(t))}{g(x(t))}-1\right| < 1$$

Put the above value in equation (1), we get

$$|\sigma(t)-1| \quad < 1$$

$$\{\sigma\} \subset B((1,0);1)$$

$\therefore$ 0 belongs to the unbounded component of $w \setminus \{\sigma\}$

$\therefore$ by definition of winding no;

$$\eta(\sigma;0) = 0 \qquad (\because \text{ 0 lies outside the curve } \sigma)$$

$$\Rightarrow \frac{1}{2\pi i}\int_{\sigma} \frac{d\omega}{\omega - 0} = 0$$

Put $\omega = \sigma(t) \quad \Rightarrow d\omega = \sigma'(t)\,dt \qquad t \in [0, 2\pi]$

$$\frac{1}{2\pi i}\int_0^{2\pi} \frac{\sigma'(t)}{\sigma(t)}dt = 0 \qquad \text{-------------------} (2)$$

Now, $\sigma(t) = \dfrac{f(x(t))}{g(x(t))}$

$$\therefore \sigma'(t) = \frac{g(x(t))\,f'(x(t))^{x'(t)} - f(x(t))\,g'(x(t))\cdot x'(t)}{[g(x(t))]^2}$$

$$\frac{\sigma'(t)}{\sigma(t)} = \frac{g(x(t))}{f(x(t))}\left[\frac{g(x(t))\,f'(x(t)) - g'(x(t))}{[g(x(t))]^2}\right]x'(t)$$

$$= \left[\frac{[g(x(t))]^2 f(x(t))}{g(x(t))[g(x(t))]^2} - \frac{g(x(t)\,f(x(t)))\,g'(x(t))}{f(x(t))[g(x(t))]^2}\right]x'(t)$$

$$\frac{\sigma'(t)}{\sigma(t)} = \left[\frac{f'(x(t))}{f(x(t))} - \frac{g'(x(t))}{g(x(t))}\right] x'(t)$$

$$0 = \frac{1}{2\pi i}\int_0^{2\pi}\left[\frac{f'(x(t))}{f(x(t))} - \frac{g'(x(t))}{g(x(t))}\right] x'(t)\,dt \qquad \text{----------------- (2)}$$

Put $z = x(t) \Rightarrow dz = x'(t)\,dt$

$$0 = \frac{1}{2\pi i}\int_x\left[\frac{f'(z)}{f(z)} - \frac{g'(z)}{g(z)}\right] dz$$

$$\Rightarrow \frac{1}{2\pi i}\int_x \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\int_x \frac{g'(z)}{g(z)}\,dz$$

$\therefore$ by the argument principle

$$Z_f - P_f = Z_g - P_g$$

Example:

(1) Use Rouche's theorem to prove that all zeros of the polynomial $z^7 - 5z^3 + 12 = 0$ lie between the circles $|z| = 1$ and $|z| = 2$.

Solution:

Consider the circle $x_1 : |z| = 1$

Let $f(z) = z^7 - 5z^3 + 12$ *and* $g(z) = 12$

$\Rightarrow g$ has no zeros inside x_1

For any point $z \in x_1$

$$|f(z) - g(z)| = |z^7 - 5z^3 + 12 - 12| = |z^7 - 5z^3| \le |z|^7 + 5|z|^3$$

$$= 1^7 + 5(1)^3 \quad = 1 + 5 = 6 < 12 \quad = |9(z)|$$

$$|f(z) - g(z)| < |g(z)| \forall z \in \{x)\}$$

$$\sigma = Fox$$

$$\therefore |f(z) - g(z)| < |g(z)| \qquad \forall z \in x_1$$

$\therefore$ by Rouche's theorem,

$$Z_f = Z_g \qquad \text{(Here, there are no poles)}$$

$\Rightarrow f$ has no zeros inside x_1

Consider the Circle $x_2 : |z| = 2$

Let $f(z) = z^7 - 5z^3 + 12$ *and* $g(z) = z^7$

$\Rightarrow g$ has 7 zeros, counting order, inside x_2

For any point $z \in x_2$

$$\left|f(z)-g(z)=\left|z^7-5z^3+12-z^7\right|\right|=\left|-5z^3+12\right|$$

$$=5|z|^3+12 \;=5(z)^3+12=5\times 8+12=40+12$$

$$=52<z^7=|g(z)|$$

$$\therefore |f(z)-g(z)|<|g(z)| \qquad \forall z \in x_2$$

Hence all zeros of the polynomial $z^7-5z^3+12=0$ lie between the circles $|z|=1$ *and* $|z|=2$

(2) Use Rouche's theorem, to prove that $e^z = az^n \;(a>e)$ has n zeros (roots) inside the circle $|z|=1$

Solution: Consider the circle $x:|z|=1$

Let $f(z)=az^n-e^z$

and $g(z)=az^n$

$\Rightarrow g$ has n zeros, counting order, inside x.

For any point $z \in x$,

$$|f(z)-g(z)|=\left|az^n-e^z-az^n\right|$$

$$=\left|e^z\right|=e^{|z|}=e^1<a=|g(z)|$$

$$\therefore |f(z)-g(z)|<|g(z)| \qquad \forall z \in x$$

$\therefore$ by Rouche's theorem $Z_f = Z_g$

$\Rightarrow$ f has n-zeros inside the circle $|z|=1$

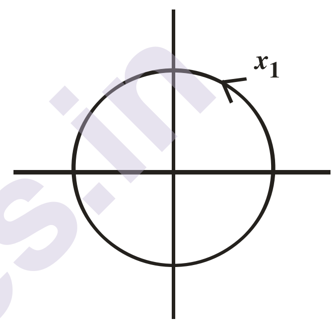

Fig 12.13

(3) Use Rouche's theorem to prove that every polynomial of degree n has n zeros.

12.6 SUMMARY

1) Residues:

Let f has an isolated singularity at $z=\alpha$ and

Let $f(z)=\sum_{n=-\infty}^{\infty} a_n (z-\alpha)^n$ be its Laurent expansion about $z=\alpha$ Res $(f;\alpha)$= Coefficient of $(z-\alpha)^{-1}$ in Laurent series = a_{-1}.

2) Calculation of Residues:

1) If f has a pole of order mat $z=\alpha$ then,

$$\text{Res }(f;\alpha)=\frac{1}{(m-1)!}\lim_{z\to\alpha}\left[\frac{d^{m-1}}{dz^{m-1}}(z-\alpha)^m f(z)\right] \text{ Where m = order}$$

2) If f has a simple pole at $z=\alpha$, then,

$$\text{Res }(f;\alpha)=\lim_{z\to\alpha}\left[(z-\alpha)f(z)\right]$$

Let f be analytic in a domain G except for the isolated singular points $z_1, z_2, ...z_m$: If x is a simple closed curve which does not pass through an of the points z_k than.

3) Cauchy – Residue Theorem:

$$\int_x f(z)d\,z = 2\pi i\sum_{k=1}^{m}\text{Re}s(f;z_k)$$

$= 2\pi i$ [sum of residue of f at its pole inside x]

Where, x is traversed in anticlockwise direction

4) Meromorphic Function (M.F):

If f is defined and analytic in an open set $G \mathbb{C}\ \mathbb{C}$ except for poles. Then f is a Meromorphic function on G.

5) The Argument Principle:

Let f be a Meromorphic function in a domain G and have only finitely many zeros and poles.

If x is a simple closed curve in G s.t. no zeroes and poles of f lie, on x, then $\boxed{2\pi i\int_x \frac{f'(z)}{f(z)}dz = Z_f - P_f}$ where, Z_f, P_f, denote respectively the number of zeros and poles of f inside x, each counted according to their order or multiplicity.

