1

MEANING OF SET AND SET THEORY

Unit Structure :

1.0 OBJECTIVES

- To Know about the need of economic models
- To Know the meaning of Set
- To study the basic set theoretic operations
- To study the De Morgan's law.

1.1 INTRODUCTION

Mathematical Economics is a group of mathematical tools & techniques which are applied to represent the various problems, theorems, conditions to resolve issues and analyze situations encountered in microeconomics, macroeconomics, public finance, international trade and so on. Just as diagrams and graphs constitute important tools to express the laws & theories in economics, in the same way mathematical techniques are also commonly used in economic literature. Diagrams can represent 2 variables at a time- one on X-axis & other on Y-axis. But in reality we know that the economy is a multiple-stakeholder & dynamic world. It includes several factors simultaneously affecting a phenomenon. In this context, equations can be used to express these multi-dimensional phenomena. The mathematical techniques form the basis for representing & reasoning the behavior of several variables at a time. They can be also utilized to represent the constraints and assumptions underlying economic laws and theories in a precise manner along with their specific implications.

Thus, the mathematical techniques perform the following functions:

1. They facilitate consideration of 'n' variables simultaneously
2. They offer a concise & precise expression of stating complex phenomena characterized by a number of inter-relationships.
3. It acts as a medium to explicitly state with clarity the assumptions to laws & theorems
4. Efficiency and optimality conditions framework is generally put forward through mathematical approach in many economic models.

To conclude, mathematical techniques refers to the application of mathematical tools and approach to the theoretical aspects of economic analysis.

1.2 ECONOMIC MODELS

Before we move to the set and its elements lets familiarize ourselves with the framework or construct of an economic model. Because mathematical approach is applied as a building block to construct and express an economic model. And the 'set' is a fundamental technique in mathematical economics. So, before we go to the set and its elements; we must review our knowledge of an economic model briefly.

There are 2 major economic models you might have read about in your under-graduate class, namely, circular flow of income and production possibility frontier.

Need for Economic Models:

(Why) The Economy is a complex and dynamic world. It has many entities that are inter related and has several players whose behaviour changes. Due to this, analyzing the economy requires a lot of discipline in thought, systematic laying down of relations among entities and application of specialized tools; such as mathematical approach.

(What is meant by models) Economists use specialised tools and techniques to think in a logical, coherent manner. These tools may be abstract and may not exist in physical form. Such thought tools are called models.

(What they do; forms) These models are then applied to the real economy to analyze the behaviour of entities and processes underlying their decisions. Models may be expressed in the form of words, diagrams or equations. Increasing use of mathematical equations is made these days. In case of 2 variables, diagram can be used. But if we are dealing with several entities then equations or such type of mathematical techniques are required.

The steps involved in building or constructing economic models in a rational framework are as follows:

a) Constructing a model-

1. Selecting an entity to study.
2. Laying down assumptions; keeping other factors as constant.
3. Identifying specific variables to be related or compared. Specific variables are considered and related to one another (direct/inverse relation).
4. Considering trade- off between the variables if any. Specifying the nature of relation among the particular variables.
5. Formulating a statement through logical reasoning in order to make generalizations.

b) Validating a model-

6. This statement or prediction is put to test. This process is called validating the model.

   So relevant data is collected. If the data collected supports the statement or prediction then the generalization made holds true.

7. If data collected does not support the statement/prediction made, then the model has to be modified.

With this brief background to economic models, lets now move on to the set and its elements.

1.3 SET: MEANING, NOTATIONS AND RELATIONSHIPS BETWEEN SETS

Set refers to a collection or group of objects or elements. These objects or elements could be numbers or entities like goods or stakeholders like consumers preferring a specific brand or students of the MA Economics class. Students enrolled in the MA Economics class together as a group is a set. Each student in this class belonging to this group (MA Economics) is called the element.

There are 2 ways of denoting a set- by enumeration i.e. by enlisting all actual elements in the set and second way is by description i.e. by stating or describing the characteristic that helps us to identify the elements in that set. Example: set that includes roll numbers of MA economics students who have opted for mathematical economics can be expressed in an enumerative manner. But is we are referring to a set that includes all odd numbers or all positive integers then in this case we may express the set in a descriptive manner as the elements are too many to enlist and it becomes convenient to express the set by describing the specific characteristic or feature common to all the elements.

Illustration 1: Set by enumeration

M = {1, 2,3,4,5,6,7,8,9,10,11,12,13,14,15}

This indicates that set M includes roll numbers 1 to 15 as elements. These form a set 'M' as these are roll numbers of students in the MA Economics class who have opted for mathematical economics. The elements in the set are enclosed in the brace brackets {….}.

Illustration 2: Set by description

If 'N' is a set including all odd numbers as elements, then it would be difficult to enlist all odd numbers or express set N by enumeration. In this case set by description is used as follows-

N = {x/ x an odd number}

In the above representation set 'N' includes all elements x, where x are odd numbers. Generally, a slant (/) or a vertical line is used to separate the symbol x i.e. the element from the description of the elements belonging to a particular set as shown in the above illustration 2. It is read as N is a set of all numbers (x) where x is an odd number.

There are 2 types of sets based on the number of elements- finite set in which case the elements in the set are always countable or denumerable. Infinite set is a case in which the elements in the set are either countable example S = {x/ 1< x < 10} i.e. S is a set of x elements that are numbers greater than 1 but less than 10 OR the elements are non-denumerable i.e. uncountable (example set N in illustration 2).

An element in a set is the member of that particular set, i.e. it belongs to the set. This is denoted by the greek letter epsilon with the symbol ϵ. Thus, for set M mentioned above, we can say that: 1 $\in$ M, 15 $\in$ M. In the same way for set S numbers or elements such as 2, 3, 4,5 6, 7, 8 and 9 $\in$ S, i.e. numbers 2 to 9 are members of set S, they belong to set S.

Let's now move on to **relationships between sets.** These are discussed in the context of comparison among sets.

1. **Equal sets:** The first one, is the case of equal sets. 2 sets, say M and N are considered as equal if the elements in set 'M' are same as elements in set 'N', irrespective of the order in which the elements or members belonging to the set are enlisted or given.

   Thus, if M = {1, 3, 5,7} and N = {7, 5, 3, 1} then set M and set N are equal. The elements in set M and set N are exactly identical. So, we can represent the relationship between set M and set N as: **(M = N), i.e. set M equal to set N.** Even if, one element is different; then the sets will not be equal.

2. **Sub-sets:** The second case of relationships between sets is when one set is a sub-set of another one. That is if set N is a sub-set of set M; then every element in set N is also a member or element of set M. Example: M = {1,3,5,7} and N = {5,7}. All elements or members of set N i.e. 5 & 7 are also members of set M. So, in this case set N is a sub-set of set M. Thus, this implies that if x $\in$ N then x $\in$ M. We can also write this relationship using set inclusion symbols like $\subset$ which means contains in and symbol $\supset$ which means includes as follows:

   N $\subset$ M (set N is contained in set M) and M $\supset$ N (set M includes set N).

   If the 2 sets happen to be sub-sets of each other, then the sets will be equal sets. So, if N $\subset$ M and M $\subset$ N, when M = N. So if set N is sub-set of set M and set M is sub-set of set N, it means that set M is equal to set N.

   The largest sub-set that can be formed from a set, is the given set itself. So, set M is the largest sub-set of set M. M = {1,3,5,7}, then set M: M = {1,3,5,7} is the largest sub-set of itself. The smallest subset of a set contains no elements i.e. it is a *null* set or *empty* set. It is represented as { } OR by symbol ∅.

Illustration 3: Forming sub-sets

If M = {1,2} then subsets are {1}, {2}, {1, 2} and { }.

In general, if a set M has 'n' elements; then 2^n i.e. 2 raised to n subsets can be formed from it.

In above illustration set M has 2 elements so $2^n = 2^2 = 4$. So, 4 subsets have been formed from it. Null set is a sub-set of any set. It should be noted that { } is different from {0}. The first implies empty or null set and does not contain any element; while as the second one is a set containing the element 0 'zero'.

3. **Disjoint sets:** Two sets are said to be disjoint sets when they have no elements in common. If the elements in set M are different from elements in set N, then set M and set N are disjoint sets. None of the elements or members belonging to set M and set N are common or same; then set M and set N are disjoint sets.

Illustration 4: Disjoint sets

M = {1,3,5,7,9} and N = {2,4,6,8,10} then set M and set N are disjoint sets.

In the above illustration, set M includes all odd numbers from 1 to 10 and set N includes all even numbers from 1 to 10. The elements of the two sets are different, none is common element and thus we say that set M and set N are disjoint sets.

 # Exercise 1

i. Enumerate the subsets for S = {a, b, c}. How many sub-sets can be formed in all?

ii. Form subsets for M = {12, 14, 16,18, 20}. How many sub-sets can be formed in all?

iii. If M = {2, 3, 4} and N = {4,3,2}; then state with reasons whether these sets are equal or disjoint?

iv. If M = {11, 13, 15, 17, 19} and N = {12, 14, 16, 18, 20}; then state with reasons whether these sets are equal or disjoint?

1.4 BASIC SET THEORETIC OPERATIONS

Like we perform operations such as addition, subtraction, multiplication, division, square or square-root on numbers, certain operations can be performed on sets. **These include- union, intersection and complement.**

1. **Union-** The operation of union implies forming a new set from the 2 given sets i.e. combining them. Thus the 'union' set formed will include elements (and only those elements) from set 1 or elements from set 2 or elements from both set 1 and set 2. It is symbolized as:

   set 1 $\cup$ set 2 and read as "set 1 union set 2".

   Illustration 5a: Union of sets

   If we have 2 sets: A = {1, 2, 3} and set B = {4, 5, 6},

   then A $\cup$ B = {1, 2, 3, 4, 5, 6}

   In the above illustration, note that set A and set B were not equal sets.

   Illustration 5b: Union of sets

   If we have 2 sets: A = {1, 2, 3} and set B = {3, 4, 5, 6},

   then A $\cup$ B = {1, 2, 3, 4, 5, 6}

   In the above illustration, note that set A and set B are not equal sets nor are they disjoint sets (element 3 is common to the 2 sets). In this case, while taking union of set A and set B, element 3 is not repeated.

2. **Intersection-** The operation of intersection involves forming a new set of elements and only those elements that belong to both the 2 given sets. So, only elements that are common to both the 2 original sets will be taken to form the new set or the intersection. It is symbolized as set 1 $\cap$ set 2 and is read as set 1 intersection set 2.

   Illustration 6a: Intersection of sets

   A = {1, 2, 3} and B = {2, 3, 4, 5, 6}, then A $\cap$ B = {2, 3}

   The intersection of set A and set B is a set containing elements 2, 3 that are common to both set A as well as set B.

   Illustration 6b: Intersection of sets

   A ={1, 2, 3} and B = {4, 5, 6}, the A $\cap$ B = {$\emptyset$ }

   In the above illustration 6b, the intersection of set A and set B is a null or empty set as none of the elements in set A and set B are common or similar. Since set A and set B are disjoint sets, their intersection is an empty or a null set.

Intersection: $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

Union: $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

Union is a broader concept since it is a set formed with elements from either set A or set B or both. While as, intersection is a restrictive concept since it is a set formed by only those elements that are common or belong to both set A as well as set B at the same time.

a. **Complement-** The concept of complement can be explained with the help of a universal set. If we have a broad set including all elements fulfilling certain feature or characteristic or condition; then it is an **all-inclusive** set. Such a set is called a universal set. Now, if we have a set A which includes some of the elements from the universal set, then its complement can be formed by taking a set of remaining elements from the universal set that are not contained in set A. It is symbolized as A' and is read as A complement.

$A' = \{x / x : \in U \;\&\; \notin A\}$ i.e. complement of A will include elements from the universal set & not belonging to set A.

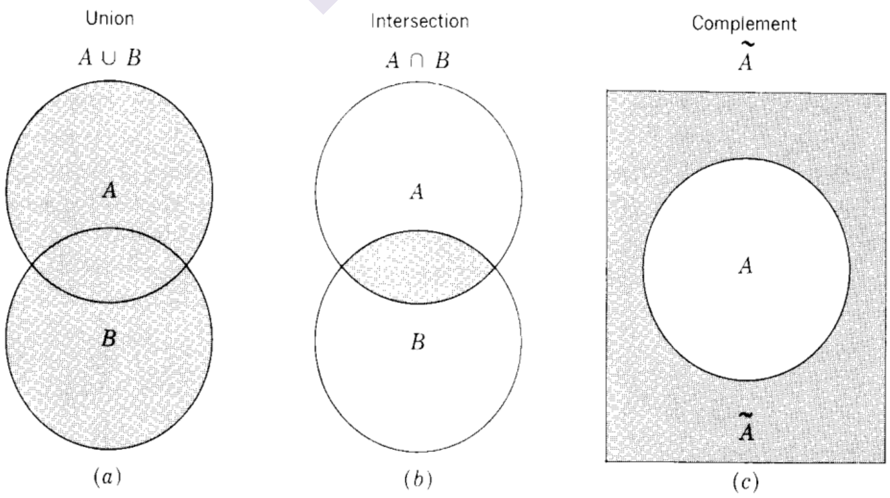
Illustration 7: Complement of a set

If we have a universal set U = {1, 2, 3 4, 5, 6} and set A = {1, 2, 3, 4}, then

A' = {5, 6}. In this illustration A complement includes elements 5 and 6 that belong to universal set U but not to set A. Thus, in simple words complement of set A = Universal set – set A.

The complement of a universal set U will be a null or empty set, since all elements are already contained in the universal set.

Diagram 1.1 Union, intersection & complement of sets



The above Venn diagrams represent the 3 set operations: union, intersection and complement. The shaded region in the diagrams a, b and c indicate the result or outcomes of union, intersection and complement respectively.

 # Exercise 2

i. Given U = {2, 4, 6, 7}, A = {2, 4, 6} and B = {7, 2, 6}. Find A $\cup$ B, A $\cap$ B, A' (A complement) and B' (B complement).

ii. Given A = {2, 4, 6}, B = {7, 2, 6}, C = {4, 2, 6} and D = {2, 4}. Find the following: A $\cup$ B, A $\cup$ C, B $\cap$ C and B $\cap$ D.

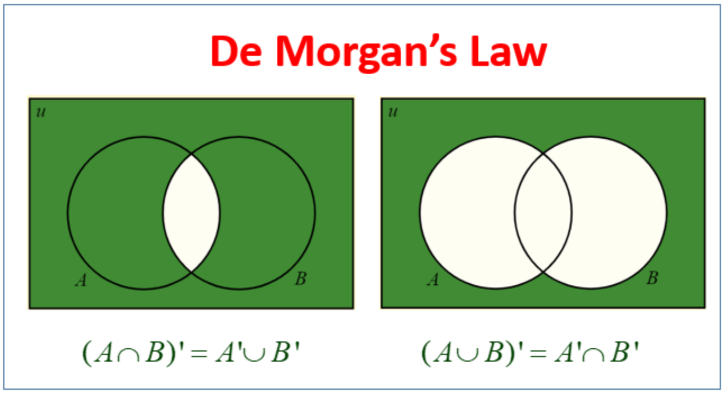
iii. Given S_1 = {11, 13, 15, 17, 19}, S_2 = {12, 14, 16, 18, 20} and U = {11, 12, 13, ………, 20}. Find $S_1 \cup S_2$. What can you observe about this outcome? Also find $S_1 \cap S_2$ and S_1' (S_1 complement) and S_2' (S_2 complement)

1.5 DE MORGAN'S LAWS OR THE LAWS OF SET OPERATIONS

De Morgan's law states that the complement of the union of two **sets** is equal to the intersection of their complements and the complement of the intersection of two **sets** is equal to the union of their complements. ... For any two finite **sets** A and B; (i) (A U B)' = A' ∩ B' (which is a **De Morgan's law** of union).

These De Morgan's laws for sets are depicted in the Venn diagrams below.

Diagram 1.2 De Morgan's law



In the above Venn diagram, left panel indicates that the **complement of A intersection B is equal to the union of A complement and B complement.** For example, if U = {1, 2, 3, 4}, A = {1, 2} and B = {3, 4}; then A ∩ B = {∅} and so its complement will be {1,2,3,4}. Now A' (A complement) = {3, 4} and B' (B complement) = {1 ,2}; so union of A' and B' will be {1,2,3,4}. Thus LHS = RHS.

In the above Venn diagram, right panel indicates that complement of A Union B is equal to intersection of A complement and B complement. Take the same composition of set U, set A and set B as in the example above. A union B = {1,2,3,4}, so LHS: {∅}.

Similarly A' is {3,4} and B' is {1,2} so intersection of A' and B' i.e. RHS: {∅ }.

Therefore, LHS = RHS.

 # Exercise 3

1) Let U = {1, 2, 3, 4, 5, 6}, A = {2, 3} and B = {3, 4, 5}. Prove the De Morgan's laws.

2) Let U = {1, 2, 3, 4, 5}, A = {2, 4} and B = {1, 2, 3, 5}. Prove De Morgan's laws.

3) Let U = {5, 10, 15, 20, 25, 30, 35, 40, 45, 50}, A = {5, 10, 15, 20, 25, 30} and B = {25, 30, 35, 40, 45, 50}. Prove the De Morgan's laws.

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Laws of set operations can be discussed as follows:

1. **Commutative law of unions and intersections-** They can be stated as follows:

   A ∪ B = B ∪ A and A ∩ B = B ∩ A. It implies that union of A and B is equal to union of B and A. The order of occurrence of sets whose union is formed does not matter, the result is the same. The same is implied in case of intersection of the 2 sets A and B. The A intersection B is equal to B intersection A.

2. **Associative law of unions and intersections-** They apply to 3 sets case. If we have 3 sets A, B and C; then while taking the union of

the 3 sets, we first take union of say A & B and then the union of this resulting set with the remaining set C. The same procedure is followed in case of the intersection of 3 sets. The order in which the sets are selected in the operation is immaterial. This is called the associative law. This can be expressed as under:

$A \cup (B \cup C) = (A \cup B) \cup C$

$A \cap (B \cap C) = (A \cap B) \cap C$

3. **Distributive law of unions and intersections-** This is applicable when operations of unions and intersections are combined. They can be expressed as follows:

   $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. It means the union of A and B intersection C (LHS) is equal to the intersection between A union B and A union C (RHS).

   And in the same way, we can also express the distributive law as:

   $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. It means intersection between A and the union of sets B & C (LHS) is equal to the union between the intersections A, B and A, C (RHS).

 # Exercise 4

1. Verify the distributive laws, given: A = {4, 5}, B = {3, 6, 7} and C = {2, 3}
2. Verify the distributive laws, given: X = {7, 8}, B = {6, 9, 10} and C = {5, 6}

1.6 REFERENCES:

1. Alpha Chiang, Fundamentals Methods of Mathematical Economics
2. Alpha Chiang and Wainwright, Fundamental Methods of Mathematical Economics
3. https://www.onlinemathlearning.com/exponent-rules-2.html

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2

FUNCTIONS

Unit Structure :

2.0 OBJECTIVES

- To study the various types of functions and rules of exponents
- To study the slop and intercept of a stray it line

2.1 SLOPE AND INTERCEPT OF A STRAIGHT LINE

Another major tool for stating and analysing the inter-relationships among variables and representing an economic model is through the means of functions and their graphical representation. Thus is one of the very lucid mathematical techniques. It helps to comprehend the nature, direction and extent of change in the variables when other variable is known. Let us review the basics about drawing graphs in order to understand how economic models work. Let us begin with the meaning of a function. Think of the following example. Suppose there are four students in the class, and let their names be Surekha, Rita, Samrin and Babita. We decide to assign roll numbers to them by arranging them in an alphabetic order. The arrangement would be like one given below.

Illustration 2.1: Meaning of function

Name of Student	Roll no
Babita	01
Rita	02
Samrin	03
Surekha	04

The rule of assigning roll numbers to students on the basis of the alphabetical arrangement of their names defines a function. It tells us what number will be assigned to each student. The function assigns a unique roll number to each student, and there is no student without a roll number. In more formal language, the set of students is referred to as the domain of the function. The set of integers/numbers is referred to as the range of the function.