6) Rouche's Theorem:

Suppose f and g are Meromorphic functions in a nbd. of $\overline{B}(\alpha;R)$ with no zeros and poles on the circle $x=\{z;|z-\alpha|=R\}$. If $z_f, z_g\left(P_f, P_g\right)$ are the no. of zeros (poles) of f and g counted according to their order and if $|f(z)-g(z)|<|g(z)|\ \forall 2\in x$ then, $Z_f - P_f = Z_g - P_g$

7) Schwarz's Lemma

Let $D=\{z\in\mathbb{C}:|z|<1\}$ be the unit disk and suppose f is analytic in D with, (i) f (0) = 0 and (ii) $|f(z)|\le 1 \; for \; z\in D$

Then, $|f(z)|\le|z| \quad \forall z\in D \quad and \; |f'(0)|\le 1$

8) A function $f:D\to D$ is said to be an Analytic automorphism or Automorphism of the Unit disc D, if f is bijective and if both f, f^{-1} are analytic in D.

12.7 UNIT END EXCERCISES

1) Show that the Polynomial $P(z)=2z^{10}+4z^2+1$ has exactly two zeroes in $|z|<1$.

Solution: Let $f(z)=4z^2, g(z)=2z^{10}+1$

$\therefore$ $|f(z)|>|g(z)|$ for every number on the Unit circle.

$\therefore$ By Rouche's theorem, the number of zeroes of $(f+g)$ inside the curve $(|Z|=1)$= the number of zeroes of ***f*** inside the curve $(|Z|=1)$.

$\therefore$ $2z^{10}+4z^2+1$ has exactly two zeroes in the curve $|z|<1$.

(Here the number of zeroes of ***f*** inside the curve $(|Z|=1)$.

$=\frac{1}{2\pi i}\int_{|z|=1}\frac{f'}{f}=\frac{1}{2\pi i}\int_{|z|=1}\frac{8z}{4z^2}dz=2$.

2) Suppose that *f* is entire and $f(z)$ is real if and only if z is real .Use the Argument Principle to show that *f* can have atmost one zero.

(Hint: Consider the image of the circle $|z|=R$. Here *f* maps the entire upper semicircle $|z|=R, y>0$ into either the upper half plane or the lower half plane .

Similarly, f maps the entire lower semicircle $|z|=R, y>0$ into either the upper or lower half plane , because $\vartriangle Arg(w)$ is atmost π in any upper/ lower half plane $\therefore$ $\vartriangle Agf(z)\le 2\pi$ as z traverses through the circle $|z|=R$.

$\therefore$ The Number of zeroes of $f(z)$ in $|z|\le R$

$=\frac{1}{2\pi}\vartriangle Argf(z)\le\frac{1}{2\pi}2\pi=1$.

3) Find the number of zeroes of $f(z)=\frac{1}{3}e^{z}-z$ in $|z|\le 1$. (Hint : Let $f(z)=z, g(z)=\frac{1}{3}e^{z}$ $\therefore$ $|f(z)|>|g(z)|\,\forall z, |z|\le 1$)

4) Find the number of zeroes of $f(z)=z^{6}-5z^{4}+3z^{2}-1$ in $|z|\le 1$ (Hint : Take $f(z)=5z^{4}, g(z)=z^{6}+3z^{2}-1$

$\therefore$ on $|z|=1, |f(z)|=|5z^{4}|=5\ge |z^{6}+3z^{2}-1|$

Also $|f(z)|=|g(z)|$ only at $\mp i$.

$\therefore$ There are 4 zeroes of f in $|z|\le 1$.

5) Show that for each $R>0$ if n is large enough then $P_n(z)=1+z+\frac{z^{2}}{2!}+...+\frac{z^{n}}{n!}$ has no zeroes in $|z|\le R$.

(Hint: $P_n(z)\to e^{z}$ as $n\to\infty$.)

6) If f is Meromorphic on G and $\tilde{f}:G\to\mathbb{C}_\infty$ is defined by
$\tilde{f}(z)=\infty$ if z is pole of f
$=f(z)$ otherwise. Show that f is continuous on G.

7) Find $\int_{-\infty}^{\infty}\frac{dx}{x^{4}+1}$.

Solution: Here $z_1=e^{\frac{i\pi}{4}}$ and $z_2=e\frac{3i\pi}{4}$

Represent the poles of $\frac{1}{z^{4}+1}$ in the upper half plane. Since each of these is a simple pole

$\therefore$ The residues are given by the values of $f'(z)=\frac{1}{4z^{3}}$ at these poles.

$\therefore \text{Res}\left(\frac{1}{z^{4}+1};e^{\frac{i\pi}{4}}\right)=-\frac{1}{8}\left(\sqrt{2}+i\sqrt{2}\right)$ and

$\text{Res}\left(\frac{1}{z^{4}+1};e^{\frac{3i\pi}{4}}\right)=-\frac{1}{8}\left(\sqrt{2}-i\sqrt{2}\right)$

$$\therefore \int_{-\infty}^{\infty}\frac{dx}{x^{4}+1}=2\pi i\left[\text{Res}\left(\frac{1}{z^{4}+1};e^{\frac{i\pi}{4}}\right)+\text{Res}\left(\frac{1}{z^{4}+1};e^{3i\pi}\right)\right]=\pi\frac{\sqrt{2}}{2}$$

8) Evaluate $\int_{-\infty}^{\infty} \frac{\sin(x)}{x} dx$.

Solution: We know that the function $\frac{e^{ix}}{x}$ has pole at $x = 0$.

$\therefore$ We modify the integral as follows:

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x} dx = \text{Im} \int_{-\infty}^{\infty} \frac{e^{ix} - 1}{x} dx.$$

$$\because \int_{-M}^{M} \frac{e^{ix} - 1}{x} dx = \int_{IM} \frac{1 - e^{iz}}{z} dz = \pi i - \int_{IM} \frac{e^{iz}}{z} dz$$

$$\therefore \lim_{M \to \infty} \int_{IM} \frac{e^{iz}}{z} dz = 0 \Rightarrow \int_{-\infty}^{\infty} \frac{e^{ix} - 1}{x} dx = \pi i$$

$$\therefore \text{Im} \int_{-\infty}^{\infty} \frac{e^{ix} - 1}{x} dx = \pi. \Rightarrow \int_{-\infty}^{\infty} \frac{\sin(x)}{x} dx = \pi.$$

9) Evaluate $\int_{0}^{\infty} \frac{dx}{1 + x^3}$

Solution: Let $f(z) = \frac{1}{1 + z^3}$. Then f has poles at $z_1 = e^{\frac{i\pi}{3}}$ and $z_2 = e^{i\pi} = -1, z_3 = e^{\frac{3i\pi}{3}}$

$$\therefore \text{Res} \left(\frac{\log(z)}{1 + z^3}; z_1 = e^{\frac{i\pi}{3}} \right) = -\frac{i\pi}{9} \left(\frac{1}{2} + \frac{i\sqrt{3}}{2} \right)$$

$$\therefore \text{Res} \left(\frac{\log(z)}{1 + z^3}; z_2 = e^{\frac{i\pi}{3}} \right) = -\frac{i\pi}{3}$$

$$\therefore \text{Res} \left(\frac{\log(z)}{1 + z^3}; z_3 = e^{\frac{si\pi}{3}} \right) = -\frac{Si\pi}{9} \left(\frac{1}{2} + \frac{i\sqrt{3}}{2} \right)$$

$$\therefore \sum_k \text{Re}\, s \left(\frac{\log(z)}{1 + z^3}; z_k \right) = -\frac{2\pi}{9} \sqrt{3}$$

$$\therefore \int_{0}^{\infty} \frac{dx}{1 + x^2} = -\sum_k \text{Re}\, s \left(\frac{\log(z)}{1 + z^3}; z_k \right) = -\frac{2\pi}{9} \sqrt{3}$$

10) Evaluate $\int_{0}^{\infty} \frac{dx}{\sqrt{x}(1 + x)}$.