Let's now take another example. Think of the set of all students in your MA Economics class as X. Think of the set of their parents as set Y. With each student in set X, we can assign a unique pair of parents in set Y. It is possible that two students in set X will have the same pair of parents in Y, but it is not possible for a student in X to be associated with more than one pair of parents in set Y. So, every element in set X is associated with a unique element in set Y. Hence, this rule is also a function, where the domain is X and range is Y.

In formal language, a function from X to Y is a rule which associates with every element of the set X a unique element of set Y. We write this as:

$$f : X \to Y$$

Now, suppose X and Y are both sets that contain numbers. Let X be the domain of the function and let Y be the range of the function. Let the function be: y = 2+3*x. That is, if x=1, the rule says that y must be equal to 2+3*1 =2 +3 = 5, that is, the function assigns the element 1 from the set X to the element 5 in the set Y. let us say X contains the elements 0,1,2,3,4,5,6.

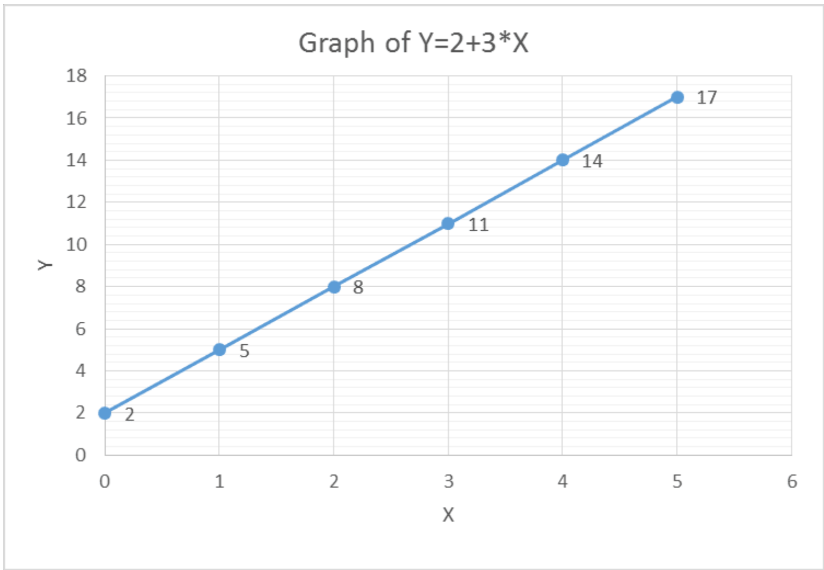
Let us represent the function in the following table:

Illustration 2.2: Linear Function

X	Y = 2 + 3x	Increase/decrease
0	2 + 3*0	
1	2 + 3*1 = 5	3
2	2 + 3*2 = 8	3
3	2 + 3*3 = 11	3
4	2 + 3*4 = 14	3
5	2 + 3*5 = 17	3
6	2 + 3*6 = 20	3

As you can see, the function y =2+3*x assigns the value 5 to x=1, 8 to x=2, 11 to x=3, 14 to x=4, 17 to x=5 and 20 to x=6. You will notice that every time x increases by 1 unit, y increases by 3 units. **The rate at which y increases when x increases by one unit is called the slope of the function.** If x = 0, then, y = 2+3*0 = 2. So, 2 represents the **value of y when x = 0. It is called the intercept.** In above example, slope is 3 and intercept is 2. This function can also be graphically illustrated as follows:

Diagram 2.1 Linear function given by straight line



In the graph above, values of Y are on the vertical axis while the values of X are on the horizontal axis. You can see that 2 is the intercept, because it is the value of Y that corresponds to X=0. The value of Y that corresponds to X=1 is 5, and so on.

Activity:

What value of Y, in the above graph, would correspond to X = 7?

Take another function: Let Y = 6 - 2*X. Let X take the values 0,1,2,3,4,5. Find Y.

X	Y= 6 - 2x	Increase/decrease
0	6	
1	4	2
2	2	2
3	0	2
4	-2	2
5	-4	2

Thus the slope of this function is 2 and the intercept is 6
(i.e. the value of Y, when x = 0).

As you can see, in the two examples above, the slope of the function is constant at various values of X. In this case, the graphical representation of the function will be a straight line i.e. linear upward if variables X and Y move in the same direction like market price and quantity supplied are directly or positively related. If the graph is downward sloping straight line, it indicates inverse relationship between variables X and Y like in the case of price and quantity demanded.

However, the variables need not change at the same rate always.

For instance, consider the function

$Y = 2 + 3 * X^2$. Let X = 0,1,2,3,4,5. Then, the following table results:

Illustration 2.3: Non- linear function

X	Y= 2+ 3x
0	2
1	5
2	14
3	29
4	50
5	77

The graph of this function is given below:

Diagram 2.2 Non-linear

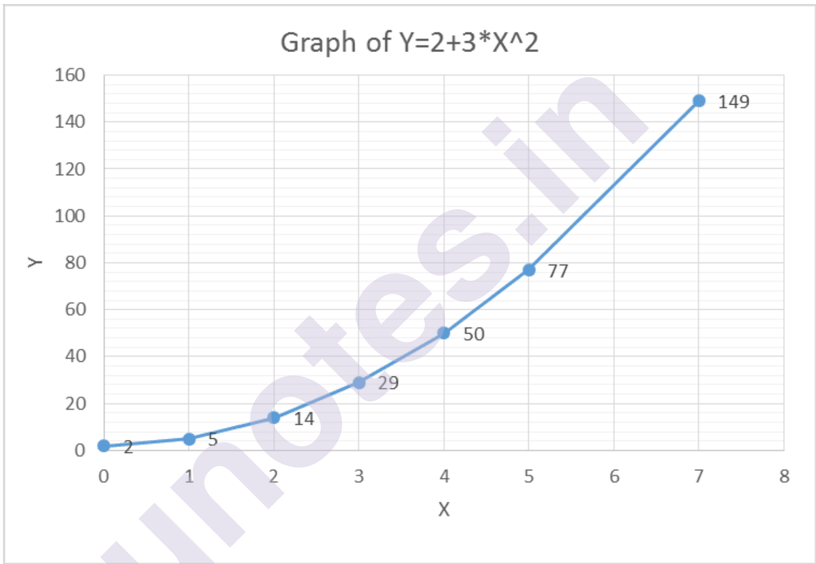
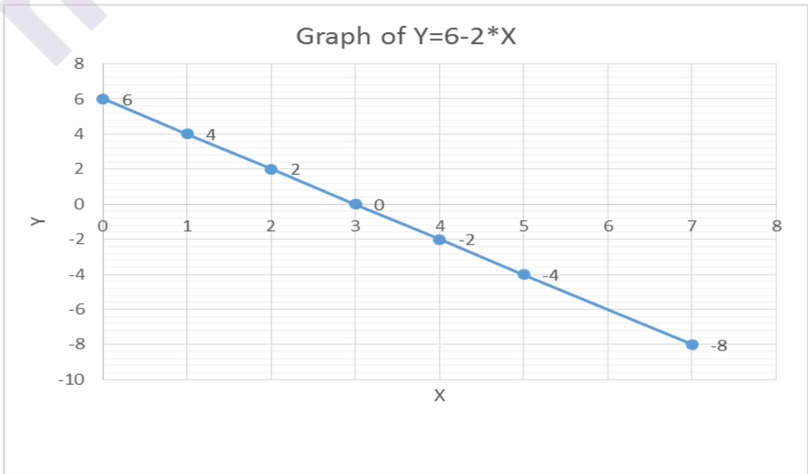


Diagram 2.5 Linear function: downward sloping



The graph of the function would look like this:

As you can see from this graph, the increase in Y is not constant. When X increases from 0 to 1, Y increases from 2 to 5, that is by 3 units. However, as X increases from 1 to 2, Y increases from 5 to 14, that is by 9 units. The slope of this function is not constant, but varies with X. This gives rise to **non-linear** curve.

 # Exercise 5

i. Find the values of Y for the function represented as: $Y = 10 - 2x$ when $x = 0, 1, 2, 3$ and 4. Graph this function. Identify the slope and intercept.

ii. Find the values of Y for the function represented as: $Y = 10 + 2x^2$ when $x = 0, 1, 2, 3$ and 4. Graph this function. Identify the intercept. Does the slope vary or is it constant? Is the function represented by a linear line or is it non-linear?

2.2 HIGHER ORDER FUNCTIONS

In case of sets, the order of appearance of the elements was immaterial. So $\{a, b\} = \{b, a\}$. But in case of ordered pairs the order of appearance of the elements matter. So, $(a, b) \neq (b, a)$

Let's take example of ordered pairs. While representing the height and weight of a sample of say 50 people; we may express this information with the help of ordered pairs with the height followed by weight of person one, then height followed by weight of person two etc. In this case in the ordered pair (x, y), the x coordinate corresponds to height and y coordinate corresponds to the weight.

Thus $(x, y) \neq (y, x)$. The set $\{(x,y) \mid y = 2x\}$ leads to ordered pairs such as (0,0), (1,2), (3, 6) and so on. In this case given the value of x, we get a single, unique corresponding value of y. But if we take a set such as: $\{(x, y) \mid y \geq x\}$, then in this case given value of x, y can take multiple values satisfying the condition
that $y \geq x$.

The ordered pairs indicating the values y will take with given values of x; implies a relation between x and y. A function is defined as a set of ordered pairs with the property that a given x value has a unique corresponding y value. Thus, a function must be a relation; however, every relation need not be a function. A function is also called 'mapping' or transformation. In this, the x values are mapped to their corresponding y values where a given x value has a specific, unique y value associated with it. This can be represented as:

$F: x \longrightarrow y$

In a function y = f (x), x is referred to as the argument of the function and y is the value of the function. In common parlance, x is the independent variable and y is the dependent variable. The set of all permissible values x can take in a given context is called as the domain of the function. The y value into which an x value is mapped is called the image of that x value. This set of images is called the range of the function. In other words, a set of all values the variable y can take is known as the range of the function and the set of values the variable x takes is called the domain of the function.

2.3 TYPES OF FUNCTION

The expression y = *f (x)* is a general statement to the effect that a mapping is possible, but the actual rule of mapping is not thereby made explicit. Now let us consider several specific types of function, each representing a different rule of mapping.

Constant Functions

A function whose range consists of only one element is called a *constant function. In this function where y = f(x), y takes only one possible value irrespective of x.*

As an example, we cite the function

y = f (x) = 10

which is alternatively expressible as y = 10 or f (x) = 10, whose value stays the same regardless of the value of x. In the coordinate plane, such a function will appear as a horizontal straight line. In national-income models, when government or public sector investment 'G' is exogenously determined, we may have an investment function of the form

G = \$100 million, or G = G_0, which exemplifies the constant function.

Polynomial Functions

The constant function is actually a "degenerate" case of what are known as *polynomial functions.*

The word "polynomial" means "multi term," and a polynomial function of a single variable x has the general form:

$y = a_0 + a_1x + a_2x^2 + \ldots\ldots\ldots + a_nx^n$

In this case each term contains a coefficient as well as a nonnegative-integer power

of the variable x. Note that, instead of the symbols *a, b, c, ... ,* we have employed thesubscripted symbols a_0, a_1, ...

*T*he subscript helps to pinpoint the location of a particular coefficient in the entire equation.

Depending on the value of the integer n (which specifies the highest power of x), we have several subclasses of polynomial function:

Case of *n* — 0: $y = a_0$ *constant* function]

Case of *n* — 1: $y = a_0 + a_1x$ *linear* function]

Case of *n*— 2: $y = a_0 + a_1x + a_2x^2$ *quadratic* function]

Case of n = 3: $y = a_0 + a_1x + a_2x^2 + a_3x^3$ [*cubic* function]

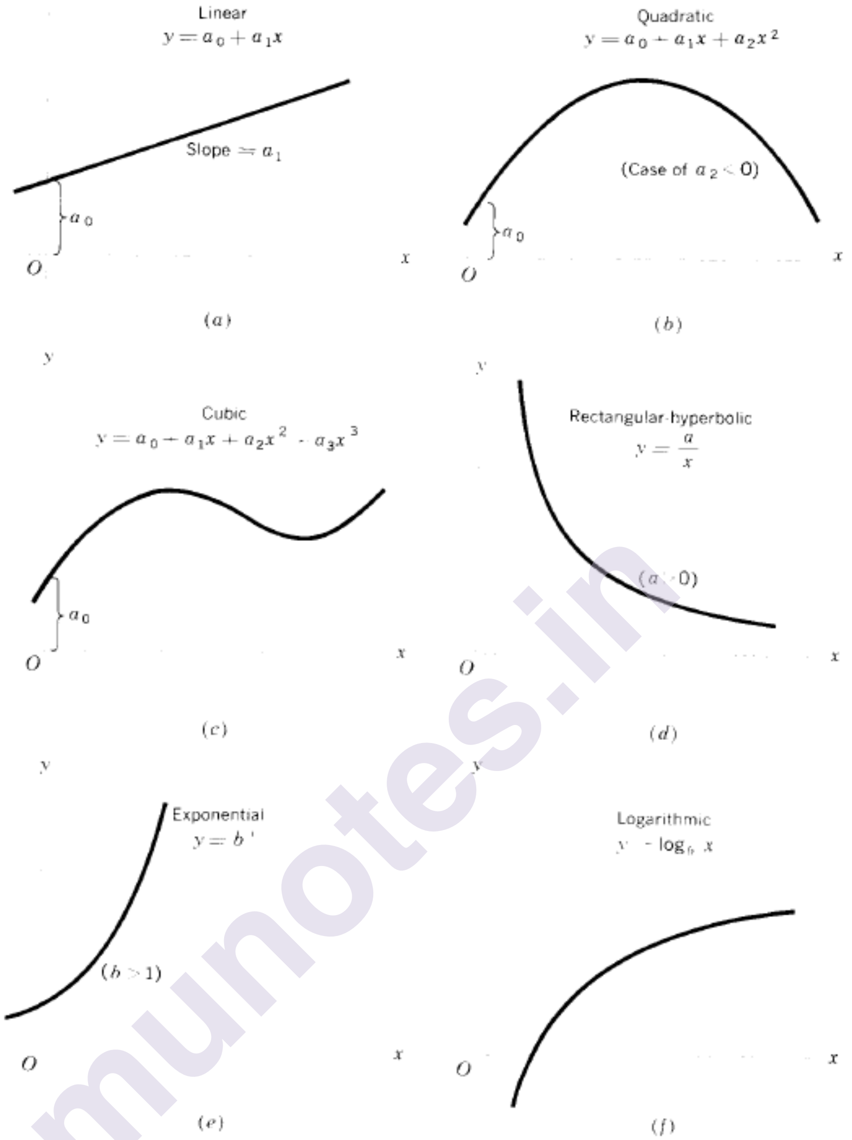
and so forth. The superscript indicators of the powers of x are called exponents. For example, in the quadratic equation the highest exponent i.e. the power taken by x is 2. The highest power involved, i.e., the value of n, is often called the degree of the polynomial function; a quadratic function, for instance, is a second-degree polynomial, and a cubic function is a third-degree polynomial. The 'n' value is non-zero. If it were non-zero, the function would become a lower degree polynomial.

When plotted in the coordinate plane, a linear function will appear as a straight line. In case of a linear function, when x = 0, it yields $y = a_0$.

Thus the ordered pair (0, a0) represents the y intercept. The other coefficient a1 represents the slope(steepness) of the line. This means that a unit increase in x will result in an increment in y in the amount of a_1. *In* the case of $a_1 > 0$, involving a positive slope and thus will be an upward-sloping line. But if $a_1 < 0$, the line will be downward-sloping.

A quadratic function, on the other hand, plots as a parabola—roughly, a curve with a single built-in bump or wiggle. The particular illustration in diagram 1.6b implies a negative a_2. In the ca*se* of $a_2 > 0$, the curve will "open" the other way, displaying a valley rather than a hill. The graph of a cubic function is generally shown by two wiggles, as illustrated in diagram 1.6c below. These functions are used quite frequently in the economic models to explain and depict the behaviour of several economic variables in relation to one another.

Diagram 1.6: Types of functions



***** (above image i.e. diagram 1.6 is taken from Alpha Chiang's book)**

Rational Functions

A function in which y is expressed as a ratio of two polynomial functions in terms of x is called a rational function. For example:

$$y = \frac{x-1}{x^2 + 3x + 5}$$

This implies that any polynomial function must itself be a rational function, since it can be always expressed as a ratio to 1.

A special rational function that has interesting applications in economics is the function:

$y = \frac{a}{x}$ OR $xy = a$

This function plots as a *rectangular hyperbola,* as shown above in diagram 1.6d. Since the product of the two variables is always a fixed constant in this case, this function may be used to represent that special demand curve—representing unitary elastic demand for which the total expenditure is constant at all levels of price. Another application is the average fixed cost (AFC) curve. With AFC on one axis and output (Q) on the other, the AFC curve is rectangular-hyperbolic because TFC (given by AFC x Q) is a fixed constant. The AFC curve slopes downwards since AFC continuously declines as TFC remains constant and Q increases.

The rectangular hyperbola drawn from xy = *a* never meets the axes. Rather, the curve approaches the axes *asymptotically: that is the curve will come closer to the X and Y axes but never meet the axes.*

Non-algebraic Functions

Any function expressed in terms of polynomials and/or roots (such as square root) of polynomials is an *algebraic function.* Accordingly, the functions discussed thus far are all algebraic. A function such as $y = \sqrt{x^2+5}$ *(in the square-root x square plus 5)* is not rational, yet it is algebraic.

However, *exponential functions* such as $y = b^x$, in which the independent

Variable 'x' appears in the exponent, are *non-algebraic functions. Non-algebraic functions are also known as transcendental functions.* The closely related *logarithmic functions,* such as $y = \log_b x$, are also non-algebraic. The general graphic shapes these functions can take are indicated in diagram 1.6 e & f. Other types of non-algebraic function are the *trigonometric* (or *circular) functions which aid in representation of* dynamic analysis.

2.4 RULES OF EXPONENTS

In discussing polynomial functions, we introduced the term *exponents* as indicators of the power to which a variable (or number) is to be raised. The expression 6^2 means that 6 is to be raised to the second power; that is, 6 is to be multiplied by itself, i.e. $6^2 = 6 \times 6 = 36$. In general, we define

$$x^n \equiv x \times x \times \cdots \times x$$

X term taken 'n' times

So, $6^3 = 6 \times 6 \times 6$. As a special case, we note that $x^1 = x$.

From the general definition. it follows that exponents obey the following rules:

Rule I: $x^m * x^n = x^{m+n}$

Illustration 2.4 : Exponentiation Rule1

$6^2 \times 6^3 = 6^{2+3} = 6^5$

Rule II: $\frac{x^m}{x^n} = x^{m-n}$

Illustration 2.5: Exponentiation Rule II

$\frac{6^3}{6^2} = 6^{3-2} = 6^1 = 6.$

Note that the case of $x = 0$ is ruled out in the statement of this rule. This is because when $x = 0$, the expression $x^m \div x^n$ would involve division by zero, which is undefined.

What if $m < n$: say, $m = 3$ and $n = 5$? In that case we get,

According to Rule II: $x^{m-n} = x^{3-5} = x^{-2}$ a *negative power* $m < n$.

Rule III : $x^{-n} = \frac{1}{x^n}$. **(where $x \neq 0$)**

To raise a (nonzero) number to a power of –n is to take the *reciprocal* of its

n th power.

Another special case of the application of Rule II is when $m = n$, *which yields*

the expression $x^{m-n} = x^0$.

Rule IV: $x^0 = 1$ **(where $x \neq 0$)**

As long as we are concerned only with polynomial functions, only non-negative integer powers are required. In exponential functions, however, the exponent is a variable that can take non-integer values as well.