(Hint: The integral has the form $\int_{0}^{\infty} \frac{x^{a-1}}{p(x)} dx$ with $0 < a = \frac{1}{2} < 1$

Use the formula $\left[\left(1 - e^{2\pi i(a-1)} \right) \int_{0}^{\infty} \frac{x^{a-1}}{p(x)} dx = \sum_k \text{Re}\, s \left(\frac{z^{a-1}}{p(x)}; z_k \right) \right]$ the sum on R.H.S. is taken over the zeroes of the function $p(z)$.

11) Evaluate $\int_0^{2\pi} \frac{d\theta}{2+\infty s(\theta)}$

Solution: Consider $\int_0^{2\pi} \frac{d\theta}{2+\cos(\theta)} = \frac{2}{i}\int_{|z|=1} \frac{dz}{z^2+4z+1}$

$= 4\pi \operatorname{Re} s\left(\frac{1}{z^2+4z+1}; \sqrt{3}-2\right) = \frac{2}{3}\pi\sqrt{3}$

12) Evaluate the integral $\int\limits_{-\infty}^{\infty} \frac{x^2}{\left(x^2+1\right)^2} dx$.

13) Use Cauchy-Residue theorem to evaluate

$\int\limits_{\gamma} \frac{\sin z}{z^2+1} dz$ where γ is the circle $|z-i| = \pi$.

14) Use Cauchy-Residue theorem to evaluate $\int\limits_0^{\infty} \frac{\cos 5x}{x^2+4} dx$.

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13

MOBIUS TRANSFORMATION

Unit structure

13.0 Objective
13.1 Introduction
13.2 Conformal Mapping
13.3 Some standard transformation
13.4 Mobius Transformation Or Bilinear Transformation Or Linear Fractional Transformation
13.5 Summary
13.6 Unit End Exercise

13.0 OBJECTIVE

After going through this unit you shall come to know about

- Special type of functions called transformation from $\mathbb{C} \to \mathbb{C}$
- The combination of special function to give rise to a transformation called Mobius Transformation
- Special properties of Mobius Transformation like fixed point and cross ratio.
- Method to find the bilinear transformation using various method.

13.1 INTRODUCTION

There are certain transformation that can be readily described in terms of geometry. In this chapter, we are mainly concerned with certain geometric interpretations of functions and finding the image of a given figure under a given bilinear function .

13.2 CONFORMAL MAPPING

A differentiable map $f : \Omega \to \mathbb{R}^2$ is said to be conformal map if

$$\det\left(Df_z\right) \neq 0 \ \forall z \in \Omega \ \& \ \measuredangle\left(Df_z(\alpha), Df_z(\beta)\right) = \measuredangle(\alpha, \beta) \ \forall \alpha, \beta \in \mathbb{C} - \{0\}$$

Thus, conformal map is preserves the angle between two intersecting curves in $\mathbb{C}$

Proposition: *Let* Ω *be a domain in* $\mathbb{C}$ *and* $f:\Omega \to \mathbb{C}$ *be a map. Then f is any analytic function with* $f'(z) \neq 0 \; \forall \; z \in \Omega$ *if and only if f is conformal map with* $det(Df_z) > 0 \; \forall \; z \in \Omega$

Proof: Let *f* be analytic, $Df_z(\alpha) = f'(z)\alpha \; \forall \; \alpha \in \mathbb{C}$. Then

$$\measuredangle\left(f'(z)\alpha, f'(z)\beta\right) = \frac{\operatorname{Re}\left(\left(f'(z)\alpha\right)\overline{f'(z)\beta}\right)}{\left|f'(z)\alpha\right|\left|f'(z)\beta\right|} = \frac{\operatorname{Re}\left(\alpha\overline{\beta}\right)}{|\alpha||\beta|} \measuredangle(\alpha,\beta) \; \forall \alpha,\beta \in \mathbb{C}/\{0\}$$

Thus *f* is conformal map.

Let $f(z)=u(z)+iv(z), z \in \Omega$. By Cauchy Riemann equation, the Jacobian of $f = (u(x,y), v(x,y))$ is $\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} u_x & -v_x \\ v_x & u_x \end{bmatrix}$.

Hence $det(Df_x) = u_x^2 + u_y^2 \geq 0$. Now $f'(z) = u_x + iv_x$ & $f'(z) \neq 0$.

Hence $det(Df_x) = u_x^2 + u_y^2 = \left|f'(z)\right|^2 > 0$.

Conversely, Fix $z \in \Omega$. Since *f* is conformal map, $\measuredangle\left(Df_z(1,0), Df_z(0,1)\right) = \measuredangle(1,i) = 0$

So, $Df_z(1,0) \perp Df_z(0,1)$. Let $Df_z(1,0) = (a,b) \in \mathbb{R}^2$. Then $Df_z(1,0) = (\pm b, a)$.

Since $\det(Df_z) > 0, Df_z(0,1) = (-b,a)$. Let $\alpha = a + ib$. Then $Df_z(\beta) = \alpha\beta$ (Verify) and *f* is complex differentiable.

13.3 SOME STANDARD TRANSFORMATION

(i) Translation: $\boxed{w = z + c}$ where c is a complex constant.

The transformation $w = z + c$ is simply a translation of the axes and as such as preserves the shape if the region in *z-plane*.

Note: in particular translation maps circle in *z-plane* onto circles in the *w-plane*.

(ii) Rotation and Reflection: $\boxed{w = cz}$ where c is a complex constant.

Let $w = \operatorname{Re}^{i\phi}, z = re^{i\theta}, c = \rho e^{i\alpha}$

Now, $\operatorname{Re}^{i\phi} = re^{i\theta} \cdot \rho e^{i\alpha} = r\rho e^{i(\alpha+\theta)}$

$\therefore$ $R = r\rho \; \& \; \phi = \alpha + \theta$

$\therefore$ Thus the transformation maps a point $P(r,\theta)$ in the z-plane onto a point $P'(r\rho, \alpha+\theta)$ in the w-plane.

i.e. the image is magnified (or diminished) by $r\rho$ and rotated by $\alpha + \theta$.

Note: in particular $w = cz$ maps circle in *z-plane* onto circles in the *w-plane.*

(iii) Inversion: $\boxed{w = \frac{1}{z}}$

Let $w = \text{Re}^{i\phi}, z = re^{i\theta}$

$\therefore \text{Re}^{i\phi} = \frac{1}{re^{i\theta}}$

$\therefore \text{R} = \frac{1}{r} \ \& \ \phi = -\theta$

i.e. transformatiom $w = \frac{1}{z}$ maps $P(r, \theta)$ in the z-plane onto a point $P'(r, -\theta)$ in the w-plane.

Note: in particular $w = \frac{1}{z}$ maps circle in *z-plane* onto circles in the *w-plane.*

13.4 MOBIUS TRANSFORMATION OR BILINEAR TRANSFORMATION OR LINEAR FRACTIONAL TRANSFORMATION

Definition: A transformation $s(z) = w = \frac{az+b}{cz+d}$, where a, b, c, d are complex constant and $ad - bc \neq 0$ is called Mobius Transformation or Bilinear Transformation or Linear fractional transformation.

Note : If $ad - bc = 0 \quad \Rightarrow \quad ad = bc \quad \Rightarrow \quad \frac{b}{a} = \frac{d}{c}$.

$$\therefore \qquad w = \frac{az+b}{cz+d} = \frac{a\left(z + b/a\right)}{c\left(z + d/a\right)} = \frac{a}{c} = \text{constant.}$$

Thus, $ad - bc \neq 0$ is a necessary condition for the Mobius Transformation : $s(z) = w = \frac{az+b}{cz+d}$

1) If *S* is a Mobius transformation, then S^{-1} is the inverse mapping of *S* i.e. $\left(S \circ S^{-1}\right)(z) = z = \left(S^{-1} \circ S\right)(z)$.

Let $S(z) = \frac{az+b}{cz+d}$ where a, b, c, d are complex constant and $ad - bc \neq 0$.

$$\therefore \qquad S^{-1}(z) = \frac{dz - b}{-cz + a}$$

$\therefore \quad \left(S \circ S^{-1}\right)(z) = S\left(S^{-1}(z)\right) = S\left(\frac{dz-b}{-cz+a}\right)$

$$= \left[a\left(\frac{dz-b}{-cz+a}\right)+b\right] \times \frac{1}{\left[c\left(\frac{dz-b}{-cz+a}\right)+d\right]}$$

$$= \frac{adz - ab - bcz + ab}{-cz+a} \times \frac{-cz+a}{cdz - bc - cdz + ad} = \frac{adz-bcz}{-bc+ad}$$

$$= z\,\frac{(ad-bc)}{(ad-bc)} = z$$

Similarly, $\left(S^{-1} \circ S\right)(z) = z$.

Consider $\frac{(\lambda a)z + (\lambda b)}{(\lambda c)z + (\lambda d)} = \frac{\lambda(az+b)}{\lambda(cz+d)} = s(z) \qquad (\lambda \neq 0)$

$\Rightarrow$ The $\omega-$efficients a, b, c, d are not unique.

$a_1 a_2 b_1 c_2 + a_1 a_2 d_1 d_2 + b_1 b_2 c_1 c_2 + b_2 c_1 d_1 d_2$

$-a_1 a_2 b_1 c_2 - a_2 b_1 c_1 d_2 - a_1 b_2 c_2 d_1 - b_2 c_1 d_1 d_2 =$

$a_1 a_2 d_1 d_2 - a_2 b_2 c_2 d_1 + b_1 b_2 c_1 c_2 - a_2 b_1 c_1 d_2$

$= a_1 d_1 \left(a_2 d_2 - b_2 c_2\right) + b_1 c_1 \left(b_2 c_2 - a_2 d_2\right) =$

$a_1 d_1 \left(a_2 d_2 - b_2 c_2\right) - b_1 c_1 \left(a_2 d_2 - b_2 c_2\right)$

$= \left(a_1 d_1 - b_1 c_1\right).\left(a_2 d_2 - b_2 c_2\right) \neq 0$. i.e. the coefficients a, b, c, d are not unique.

3) Let S be a Mobius transformation on $\mathbb{C}_\infty$.

$S(z) = \frac{az+b}{cz+d}$ where a, b, c, d are complex constant and $ad - bc \neq 0$.

$$S(z) = \frac{z\left(a + b/z\right)}{z\left(c + d/z\right)} = \frac{a + b/z}{c + d/z}$$

$S(\infty) = \frac{a}{c}$ when $c \neq 0$

$\quad = \infty$ when $c = 0$

Again, $S(z) = \frac{az+b}{cz+d} = \frac{az+b}{c\left(z + d/c\right)}$

$S\left(-\frac{d}{c}\right) = \infty$ when $c \neq 0$

4) If S and T are Mobius Transformations, then $(T \circ S)$, (composition of T and S) is also a Mobius Transformation.

Let $\xi = S(z) = \dfrac{a_1 z_1 + b_1}{c_1 z + d_1}$ where a, b, c, d are complex constant and $a_1 d_1 - b_1 c_1 \neq 0$ and $\omega = T(\xi) = \dfrac{a_2 \xi + b_2}{c_2 \xi + d_2}$ where a_2, b_2, c_2, d_2 are complex constant and $a_2 d_2 - b_2 c_2 \neq 0$.

$$[T \circ S](z) = T[S(z)] = T\left[\frac{a_1 z + b_1}{c_1 z + d_1}\right]$$

$$(T \circ S)(z) = a_2\left[\frac{a_1 z + b_1}{c_1 z + d_1}\right] + b_2 \times \frac{1}{c_2\left[\frac{a_1 z + b_1}{c_1 z + d_1}\right] + d_2}$$

$$= \frac{a_1 a_2 z + a_2 b_1 + b_2 c_1 z + b_2 d_1}{\cancel{c_1 z + d_1}} \times \frac{\cancel{c_1 z + d_1}}{a_1 c_2 z + b_1 c_2 + c_1 d_2 z + d_1 d_2}$$

$$= \frac{a_1 a_2 z + a_2 b_1 + b_2 c_1 z + b_2 d_1}{a_1 c_2 z + b_1 c_2 + c_1 d_2 z + d_1 d_2} = \frac{(a_1 a_2 + b_2 c_1) z + (a_2 b_1 + b_2 d_1)}{(a_1 c_2 + c_1 d_2) z + (b_1 c_2 + d_1 d_2)} = \frac{\alpha z + \beta}{x z + \delta}$$

where, $\alpha = a_1 a_2 + b_2 c_1$, $\beta = a_2 b_1 + b_2 d_1$, $x = a_1 c_2 + c_1 d_2$, $\delta = b_1 c_2 + d_1 d_2$

Now, $\alpha\delta - \beta x = (a_1 a_2 + b_2 c_1)(b_1 c_2 + d_1 d_2) - (a_2 b_1 + b_2 d_1)(a_1 c_2 + c_1 d_2)$

$$= a_1 a_2 b_1 c_2 + a_1 a_2 d_1 d_2 + b_1 b_2 c_1 c_2 + b_2 c_1 d_1 d_2 - (a_1 a_2 b_1 c_2 + a_2 b_1 c_1 d_2 + a_1 b_2 c_2 d_1 + b_2 c_1 d_1 d_2)$$

$$\neq 0$$

Hence, composition of S and T is also a Mobius transformation.

Proposition : *If S is a M.T. then, S is a composition of translation rotation, inversion and Magnification.*

Proof : Consider a Mobius Transformation, $S(z) = \dfrac{az + b}{cz + d}$ where, a, b, c, d are complex constant and $ad - bc \neq 0$

Case I : When $c = 0$

$$S(z) = \frac{az + b}{d} = \frac{a}{d} z + \frac{b}{d}$$

If $S_1(z) = \dfrac{a}{d} z$ and $S_2(z) = z + \dfrac{b}{d}$

then, $\left(S_2 \circ S_1\right) = S_2\left[S_1(z)\right] = S_2\left[\frac{a}{d} z\right] = \frac{a}{d} z + \frac{b}{d} = S(z)$

$S = S_2 \circ S_1$

In this case, Mobius transformation is a composition of translation, rotation and Magnification.