Rule V: $x^{1/n} = n\sqrt{x}$ **(i.e. nth root of x)**

Two other rules obeyed by exponents are:

Rule VI: $(x^m)^n = x^{mn}$

Illustration 2.6: Exponentiation Rule VI

$(6^2)^3 = 6^{2*3} = 6^6$ (that will yield the result 6 taken 6 times)

Rule VII: $x^m \times y^m = (xy)^m$

Illustration 2.7 : Exponentiation Rule VII

$2^3 \times 4^3 = (2 \times 4)^3 = 8^3 = 8*8*8 = 512$

Illustration 2.8: Summary of all Rules of exponents

Exponential Rules

Product Rule

$$a^x \times a^y = a^{x+y}$$
$$a^2 \times a^3 = a^5$$

Quotient Rule

$$a^x \div a^y = a^{x-y}$$
$$a^7 \div a^3 = a^4$$

Power Rule

$$\left(a^x\right)^y = a^{xy}$$
$$\left(a^7\right)^2 = a^{14}$$

Negative Rule

$$a^{-x} = \frac{1}{a^x}$$
$$a^{-4} = \frac{1}{a^4}$$

Zero Rule

$$a^0 = 1$$

Source: https://www.onlinemathlearning.com/exponent-rules-2.html

Credit: onlinemathlearning.com

 # Exercise 6

i. Graph the functions:

a. $y = 10 + 2x$

b. $y = 10 - 2x$

c. $y = 2x + 8$

(In each case, consider the domain as consisting of nonnegative real numbers only.)

Comment on the difference between *(a)* and (b*)* above? How is this difference *depicted in the graphs*? What is the major difference between (a) and (c)? How do their graphs reflect it?

ii. Graph the functions

(a) $y = -x^2 + 5x - 2$ (b) $y = x^2 + 5x + 2$

with the set of values $-5 \leq x \leq 5$ as the domain.

iii. Graph the function $y = 36/x$, assuming that x and y can take positive values only.

iv. Condense the following expressions:

(a) $x^4 \times x^3$ (b) $x^2 \times y^2$ (c) $(x^3)^5$

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2.5 LIMITS OF SEQUENCES

A **sequence** is an ordered list of numbers (or other elements like geometric objects), that often follow a specific pattern or function. **Sequences** can be both finite and infinite. A **sequence** is an ordered list of numbers . The three dots mean to continue forward in the pattern established. Each number in the **sequence** is called a term. In the **sequence** 1, 3, 5, 7, 9, …, 1 is the first term, 3 is the second term, 5 is the third term, and so on. **The 4 types of sequence are:**

- Arithmetic **sequence**.
- Geometric **sequence**.
- Harmonic **sequence**.
- Fibonacci **sequence**.

In **mathematics**, a sequence is an enumerated collection of objects in which repetitions are allowed and order matters. Like a set, it contains members (also called elements, or terms). ... For example, (M, A, R, Y) is a sequence of letters with the letter 'M' first and 'Y' last. This sequence differs from (A, R, M, Y).

The nth (or **general**) **term of a sequence** is usually denoted by the symbol an. Example 1: In the **sequence** 2,6,18,54,... the first **term** is. a1=2, the second **term** is a2=6 and so forth.

Finite and Infinite Sequences

A more formal definition of a finite sequence with terms in a set S is a function from {1,2,⋯,n} to S for some n>0. An infinite sequence in S is a function from {1,2,⋯}to S. For example, the sequence of prime numbers (2,3,5,7,11,⋯) is the function

1→2,2→3,3→5,4→7,5→11⋯

A sequence of a finite length n is also called an n-tuple. Finite sequences include the empty sequence ( ) that has no elements.

Arithmetic Sequences

An arithmetic (or linear) sequence is a sequence of numbers in which each new term is calculated by adding a constant value to the previous term. An example is (10,13,16,19,22,25). In this example, the first term (which we will call a1) is 10, and the *common difference* (d)—that is, the difference between any two adjacent numbers—is 3.

Another example is (25,22,19,16,13,10). In this example a1=25, and d = −3.

Geometric Sequences

A geometric sequence is a list in which each number is generated by multiplying a constant by the previous number. An example is (2,6,18,54,162). In this example, a1=2, and the *common ratio* (r)—that is, the ratio between any two adjacent numbers—is 3. Another example is (162, 54,18,6,2). In this example a1=162, and r =1/3.

The limit of a sequence is the value the sequence approaches as the number of terms goes to infinity. Not every sequence has this behaviour those that do are called convergent, while those that don't are called divergent. Limits capture the long-term behaviour of a sequence and are thus very useful in bounding them. In mathematics, the limit of a sequence is the value that the terms of a sequence "tend to". If such a limit exists, the sequence is called convergent. A sequence that does not converge is said to be divergent. A sequence is "converging" if its terms approach a specific value at infinity.

2.6 REFERENCES:

1. Alpha Chiang, Fundamentals Methods of Mathematical Economics
2. Alpha Chiang and Wainwright, Fundamental Methods of Mathematical Economics
3. https://www.onlinemathlearning.com/exponent-rules-2.html

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3

DERIVATIVES

Unit Structure

3.0 OBJECTIVES

- To understand the concept of derivatives.
- To find out derivative of a function.
- To study the addition rule, multiplication rule and chain rule of derivatives.
- To know the applications of derivatives in economics.
- To study unconstrained optimisation in economics.

3.1 INTRODUCTION

The process of obtaining derivative is called 'Differentiation'. The concept of derivative involves small change in the dependent variable with reference to a small change in independent variables.

The problem is to find a function derived from a given relationship between the two variables so as to express the idea of change. This derived function is called 'Derivative' of a given function.

If a function has a derivative, then it is known to be differentiable.

Therefore,

- Marginal cost is the rate of change of total cost with change in the quantity produced.
- Marginal revenue is the rate of change of total revenue with the change in quantity produced.

- Marginal utility is the rate of change of total utility with the change in quantity consumed.
- Marginal productivity is the rate of change of total productivity with the change in factors of production.

There are mainly three types of derivatives as Simple Derivative, Partial Derivative and Total Derivative.

Simple Derivative : Simple derivative is a derivative of a single variable function.

Partial Derivative : Partial Derivative of a function of various variables is a derivative with respect to one of those variables with the others held constant.

Total Derivative : Total Derivative Is a derivative of all variables, not just a single one.

On the basis of order, derivatives can be classified such as first order derivative and second order derivative.

First order derivatives shows the direction of the function whether the function is increasing or decreasing. Second order derivatives are used to know the shape of the graph for the given function.

3.2 DERIVATIVE OF A FUNCTION

Suppose 'y' is a continuous and single valued function of 'x'.

$$y=f(x) \qquad \text{------------------} \qquad (1)$$

If value of x increase or decrease, then the value of y increase or decrease. y is dependent variable and x is an independent variable.

The derivative of y with respect to x exists only if the function is a single valued function of a continuous variable.

An increase in the value of x which will produce a corresponding increase or & decrease in the value of y. These increments or changes are denoted the symbols Δx, Δy respectively.

$$y + \Delta y = f(x + \Delta x) \qquad \text{------------------} \qquad (2)$$

By equation (1) and (2)

$$\Delta y = f(x + \Delta x) - y$$

$$\Delta y = f(x + \Delta x) - f(x) \qquad \text{------------------} \qquad (3)$$

$$(y = f(x))$$

Dividing by Δx to both sides of the equation (3)

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} \qquad \text{------------------} \qquad (4)$$

The average rate of change $\left(\frac{\Delta y}{\Delta x}\right)$ tends to a definite limit as Δx tends to zero (Δx -> 0)

$$\lim_{(\Delta x -> 0)} \frac{\Delta y}{\Delta x} = \Delta x -> 0 \frac{f(x+\Delta x) - f(x)}{\Delta x} \quad \text{------------------} \quad (5)$$

If in the right hand side, the limit exists, then left hand side limit written as $\frac{dy}{dx}$, it is called "Differential Co-efficient".

The differential Co-efficient is also called the 'Derivative'.

$\frac{dy}{dx}$ indicates derivative of y with respect to x.

$$\frac{dy}{dx} = \frac{d(f(x))}{dx}$$

$$\frac{dy}{dx} = \lim_{(\Delta x -> 0)} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Therefore, $\frac{dy}{dx}$ is the rate of change of y with respect to change in x, it is called the derivative of the function y with respect to x.

3.3 RULES OF DIFFERENTIATION

Some basic rules of differentiation are as follows –

Rule I- Polynomial functions Rule or power Functions Rule:

If $y=x^n$ is any is a power function where n is any real number, the derivative of a power function is nx^{n-1}.

Symbolically,

- If $y=x^n$, $\frac{dy}{dx} = nx^{n-1}$
- If $y=x$, $\frac{dy}{dx} = x^{1-1} = x^0 = 1$

(Anything to the power of zero is 1)

Examples :

i) If $y=x^9$, $find \frac{dy}{dx}$

Solution - $\frac{dy}{dx} = \frac{d(x^9)}{dx} = 9x^{9-1} = 9x^8$

ii) If $y=x^{-8}$, $find \frac{dy}{dx}$

Solution - $\frac{dy}{dx} = \frac{d(x^{-8})}{dx} = -8x^{-8-1} = -8x^{-9}$

iii) If $y=\sqrt{x}$, $find \frac{dy}{dx}$

Solution - $\frac{dy}{dx} = \frac{d(\sqrt{x})}{dx} = \frac{d(x^{\frac{1}{2}})}{dx} = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}}$

Rule II- Constant function Rule

A) Derivative of a Constant

Derivative of a constant function (y=f(x)=C) is zero.

If y = C, where C is constant, then

$\frac{dy}{dx} = \frac{d(C)}{dx} = 0$

Examples :

i) If y=7, $find\ \frac{dy}{dx}$

Solution - $\frac{dy}{dx} = \frac{d(7)}{dx} = 0$

ii) If y=120, $find\ \frac{dy}{dx}$

Solution - $\frac{dy}{dx} = \frac{d(120)}{dx} = 0$

iii) If y=75, $find\ \frac{dy}{dx}$

Solution - $\frac{dy}{dx} = \frac{d(75)}{dx} = 0$

Rule III- Linear function Rule :

If, y = mx + c where m and c are constants, then

$\frac{dy}{dx} = \frac{d(mx+c)}{dx} = m$

Examples :

i) If y= 5x + 7, find $\frac{dy}{dx}$

Solution - $\frac{dy}{dx} = \frac{d(5x+7)}{dx} = 5$

ii) If y= 7x + 8, find $\frac{dy}{dx}$

Solution - $\frac{dy}{dx} = \frac{d(7x+8)}{dx} = 7$

iii) If y= 13x + 8, find $\frac{dy}{dx}$

Solution - $\frac{dy}{dx} = \frac{d(13x+8)}{dx} = 13$

Exercise :

i) If y= x^{15}, find $\frac{dy}{dx}$

ii) If y= 170, find $\frac{dy}{dx}$

iii) If y= 13x +10, find $\frac{dy}{dx}$

3.4 ADDITION RULE

Addition rule is known as derivative of a sum or sum rule.

The derivative of a sum of two functions is equal to the sum of the separate derivatives.

If, y=u+v

where, u and v are the differentiable functions. of x,

then the derivative of sum is

$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ (i.e. derivative of first function + derivative of second function)

Q1. If y = $x^5 + x^8$, then find $\frac{dy}{dx}$

Solution –

$$\frac{dy}{dx} = \frac{d(x^5)}{dx} + \frac{d(x^8)}{dx}$$

$$\frac{dy}{dx} = 5x^4 + 8x^7$$

Q2. If y = $11x^{-3} + 4x^{-9} + 3x + 7$, then find $\frac{dy}{dx}$

Solution –

$$\frac{dy}{dx} = \frac{d(11x^{-3})}{dx} + \frac{d(4x^{-9})}{dx} + \frac{d(3x)}{dx} + \frac{d(3x)}{dx}$$

$$= -33x^{-4} + (-36x^{-10}) + 3 + 0$$

$$= -33x^{-4} - 36x^{-10} + 3$$

Q3. If y = $11x^2 + 5x^2 + 7x^2$, then find $\frac{dy}{dx}$

Solution –

$$\frac{dy}{dx} = \frac{d(11x^2)}{dx} + \frac{d(5x^2)}{dx} + \frac{d(7x^2)}{dx}$$

$$= 22x + 10x + 14x$$

$$= 46x$$

Q4. If $y = 7x^4 + x^9 + 6x + 11$, then find $\frac{dy}{dx}$

Solution –

$$\frac{dy}{dx} = \frac{d(7x^4)}{dx} + \frac{d(x^9)}{dx} + \frac{d(6x)}{dx} + \frac{d(11)}{dx}$$

$$= 28x^3 + (9x^8) + 6 + 0$$

$$= 28x^3 + (9x^8) + 6$$

Exercise:

Q1. If $y = 5x^2 + 11x^{10} + 13x^5$, then find $\frac{dy}{dx}$

Q2. If $y = 7x^2 + 7x^3 + 7x^4$, then find $\frac{dy}{dx}$

3.5 MULTIPLICATION RULE

Multiplication Rule of differentiation is also known as product rule or derivativative of a product.

Derivative of the product of two functions is equal to first function multiplied by the derivative of the second Function plus the second function multiplied by the derivative of the first function.

If, y = uv, where u and v are differentiable functions of x, then

$$\frac{dy}{dx} = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$$

i.e. first function x derivative of second function + second function x derivative of first function.

Examples :

Q1. If $y = (2x^3 + 9)(x^3 + 2x)$, then find $\frac{dy}{dx}$

Solution –

$(2x^3 + 9) = u$, $(x^3 + 2x) = v$

$$\frac{dy}{dx} = (2x^3 + 9) \cdot \frac{d(x^3 + 2x)}{dx} + (x^3 + 2x) \cdot \frac{d(2x^3 + 9)}{dx}$$

$= (2x^3 + 9)\ (3x^2 + 2) + (\ x^3 + 2x\ )\ \ (6x^2 + 0)$

$= 6x^5 + 4x^3 + 27x^2 + 18 + 6x^5 + 12x^3$

$= 6x^5 + 6x^5 + 4x^3 + 12x^3 + 27x^2 + 18$

$= 12x^5 + 16x^3 + 27x^2 + 18$

Q2. If y = $5x^2 \cdot 12x^3$, then find $\frac{dy}{dx}$

Solution –

u = $5x^2$, $v = 12x^3$

$\frac{dy}{dx} = 5x^2 \cdot \frac{d(12x^3)}{dx} + 12x^3 \cdot \frac{d(5x^2\ )}{dx}$

$= 5x^2\ x\ 36x^2 + 12x^3$ x $10x$

$= 180x^4 + 120x^4$

$= 300x^4$

Q3. If y = $(3x^2 + 2x)\ (\ x + 9)$, then find $\frac{dy}{dx}$

Solution –

u = $(3x^2 + 2x)$, v = $(\ x + 9)$

$\frac{dy}{dx} = (3x^2 + 2x) \cdot \frac{d(\ x + 9)}{dx} + (\ x + 9)\ \ \cdot \frac{d(3x^2 + 2x)}{dx}$

$= (3x^2 + 2x)\ (1) + (\ x + 9)\ \ (6x\ + 2)$

$= 3x^2 + 2x + 6x^2 + 2x + 54x + 18$

$= 3x^2 + 6x^2 + 2x\ + 2x + 54x + 18$

$= 9x^2 + 58x + 18$

Exercise:

Q1. If y = $(5x^2 + 3x)\ (\ 6x^2 - 4x^3\ )$, then find $\frac{dy}{dx}$

Q2. If y = $(6x^3 + 4x)(6x^2 + 3x)$, then find $\frac{dy}{dx}$

3.6 CHAIN RULE

Chain rule is also known as Function of a Function Rule or Derivative of a composite function.

If, y = $[f(x)]^n$, the derivative of this function is,

$$\frac{dy}{dx} = n[f(x)]^{n-1} \cdot \frac{d(f(x))}{dx}$$

Example :

i) If, y = $(x^2 + 1)^3$, find $\frac{dy}{dx}$

Solution –

$$\frac{dy}{dx} = \frac{d(x^2 + 1)^3}{dx}$$

$$= 3(x^2 + 1)^{3-1} \cdot \frac{d(x^2 + 1)}{dx}$$

$$= 3(x^2 + 1)^2 \cdot (2x + 0)$$

$$= 3(x^2 + 1)^2 \cdot 2x$$

$$= 6x(x^2 + 1)^2$$

$$= (6x^3 + 6x)(6x^3 + 6x)$$

$$= 6x^3(6x^3 + 6x) + 6x(6x^3 + 6x)$$

$$= 36x^{3+3} + 36x^{3+1} + 36x^{1+3} + 36x^{1+1}$$

$$= 36x^6 + 72x^4 + 36x^2$$

ii) If, y = $(x^3 + x^2 + 4)^2$, find $\frac{dy}{dx}$

Solution –

$\frac{dy}{dx} = 2(x^3 + x^2 + 4)^{2-1} \cdot \frac{d(x^3 + x^2 + 4)}{dx}$

$\frac{dy}{dx} = 2(x^3 + x^2 + 4) \cdot (3x^2 + 2x)$

$= (2x^3 + 2x^2 + 4) \cdot (3x^2 + 2x)$

$= 3x^2(2x^3 + 2x^2 + 4) + 2x(2x^3 + 2x^2 + 4)$

$= 6x^{2+3} + 6x^{2+2} + 24x^2 + 4x^{1+3} + 4x^{1+2} + 16x$

$= 6x^5 + 6x^4 + 24x^2 + 4x^4 + 4x^3 + 16x$

$= 6x^5 + 10x^4 + 4x^3 + +24x^2 + 16x$

iii) If, y = $(2x^2 + 3x^3 + 4x^4 + 20)^2$, find $\frac{dy}{dx}$

Solution –

$\frac{dy}{dx} = 2(2x^2 + 3x^3 + 4x^4 + 20)^{2-1} \cdot \frac{d}{dx}(2x^2 + 3x^3 + 4x^4 + 20)$

$= 2(2x^2 + 3x^3 + 4x^4 + 20) \cdot (4x + 9x^2 + 16x^3)$

$= (4x^2 + 6x^3 + 8x^4 + 40) \cdot (4x + 9x^2 + 16x^3)$

$= 4x(4x^2 + 6x^3 + 8x^4 + 40) + 9x^2(4x^2 + 6x^3 + 8x^4 + 40) + 16x^3(4x^2 + 6x^3 + 8x^4 + 40)$

$= 16x^3 + 24x^4 + 32x^5 + 160x + 36x^4 + 54x^5 + 72x^6 + 360x^2 + 64x^6 + 96x^6 + 128x^7 + 640x^3$

$= 128x^7 + 64x^6 + 96x^6 + 72x^6 + 54x^5 + 32x^5 + 24x^4 + 36x^4 + 16x^3 + 640x^3 + 360x^2 + 160x$

$= 128x^7 + 232x^6 + 86x^5 + 60x^4 + 650x^3 + 360x^2 + 160x$

Exercise:

i) If, y = $(x^2 + x + 1)^2$,then find $\frac{dy}{dx}$

ii) If, y = $(5x + 6)^3$,then find $\frac{dy}{dx}$

iii) If, y = $(6x^2 + 3x^2 + x + 1)^4$,then find $\frac{dy}{dx}$

iv) If, y = $(10x^2 + 5x^2 + x^2 + 2)^2$,then find $\frac{dy}{dx}$

3.7 APPLICATION OF SIMPLE DERIVATIVES IN ECONOMICS

1. Marginal Cost –

Marginal cost is the change in the total cost incurred from the production of an additional unit.

If, total cost function TC = TC (q) which is continuous and differentiable function, the first derivative of the total cost function is a marginal cost function.

Example : If TC = $3q + 4q^2 + 20$ is total cost function, find marginal cost function.

Solution-

for calculating the marginal cost function, we have to take first order simple derivative of total cost function with respect to q.

$\frac{dTC}{dq} = \frac{d}{dq}(3q + 4q^2 + 20)$

$= 3 + 8q$

2. Marginal Revenue –

Marginal Revenue as the change in total revenue brought about by selling one extra unit of output.

If total revenue function is TR = TR (q), which is continuous and differential function, then the first order simple derivative of the total revenue function is a marginal revenue function.

Example:

If, TR = 100Q – 4Q² is total revenue function, calculate Marginal revenue function.