Case II : When $c \neq 0$

$$S(z) = \frac{c}{c}\left(\frac{az+b}{cz+d}\right) = \frac{az+bc+ad-ad}{(ccz+d)} = \frac{(bc-ad)}{c(cz+d)} + \frac{a}{c}\frac{(cz+d)}{(cz+d)}$$

$$= \frac{bc-ad}{c^2} \cdot \frac{1}{z+d/c} + \frac{a}{c}$$

If $S_1(z) = z + d/c$, $S_2(z) = 1/z$, $S_3(z) = \left(\frac{bc-ad}{c^2}\right) z$ and $S_4(z) = z + \frac{a}{c}$

$$\left(S_4 \circ S_3 \circ S_2 \circ S_1\right)(z) = S_4\left[S_3\left(S_2\left(S_1(z)\right)\right)\right] = S_4\left[S_3\left(S_2\left(z + \frac{d}{c}\right)\right)\right]$$

$$= S_4\left[S_3\left(\frac{1}{z+d/c}\right)\right]$$

$$= S_4\left[S_3\left(\frac{c}{cz+d}\right)\right] = S_4\left[\frac{(bc-ad)}{c^2} \times \frac{c}{c_2+d}\right] = S_4\left[\frac{bc-ad}{c(cz+d)}\right]$$

$$= \frac{bc-ad}{c(cz+d)} + \frac{a}{c}$$

$$= \frac{a+bc-ad}{c(cz+d)} = S(z)$$

$\therefore \quad S = S_4 \circ S_3 \circ S_2 \circ S_1$

In this case, M.T. is a composition of translation, rotation, inversion and magnification.

Fixed Points :

Definition : Let G be a subset of $\mathbb{C}_\infty$ and $f : G \to \mathbb{C}_\infty$. Then point $z_0 \in G$ is said to be a fixed point of f if $f(z_0) = z_0$.

e.g. i) Let $f(z) = z^2$.Here, f has fixed points 0, 1 and ∞.

ii) Let $f(z) = 1/z$. Here, f has fixed point 1 and – 1.

iii) Let $f(z) = z + (3+i)$.Here, f has fixed point ∞.

Example 1 : What are the fixed points of Mobius transformation?

Solution : Consider a M.T., $S(z)=\frac{az+b}{cz+d}$ where, a,b,c,d are complex constant and $ad-bc\neq 0$.
For fixed points,
Put

$S(z)=z \Rightarrow \frac{az+b}{cz+d}=z \Rightarrow \quad az+b=cz^2+dz$

$\Rightarrow \quad cz^2+(d-a)z-b=0$

$\Rightarrow \quad z=\frac{-(d-a)+\sqrt{(d-a)^2-4.b.c}}{2c} \Rightarrow$

$z=\frac{(a+d)\pm\sqrt{(d-a)^2-4bc}}{2c}$

$\therefore$ A. M.T. can have atmost 2 fixed points unless $S(z)=z \quad \forall z$

2) $S(z)=\frac{iz+2}{z+1}$

Solution : For fixed points

Put $S(z)=z \quad \Rightarrow \quad \frac{iz+2}{z+1}=z \Rightarrow \quad iz+2=z^2+z \Rightarrow \quad z^2+(1-i)z-2=0$

$\Rightarrow \quad z=\frac{-(1-i)\pm\sqrt{(1-i)^2-8}}{2}$

Definition : For any three distinct points z_2, z_3, z_4 in C_∞, the cross ratio of four points z_1, z_2, z_3, z_4 is defined to be $(z_1, z_2, z_3, z_4)=\frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)}$.

For $z_1=z_2$, $\quad (z_2, z_2, z_3, z_4)=\frac{(z_2-z_3)(z_2-z_4)}{(z_2-z_4)(z_2-z_3)}=1$

For $z_1=z_3$, $\quad (z_3, z_2, z_3, z_4)=\frac{(z_3-z_3)(z_2-z_4)}{(z_3-z_4)(z_2-z_3)}=0$

For $z_1=z_4$, $\quad (z_4, z_2, z_3, z_4)=\frac{(z_4-z_3)(z_2-z_4)}{(z_4-z_4)(z_2-z_3)}=\infty$

Definition : If $z_1\in\mathbb{C}_\infty$ then the cross ratio (z_1, z_2, z_3, z_4) is the image of z_1 under the unique. Mobius transformation which takes z_2 to 1, z_3 to 0 and z_4 to ∞. i.e. $S(z_1)=(z_1, z_2, z_3, z_4)$.

Note : If M is any Mobius transformation and w_2, w_3, w_4 are complex number s.t. $M(w_2) = 1,\ M(w_1) = 0$ and $M(w_4) = \infty$ then $M(z) = (z, w_2, w_3, w_4)$.

Proposition : *If z_2, z_3, z_4 are distinct points in $\mathbb{C}_\infty$ and T is any Mobius transformation then, $(z_1, z_2, z_3, z_4) = (Tz_1, Tz_2, Tz_3, Tz_4)$ for any fix z_1.*

Proof : Let $S(z) = (z_1, z_2, z_3, z_4)$

$\Rightarrow$ S is a M.T. (1)

and $S(z_2) = 1,\ S(z_3) = 0$ and $S(z_4) = \infty$

Given that, T is any M.T.

Put $M = ST^{-1}$

$\therefore$ $M(Tz_2) = S\left(T^{-1}(Tz_2)\right) = S(z_2) = 1$

$\therefore$ $M(Tz_3) = S\left(T^{-1}(Tz_3)\right) = S(z_3) = 0$

$\therefore$ $M(Tz_4) = S\left(T^{-1}(Tz_4)\right) = S(z_4) = \infty$

$\therefore$ $M(z) = (z, Tz_2, Tz_3, Tz_4)$

Put $z = Tz_1$

$\therefore$ $M(Tz_1) = (Tz_1, Tz_2, Tz_3, Tz_4)$

$S\left(T^{-1}(Tz_1)\right) = (Tz_1, Tz_2, Tz_3, Tz_4)$ $\left(\because M = ST^{-1}\right)$

$S(z_1) = (Tz_1, Tz_2, Tz_3, Tz_4)$

$\therefore$ $(z_1, z_2, z_3, z_4) = (Tz_1, Tz_2, Tz_3, Tz_4)$ ($\because$ S is M.T.)

i.e. The cross ratio is invariant under Mobius Transition.

Proposition : *If z_2, z_3, z_4 are distinct points in $\mathbb{C}_\infty$ and w_2, w_3, w_4 are also distinct points in $\mathbb{C}_\infty$ then, there is a unique M.T. S s.t. $S(z_2) = w_2$, $S(z_3) = w_3$ and $S(z_4) = w_4$.*

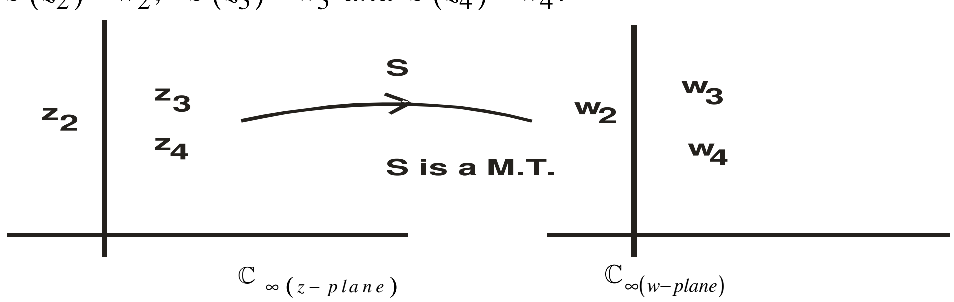

Fig 13.1

Proof : Let $T(z)=(z, z_2, z_3, z_4)$ and $M(z)=(z, w_2, w_3, w_4)$.

$\Rightarrow$ T and M are Mobius Transformations and $T(z_2)=1$, $T(z_3)=0$ and $T(z_4)=\infty$, $M(w_2)=1$, $M(w_3)=0$ and $M(w_4)=\infty$.

Put $S=M^{-1}T$

$\therefore$ $S(z_2)=M^{-1}(Tz_2)=M^{-1}(1)=w_2$

$S(z_3)=M^{-1}(Tz_3)=M^{-1}(0)=w_3$

$S(z_4)=M^{-1}(Tz_4)=M^{-1}(\infty)=w_4$

Let R be another M.T. s.t.

$R(z_j)=w_j$ for $j=2,3,4$

$(R^{-1}\circ S)(z_2)=R^{-1}(S(z_2))=R^{-1}(w_2)=z_2$

$(R^{-1}\circ S)(z_3)=R^{-1}(S(z_3))=R^{-1}(w_3)=z_3$

$(R^{-1}\circ S)(z_4)=R^{-1}(S(z_4))=R^{-1}(w_4)=z_4$

Here, $(R^{-1}\circ S)$ composite map of S and R^{-1} has 3 fixed points.

$\therefore$ $R^{-1}\circ S=I$ (Identify map)

$\Rightarrow$ $S=R$

Hence, S is the unique transformation.

Proposition : *Let z_1, z_2, z_3, z_4 be distinct points in $\mathbb{C}_\infty$ then the cross ratio (z_1, z_2, z_3, z_4) is a real number $\Leftrightarrow$ all four points lies on a circle (or a straight line).* *(2009)*

Proof : Let $S:\mathbb{C}_\infty \to \mathbb{C}_\infty$ be defined by $S(z)=(z, z_2, z_3, z_4)$ = Real number.

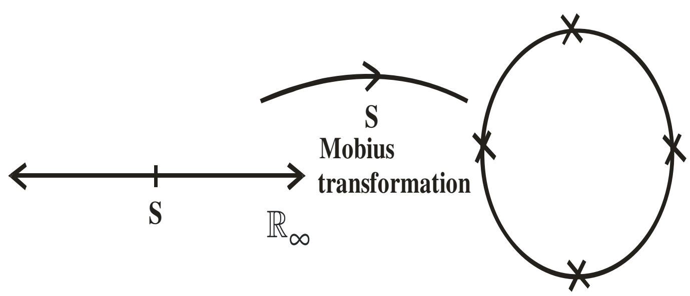

Fig 13.2

Then, $S^{-1}\{\mathbb{R}\}=\{z:(z, z_2, z_3, z_4) \text{ is real}\}$ i.e. image of $\mathbb{R}_\infty$ under the Mobius Transformation is a circle.

We will prove that the image of $\mathbb{R}_\infty$ under the Mobius transformation is a circle.

Let $S(z)=\dfrac{az+b}{cz+d}$ where a,b,c,d are complex constant and $ad-bc\neq 0$.

If $z=x\in\mathbb{R}$ and $w=S^{-1}(x)$, then $S(w)=x\in\mathbb{R}$.

$\because$ $S(w)$ is purely real number.

$\therefore$ $S(w)=\overline{s(w)}$

$\therefore$ $\dfrac{aw+b}{cw+d}=\dfrac{\bar{a}.\bar{w}+\bar{b}}{\bar{c}.\bar{w}+\bar{d}}$

$\therefore$ $(aw+b)(\bar{c}\,\bar{w}+\bar{d})=(\bar{a}\,\bar{w}+\bar{b})(cw+d)$

$\Rightarrow$ $a\bar{c}|w|^2+a\bar{d}w+b\bar{c}\bar{w}+b\bar{d}=\bar{a}c|w|^2+\bar{a}\bar{w}d+\bar{b}cw+\bar{b}d$

$(a\bar{c}-\bar{a}c)|u|^2+(a\bar{d}-\bar{b}c)w+(b\bar{c}-\bar{a}d)\bar{w}+(b\bar{d}-\bar{b}d)=0$ (1)

Case I : If $a\bar{c}$ is not real, then $a\bar{c}-\bar{a}c\neq 0$.

If $a\bar{c}$ is real then $a\bar{c}=\overline{a\bar{c}}=\bar{a}c$

$\Rightarrow$ $a\bar{c}=\bar{a}c=0$

From equation (1),

$$|w|^2+\frac{(a\bar{d}-\bar{b}c)}{(a\bar{c}-\bar{a}c)}w+\frac{(\bar{a}d-b\bar{c})}{(\bar{a}c-a\bar{c})}w-\frac{(b\bar{d}-\bar{b}d)}{(\bar{a}c-a\bar{c})}=0$$

Put $x=\dfrac{\bar{a}d-b\bar{c}}{\bar{a}c-a\bar{c}}$ and $\delta=\dfrac{b\bar{d}-\bar{b}d}{\bar{a}c-a\bar{c}}$

$|w|^2+\bar{x}w+x\bar{w}-\delta=0$

$|w|^2+\bar{x}w+x\bar{w}+|x|^2=|x|^2+\delta$

$|w+x|=R$ $\Rightarrow$ $|w-(-x)|=R$ which is the equation of the circle with centre at (-*x*, 0) and radius equal to R.

Where, $R=\sqrt{|x|^2+\delta}$

$$=\sqrt{\left|\frac{\bar{a}d-b\bar{c}}{\bar{a}.c-a\bar{c}}\right|^2+\frac{b\bar{d}-\bar{b}d}{\bar{a}c-a\bar{c}}}=\left|\frac{\bar{a}d-b\bar{c}}{\bar{a}c-a\bar{c}}\right|\neq 0$$

Case II : If $a\bar{c}$ is real then $a\bar{c}-\bar{a}c=0$

From equation (1)

$$\left(a\bar{d}-\bar{b}c\right)w+\left(b\bar{c}-\bar{a}d\right)\bar{w}+\left(b\bar{d}-\bar{b}d\right)=0$$

Put $\alpha=a\bar{d}-\bar{b}c$ & $\beta=i\left(b\bar{d}-\bar{b}d\right)$

$$\Rightarrow \frac{\alpha}{2}\bar{w}-\frac{\bar{\alpha}}{}\bar{w}-i\beta=0.$$

$$\therefore \text{Im }(\alpha\omega)-i\beta=0.$$

$$\Rightarrow \text{Im }(\alpha\omega-\beta)=0.$$

or $$\text{Im}\left(\frac{\left(w-\beta/\alpha\right)}{1/\alpha}\right)=0 \qquad (2)$$

Note : Consider a straight line L in the Complex Plane $\mathbb{C}$. If a is any point in L and b is its direction vector then,

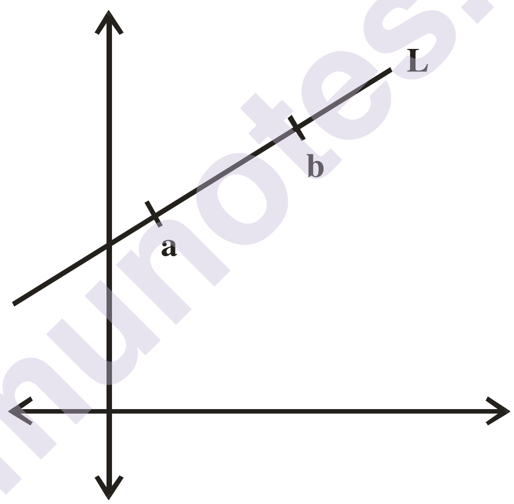

Fig 13.3

$$L=\left\{z:\text{Im}\left(\frac{z-a}{b}\right)=0\right\}$$

$$L=\left\{z:\text{Im}\left(\frac{z-a}{b}\right)=0\right\}$$

$$=\left\{z=a+bt:t\in\mathbb{R},\ \text{i.e.}-\infty<t<\infty\right\}$$

$\therefore$ The point w lies on a line determined by equation (2) for fixed α and β.