Solution –

Marginal Revenue = $\frac{dTR}{dQ} = \frac{d}{dQ}(100Q - 4Q^2)$

$= 100 - 8Q$

Exercise:

i) If TC = $100q^2 + 25q^4 + 400q + 60$, find $\frac{dTC}{dq}$

ii) If TR = $5Q^3 + 6Q^4 - 60Q + 100$, find $\frac{dTR}{dQ}$

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3.8 UNCONSTRAINED OPTIMISATION IN ECONOMICS

Basically there are two conditions for unconstrained optimization with one explanatory variable x. One condition is necessary and one is sufficient.

Suppose. Y= f(x) i.e, y is a function of x (explanatory variable).

Conditions for Maximization –

i) $\frac{dy}{dx} = 0$

i.e., First order derivative should be zero. It is necessary condition.

ii) $\frac{d^2y}{dx^2} < 0$

i.e., second order derivative should be less than zero means it should be negative. It is sufficient condition.

Conditions for Minimization -

i) $\frac{dy}{dx} = 0$ and ii) $\frac{d^2y}{dx^2} > 0$

It means first order derivative should be zero $\left(\frac{dy}{dx} = 0\right)$ which is necessary

condition and second order derivative should be more or greater than zero means it should be positive number.

Examples –

Q. If Total Revenue Function, TR = 29Q-3Q^2 and total cost function, TC = $\frac{1}{3}Q^3 - 6Q^2 + 2Q$ + 40, find profit maximizing output and maximum profit.

Solution - Profit is denoted by π and calculated by the following formula.

π = TR-TC

In this problem, we are asked to find out the profit maximizing output which means at what value of Q (Q = output) the profit of the firm will be maximum.

So for that we have to find out the value of Q and after getting the value of Q maximum profit will not be difficult to find out.

TR = 29Q-3Q^2 --------------(1)

TC = $\frac{1}{3}Q^3 - 6Q^2 + 2Q$ + 40 --------------(2)

By equation (1) and (2) profit is

Profit (π) = TR-TC

$$\pi = (29Q-3Q^2) - (\tfrac{1}{3}Q^3 - 6Q^2 + 2Q + 40)$$

$$= 29\,Q - 3Q^2 - \tfrac{1}{3}Q^3 + 6Q^2 - 2Q - 40$$

$$\pi = -\tfrac{1}{3}Q^3 + 3Q^2 + 27Q - 40 \quad \text{--------------(2)}$$

Necessary condition –

First order condition of maximization of profit is

$$\frac{d\pi}{dQ} = 0$$

$$\frac{d}{dQ}\left(-\tfrac{1}{3}Q^3 + 3Q2 + 27Q - 40\right) = 0$$

$$-Q^2 + 6Q + 27 = 0$$

Q^2 - 6Q - 27 = 0

Q^2 –(9-3)Q - 27 = 0

Q^2 -9Q +3Q -27 = 0

Q(Q-9) + 3(Q-9)=0

(Q-9) (Q+3)=0

Therefore, Q = 9 or Q = -3

Sufficient Condition

$\frac{d^2\pi}{dQ^2} = \frac{d}{dQ}(-Q^2 + 6Q + 27)$

$= -2Q + 6$

By putting values of Q (Q = 9 or Q = -3) in the second order condition

If Q=9, $\frac{d^2\pi}{dQ^2}$ = -2(9)+6 = -18 + 6 = -12 < 0

If Q=-3, $\frac{d^2\pi}{dQ^2}$ = -2(-3)+6 = 6 + 6 = 12 > 0

Therefore, the necessary condition ($\frac{d\pi}{dQ} = 0$) of profit maximization is satisfied when Q=9. So, profit maximizing output is 9.

For getting maximum level of output the value Q=9, put in the equation (3)

$\pi = -\frac{1}{3}Q^3 + 3Q^2 + 27Q - 40$

$\pi = -\frac{1}{3}Q^3 + 3Q^2 + 27Q - 40$

$\pi = -\frac{1}{3}(9)^3 + 39^2 + 27Q - 40$

$\pi = 203$

3.9 Summary

- If y= e^x, $\frac{dy}{dx}$ =?

$$\frac{dy}{dx} = \frac{d(e^x)}{dx} = e^x$$

- If y= e^{-x}, $\frac{dy}{dx}$ =?

$$\frac{dy}{dx} = \frac{d(e^{-x})}{dx} = e^{-x}$$

- If y= e^{ax}, $\frac{dy}{dx}$ =?

$$\frac{dy}{dx} = \frac{d(e^{ax})}{dx} = e^{ax}$$

- If y= e^u, u =f(x), $\frac{dy}{dx}$ =?

$$\frac{dy}{dx} = \frac{d(e^u)}{dx} \cdot \left(\frac{du}{dx}\right) = e^u \left(\frac{du}{dx}\right)$$

- Inverse Function Rule -

If x=f(y), find $\frac{dx}{dy}$

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

- y = log x, find $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{d(\log x)}{dx}$$

- if y = log u, and u=f(x), find $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{d(\log u)}{dx} \cdot \left(\frac{du}{dx}\right) = \frac{1}{u} \cdot \left(\frac{du}{dx}\right)$$

3.10 QUESTIONS

Q1. Explain the derivative of a function.

Q2. Explain addition rule of differentiation with the help of any two examples.

Q3. Explain multiplication rule of differentiation by giving any two examples.

Q4. Explain chain rule of differentiation by giving any two examples.

Q5. What are the application of simple derivative in economics.

Q6. Explain unconstrained optimization in Economics.

Q7. If $TR = 5Q^2 + 3Q – 40$ and $TC = Q^3 +Q^2 -60$, find profit maximizing output.

3.11 REFERENCES

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4

PARTIAL DERIVATIVES AND INTEGRATION

Unit Structure :

4.0 OBJECTIVES

- To know the meaning and the concept of partial derivative.
- To study the applications of partial derivatives in Economics.
- To introduce the concept of integration.
- To study the applications of integration in Economics.

4.1 INTRODUCTION TO PARTIAL DERIVATIVE

In the mathematical economics, partial derivative of a function of several variables is its derivative with respect to one of those variable and other variables held constant. Partial derivative is opposed to the total derivative in which all variables are allowed to vary.

Partial derivative concept is mostly used in vector calculus and differential geometry.

Partial derivative of a function $f(x, y_2...)$ with respect to the variable x is denoted by various symbols as

$f'x$, ∂xf, Dxf, D_1f, $\frac{\partial}{\partial x}f$, or $\frac{\partial f}{\partial x}$

If $z = f(x, y_2...)$, then the partial derivative of z with respect to x is denoted as $\frac{\partial z}{\partial x}$

In this unit we will use $\frac{\partial z}{\partial x}$ for partial derivative. The symbol ∂ is used to denote partial derivatives. Marquis de Condorcet used this symbol first in mathematics in 1770 for partial differences.

If, z = f(x, y) then Partial Derivative of first order of z with respect to x

$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(f(x,y)) \quad \rightarrow$ y is constant

Partial Derivative of first order of z with respect y

$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(f(x,y)) \quad \rightarrow$ x is constant

Second order partial derivative of z with respect to x

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial\left(\frac{\partial z}{\partial x}\right)}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}(f(x,y))\right)$$

Second order partial derivative of z with respect to y

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial\left(\frac{\partial z}{\partial y}\right)}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}(f(x,y))\right)$$

Examples :

Q1. If $z = 4x^2 + 4xy + y^2$, find $\frac{\partial z}{\partial x}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial y^2}$

Solution –

$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(4x^2 + 4xy + y^2)$

$= 8x + 4y + 0$

$= 8x + 4y$

$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(8x + 4y)$

$= 8$

$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(4x^2 + 4xy + y^2)$

$= 0 + 4x + 2y$

$= 4x + 2y$

$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}(4x + 2y)$

$= 0 + 2$

$=2$

Q.2 If, $z = 3x^2 + 5xy^2 + 6x^2y^2 + 5y + 6x$, find first and second order partial derivatives of z with respect to x and with respect to y.

Solution –

First order partial derivative of z with respect to x –

$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(3x^2 + 5xy^2 + 6x^2y^2 + 5y + 6x)$

$= 6x + 5y^2 + 12xy^2 + 0 + 6$

$= 6x + 5y^2 + 12xy^2 + 6$

Second order partial derivatives of z w.r.t. x

$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(6x + 5y^2 + 12xy^2 + 6)$

$=12y^2 + 6$

First order partial derivative of z w.r.t. y

$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(3x^2 + 5xy^2 + 6x^2y^2 + 5y + 6x)$

$= 0 + 10xy + 12x^2y + 5 + 0$

$= 10xy + 12x^2y + 5$

Second order partial derivative of z w.r.t. y

$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}(10xy + 12x^2y + 5)$

$= 10x + 12x^2$

Q 3. If $z = 5xy + 10x^2y^2 + 20x^4y^4 - 40$, find $\frac{\partial z}{\partial x}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial y^2}$

Solution -

$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(5xy + 10x^2y^2 + 20x^4y^4 - 40)$

$= 5y + 20xy^2 + 80x^3y^4 – 0$

$= 80x^3y^4 + 20xy^2 + 5y$

$\frac{\partial^2 z}{\partial x^2} = \frac{\partial\left(\frac{\partial z}{\partial x}\right)}{\partial x}$

$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(80x^3y^4 + 20xy^2 + 5y)$

$= 240x^2y^4 + 20y^2 + 0$

$= 240x^2y^4 + 20y^2$

$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(5xy + 10x^2y^2 + 20x^4y^4 - 40)$

$= 5x + 20x^2y + 80x^4y^3 - 0$

$= 80x^4y^3 + 20x^2y + 5x$

$\frac{\partial^2 z}{\partial y^2} = \frac{\partial\left(\frac{\partial z}{\partial y}\right)}{\partial y}$

$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}(80x^4y^3 + 20x^2y + 5x)$

$= 240x^4y^2 + 20x^2 + 0$

$= 240x^4y^2 + 20x^2$

Exercise :

Q. Find the $\frac{\partial z}{\partial x}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial z}{\partial y}$ *and* $\frac{\partial^2 z}{\partial y^2}$ from following functions.

i) $z = 3x^3 + 5x^2y^2 + 12x^3y^3 - 30y^2 - 30x^2 + 20$

ii) $z = x^3 + y^3 - y^2 - x^2 - x + y$

iii) $z = 3x^4y^4 + 4x^5y^5 + 5x^6y^6$

iv) $z = 12x^3 - 12y^3 + 24x^3y^3$

4.2 APPLICATIONS OF PARTIAL DERIVATIVES IN ECONOMICS

1. Measuring Marginal functions –

Utility Function :

u=f(x,y) is a utility function which is differentiable function and a function of two commodities, x and y.

Marginal utility x i.e. MU_x, is the first order derivative with respect to y.

Example:

If u = $x^{0.4}y^{0.6}$, is the utility function: find the marginal utility of x (MU_x) and marginal utility of y (MU_y).

Solution -

u = $x^{0.4}y^{0.6}$

$MU_x = \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x^{0.4}y^{0.6})$

$=0.4x^{0.4-1}y^{0.6}$

$=0.4x^{-0.6}y^{0.6}$

$MU_y = \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(x^{0.4}y^{0.6})$

$=0.6x^{0.4}\,y^{0.6-1}$

$=0.6x^{0.4}\,y^{-0.4}$

Production Function :

Similarly, if the production function is a function of capital (K) and Labour (L), Q = f(K, L), then the marginal productivity of capital (MP_K) is measured by taking the first order partial derivatives with respect to k and the marginal productivity of Labour (MP_L) is mesured by taking the first order partial derivative with respect to L.

Example :

If, Q = $10K^{0.5}L^{0.5}$, is production function, then find the marginal productivity of capital (MP_K) and marginal productivity of labour (MP_L).

Solution -

Q = $10K^{0.5}L^{0.5}$

Marginal Productivity of Capital (MP_K)

$MP_K = \frac{\partial Q}{\partial K} = \frac{\partial}{\partial K}(\ 10K^{0.5}L^{0.5})$

$= \ 10\ x\ 0.5K^{0.5-1}L^{0.5}$

$= \ 5K^{-0.5}L^{0.5}$

Marginal Productivity of Labour (MP_L)

$MP_L = \frac{\partial Q}{\partial L} = \frac{\partial}{\partial L}(\ 10K^{0.5}L^{0.5})$

$= \ 10\ x\ 0.5K^{0.5}L^{0.5-1}$

$= \ 5K^{0.5}L^{-0.5}$

2. Measuring the slope of indifference curves -

An indifference curve shows the locus of all possible combinations of two goods which yield the same level of satisfaction for the consumer.

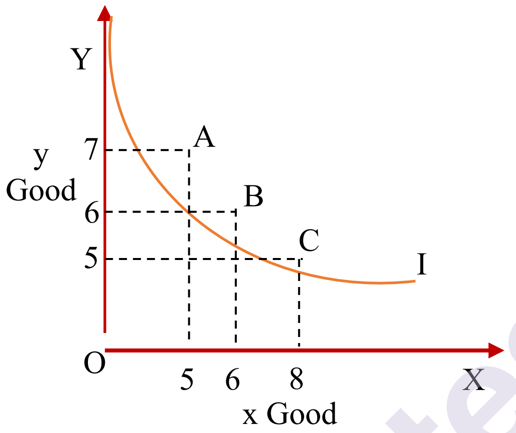
Suppose, consumer consumes two goods as x and y.

An indifference curve indicates a set of various combinations of x and y and from each combination consumer gets equal satisfaction (for example, K).

u(x, y) = k where, K = constant

An indifference curve has been shown as below –

Figure No. 4.1



If the consumer moves from A to B at the same indifference curve, this movement leads to a change in x (dx) and a change in y (dy).

The change in the utility due to the change in x is $MU_x dx$.

The change in the utility due to the change in y is $MU_y dy$. However, there is no change in the total utility, because the consumer remains on the same indifference curve.

Thus, on the same indifference curve :

(Marginal utility of x) x (change in x) +

(Marginal utility of y) x (change in y) = 0

$MU_x dx + MU_y dy = 0$

$MU_y dy = -MU_x dx$

$$\frac{dy}{dx} = -\frac{MU_x}{MU_y}$$

Hence, $\frac{dy}{dx} = -\frac{MU_x}{MU_y}$, is the slope of the indifference curve.

$MU_x = \frac{\partial u(x,y)}{\partial x}$ and $MU_y = \frac{\partial u(x,y)}{\partial y}$

Example:

If the equation of an indifference curve is u(x, y) = xy = 10, calculate the slope of the indifference curve.

Solution - Slope of the Indifference Curve

$$\frac{dy}{dx} = -\frac{MU_x}{MU_y}$$

$$= \frac{\frac{\partial(xy)}{\partial x}}{\frac{\partial(xy)}{\partial y}}$$

$$= -\frac{y}{x}$$

$\frac{dy}{dx} = -\frac{y}{x}$ (i.e., slope of indifference curve)

3. Measuring the slope of isoquants –

An isoquant is a curve showing different combinations of factors of production capital (K) and labour (L) which yield the same level of production.

Similarly, we can find the slope of an isoquant as indifference curve.

Slope of isoquant is $\frac{dK}{dL} = -\frac{MP_L}{MP_K}$

$MP_L = \frac{\partial(Q(K,L))}{\partial L}$ and $MP_K = \frac{\partial(Q(K,L))}{\partial K}$

Example:

If the equation of an isoquant is (Q(K, L) $= K^{0.5}L^{0.5}$ =200 calculate the slope of the isoquant.

Solution - slope of the Isoquant

$$\frac{dK}{dL} = -\frac{MP_L}{MP_K}$$

$$= -\frac{\frac{\partial(K^{0.5}L^{0.5})}{\partial L}}{\frac{\partial(K^{0.5}L^{0.5})}{\partial K}}$$

$$= -\frac{0.5K^{0.5}L^{0.5-1}}{0.5K^{0.5-1}L^{0.5}}$$

$$= -\frac{0.5K^{0.5}L^{-0.5}}{0.5K^{-0.5}L^{0.5}}$$

$$= -\frac{0.5}{0.5}\frac{K^{0.5}K^{0.5}}{L^{0.5}L^{0.5}}$$

$$= -1\ \frac{K}{L}$$

$$= -\frac{K}{L}$$

4. Marginal Rate of substitution (MRS) -

The marginal rate of substitution (MRS) measures a consumer's willingness to substitute one good for another goods while remaining on the same indifference curve.

Example:

If, the equation of an indifference curve is u(x, y) = xy = K, calculate the MRS_{xy} at the following points.

A(2,8) and B (8,2).

Solution - Marginal Rate of Substitution of x for y (MRS_{xy}) at pont A(2,8)

$$MRS_{xy} = \frac{dy}{dx} = -\frac{MU_x}{MU_y}$$

$$= \frac{\frac{\partial(xy)}{\partial x}}{\frac{\partial(xy)}{\partial y}} = -\frac{y}{x}$$

$$MRS_{xy}(2,8) = \frac{y}{x} = \frac{8}{2} = 4$$

$$MRS_{xy}(8,2) = \frac{y}{x} = \frac{2}{8} = 0.25$$

5. Marginal Rate of Technical substitution (MRTS)

Example:

If, the equation of an isoquant is

Q(K, L) $=K^{0.5}L^{0.5}$ =K, calculate the $MRTS_{LK}$

when, L=4 and K = 4

Solution -

$$MRTS_{LK} = \frac{MP_L}{MP_K}$$

$$= \frac{\frac{\partial(K^{0.5}L^{0.5})}{\partial L}}{\frac{\partial(K^{0.5}L^{0.5})}{\partial K}} = \frac{0.5K^{0.5}L^{0.5-1}}{0.5K^{0.5-1}L^{0.5}}$$

$$= \frac{0.5K^{0.5}L^{-0.5}}{0.5K^{-0.5}L^{0.5}} = \frac{K^{0.5}K^{0.5}}{L^{0.5}L^{0.5}}$$

$$= \frac{K}{L}$$

$$MRTS_{LK} = \frac{4}{4} = 1$$

Exercise

i. If u = $x^{0.7}y^{0.3}$, is the utility function, calculate MU_x and MU_y.

ii. If, Q = $5K^{0.3}L^{0.7}$, is production function, then find the MU_K and MU_L.

iii. If the equation of indifference curve is u(x, y) = 2xy = 20 then find the slope of indifference curve.

iv. If the equation of isoquant is Q(K, L) =$5K^{0.7}L^{0.3}$ = 100, then calculate the slope of isoquant.

v. If the equation of an indifference curve is u(x, y) = 2xy = K. then find the MRS_{xy} at the point A (5,5) and Point B (7,3).

vi. If the equation of an isoquants is Q(K, L) =$K^{0.4}L^{0.6}$ =K, then calculate the $MRTS_{KL}$ when, K=5 and L = 5

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4.3 INTRODUCTION TO INTEGRATION

The reverse or inverse process of ‘Differentiation' is known as ‘Indefinite Integration’ or Integral Calculus' or 'Antidiffertication’. Differentiation or derivatives used to find marginal from total. And integration used to find total from marginal. The process of the integration is finding the function whose derivative is given.

Differentiation-

If, the total utility function is q^2, the marginal utility function is 2q.

u=q^2

$$\frac{du}{dq} = \frac{dK}{dq}(q^2) = 2q$$

Integration (Indefinite Integration) -

If, the marginal utility function is 2q, the total utility function is q^2. That is, if derivative is given, our problem is to find out the function.

Definition -

If, differential co-efficient of F(x) with respect to x is f(x), an integral of f(x) with respect to x is F(x).

Example

We know, differentiation means finding the rate at which a variable quantity is changing, it is known as derivative. If, total utility function is U $=q^2$, where u is the total utility and q is the quantity consumed, $\frac{du}{dq} = 2q$. It is the rate of change of total utility (U) with respect to quantity consumed (q) and is the marginal utility function.

If, the marginal utility function is 2q, then what is the total utility function?

The total utility function, $U=q^2$.

The problem is reversed here, between differentiation and integration.

In the case of integration, we are now given the marginal utility function and we have to find the total utility function. This reverse process involves the summation of the differences, it is called as 'Integration'.

Notation-

If $\frac{d}{dx}$ F(x) = f(x), the usual notation for the indefinite integration is,

$\int f(x)dx$ or $\int ydx$ = F(x)

Where,

$\int$ = Integral sign

f(x) = Integrand or the function to be integrated.

Dx = suggests that the operation of integration is to be with respect to the variable x.