$\therefore$

Theorem : *A Mobius transformation takes circles onto circles.*

Proof : Let Γ be any circle in $\mathbb{C}_\infty$ (z-plane). Let S be any Mobius Transformation. Suppose z_2, z_3, z_4 are distinct points on the circle Γ.

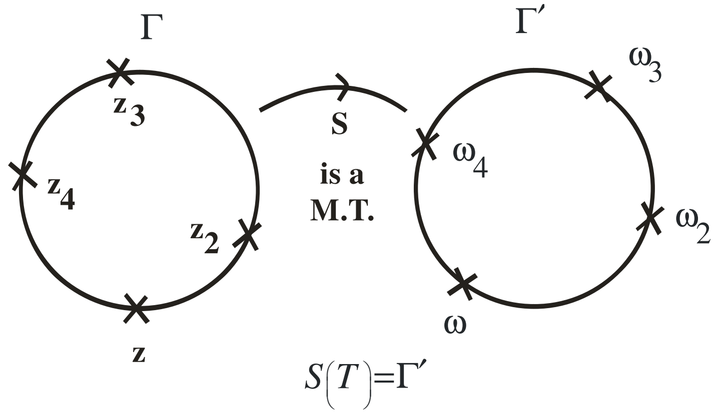

Fig 13.4

Put $Sz_j = w_j$ for $j = 2, 3, 4$.

$\Rightarrow$ w_2, w_3, w_4 are distinct points in $\mathbb{C}_\infty$ (w-plane).

$\therefore$ These three distinct points w_2, w_3, w_4 determine a circle in w-plane.

T.P.T. $S(\Gamma) = \Gamma'$

Since, z_2, z_3, z_4 are distinct points in $\mathbb{C}_\infty$ and S in a M.T.

$\therefore$ $(z, z_2, z_3, z_4) = (Sz, Sz_2, Sz_3, Sz_4)$ for any point z.

$$= (Sz, w_2, w_3, w_4) \ldots \quad (1)$$

$\therefore$ If $z \in \Gamma$ then the cross ratio (z, z_2, z_3, z_4) is a real number.

$\because$ z, z_2, z_3, z_4 all lie on a circle …

$\therefore$ (Sz, w_2, w_3, w_4) is a real number. (by equation (1))

$\Rightarrow$ These four points Sz, w_2, w_3, w_4 lie on a circle Γ'.

Put $Sz = w$

As z moves on the circle Γ, then the corresponding point w moves on the circle Γ' under a M.T. 'S'.

$\Rightarrow$ $S(\Gamma) = \Gamma'$.

Hence, a Mobius Transformation takes circles onto circles.

13.6 UNIT END EXERCISE

Example : Find a M.T. which maps points $z = -1, 0, 1$ onto the points $w = -1, -i, 1$. Also find the image of unit circle $|z| = 1$ in the $\mathbb{C}$-plane under this M.T.

Solution : Given points $z = z,\ z_2 = -1,\ z_3 = 0,\ z_4 = 1$ and $w = w$, $w_2 = -1\ \ w_3 = -i,\ w_4 = 1$.

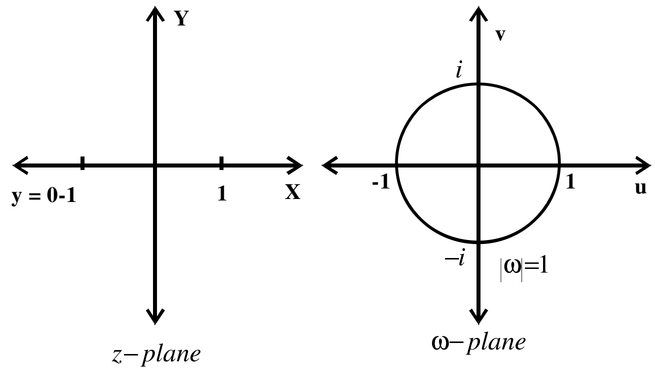

Fig 13.5

$\therefore$ M.T. is given by $(z, z_2, z_3, z_4) = (w, w_2, w_3, w_4)$

(Proposition on page no.81(2))

$$\Rightarrow \quad \frac{(z-z_3)(z_2-z_4)}{(z-z_4)(z_2-z_3)} = \frac{(w-w_3)(w_2-w_4)}{(w-w_4)(w_2-w_3)} \Rightarrow$$

$$\frac{(z-0)(-1-1)}{(z-1)(-1-0)} = \frac{(w+i)(-1-1)}{(w-1)(-1+i)}$$

$$\Rightarrow \quad (-z+iz)(w-1) = (1-z)(w+i) \quad \Rightarrow$$

$$-\cancel{zw} + z + izw - \cancel{iz} = w + i - \cancel{zw} - \cancel{iz}$$

$$\Rightarrow \quad z + izw = w + i \quad \Rightarrow$$

$$S(z) = w = i\left[\frac{z-i}{z+i}\right] \qquad (1)$$

which is a required bilinear transformation.

$\therefore$ From equation (1), $w = i\left[\dfrac{z-i}{z+i}\right]$

$$\Rightarrow \quad wz + iw = iz + 1 \Rightarrow wz - iz = 1 - iw \Rightarrow z(w-i) = 1 - iw$$

$$\Rightarrow \quad z = \frac{1-iw}{w-i}$$

Given equation of unit circle is $|z| = 1$.

$$\therefore \quad |z| = \frac{|1-iw|}{|w-i|} = 1 \quad \Rightarrow \quad |1-iw| = |w = i| \Rightarrow \quad |1-iw|^2 = |w-i|^2$$

$$\Rightarrow \quad (1-iw)\left(\overline{1-iw}\right) = (w-i)\left(\overline{w-i}\right)$$

$$\Rightarrow \quad (1-iw)\left(\bar{1} - \bar{i}\bar{w}\right) = (w-i)\left(\bar{w} - \bar{i}\right)$$

$$\frac{w}{-i}=\frac{i}{-z}\Rightarrow w=\frac{-1}{z}\ \therefore$$

$$\because z_1=\infty,\quad \frac{z}{z_1}=0\quad \&\quad \frac{z_2}{z_1}=0$$

$$\because w_3=\infty,\quad \frac{w}{w_3}=0\quad \&\quad \frac{w_2}{w_3}=0$$

$\Rightarrow\ (1-iw)\left(1+i\overline{w}\right)=(w-i)\left(\overline{w}+i\right)$

$\Rightarrow\ 1+i\overline{w}-iw+|w|^2=|w|^2+iw-i\overline{w}+1$

$\Rightarrow\ 2iw-2i\overline{w}=0\ \Rightarrow\ w-\overline{w}=0\ \Rightarrow\ w=\overline{w}$

Put $w=u+iv$,

$\therefore\ w=\overline{w}\ \Rightarrow\ u+iv=\overline{u+iv}\ \Rightarrow\ u+iv=u-iv\ \Rightarrow\ 2iv=0$

$\Rightarrow\ v=0$

which is the equation of real axis.

1) Find a Mobius transformation , which send 1, $i,-1$ onto $-1,i,1$ respectively.

Solution : $f(1)=\dfrac{a+b}{c+d}=-1,\ f(i)=\dfrac{ai+b}{ci+d}=i,\ f(-1)=\dfrac{-a+b}{-c+d}=1.$.