$\int f(x)dx$ means antiderivative or integration of f(x).

Rules of Integration –

Some basic rules of integration are as follows –

1. Rule I -

$\int dx$ = x + c

2. Rule II -

If a function is multiplied by a constant, then the integral of that function is also multiplied by the same constant.

Examples:

1) $\int 5dx = 5\int dx =$ 5x+C

2) $\int 15dx = 15\int dx =$ 15x+C

3. Rule III –

$\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$

Examples:

i. $\int x^6 dx = \frac{x^{6+1}}{6+1} + C = \frac{x^7}{7} + C$

ii. $\int \frac{1}{x^8} dx = \int x^{-8} dx = \frac{x^{-8+1}}{-8+1} + C = \frac{x^{-7}}{-7} + C$

iii. $\int 1 dx = \int x^0 dx = x + C$

iv. $\int \sqrt{x} dx = \int x^{\frac{1}{2}} dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C$

v. $\int x^7 dx = \frac{x^{7+1}}{7+1} + C = \frac{x^8}{8} + C$

4. Rule IV

Integral of Sum or Difference

Integral of sum or difference of a number of functions is equal to the sum or difference of the integrals of the separate functions.

Symbolically,

$\int (dx_1 + dx_2 + \ldots\ldots + dx_n) = \int dx_1 + \int dx_2 + \ldots. + \int dx_n + C$

$\int (dx_1 - dx_2 - \ldots\ldots - dx_n) = \int dx_1 - \int dx_2 - \ldots. - \int dx_n + C$

Examples:

i) $\int (x^3 - x + 1)\, dx$

$= \int x^3 dx - \int x + \int 1\, dx$

$= \frac{x^4}{4} - \frac{x^2}{2} + x + c$

ii) $\int (x^5 - \frac{1}{x^9} - x + 1)\, dx$

$= \int x^5 dx - \int \frac{1}{x^9} dx + \int x\, dx + \int 1\, dx$

$= \frac{x^6}{6} + \frac{x^{-8}}{-8} - \frac{x^2}{2} + x + c$

iii) $\int (8x^3 - 5x^2 + x - 2)\, dx$

$= \int 8x^3 dx - \int 5x^2\, dx + \int x\, dx - \int 2\, dx$

$= 8\frac{x^4}{4} - 5\frac{x^3}{3} + \frac{x^2}{2} - 2x + c$

$= 2x^4 - 5\frac{x^3}{3} + \frac{x^2}{2} - 2x + c$

5. Rule V

Integral of a multiple by a constant

When a function is multiplied by constant number, this number will remain a multiple of the integral of the function.

Examples:

i. $\int 5x^7dx = 5\int x^7dx$

$= 5\frac{x^{7+1}}{7+1} + C$

$= 5\frac{x^8}{8} + C$

ii. $\int 7x^{12}dx = 7\int x^{12}dx$

$= 7\frac{x^{12+1}}{12+1} + C$

$= 7\frac{x^{13}}{13} + C$

iii. $\int 6x^5dx = 6\int x^5dx$

$= 6\frac{x^{5+1}}{5+1} + C$

$= 6\frac{x^6}{6} + C$

$= x^6 + C$

Definite Integration-

We knew the indefinite integration, now we study the definite integration. Notation for Definite Integration -

$\int_a^b ydx$ or $\int_a^b f(x)dx$

It is to be read as 'definite integral' of y or f(x) from x= a to x=b

The value of definite integral from a to b is written as

$\int_a^b ydx = \int_a^b f(x)dx = [F(x)]_a^b$

=F(b)-F(a) ………….. b > a

where,

$\int_a^b ydx$ or $\int_a^b f(x)dx$ = definite integral of y or f(x)

a = the lower limit of the integral

b = Upper limit of the integral.

Examples:

i) Evaluate $\int_1^2 x^3 dx$

Solution - $\int_1^2 x^3 dx = \left[\frac{x^{3+1}}{3+1}\right]_1^2 = \left[\frac{x^4}{4}\right]_1^2$

$= \left[\frac{2^4}{4} - \frac{1^4}{4}\right] = \frac{16}{4} - \frac{1}{4} = \frac{15}{4}$

ii) $\int_2^3 4x^2 dx$ Evaluate

Solution - $\int_2^3 4x^2 dx = \left[4\frac{x^{2+1}}{2+1}\right]_2^3 = \left[4\frac{x^3}{3}\right]_2^3$

$= \left[4\frac{3^3}{3} - 4\frac{2^3}{3}\right]$

$= \left[4\frac{27}{3} - 4\frac{8}{3}\right]$

$= \frac{108}{3} - \frac{32}{3} = \frac{76}{3}$

iii) $\int_1^2 (x^3 - 2x - 4)\, dx$ Evaluate

Solution - $\int_1^2 (x^3 - 2x - 4)\, dx = \left[\frac{x^{3+1}}{3+1} - 2\frac{x^{1+1}}{1+1} - 4x\right]_1^2$

$= \left[\frac{x^4}{4} - 2\frac{x^2}{2} - 4x\right]_1^2$

$= \left[\frac{x^4}{4} - x^2 - 4x\right]_1^2$

$= \left(\frac{2^4}{4} - 2^2 - 4 \times 2\right) - \left(\frac{1^4}{4} - 1^2 - 4 \times 1\right)$

$= \left(\frac{16}{4} - 4 - 8\right) - \left(\frac{1}{4} - 1 - 4\right)$

$= \left(\frac{16}{4} - 12\right) - \left(\frac{1}{4} - 5\right)$

$= \frac{16}{4} - \frac{1}{4} - 12 + 5$

$= \frac{15}{4} - 7$

Exercise :

Q1. Evaluate $\int_1^2 6x^3 dx$

Q2. Evaluate $\int_2^3 (6x^3 + 6x)\, dx$

Q3. Evaluate $\int_0^1 7x^6 dx$

4.4 APPLICATIONS OF INTEGRATION IN ECONOMICS

The concept of integration is widely used in economics. Some important applications of integration in economics are as following applications in economics are as follows -

1) For calculating total revenue from marginal revenue.
2) For calculating total cost from marginal cost
3) For calculating total profit from marginal profit
4) Capital accumulation over a specified period of time.
5) Consumer Surplus.
6) Producer Surplus.
7) Lorenze curve.
8) Gini coefficient.

1) For Calculating Total Revenue from Margin Revenue :

Marginal revenue is the revenue which is gained by producing one more 4 unit of a product or service.

If total revenue function is given, we can calculate marginal revenue function by taking first order derivative of total revenue function.

MR = $\frac{dTR}{dQ}$

Where,

MR = Marginal Revenue

TR = Total Revenue

Q = Quantity produced

If, marginal revenue function MR(Q) is given, then the total revenue can be calculated ny integrating the marginal revenue function.

TR(Q) = $\int MR(Q)\, dQ$

Example :

If marginal revenue function MR = MR = 200 + 40Q +6Q^2 where Q is amount of units sold for a period, then find the total revenue function if at Q=2 and it is equal to 520.

Solution -

MR = 200 + 40Q +6Q^2

By integrating above marginal revenue function, we can find the total revenue function as below

TR(Q) = $\int MR(Q)\, dQ = \int (200 + 40Q + 6Q^2)\, dQ$

= 200Q + $20Q^2$ +$2Q^3$ + C

The value of C (constant) can be determined by using TR(Q=2) = 520. So,

200(2) + 20$(2)^2$ + 2$(2)^3$ +C = 520

596 + C = 520

C = 24

Therefore, the total revenue function is,

TR(Q) = 200Q + $20Q^2$ +$2Q^3$ + 24

2) For calculating Total cost from Marginal cost:

Marginal cost is a cost of last unit which used in the production.

There is similar relationship between the marginal cost (MC) and total cost (TC) as Total Revenue (TR) and Marginal Revenue (MR).

MR = $\frac{dTC}{dQ}$

TC(Q) = $\int MC(Q)\, dQ$

Example –

If MC = $5Q^2$+ $2Q$, then find TC(Q)

Solution –

TC(Q) = $\int (5Q^2 + 2Q)\, dQ$

= $\frac{5Q^3}{3}$ + $\frac{2Q^2}{2}$ + C

TC(Q) =$\frac{5}{3}Q^3$ + Q^2 + C

3) For calculating Marginal profit from total output :

TP = TR – TC

Marginal Profit (MP) –

MP = MR – MC or $\frac{dTP}{dQ}$ = $\frac{dTR}{dQ}$ - $\frac{dTC}{dQ}$

Example –

If MR = 200 + 40Q +6Q^2 and MC = $5Q^2$+ 2Q then find total profit function.

Solution – Total Profit = Total Revenue – Total Cost

Therefore,

TR(Q) = $\int MR(Q)\, dQ$

= $\int (200 + 40Q + 6Q^2)\, dQ$

$= 200Q + 20Q^2 + 2Q^3 + C$

$TC(Q) = \int MC(Q)\, dQ$

$= \int (5Q^2 + 2Q)\, dQ$

$= \frac{5}{3}Q^3 + Q^2 + C$

Total Profit TP(Q) = TR(Q) – TC(Q)

$= 200Q + 20Q^2 + 2Q^3 + C - \frac{5}{3}Q^3 - Q^2 - C$

$TP(Q) = 200Q + 19Q^2 + 2Q^3$

4) Capital Accumulation over a Specified Period of Time :

If, I(t) is the rate of investment, then the total capital decumulation K during time interval (a,b) is calculated by

$$K = \int_a^b I(t)dt$$

Example -

If the rate of investment is, I(t) = ln t, then compute the total capital accumulation between the 1st and the 5th years.

Solution.

Capital accumulation is calculated by

$$K = \int_a^b I(t)dt = \int_1^5 \ln t\, dt$$

$$= \begin{bmatrix} u = \ln t \\ dv = dt \\ du = \frac{dt}{t} \\ v = t \end{bmatrix}$$

$$= t \ln t - \int t \frac{dt}{t} = t \ln t - t$$

Thus

$$K = \Big[t \ln t - t\Big]_1^5 = (5 \ln 5 - 5) - (\ln 1 - 1)$$

$$= 5 \ln 5 - 4 \approx 4.05$$

5) Consumer Surplus :

The consumer surplus is the difference between the price which consumer is willing to pay for a good and the actual price to he pay for the good.

So,

Consumer surplus = What a person is willing to pay - what he actually pays.

OR

Consumer Surplus = Potential Price - Actual Price.

According to Alfred Marshall, "The excess of price which a person would be willing to pay for a thing rather than go without the thing, over that which he actually does pay is the economic measure of this surplus satisfaction. It may be called as Consumer's Surplus."

According to Taussig, "Consumer's surplus is the difference between the potential price and actual price."

Assumptions :

- Marginal utility money is constant.
- All consumers of have same utility function.
- A demand curve or a demand function for a good shows the amount of that commodity or good that will be bought by people at given price 'P'.

Suppose, p= f(x), is the demand function and a consumer purchases x_0 quantity, at P_0 price. So,

$$P_0 = f(x_0)$$

It implies that at price P_0 the consumer is willing to buy and the producer is willing to sell a quantity x_0

Therefore, the total expenditure of the consumer = $P_0 x_0$ -----(1)

However, there are buyers who would be willing to pay a price higher than P_0

Having already purchased a quantity x, he would be willing to purchase of the quantity dx at the price f(x). So, the expenditure would be f(x)dx. Thus, the total expenditure he would have been willing pay for the quantity x_0, is

$\int_0^{x_0} f(x)dx$ ---------- (2) The difference between these two is called consumer's surplus.

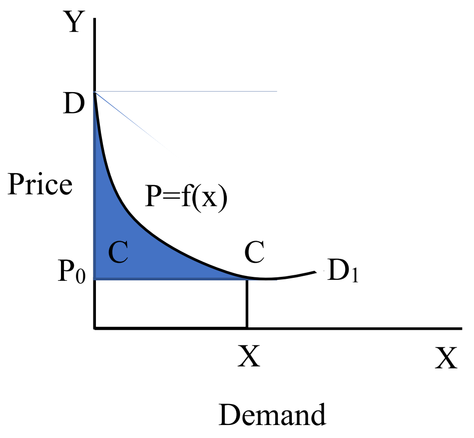
Therefore,

Consumer's surplus = $\int_0^{x_0} f(x)dx - P_0 x_0$

The consumer's surplus can be expressed as shown in the below diagram.

Figure No. 4.2

Consumer's surplus



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In the above figure, Demand is shown on the x axis and price is shown on the y axis. DD_1 is the demand curve. Demand x_0 corresponds to P_0. Then the consumer's surplus is,

DCP_0 = Area DCx_00 - Area P_0Cx_00

$DCP_0 = \int_0^{x_0} f(x)dx - P_0 x_0 \quad = \int_0^{x_0} pdx - P_0 x_0$

This difference is known as the consumer surplus. Consumer surplus is denoted CS

Examples :

1) If demand function is $p = 30\text{-}4x\text{-}3x^2$ the demand x_0 is 3, then what will be the consumer surplus ?

Solution –

$x_0 = 3$

$p_0 = 30 - 4(3) - 3(3)^2$ (By putting the value $x_0 = 3$ in demand function)

$p_0\ x_0 = 30 – 12 – 27 = 30 – 39 = -9$

$x_0 = 3,\ p_0 = -9$

$p_0\ x_0 = 3 \times (-9) = -27$

Consumer's surplus = $\int_0^{x_0} pdx - P_0 x_0$

$= \int_0^{3} (30 - 4x - 3x2\ )dx - (-9)$

$= \left[30x + \frac{4x^2}{2} - x^3\right]_0^3 + 9$

$= \left[30(3) + \frac{4(3)^2}{2} - (3)^3\right]_0^3 + 9$

$=90 + \frac{4(9)}{2} - 27 + 9$

$= 90 – 18 -27 +9$

$= 54$

$C.S = 54$

2) If, demand function is $P = 30 -2x$ and the supply function $2P = 5 + x,$ find the consumer surplus?

Solution –

Demand function $P = 30 -2x$

Supply function $2P = 5 + x$

$P = \frac{5+x}{2}$

For getting the value of x_0

Demand = Supply

$30 - 2x = \frac{5}{2} + \frac{x}{2}$

$-2x - \frac{x}{2} = 2.5 - 30$

$2x + \frac{x}{2} = -(2.5 - 30) = -2.5 + 30$

$\frac{4x + x}{2} = 27.5$

$\frac{5x}{2} = 27.5$

$x = 27.5 \cdot \frac{2}{5} = \frac{55}{5}$

$x = 11$

By putting x = 11 in the demand function

$P = 30 - 2x$

$P = 30 - 2(11) = 30 - 22$

$P = 8$

$\therefore x_0 = 11, P_0 = 8$

C.S. $= \int_0^{x_0} p\,dx - P_0\, x_0$

$= \int_0^{x_0} (30 - 2x)\,dx - (8 \times 11)$

$= \left[30x - x^2\right]_0^{11} - 88$

$= 30(11) - (11)^2 - 88$

$= 330 - 121 - 88$

$= 121$

Thus, Consumer's Surplus = 121.

Exercise :

i) If demand function $p = 35-2x-x^2$, then find the consumer's surplus at $x = 3$.

ii) If demand function $P=85-4x-x^2$ find the consumer's surplus at $x_0 = 5$.

6) Producer's Surplus :

In this case, it is assumed that marginal utility of money is constant and all producers have the same production function.

Suppose, p = f(x) so the supply function or supply curve which represents the amount of commodity that can be supplied at given price P i.e., market price. Let us suppose a producer sells a quantity x_0 at price P_0 so, $P_0 = f(x_0)$. It implies that at this price consumer is willing to sell a quantity x_0.

Therefore , the producer's revenue = $P_0\, x_0$

producer's revenue = $P_0\, x_0$ ---------- (1)

However, there are producers who are willing to sell or supply the good at a price lower than P_0. Having sold already quantity x, he would be willing to sell quantity dx at price f(x). Therefore, producer's revenue from dx would be f(x) dx. So, his real revenue would have been,

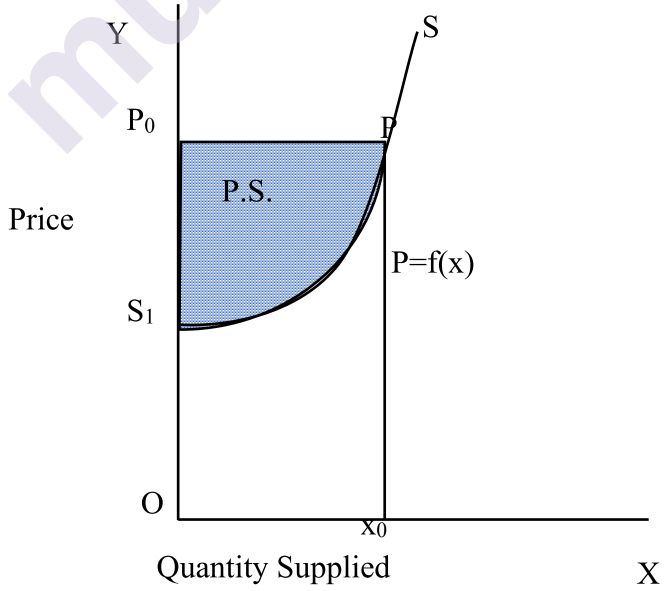
$\int_0^{x_0} f(x)\,dx$ ---------- (2)

So, the difference between the two, It is :known as 'Producer's Surplus'. It is denoted by PS.

Producer's surplus (P.S.) = $P_0\, x_0 - \int_0^{x_0} f(x)\,dx$

The producer's surplus has been shown in the below diagram.

Figure No. 8.
Producer's Surplus.



In the above diagram, quantity supplied shown on x axis and price shown on the y axis. SS is market supply curve x_0 corresponds to P_0. OP_0 is the market price. OS_1 is the producer's willingness to sell or producer's willingness price. Thus, Producer's Surplus = Area of the whole rectangular P_0Ox_0P – Area under the supply curve (P_0S_1P)

$= P_0\, x_0 – f(x)\, dx$

$= P_0\, x_0 - \int_0^{x_0} P\, dx$

This difference is called producer's surplus.

Examples-

1) If, supply function of the commodity $P = 2 + D^2$. Then find the producer's surplus when price is 18.

Solution

$P = 2 + D^2$.

By substituting the value of P (18) in the above supply function.

$18=2+D^2$

$2+D^2 = 18$

$D^2 = 18 – 2$

$D^2 = 16$

$D = \sqrt{16}$

$D= \pm 4$

But, the supply can not be negative,

so D=4

$P_0 = 18$ and $D_0 = 4$

producer's surplus $= P_0\, D_0 - \int_0^{D_0} P\, dD$

$= 18 \times 4 - \int_0^4 (2 + D^2)\, dD$

$= 72 - \left[2D + \frac{D^3}{3}\right]_0^4$

$= 72 - \left[2(4) + \frac{(4)^3}{3}\right]$

$= 72 - \left[8 + \frac{64}{3}\right]$

$= 72 – 29.3 = 42.7$

producer's surplus = 42.7

2) When the supply function $P = 2x^2 + x + 5$ where p = price and x = quantity supplied; find the producer's surplus if $x_0=3$.

Solution

$p=2x^2 + x + 5$

By substituting the value of x_0 ($x_0=3$) in the place of x in the supply function.

$P = 2(3)^2 + 3 + 5$

$P = 18 + 3 + 5$

$P=26$

Thus, $x_0=3$ and $P_0 = 26$

Producer's surplus (P.S.) = $P_0\, x_0 - \int_0^{x_0} P\, dx$

$= 26 \times 3 - [2x^2 + x + 5]_0^3$

$= 78 - \left[\frac{2x^3}{3} + \frac{x^2}{2} + 5x\right]_0^3$

$= 78 - \left[\frac{2(3)^3}{3} + \frac{(3)^2}{2} + 5(3)\right]$

$= 78 - \left[\frac{2 x\, 27}{3} + \frac{9}{2} + 15\right]$

$= 78 - (2x9 + 4.5 + 15)$

$= 78 - (18 + 19.5)$

$=78 - 37.5$

P.S. $= 40.5$

Thus, consumer's surplus is 40.5

Exercise:

1) If $p= 4 + x + x^2$, is a supply function for a commodity. Then find the producer's surplus if the price is 6.