$\Rightarrow a+b=-c-d$,

$ai+b=di-c\Rightarrow a=d \ and\ b=-c$,

$b-a=d-c\Rightarrow -c-d=d-c\Rightarrow d=0\Rightarrow a=0$

$\therefore f(z)=\dfrac{b}{cz}=\dfrac{c}{cz}=-\dfrac{1}{z}$, since $c\neq 0$.

$\therefore f(z)=-\dfrac{1}{z}$ is the required bilinear transformation.

2) Find the fixed points of the mapping $w=\dfrac{z}{z+1}$.

Solution: Let z_0 be the fixed point of the mapping $w=f(z)=\dfrac{z}{z+1}\Rightarrow z_0=\left(\dfrac{z_0}{z_0+1}\right)\Rightarrow z_0^2=0$, let $z_0=x_0+iy_0\Rightarrow x_0=0$ and $y_0=0$.

$\therefore z_0=0$ is the fixed point of the mapping $w=\dfrac{z}{z+1}$.

3) Find the Mobius Transformation bilinear mapping sending $-i,i,2i$ onto $\infty,0,\dfrac{1}{3}$ respectively.

Solution: Let $f(z)=\frac{az+b}{cz+d'}(ad-bc)\neq 0$ be the required bilinear mapping . We know that f maps $-\frac{d}{c}$ onto $\infty \Rightarrow -\frac{d}{c}=-i,$

$\therefore d=ic \Rightarrow f(z)=\frac{az+b}{cz+ic}$

Now $f(i)=0 \Rightarrow f(i)=\frac{ai+b}{ci+ic}=\frac{ai+b}{2ci}=0 \Rightarrow b=-ai$

$\therefore f(z)=\frac{az-ai}{cz+ic}=\frac{a(z-i)}{c(z+i)}.\ \because f(2i)=\frac{1}{3} \Rightarrow f(2i)=\frac{a}{3c}=\frac{1}{3}$

$\Rightarrow a=c$.

$\therefore f(z)=\frac{a(z-i)}{c(z+i)}=\frac{z-i}{z+i},$ here $a\neq 0$

(Alternate method for problems involving infinity)

4) Find the bilinear transformation which maps the points $z=\infty,i,0$ onto the points $z=0,i,\infty$

Solution: We have transformation $\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)}=\frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$

Since, $z_1=\infty$ and $w_3=\infty$, we define N and D of LHS and RHS by w_3 and z_1 respectively.

$\because z_1=\infty,\quad \frac{z}{z_1}=0 \quad \& \quad \frac{z_2}{z_1}=0$ and $\because w_3=\infty,\quad \frac{w}{w_3}=0 \quad \& \quad \frac{w_2}{w_3}=0$

$$\frac{(w-w_1)(-1)}{(w_1-w_2)}=\frac{(-1)(z_2-z_3)}{(z_3-z)}$$

Put $z_2=i,\ z_3=0$ *and* $w_1=0,\ w_2=i$

$\therefore \frac{w}{-i}=\frac{i}{-z} \Rightarrow w=\frac{-1}{z}$

5) Find a Mobius transformation , which send $1,\ i,-1$ onto $-1,i,1$ respectively.

Solution : $f(1)=\frac{a+b}{c+d}=-1,\ f(i)=\frac{ai+b}{ci+d}=i,\ f(-1)=\frac{-a+b}{-c+d}=1..$

$\Rightarrow a+b=-c-d,$

$ai+b=di-c \Rightarrow a=d \text{ and } b=-c$,

$b-a=d-c \Rightarrow -c-d=d-c \Rightarrow d=0 \Rightarrow a=0$

$\therefore f(z)=\frac{b}{cz}=\frac{c}{cz}=-\frac{1}{z},$ since $c\neq 0$.

$\therefore f(z)=-\frac{1}{z}$ is the required bilinear transformation.

6) Find the fixed points of the mapping $w=\frac{z}{z+1}$.

Solution: Let z_0 be the fixed point of the mapping

$w=f(z)=\frac{z}{z+1} \Rightarrow z_0=\left(\frac{z_0}{z_0+1}\right) \Rightarrow z_0^2=0,$ let $z_0=x_0+iy_0 \Rightarrow x_0=0$ and $y_0=0$.

$\therefore z_0=0$ is the fixed point of the mapping $w=\frac{z}{z+1}$.

7) Find the Mobius Transformation/ bilinear mapping sending $-i,i,2i$ onto $\infty,0,\frac{1}{3}$ respectively.

Solution: Let $f(z)=\frac{az+b}{cz+d},(ad-bc)\neq 0$ be the required bilinear mapping . We know that f maps $-\frac{d}{c}$ onto $\infty \Rightarrow -\frac{d}{c}=-i,$

$\therefore d=ic \Rightarrow f(z)=\frac{az+b}{cz+ic}$

Now $f(i)=0 \Rightarrow f(i)=\frac{ai+b}{ci+ic}=\frac{ai+b}{2ci}=0 \Rightarrow b=-ai$

$\therefore f(z)=\frac{az-ai}{cz+ic}=\frac{a(z-i)}{c(z+i)}.$ $\because f(2i)=\frac{1}{3} \Rightarrow f(2i)=\frac{a}{3c}=\frac{1}{3}$

$\Rightarrow a=c$.

$\therefore f(z)=\frac{a(z-i)}{c(z+i)}=\frac{z-i}{z+i},$ here $a\neq 0$

8) Let z_1,z_2,z_3,z_4 be four distinct points in $\mathbb{C}_\infty$. Then show that (z_1,z_2,z_3,z_4) is a real number iff all four points lie on a circle.

(Hint: Define $s:\mathbb{C}_\infty \to \mathbb{C}_\infty$ by $s(z)=(z,z_2,z_3,z_4)$ Show that $s(\mathbb{R}_\infty)$ is a circle. Here $s^{-1}(\mathbb{R})=$ the set of all z such that (z,z_2,z_3,z_4) is real.

9) Prove that all the points $z\in C$ satisfying $\left|\frac{z+1}{z+4}\right|=2$ lie in a circle. Find its radius and centre (2009)

10) Find the image of the circle $x^2+ y^2+2x=0$ in the complex plane under the transformation $w=\frac{1}{z}$ (2008)

11) Find the Mobius transform which maps the points $z=1,\ I\ ,-1$ onto the points Mobius transformation (2008)

12) Let $H=\{z\in C / \text{Im}(z)>0\}$ and let $D=\{z\in C / |z|<1\}$. Find the Mobius transformation g *s.t .g(H)=D and g(i)=0*. Justify your claims (2007)

13) Show that Mobius transformation has 0 and ∞ as its only fixed points if and only if it is dilation (magnification) (2007)

14) Show that Mobius transformation has ∞ as its only fixed points if and only if it is a translation (2007)

15) Find the Mobius transform which maps the real axis $R\bigcup\infty$ onto the circle $|z|=1$. (2006)

16) Fix a,b,c,d$\in C$ with $c\neq 0$.show that $\frac{az+b}{cz+d}\to\frac{a}{c}$ *as* $z\to\infty$

17) Verify that the Mobius transformation $w=\frac{1+iz}{i+z}$ maps the exterior of the circle $|z|=1$ in the z-plane into the upper half plane $\text{Im}(w)>0$ in the w-plane.

18) Find the image of the circle $|z-3i|=3$ in the complex plane under the transformation $w=\frac{1}{z}$. Illustrate the results graphically.

19) Find the image of an infinite strip $\frac{1}{4}\leq y\leq\frac{1}{2}$ in the complex plane under the transformation $w=\frac{1}{z}$

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