2) If, supply function of a commodity is $P = 12 + x^3 + 2x^2$, find the producer's surplus when $P = 7$.

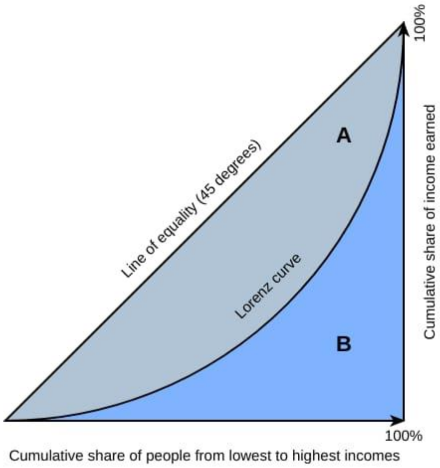
7) Lorenz Curve and Gini Coefficient :

Lorenz curve is a curve which indicates the graphical representation of income or wealth distribution among a population.

Lorenz curve is shown in the below. figure.

Figure No.4.4

Lorenz Curve



Lorenz curve is represented by a convex curve.

More convex Lorenz curve → more inequality in income distribution.

45^0 line is a line of equality.

The area between the 45^0 line and Lorenz curve is used as a measure of inequality. This measure is known as Gini Coefficient which is denoted by G.

Gini Coefficient is measured by below formula.

$$G= \frac{A}{A+B} = 2\int_0^1 [x - L(x)]\, dx$$

The Gini Coefficient is the relative measure of inequality. The value of Gini Coefficient lies between 0 to 1. 0 Gini Coefficient represents perfect equality and 1 Gini Coefficient represents perfect inequality.

Examples :

1) Calculate the Gini Coefficient for the Lorenz function, $L(x) = x^3$

Solution –

$$G= \frac{A}{A+B} = 2\int_0^1 [x - L(x)]\, dx$$

$$= 2\int_0^1 [x - x^3]\, dx$$

$$= 2\left[\frac{x^2}{2} + \frac{x^4}{4}\right]_0^1$$

$$= 2\left[\frac{1^2}{2} + \frac{1^4}{4}\right] = 2\left(\frac{1}{2} + \frac{1}{4}\right)$$

$$= 2\left(\frac{4-2}{2 x 4}\right) = = 2\left(\frac{2}{8}\right)$$

$$=2\frac{1}{4} = \frac{2}{4} = \frac{1}{4}$$

G = 0.25

Exercise :

i) Calculate the Gini Coefficient for the Lorenz Function, $L(x) = 5x^2 + 6x$

ii) Calculate the Gini Coefficient for the Lorenz function, $L(x) = 7x^2 + x$

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4.5 SUMMARY

- Partial derivative of the function of several variables is the derivative of that variable with respect to one variable out of several variables and other variables held constant. Partial derivative is generally denoted by $\frac{\partial y}{\partial x}$
- Producer's surplus (P.S.) = $P_0 x_0 - \int_0^{x_0} f(x)\,dx$
- Consumer's surplus(C.S.) = $\int_0^{x_0} f(x)dx - P_0 x_0$
- Differentiation or derivatives used to find marginal function from total function.
- Integration used to find total function from marginal function. Integration Is the reciprocal concept of differentiation.

4.6 QUESTIONS

Q1. Explain the consumer's surplus with the help of diagram and example.

Q2. Explain the producer's surplus with the help of diagram and example.

Q3. What is the partial derivative? Give any two applications of partial derivative in economics.

Q4. What is the meaning of integration? Give any two applications of integration in economics.

4.7 REFERENCES

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5

CONSTRAINED OPTIMISATION IN ECONOMICS - I

Unit Structure

5.0 OBJECTIVES

- To know the Concept of constraint optimisation in Economics
- To know the Substitution method and their example to solving economic problems.
- To know the Lagrange multiplier and equality constraint and their example to solving economic problems.

5.1 INTORDUCTION

Managers of the firm have to take decision regarding the production level of a product to be produced, the price at which the product is to be sold, the number of people to be employed in the sales force, the production technique to be used, and the amount to be spend on advertising and many other such things. In business decision making a manager faces a problem of choice where he has to choose the best among the various options available to him taking in to account his limited resources. The best choice helps the firm in achieving its desired objective or goal of the firm. For example, a manager might take into account the output level of the product. Here he manager will decide to produce that level of output which would maximize the profit of the firm, if his objective is profit maximization. Thus, the manager's aim here is to solve the profit maximization problem. Here, the manager will have to consider choosing various combinations of inputs used in the production process used to achieve that output level. Here in order to maximize profit he will have to choose those combinations of inputs which would minimize the cost of production for that level of output. So apparently he will have to solve minimization problems. Over here we see that in all such cases when the individuals face constraint in their decision making process of either to maximize or to minimize their objective functions then they have a problem of constrained optimization.

In all these cases when individuals face constraints in their decision making to maximise or minimise their objective functions we have a problem of constrained optimisation.

5.2 CONSTRAINED OPTIMISATION IN ECONOMICS

Importance of constrained optimization can be seen both in economics as well as in our real life. If we have a look at the economic aspect, the consumers try to maximize their utility which is subject to a number of constraints, and the most significant among them being the budget constraint. In reality, even the big business tycoons like Bill Gates, Mukesh Ambani etc. cannot also get whatever they want. The same situation exists in business too and whiles either maximizing profits or minimizing cost, the entrepreneurs face lot of challenges or constraints which can be both economic as well as natural in their real lives similarly, while maximizing profit or minimizing costs, the producers face several economic constraints in real life, for examples, resource constraints, production constraints, etc.

Thus, many economic decisions are face to optimization problem subject to at least one or a series of constraints:

- Consumers make decisions on what to purchase constrained by their choice and budget must be affordable.
- Firms make production decisions to maximize their profits subject to the constraint that they need limited production capacity.
- Households make decisions on how much to work or play with the constraint that there are only few hours in the day.
- The cost reduction done by the firms are subject to the constraint that they have orders to complete.

All of these problems can be categorized as constrained optimization. Fortunately, there is a same process that we can use to solve these problems.

When we talk about constrained optimization in mathematical optimization it refers to the process of optimizing an objective function with respect to some variables in the presence of constraints on those variables. The minimization of either cost function or energy function is the objective function; on the other hand it can also be reward function or utility functions which have to be maximized. Constraints can be either hard constraints or soft constraints, hard constraints are which set conditions for the variables that are required to be satisfied, or soft constraints, which have some variable values that are penalized in the objective function if, and based on the extent that, the conditions on the variables are not satisfied.

A general constrained minimization problem may be written as follows:

min f (x)

subject to $g_i(x) = c_i$ for i = 1, 2, 3 ………..,n Equality constraint

$h_j(x) \geq d_j$ for j = 1, 2, 3 ………..,m Inequality constraint

Where, $g_i(x) = c_i$ for i = 1,2 ……..n and $h_j(x) \geq d_j$ for j = 1,2 ……… m are constraints that are required to be satisfied (these are called hard constraint) and
f (x) is the objective function that needs to be optimized subject to the constraint.

In some problems, which is called constraint optimization problems, the objective function is actually the sum of cost functions, each of which penalizes the extent (if any) to which a soft constraint (a constraint which is preferred but not essentially to be satisfied) is violated.

5.2.1 Maximizing Subject to a set of constraints:

$$\underset{x,\,y}{\text{Max.}}\ f(x, y)$$

$$\text{subject to } g(x,y) \geq 0$$

Step I: Set up the problem

Here comes the challenging part, we always want that the problem should be structured or designed in a particular manner. Now, here we have decided to maximize f (x, y) by choice of x and y. The function g (x, y) represents a restriction or a series of restrictions on our actions.

$$\mathcal{L}(x,y) = f(x, y) + \lambda g(x, y)$$

For example, a common economic problem is the consumer choice problem. Households are selecting consumption of various goods. However, consumers are not allowed to expenditure more than their income. Let's set up the consumer's problem:

Suppose that consumers are choosing between Mango (A) and Bananas (B). We have a utility function that describes levels of utility for every combination of Mango and Bananas.

$A^{1/2} B^{1/2}$ = Wellbeing from consuming (A) Mango and (B) Bananas.

Next we need set prices. Suppose that Mango cost Rs.4 apiece and Bananas cost Rs.2 apiece. Further, assume that this consumer has Rs.120 available to spend. They the income constraint is

$$\text{Rs.}2B + \text{Rs.}4A \leq \text{Rs.}120$$

However, they problem needs that the constraint be in the form $g(x,y) \geq 0$.

In the above expression, subtract Rs.2B and Rs.4A from both sides.

Now we have

$$g(x,y) \geq 0$$

$$(\text{Rs.}120 - \text{Rs.}2B - \text{Rs.}4A) \geq 0$$

Now, we can write out the lagrangian

$$\mathcal{L}(x,y) = f(x, y) + \lambda g(x, y)$$

$$\boldsymbol{\mathcal{L}(A,B) = A^{1/2} B^{1/2} + \lambda(\text{Rs.}120 - \text{Rs.}2B - \text{Rs.}4A)}$$

Step II: Taking the partial derivative with reference to each variable

We have a function of two variables which have to maximize. Hence due to this, there will be two first order conditions (two partial derivatives that are set equal to zero).

In this case, our function is:

$$\mathcal{L}(A,B) = A^{1/2} B^{1/2} + \lambda(\text{Rs.}120 - \text{Rs.}2B - \text{Rs.}4A)$$

Taking the derivative with respect to A (treating B as a constant) and then take the derivative with respect to B (treating A as a constant).

$$\mathcal{L}(A, B) = ½ A^{½} B^{½} - 4\lambda = 0$$

$$\mathcal{L}(A,B) = ½ A^{½} B^{½} - 2\lambda = 0$$

III: Solve the First order conditions for lambda

If we solve the above two equations for λ we get

$$\lambda = \frac{A^{½} B^{½}}{8}$$

$$\lambda = \frac{A^{½} B^{½}}{4}$$

IV: Set the two terms for lambda equal to each other

$$\frac{A^{½} B^{½}}{8} = \frac{A^{½} B^{½}}{4}$$

If we simplify this down a bit:

$$\frac{A^{½} B^{½}}{8} = \frac{A^{½} B^{½}}{4} \quad \text{(Multiply both sides by 8)}$$

$$A^{½} B^{½} = \frac{8A^{½} B^{½}}{4} \quad \text{(Divide both sides by } B^{½}\text{)}$$

$$A^{½} B = \frac{8A^{½}}{4} \quad \text{(Divide both sides by } A^{½}\text{)}$$

$$B = \frac{8A}{4} \quad \text{(Simplify)}$$

$$\mathbf{B = 2A}$$

This explains us that if we are acting optimally, we should always buy twice as many bananas as mangoes (which make sense because they cost twice as much!). At this step, we should always have an equation that relates one variable to the other.

V: Solve for the two variables separately using the constraint

Now, we can notice that the income constraint will always be met with equality (utility always increases as we consume more and more). Therefore, we know

$$2B + 4A = 120$$

These can be used by us to solve the rest of the problem.

$B = 2A, \quad 2B + 4A = 120$

$$2(2A) + 4A = 120 \Rightarrow 8A = 120 \Rightarrow A = 15$$

We now know that we will buy 15 Mango and 30 Bananas ($B = 2A$). Notice that the income constraint is satisfied.

Rs. 2(30) + Rs. 4(15) = Rs. 120

5.2.2 Minimizing Subject to a set of constraints:

$$\min_{x,\,y}\; f(x, y)$$

$$\text{Subject to } g(x, y) \geq 0$$

Step I: Set up the problem

It basically works in the same way as the problem above. In this case here, we choose to minimize $f(x, y)$ by choice of x and y. Over here the function $g(x, y)$ represents a restriction or a series of restrictions on our possible actions.

The setup for this problem is written as

$$\mathcal{L}(x,y) = f(x, y) - \lambda g(x, y)$$

Over here it can be seen that there is an identical setup except that the second term in the above expression is being subtracted rather than being added.

Normally the problem which any firm faces is that as it tries to minimize cost it also has to see that a certain level of output is maintained.

Let us suppose that a firm chooses a certain level of labour and capital (L and K). The output is produced according to the following process.

$K^{\frac{1}{2}} L^{\frac{1}{2}}$ = Firm Output (I chose the same function as above to simplify things)

Now the next step is to set the prices. Let us suppose that the per unit cost of capital is Rs.3 and per hour cost of labour is Rs.9. So now we can write the total cost for the firm as a total of capital costs and labour cost.

Total Costs = Rs. 3K + Rs. 9L

The firm wants to minimize the total costs of producing (at least) 100 units of output.

$K^{1/2} L^{1/2} \geq 100$

Therefore, the problem we face is

min. (3K + 9L)

x, y

Subject to $K^{1/2} L^{1/2} \geq 100$

Now again, the requirement of the problem is that the constraint be in the form of g(x, y)) ≥ 0 in the above expression, subtracting 100 from both the sides we get.

$$g(K,L) \geq 0$$

$$K^{1/2} L^{1/2} - 100 \geq 0$$

We can write the problem as

$$\mathbf{\mathcal{L}(K, L) = 3K + 9L - \lambda(K^{1/2} L^{1/2} - 100)}$$

Over here it can be seen that there is an identical setup except that the second term in the above expression is being subtracted rather than being added.

Step II: Take the partial derivative with respect to each variable

We have a function of two variables that we wish to minimize. So, there will be two first order conditions (two partial derivatives that are set equal to zero).

In this case, our function is

$$\mathcal{L}(K, L) = 3K + 9L - \lambda(K^{1/2} L^{1/2} - 100)$$

Take the derivative with respect to A (treating B as a constant) and then take the derivative with respect to B (treating A as a constant).

$$\mathcal{L}_K(K, L) = 3 - \lambda(1/2\, K^{1/2} L^{1/2}) = 0$$

$$\mathcal{L}_L(K, L) = 9 - \lambda(1/2\, K^{1/2} L^{1/2}) = 0$$

III: Solve the First order conditions for lambda

If we solve the above two equations for λ we get

$$\lambda = \frac{3}{\frac{1}{2} K^{\frac{1}{2}} L^{\frac{1}{2}}}$$

$$\lambda = \frac{9}{\frac{1}{2} K^{\frac{1}{2}} L^{\frac{1}{2}}}$$

IV: Set the two expressions for lambda equal to each other

$$\frac{3}{\frac{1}{2} K^{\frac{1}{2}} L^{\frac{1}{2}}} = \frac{9}{\frac{1}{2} K^{\frac{1}{2}} L^{\frac{1}{2}}}$$

If we simplify this down a bit:

$$\frac{3}{\frac{1}{2} K^{\frac{1}{2}} L^{\frac{1}{2}}} = \frac{9}{\frac{1}{2} K^{\frac{1}{2}} L^{\frac{1}{2}}}$$ (Cancel out the 1/2s and divide both sides by 3)

$$\frac{1}{K^{\frac{1}{2}} L^{\frac{1}{2}}} = \frac{9}{3 K^{\frac{1}{2}} L^{\frac{1}{2}}}$$ (Multiply both sides by $K^{\frac{1}{2}}$)

$$\frac{K}{L^{\frac{1}{2}}} = \frac{9}{3 L^{\frac{1}{2}}}$$ (Multiply both sides by $L^{\frac{1}{2}}$)

$$\frac{K}{L} = \frac{9}{3}$$ (Simplify)

$$\frac{K}{L} = 3$$

This tells us that if we are acting optimally, we should always employ three times as much capital as labour (which makes sense because labour costs three times as much!). At this step, we should always have an expression that relates one variable to the other.

V: Use the constraint to solve for the two variables separately

Next, notice that the production constraint will always be met with equality (your costs will always go down if you produce less). Therefore, we know

$$K^{\frac{1}{2}} L^{\frac{1}{2}} = 100$$

We can use these to solve the rest of the problem.

$K = 3L$, $K^{½} L^{½} = 100$

$$\mathbf{(3L)^{½} L^{½} = 100 \Rightarrow L = 100 / \sqrt{3} = 57.8}$$

Therefore, L = 57.8, K = 3L = 173.41, and (if you check, total production does equal 100).

In all these cases when individuals face constraints in their decision making to maximise or minimise their objective function we have a problem of constrained optimisation. It is worth nothing that the existence of constraints prevents the achievement of the unconstrained optimal. There are two techniques of solving the constrained optimisation problem. They are:

1. Substitution Method
2. Lagrangian Multiplier technique

5.3 CONSTRAINED OPTIMISATION: SUBSTITUTION METHOD

Substitution method to solve constrained optimisation problem is used when equation is simple and not too complex. For example substitution method to maximise or minimise the objective function is used when it is subject to only one constraint equation of a very simple nature. In this method, we solve the constraint equation for one of the decision variable and substitute that variable in the objective function that is to be maximised or minimised. In this way this method converts the constrained optimisation problem into one of unconstrained optimisation problems of maximisation or minimisation.

An example will clarify the use of substitution method to solve constrained optimisation problem. Suppose a manager of a firm which is producing two products x and y, seeks to maximise total profits function which is given by the following equation

$$\pi = 50x - 2x^2 - xy - 3y^2 + 95y$$

Where, x and y represent the quantities of the two products. The manager of the firm faces the constraints that the total output of the two products must be equal to 25. That is, according to the constraint,

$$x + y = 25$$

To solve the constrained optimisation problem through substitution we first solve the constraint equation for x. Thus,

$$x = 25 - y$$

The next step in the substitution method is to substitute this value of x = 25 – y in the objective function (i.e. the given profit function) which has to be maximised. Thus substituting the constrained value of x in the profit function we have

$$\pi = 50(25 - y) - 2(25 - y)^2 - (25 - y)\,y - 3y^2 + 95y$$
$$= 1250 - 50y - 2(625 - 50y - y^2) - 25y + y^2 - 3y^2 + 95y$$
$$= 1250 - 50y - 1250 + 100y - 2y^2 - 25y + y^2 - 3y^2 + 95y$$
$$= 120y - 4y^2$$

To maximise the above profit function converted into the above unconstrained form we differentiate it with respect to y and set it equal to zero and solve for y. thus,

$$\frac{d\pi}{dy} = 120 - 8y = 0$$
$$8y = 120$$
$$Y = 15$$

Substituting y = 15 in the given constraint equation (x + y = 25) we have

$$x + 15 = 25$$
$$x = 25 - 15 = 10$$

Thus, the given constraint, profit will be maximised if the manager of the firm decides to produce 10 units of the product x and 15 units of the product y. We can find the total profits in this constrained optimum situation by substituting the values of x and y obtained in the given profit function. Thus,

$$\pi = 50x - 2x^2 - xy - 3y^2 + 95y$$
$$= 50(10) - 2(10)^2 - (10)(15) - 3(15)^2 + 95(15)$$
$$= 500 - 200 - 150 - 675 + 1425$$
$$= 1925 - 1025 = \mathbf{900}$$

5.4 LAGRANGE MULTIPLIER AND EQUALITY CONSTRAINTS

A typical economic example of a constrained optimization problem concerns a consumer who chooses how much of the available income m to spend on a good x whose price is p, and how much income to leave over for expenditure y on other goods. Note that the consumer then faces the budget constraint px + y = m. Suppose that preferences are represented by the utility function u(x, y). In mathematical terms the consumer's problem can be expressed as

$$\max u(x, y) \text{ subject to } px + y = m$$

When, however, the constraint involves a complicated function, or there are several equality constraints to consider, this substitution method might be difficult or even impossible to carry out in practice. In such cases, economists make much use of the Lagrange multiplier method. It is discovered by the Italian-born French mathematician J. L. Lagrange (1736–1813). The Danish economist Harald Westergaard seems to be the first who used it in economics, in 1876. Actually, this method is often used even for problems that are quite easy to express as unconstrained problems. One reason is that Lagrange multipliers have an important economic interpretation. In addition, a similar method works for many more complicated optimization problems, such as those where the constraints are expressed in terms of inequalities. All the following problems have only one solution candidate, which is the optimal solution.

We start with the problem of maximizing (or minimizing) a function $f(x, y)$ of two variables, when x and y are restricted to satisfy an equality constraint $g(x, y) = c$. This can be written as

$$\text{Max.(min) } f(x, y) \text{ subject to } g(x, y) = c \tag{1}$$

The first step of the method is to introduce a **Lagrange multiplier**, often denoted by λ, which is associated with the constraint $g(x, y) = c$. Then we define the **Lagrangian** $\mathscr{L}$ by

$$\mathscr{L}(x, y) = f(x, y) - \lambda(g(x, y) - c) \tag{2}$$

In which the expression $g(x, y) - c$, which must be 0 when the constraint is satisfied, has been multiplied by λ. Note that $\mathscr{L}(x, y) = f(x, y)$ for all (x, y) that satisfy the constraint $g(x, y) = c$.

The Lagrange multiplier λ is a constant, so the partial derivatives of $\mathscr{L}(x, y)$ w.r.t. x and y are $\mathscr{L}'_1(x, y) = f'_1(x, y) - \lambda g'_1(x, y)$ and $\mathscr{L}'_2(x, y) = f'_2(x, y) - \lambda g'_2(x, y)$, respectively. As will be explained algebraically and geometrically except in rare cases a solution of problem (1) can only be a point (x, y) where, for a suitable value of λ, the first-order partial derivatives of $\mathscr{L}$ vanish, and also the constraint $g(x, y) = c$ is satisfied.

Here is a simple economic application.

EXAMPLE 1

A consumer has the utility function $U(x, y) = xy$ and faces the budget constraint $2x + y = 100$. Find the only solution candidate to the consumer demand problem

$$\text{Maximize } xy \text{ subject to } 2x + y = 100.$$

Solution: The Lagrangian is $\mathscr{L}(x, y) = xy - \lambda(2x + y - 100)$. Including the constraint, the first-order conditions for the solution of the problem are

$$\mathscr{L}'_1(x, y) = y - 2\lambda = 0, \qquad \mathscr{L}'_2(x, y) = x - \lambda = 0, \qquad 2x + y = 100$$

The first two equations imply that $y = 2\lambda$ and $x = \lambda$. So $y = 2x$. Inserting this into the constraint yields $2x + 2x = 100$. So $x = 25$ and $y = 50$, implying that $\lambda = x = 25$.

This solution can be confirmed by the **substitution method**. From $2x + y = 100$ we get $y = 100 - 2x$, so the problem is reduced to maximizing the unconstrained function $h(x) = x(100 - 2x) = -2x2 + 100x$. Since $h'(x) = -4x + 100 = 0$ gives $x = 25$, and $h''(x) = -4 < 0$ for all x, this shows that $x = 25$ is a maximum point.

5.4.1 The Lagrange Multiplier Method

To find the only possible solutions of the problem maximize (minimize) $f(x, y)$ subject to $g(x, y) = c$

Proceed as follows:

1. Write down the Lagrangian

$$\mathcal{L}(x, y) = f(x, y) - \lambda(g(x, y) - c)$$

Where λ is a constant.

2. Differentiate $\mathcal{L}$ w.r.t. x and y, and equate the partial derivatives to 0.

3. The two equations in (II), together with the constraint, yield the following three equations:

$$\mathcal{L}'_1(x, y) = f'_1(x, y) - \lambda g'_1(x, y) = 0$$

$$\mathcal{L}'_2(x, y) = f'_2(x, y) - \lambda g'_2(x, y) = 0$$

$$g(x, y) = c$$

4. Solve these three equations simultaneously for the three unknowns x, y, and λ. These triples (x, y, λ) are the solution candidates, at least one of which solves the problem (if it has a solution).

The conditions in (III) are called the first-order conditions for problem (1).

NOTE 1 Some economists prefer to consider the Lagrangian as a function $L(x, y, \lambda)$ of three variables. Then the first-order condition $\partial L/\partial\lambda = -(g(x, y) - c) = 0$ yields the constraint. In this way all the three necessary conditions are obtained by equating the partial derivatives of the (extended) Lagrangian to 0. However, it does seem somewhat unnatural to perform a differentiation to get an obvious necessary condition, namely the constraint equation. Also, this procedure can easily lead to trouble when treating problems with inequality constraints, so we prefer to avoid it.

EXAMPLE 2

A single-product firm intends to produce 30 units of output as cheaply as possible. By using K units of capital and L units of labour, it can produce $\sqrt{K} + L$ units. Suppose the prices of capital and labour are, respectively, 1 and 20. The firm's problem is then:

$$\text{minimize } K + 20L \text{ subject to } \sqrt{K} + L = 30$$

Find the optimal choices of K and L.

Solution: The Lagrangian is $\mathscr{L} = K+20L-\lambda(\sqrt{K}+L-30)$, so the first-order conditions are:

$$\mathscr{L}'_K = 1 - \lambda/2\sqrt{K} = 0, \qquad \mathscr{L}'_L = 20 - \lambda = 0, \qquad \sqrt{K} + L = 30$$

The second equation gives $\lambda = 20$, which inserted into the first equation yields $1 = 20/2\sqrt{K}$. It follows that $\sqrt{K} = 10$, and hence $K = 100$. Inserted into the constraint this gives $\sqrt{100} + L = 30$, and hence $L = 20$. The 30 units are therefore produced in the cheapest way when the firm uses 100 units of capital and 20 units of labour. The associated cost is $K + 20L = 500$.

An economist would be inclined to ask: What is the additional cost of producing 31 rather than 30 units? Solving the problem with the constraint $\sqrt{K} + L = 31$, we see that still $\lambda = 20$ and $K = 100$, while $L = 31 - 10 = 21$. The associated minimum cost is $100 + 20*21 = 520$, so the additional cost is $520 - 500 = 20$. This is precisely equal to the Lagrange multiplier! Thus, in this case the Lagrange multiplier tells us by how much costs increase if the production requirement is increased by one unit from 30 to 31.

EXAMPLE 3

A consumer who has Cobb–Douglas utility function $U(x, y) = A\,x^a\,y^b$ faces the budget constraint $px + qy = m$, where A, a, b, p, q, and m are all positive constants. Find the only solution candidate to the consumer demand problem

$$\max A\,x^a\,y^b \text{ subject to } px + qy = m \tag{1}$$

Solution: The Lagrangian is $L(x, y) = A\,x^a\,y^b - \lambda(px + qy - m)$, so the first-order conditions are

$\mathscr{L}'_1(x, y) = aAx^{a-1}\,y^b - \lambda p = 0, \quad \mathscr{L}'_2(x, y) = bAx^a\,y^{b-1} - \lambda q = 0, \quad px+ qy = m$

Solving the first two equations for λ yields

$$\lambda = aAx^{a-1}y^b/p = bAx^ay^{b-1}/q$$

Cancelling the common factor $Ax^{a-1}y^{b-1}$ from the last two fractions gives

$$ay/p = bx/q$$

Solving this equation for qy yields qy = (b/a)px, which inserted into the budget constraint gives px + (b/a)px = m. From this equation we find x and then y.

The results are the following demand functions:

$$x = x(p, q, m) = \frac{a}{a+b}\frac{m}{p}, \qquad y = y(p, q, m) = \frac{b}{a+b}\frac{m}{q} \qquad (2)$$

(It follows from (2) that for all t one has x(tp, tq, tm) = x(p, q, m) and y(tp, tq, tm) = y(p, q, m), so the demand functions are homogeneous of degree 0. This is as one should expect because, if (p, q, m) is changed to (tp, tq, tm), then the constraint in (1) is unchanged, and so the optimal choices of x and y are unchanged.

The solution we have found makes good sense. In the utility function Axayb, the relative sizes of the coefficients a and b indicate the relative importance of x and y in the individual's preferences. For instance, if a is larger than b, then the consumer values a 1% increase in x more than a 1% increase in y. The product px is the amount spent on the first good, and (2) says that the consumer should spend the fraction a/(a+b) of income on this good and the fraction b/(a + b) on the second good.

Formula (2) can be applied immediately to find the correct answer to thousands of exam problems in mathematical economics courses given each year all over the world! But note that the utility function has to be of the Cobb–Douglas type Ax^ay^b. For the problem max $x^a + y^b$ subject to px + qy = m, the solution is not given by (2).

EXAMPLE 4

Examine the general utility maximizing problem with two goods:

$$\text{maximize } u(x, y) \text{ subject to } px + qy = m \qquad (3)$$

Solution: The Lagrangian is L(x, y) = u(x, y) − λ(px + qy − m), so the first-order conditions are:

$$\mathscr{L}'_x(x, y) = u'_x(x, y) - \lambda p = 0 \qquad \text{(i)}$$

$$\mathscr{L}'_y(x, y) = u'_y(x, y) - \lambda q = 0 \qquad \text{(ii)}$$

$$px + qy = m \qquad \text{(iii)}$$

From equation (i) we get $\lambda = u'_x(x, y)/p$, and from (ii), $\lambda = u'_y(x, y)/q$. Hence,

$$\frac{u'_x(x, y)}{p} = \frac{u'_y(x, y)}{q}, \text{ which we can rewritten as } \frac{u'_x(x, y)}{u'_y(x, y)} = \frac{P}{q} \qquad (4)$$

The left-hand side of the last equation is the *marginal rate of substitution* (MRS).

Utility maximization thus requires equating the MRS to the price ratio p/q.

A geometric interpretation of (4) is that the consumer should choose the point on the budget line at which the slope of the level curve of the utility function, $-u'_x(x,y)/u'_y(x, y)$, is equal to the slope of the budget line, $-p/q$. (See Section 12.3.) Thus at the optimal point the budget line is tangent to a level curve of the utility function, illustrated by point P in Figure 1. The level curves of the utility function are the *indifference curves*, along which the utility level is constant by definition. Thus, utility is maximized at a point where the budget line is tangent to an indifference curve. The fact that $\lambda = u'_x(x, y)/p = u'_y(x, y)/q$ at point P means that the marginal utility per dollar is the same for both goods. At any other point (x, y) where, for example, $u'_x(x, y)/p > u'_y(x, y)/q$, the consumer can increase utility by shifting expenditure away from y toward x. Indeed, then the increase in utility per extra dollar spent on x would equal $u'_x(x, y)/p$; this exceeds the decrease in utility per dollar reduction in the amount spent on y, which equals $u'_y(x, y)/q$.

As in Example 3, the optimal choices of x and y can be expressed as **demand functions** of (p, q, m), which must be homogeneous of degree zero in the three variables together.

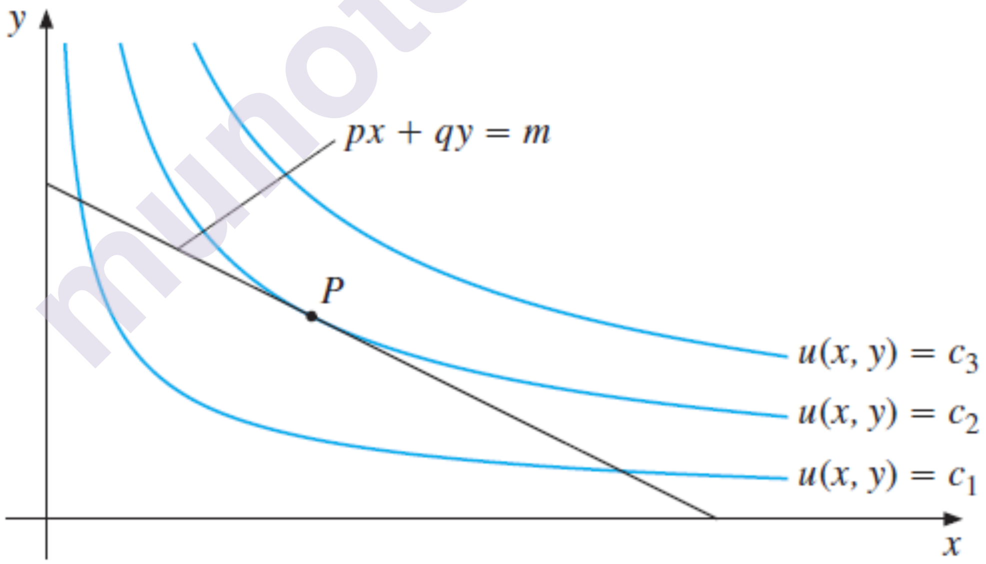

Figure 1 Assuming that $c_1 < c_2 < c_3 < \cdots$, the solution to problem (3) is at P.

PROBLEMS FOR SECTION

1. (a) Use Lagrange's method to find the only possible solution to the problem:

   max xy subject to $x + 3y = 24$

   (b) Check the solution by using the results in Example 3.

2. Use the Lagrange's method to solve the problem

   $\min -40Q_1 + Q_1^2 - 2Q_1Q_2 - 20Q^2 + Q_2^2$ subject to $Q_1 + Q_2 = 15$

3. Solve the following problems:

   (a) min $f(x, y) = x^2 + y^2$ subject to $g(x, y) = x + 2y = 4$

   (b) min $f(x, y) = x^2 + 2y^2$ subject to $g(x, y) = x + y = 12$

   (c) max $f(x, y) = x^2 + 3xy + y^2$ subject to $g(x, y) = x + y = 100$

4. A person has utility function

   $u(x, y) = 100xy + x + 2y$

Suppose that the price per unit of x is \$2, and that the price per unit of y is \$4. The person receives \$1000 that all has to be spent on the two commodities x and y. Solve the utility maximization problem.

5.4.2 Interpreting the Lagrange Multiplier

Consider again the problem

$$\max (\min) f(x, y) \text{ subject to } g(x, y) = c$$

Suppose x^* and y^* are the values of x and y that solve this problem. In general, x^* and y^* depend on c. We assume that $x^* = x^*(c)$ and $y^* = y^*(c)$ are differentiable functions of c. The associated value of $f(x, y)$ is then also a function of c, with

$$f^*(c) = f(x^*(c), y^*(c)) \qquad (1)$$

Here $f^*(c)$ is called the (optimal) value function for the problem. Of course, the associated value of the Lagrange multiplier also depends on c, in general. Provided that certain regularity conditions are satisfied, we have the remarkable result that

$$\frac{df*(c)}{dc} = \lambda(c) \qquad (2)$$

Thus, the Lagrange multiplier $\lambda = \lambda(c)$ is the rate at which the optimal value of the objective function changes with respect to changes in the constraint constant c.

In particular, if dc is a small change in c, then

$$f^*(c + dc) - f^*(c) \approx \lambda(c)\, dc \qquad (3)$$

In economic applications, c often denotes the available stock of some resource, and f (x, y) denotes utility or profit. Then $\lambda(c)$ dc measures the approximate change in utility or profit that can be obtained from dc units more (or −dc less, when dc < 0). Economists call λ a **shadow price** of the resource. If $f^*(c)$ is the maximum profit when the resource input is c, then (3) says that λ indicates the approximate increase in profit per unit increase in the resource.

EXAMPLE 1

Consider the following generalization of Example

$$\max xy \text{ subject to } 2x + y = m$$

The first-order conditions again give $y = 2x$ with $\lambda = x$. The constraint now becomes $2x + 2x = m$, so $x = m/4$. In the notation introduced above, the solution is

$$x^*(m) = m/4, \qquad y^*(m) = m/2, \qquad \lambda(m) = m/4$$

The value function is therefore $f^*(m) = (m/4)(m/2) = m^2/8$. It follows that $df^*(m) / dm = m/4 = \lambda(m)$. Hence equ.(2) is confirmed. Suppose in particular that m = 100. Then $f^*(100) = 100^2/8$. What happens to the value function if m = 100 increases by 1?

The new value is $f^*(101) = 101^2/8$, so $f^*(101) - f^*(100) = 101^2/8 - 100^2/8 = 25.125$. Note that formula (3) with dc = 1 gives $f*(101) - f^*(100) \approx \lambda(100) \cdot 1 = 25 \cdot 1 = 25$, which is quite close to the exact value 25.125.

EXAMPLE 2

Suppose Q = F(K, L) denotes the output of a state-owned firm when the input of capital is K and that of labour is L. Suppose the prices of capital and labour are r and w, respectively, and that the firm is given a total budget of m to spend on the two input factors. The firm wishes to find the choice of inputs it can afford that maximizes output. So it faces the problem

$$\max F(K, L) \text{ subject to } rK + wL = m$$

Solving this problem by using Lagrange's method, the value of the Lagrange multiplier will tells us approximately the increase in output if m is increased by 1 dollar.

Consider, for example, the specific problem

$$\max 120KL \text{ subject to } 2K + 5L = m$$

We find the solution

$$K* = 1/4m, \qquad L* = 1 / 10m, \qquad \text{with } \lambda = 6m$$

The optimal output is $Q^*(m) = 120K^*L^* = 120 \ ¼ \ m \ 1/10 \ m = 3m^2$, so $dQ^*/dm = 6m = \lambda$, and equ. (2) is confirmed.

PROBLEMS FOR SECTION

1. Verify that equation (2) holds for the problem: max x3y subject to $2x + 3y = m$.

2. Solve the utility maximization problem

$$\max U(x, y) = \sqrt{x} + y \text{ subject to } x + 4y = 100$$

using the Lagrange method, i.e. find the quantities demanded of the two goods.

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5.4.3 Several Solution Candidates

In all our examples and problems so far, the method for solving constrained optimization problems has produced only one solution candidate. In this section we consider a problem where there are several solution candidates. In such cases we have to decide which of the candidates actually solves the problem, assuming it has any solution at all.

EXAMPLE 1

Solve the problem

$$\max (\min) f(x, y) = x^2 + y^2 \text{ subject to } g(x, y) = x^2 + xy + y^2 = 3$$

Solution: The Lagrangian in this case is $\mathcal{L}(x, y) = x^2 + y^2 - \lambda(x^2 + xy + y^2 - 3)$, and the three equations to consider are

$$\mathcal{L}'_1(x, y) = 2x - \lambda(2x + y) = 0 \qquad \text{(i)}$$

$$\mathcal{L}'_2(x, y) = 2y - \lambda(x + 2y) = 0 \qquad \text{(ii)}$$

$$x^2 + xy + y^2 - 3 = 0 \qquad \text{(iii)}$$

Let us eliminate λ from (i) and (ii). From (i) we get $\lambda = 2x/(2x + y)$ provided $y \neq -2x$. Inserting this value of λ into (ii) gives

$$2y = \frac{2x}{2x+y}(x + 2y), \qquad \text{or} \quad y2 = x^2, \quad \text{and so} \quad y = \pm x$$

Suppose $y = x$. Then (iii) yields $x^2 = 1$, so $x = 1$ or $x = -1$. This gives the two solution candidates $(x, y) = (1, 1)$ and $(-1,-1)$, with $\lambda = 2/3$.

Suppose $y = -x$. Then (iii) yields $x^2 = 3$, so $x = \sqrt{3}$ or $x = -\sqrt{3}$. This gives the two solution candidates $(x, y) = (\sqrt{3},-\sqrt{3})$ and $(-\sqrt{3},\sqrt{3})$, with $\lambda = 2$.

It remains to consider the case $y = -2x$. Then from (i), $x = 0$ and so $y = 0$. But this contradicts (iii), so this case cannot occur.

We have found the only four points (x, y) that can solve the problem. Furthermore,

$$f(1, 1) = f(-1,-1) = 2, \qquad f(\sqrt{3},-\sqrt{3}) = f(-\sqrt{3},\sqrt{3}) = 6$$

We conclude that if the problem has solutions, then $(1, 1)$ and $(-1, -1)$ solve the minimization problem, whereas $(\sqrt{3}, -\sqrt{3})$ and $(-\sqrt{3}, \sqrt{3})$ solve the maximization problem.

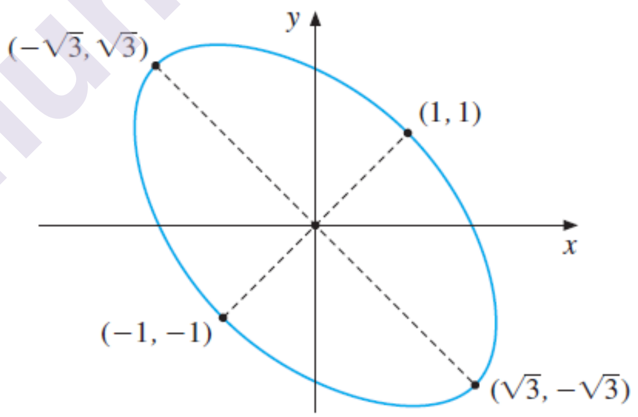

Figure 1 The constraint curve in Example 1

Geometrically, the equality constraint determines an ellipse. The problem is therefore to find what points on the ellipse are nearest to or furthest from the origin. See Fig. 1. It is "geometrically obvious" that such points exist.

Here is an alternative way of proving that $(x, y) = (1, 1)$ minimizes $f(x, y) = x^2 + y^2$ subject to the constraint $x^2 + xy + y^2 = 3$. (The other points can be treated in the same way.) Note, however, that this method works only in special cases.

Let x = 1 + h and y = 1 + k. Thus h and k measure the deviation of x and y, respectively, from 1. Then f (x, y) takes the form

$f(x, y) = (1 + h)^2 + (1 + k)^2 = 2 + 2(h + k) + h^2 + k^2$ (1)

If $(x, y) = (1+h, 1+k)$ satisfies the constraint, then $(1+h)^2 + (1+h)(1+k) + (1+k)^2 = 3$, so $h + k = -hk/3 - (h^2 + k^2)/3$. Inserting this expression for h + k into (1) yields

$f(x, y) = 2 + 2 [-1/3\, hk - 1/3\, (h^2 + k^2)] + h^2 + k^2 = 2 + 1/3\, (h - k)^2$

Because $1/3\, (h-k)^2 \geq 0$ for all (h, k), it follows that $f(x, y) \geq 2$ for all values of (x, y). But because f (1, 1) = 2, this means that (1, 1) really does minimize f (x, y) subject to the constraint.

PROBLEMS FOR SECTION

1. Solve the problem

   (a) max(min) 3xy subject to x2 + y2 = 8

   (b) max(min) x + y subject to x2 + 3xy + 3y2 = 3

2. Solve the problem

   (a) max x2 + y2 − 2x + 1 subject to x2 + 4y2 = 16

   (b) min ln(2 + x2) + y2 subject to x2 + 2y = 2

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5.5 SUMMARY

1. Constrained optimization is the process of optimizing an objective function with respect to some variables in the presence of constraints on those variables like profit maximisation, Cost minimization, production maximisation at minimisation of cost, budget constraint, utility maximisation and so on.
2. Substitution method to solve constrained optimisation problem is used when equation is simple and not too complex. For example substitution method to maximise or minimise the objective function is used when it is subject to only one constraint equation of a very simple nature.
3. In Lagrangian multiplier technique of solving constrained optimisation problem, a combined equation called Lagrangian

function is formed which incorporates both the original objective function and constraint equation.

4. This Lagrangian function is formed in a way which ensures that when it is maximised or minimised, the original given objective function is also maximised or minimised, the original given objective function is also maximised or minimised and at the same time it fulfils all the constraint requirements.

5. In equality constraint, Lagrangian multiplier by the given constraint function having been set equal to zero.

5.6 QUESTIONS

1. Explain Constrained Optimisation Problem in Economics with suitable example.

2. What do you understand by substitution method in constrained optimisation? Explain it with suitable example.

3. Explain the concept of Lagrange multiplier technique to solve the constrained optimisation problem with suitable example.

4. Interpreting the Lagrange Multiplier with suitable example.

5. Explain method for solving constrained optimisation problem where there are several solution candidates with suitable example.

6. A person has utility function

$$u(x, y) = 100xy + x + 2y$$

   Suppose that the price per unit of x is \$2, and that the price per unit of y is \$4. The person receives \$1000 that all has to be spent on the two commodities x and y. Solve the utility maximization problem.

7. Find the solutions to the necessary conditions for the problem
   max(min) f (x, y) = x + y subject to g(x, y) = x2 + y = 1.

8. Solve the problem

max f (x, y) = 24x − x2 + 16y − 2y2 subject to g(x, y) = x2 + 2y2 = 44

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6

CONSTRAINED OPTIMISATION IN ECONOMICS - II

Unit Structure

6.0 OBJECTIVES

- To know the Concept of constraint optimisation with inequality constraints
- To know the multiple inequality constraints method and their example to solving economic problems.
- To know the Non-negativity constraint and their example to solving economic problems.
- To know the applications in Economics

6.1 INTORDUCTION

Many models in economics can be expressed as inequality constrained optimization problems. Problem created in the process of optimum utilization of resources due to the limitation or constraint created by the inequality of the variables is called inequality constraint for example budget constraint can be taken as an universal example. Whenever we plan to achieve a certain goal we may surplus of certain resources while we may face a deficit as far as the availability of other resources are concerned. In order to purchase such scarce resources we have to make use of our financial resources. However, financial resources may not be enough to match the amount of resources we need leading to problem of optimum utilization of existing resources due to inequality in the existing resources. Inequality constraints define the boundary of a region over which we seek to optimize the function. This makes inequality constraints more challenging because we do not know if the maximum/minimum lies along one of the constraints (the constraint binds) or in the interior of the region.

6.2 CONSTRAINED OPTIMISATION WITH INEQUALITY CONSTRAINTS

So far this chapter has considered how to maximize or minimize a function subject to equality constraints. The final two sections concern nonlinear programming problems, which involve *inequality* constraints. Some particularly simple inequality constraints are those requiring certain variables to be nonnegative. These often have to be imposed for the solution to make economic sense. In addition, bounds on resource availability are often expressed as inequalities rather than equalities.

In this section we consider the simple **nonlinear programming problem.**

$$\max f(x, y) \text{ subject to } g(x, y) \leq c \qquad (1)$$

with just one inequality constraint. Thus, we seek the largest value attained by $f(x, y)$ in the **admissible** or **feasible** set S of all pairs (x, y) satisfying $g(x, y) \leq c$.

Problems where one wants to minimize $f(x, y)$ subject to $(x, y) \in S$ can be handled by instead studying the problem of maximizing $-f(x, y)$ subject to $(x, y) \in S$.

Since the 1950s, economists have generally tackled such problems by using an extension of the Lagrangian multiplier method due originally to H.W. Kuhn and A.W. Tucker. To apply their method, we begin by writing down a recipe giving all the points (x, y) that can possibly solve problem (1), except in some bizarre cases. The method closely resembles the one we used to solve the Lagrange problem $\max f(x, y)$ subject to $g(x, y) = c$.

6.2.1 Method for solving problem

Associate a constant Lagrange multiplier λ with the constraint $g(x, y) \leq c$, and define the Lagrangian

$$\mathscr{L}(x, y) = f(x, y) - \lambda [g(x, y) - c]$$

A. Find where $\mathscr{L}(x, y)$ is stationary by equating its partial derivatives to zero:

$$\mathscr{L}_1(x, y) = f'_1(x, y) - \lambda g'_1(x, y) = 0$$

$$\mathscr{L}_2(x, y) = f'_2(x, y) - \lambda g'_2(x, y) = 0 \qquad (2)$$

B. Introduce the complementary slackness condition

$$\lambda \geq 0, \text{ and } \lambda = 0 \text{ if } g(x, y) < c \qquad (3)$$

C. Require (x, y) to satisfy the constraint

$$g(x, y) \leq c \qquad (4)$$

Find all the points (x, y) that, together with associated values of λ, satisfy all the conditions B, C, and D. These are the solution candidates, at least one of which solves the problem (if it has a solution).

Note that the conditions (2) are exactly the same as those used in the Lagrange multiplier method of previous Section. Condition (4) obviously has to be satisfied, so the only new feature is condition (3). In fact, condition (3) is rather tricky. It requires that λ is nonnegative, and moreover that $\lambda = 0$ if $g(x, y) < c$. Thus, if $\lambda > 0$, we must have $g(x, y) = c$. An alternative formulation of this condition is that

$$\lambda \geq 0, \qquad \lambda \cdot [g(x, y) - c] = 0 \tag{5}$$

Later we shall see that even in nonlinear programming, the Lagrange multiplier λ can be interpreted as a "price" per unit associated with increasing the right-hand side c of the "resource constraint" $g(x, y) \leq c$. With this interpretation, prices are nonnegative, and if the resource constraint is not binding because $g(x, y) < c$ at the optimum, this means that the price associated with increasing c by one unit is 0.

The two inequalities $\lambda \geq 0$ and $g(x, y) \leq c$ are complementary in the sense that at most one can be "slack"—that is, at most one can hold with inequality. Equivalently, at least one must be equality.

Parts B and C of the above rule are together called the **Kuhn–Tucker conditions**. Note that these are (essentially) *necessary* conditions for the solution of problem (1). In general, they are far from sufficient. Indeed, suppose one can find a point (x_0, y_0) at which f is stationary and $g(x_0, y_0) < c$. Then the Kuhn–Tucker conditions will automatically be satisfied by (x_0, y_0) together with the Lagrange multiplier $\lambda = 0$. Yet then (x_0, y_0) could be a local or global minimum or maximum, or a saddle point.

NOTE 1 We say that conditions B and C are essentially necessary because the Kuhn–Tucker conditions may not hold for some rather rare constrained optimization problems that fail to satisfy a special technical condition called the "constraint qualification".

NOTE 2

With equality constraints, setting the partial derivative $\partial\mathscr{L}/\partial\lambda$ equal to zero just recovers the constraint $g(x, y) = c$. With an inequality constraint, however, one can have $\partial L/\partial\lambda = -g(x, y) + c > 0$ if the constraint is slack or inactive at an optimum. For this reason, we advise against differentiating the Lagrangian w.r.t. the multiplier λ, even though several other books advocate this procedure.

6.2.2 Sufficient Conditions

Consider problem (1) and suppose that (x_0, y_0) satisfies conditions (2)–(4). If the Lagrangian $\mathscr{L}(x, y)$ is concave, then (x_0, y_0) solves the problem.

Proof: Any pair (x_0, y_0) that satisfies conditions (2) must be a stationary point of the Lagrangian. If the Lagrangian is concave, this (x_0, y_0) will give a maximum. So

$$\mathscr{L}(x_0, y_0) = f(x_0, y_0) - \lambda(g(x_0, y_0) - c) \geq \mathscr{L}(x, y) = f(x, y) - \lambda(g(x, y) - c)$$

Rearranging the terms, we obtain

$$f(x_0, y_0) - f(x, y) \geq \lambda[g(x_0, y_0) - g(x, y)] \tag{1}$$

If $g(x_0, y_0) < c$, then by (3), we have $\lambda = 0$, so (1) implies that $f(x_0, y_0) \geq f(x, y)$ for all (x, y). On the other hand, if $g(x_0, y_0) = c$, then $\lambda[g(x_0, y_0) - g(x, y)] = \lambda[c - g(x, y)]$. Here $\lambda \geq 0$, and $c - g(x, y) \geq 0$ for all (x, y) satisfying the inequality constraint. Hence, (x0, y0) solves problem (1)

EXAMPLE 1

A firm has a total of L units of labour to allocate to the production of two goods. These can be sold at fixed positive prices a and b respectively. Producing x units of the first good requires αx^2 units of labour, whereas producing y units of the second good requires βy^2 units of labour, where α and β are positive constants. Find what output levels of the two goods maximize the revenue that the firm can earn by using this fixed amount of labour.

Solution: The firm's revenue maximization problem is

$$\max ax + by \text{ subject to } \alpha x^2 + \beta y^2 \leq L$$

The Lagrangian is L(x, y) = ax + by − λ (αx2 + βy2 − L), and the necessary conditions for (x^*, y^*) to solve the problem are

(i) $\mathscr{L}_x = a - 2\lambda\alpha x^* = 0$,

(ii) $\mathscr{L}_y = b - 2\lambda\beta y^* = 0$

(iii) $\lambda \geq 0$, and $\lambda = 0$ if $\alpha(x^*)^2 + \beta(y^*)^2 < L$

We see that λ, x^*, and y^* are all positive, and $\lambda = a/2\alpha x^* = b/2\beta y^*$. So

$$x^* = a / 2\alpha\lambda, \qquad y^* = b / 2\beta\lambda \tag{1}$$

Because $\lambda > 0$, condition (iii) implies that $\alpha(x^*)^2 + \beta(y^*)^2 = L$. Inserting the expressions for x^* and y^* into the resource constraint yields $a^2 / 4\alpha\lambda^2 + b^2 / 4\beta\lambda^2 = L$. It follows that

$$\lambda = \tfrac{1}{2} L^{-\frac{1}{2}} 4 \sqrt{\frac{a^2}{\alpha} + \frac{b^2}{\beta}} \tag{2}$$

Our method has produced the solution candidate with x^* and y^* given by (1), and λ as in (2). The Lagrangian $\mathscr{L}$ is obviously concave, so we have found the solution.

EXAMPLE 2

Solve the problem

max f (x, y) = x2 + y2 + y − 1 subject to g(x, y) = x2 + y2 ≤ 1

Solution: The Lagrangian is $\mathscr{L}(x, y) = x^2 + y^2 + y - 1 - \lambda(x^2 + y^2 - 1)$. Here the first-order conditions are:

(i) $\mathscr{L}_1(x, y) = 2x - 2\lambda x = 0$

(ii) $\mathscr{L}_2(x, y) = 2y + 1 - 2\lambda y = 0$

The complementary slackness condition is

(iii) $\lambda \geq 0$, and $\lambda = 0$ if $x^2 + y^2 < 1$

We want to find all pairs (x, y) that satisfy these conditions for some suitable value of λ.

Conditions (i) and (ii) can be written as $2x(1-\lambda) = 0$ and $2y(1-\lambda) = -1$, respectively. The second of these implies that $\lambda \neq 1$, so the first implies that $x = 0$.

Suppose $x^2 + y^2 = 1$ and so $y = \pm 1$ because $x = 0$. Try $y = 1$ first. Then (ii) implies $\lambda = 3/2$ and so (iii) is satisfied. Thus, (0, 1) with $\lambda = 3/2$ is a first candidate for optimality (because all the conditions (i)–(iii) are satisfied). Next, try $y = -1$. Then condition (ii) yields $\lambda = 1/2$ and (iii) is again satisfied. Thus, $(0,-1)$ with $\lambda = 1/2$ is a second candidate for optimality.

Consider, finally, the case when $x = 0$ and also $x^2+y^2 = y^2 < 1$ that is, $-1 < y < 1$. Then (iii) implies that $\lambda = 0$, and so (ii) yields $y = -1/2$. Hence, $(0,-1/2)$ with $\lambda = 0$ is a third candidate for optimality.

We conclude that there are three candidates for optimality. Now

$$f(0, 1) = 1, \qquad f(0,-1) = -1, \qquad f(0,-1/2) = -5/4$$

Because we want to maximize a continuous function over a closed, bounded set, by the extreme value theorem there is a solution to the problem. Because the only possible solutions are the three points already found, we conclude that (x, y) = (0, 1) solves the maximization problem. (The point $(0,-1/2)$ solves the corresponding minimization problem.)

PROBLEMS FOR SECTION

A. (a) Solve the problem max $-x^2 - y^2$ subject to $x - 3y \leq -10$.

(b) The pair (x^*, y^*) that solves the problem in (a) also solves the minimization problem min $(x^2 + y^2)$ subject to $x - 3y \leq -10$.

B. (a) Solve the consumer demand problem

$$\max \sqrt{x} + \sqrt{y} \text{ subject to } px + qy \leq m$$

(b) Are the demand functions homogeneous of degree 0?

C. (a) Write down the Kuhn–Tucker conditions for the problem

$$\max 4 - 1/2\, x^2 - 4y \text{ subject to } 6x - 4y \leq a$$

(b) Solve the problem.

6.3 MULTIPLE INEQUALITY CONSTRAINTS

A general nonlinear programming problem is the following:

$$\max f(x_1, \ldots, x_n) \text{ subject to } \begin{cases} g1(x1 \ldots\ldots\ldots xn) \leq c1) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. \\ gm\ (x1, \ldots\ldots\ \ \ xn) \leq cm \end{cases} \qquad (1)$$

The set of vectors x = (x1, . . . , xn) that satisfy all the constraints is called the admissible set or the feasible set. Here is a method for solving problem (1):

6.3.1 Method for solving the General Non-linear Programming Problem:

Consider the problem

$$\max f(x) \text{ subject to } g_j(x) \le c_j, \qquad j = 1, \dots, m$$

Where x denotes (x1, . . . , xn).

A. Write down the Lagrangian

$$\mathscr{L}(x) = f(x) - \sum_{j=1}^{m} \lambda_j (g_j(x) - c_j)$$

With $\lambda_1, \dots, \lambda_m$ as the Lagrange multipliers associated with the m constraints.

B. Equate all the first-order partial derivatives of $\mathscr{L}(x)$ to 0:

$$\frac{\partial \mathcal{L}(x)}{\partial x_i} = \frac{\partial f(x)}{\partial x_i} - \sum_{j=1}^{m} \lambda_j \frac{\partial g_j(x)}{\partial x_i} = 0, \qquad i = 1, \dots, n$$

C. Impose the complementary slackness conditions:

$\lambda_j \ge 0$, and $\lambda_j = 0$ if $g_j(x) < c_j$, $\qquad j = 1, \dots, m$

D. Require x to satisfy the constraints

$g_j(x) \le c_j$, $\qquad j = 1, \dots, m$

Find all the vectors x that, together with associated values of $\lambda_1, \dots, \lambda_m$, satisfy conditions B, C, and D. These are the solution candidates, at least one of which solves the problem (if it has a solution). If the Lagrangian is concave in x, then the conditions are sufficient for optimality. Even if L(x) is not concave, still any vector x which happens to maximize the Lagrangian while also satisfying C and D must be an optimum.

NOTE 1 In order for the conditions to be truly necessary, a constraint qualification is needed. The conditions in B and C are called the Kuhn–Tucker conditions. Note that minimizing f (x) is equivalent to maximizing $-f(x)$. Also an inequality constraint of the form $g_j(x) \ge c_j$ can be rewritten as $-g_j(x) \le -c_j$, whereas an equality constraint $g_j(x) = c_j$ is equivalent to the double inequality constraint $g_j(x) \le c_j$ and $-g_j(x) \le -c_j$. In this way, most constrained optimization problems can be expressed in the form (1).

EXAMPLE 1

Consider the nonlinear programming problem

$$\text{maximize } x + 3y - 4e^{-x-y} \text{ subject to } \begin{cases} 2 - x \ge 2y \\ x - 1 \le -y \end{cases}$$

(a) Write down the necessary Kuhn–Tucker conditions for a point (x∗, y∗) to be a solution of the problem. Are the conditions sufficient for optimality?

(b) Solve the problem.

Solution: (a) The first (important) step is to write the problem in the same form as (1):

$$\text{maximize } x + 3y - 4e^{-x-y} \text{ subject to } \begin{cases} 2 - x \geq 2y \\ x - 1 \leq -y \end{cases}$$

The Lagrangian is $\mathscr{L}(x, y) = x + 3y - 4e^{-x-y} - \lambda_1(x + 2y - 2) - \lambda_2(x + y - 1)$. Hence, the Kuhn–Tucker conditions for (x^*, y^*) to solve the problem are:

$$\mathscr{L}_1 = 1 + 4e^{-x*-y*} - \lambda_1 - \lambda_2 = 0 \qquad \text{(i)}$$

$$\mathscr{L}_2 = 3 + 4e^{-x*-y*} - 2\lambda_1 - \lambda_2 = 0 \qquad \text{(ii)}$$

$$\lambda_1 \geq 0, \text{ and } \lambda_1 = 0 \text{ if } x^* + 2y^* < 2 \qquad \text{(iii)}$$

$$\lambda_2 \geq 0, \text{ and } \lambda_2 = 0 \text{ if } x^* + y^* < 1 \qquad \text{(iv)}$$

These conditions are sufficient for optimality because the Lagrangian is easily seen to be concave. (Look at the Hessian matrix of $\mathscr{L}$.)

(b) Subtracting (ii) from (i) we get $-2+\lambda_1 = 0$ and so $\lambda_1 = 2$. But then (iii) together with $x^* + 2y^* \leq 2$ yields

$$x^* + 2y^* = 2 \qquad \text{(v)}$$

Suppose $\lambda_2 = 0$. Then from (i), $4e^{-x^*-y^*} = 1$, so $-x^*-y^* = \ln(1/4)$, and then $x^*+y^* = \ln 4 > 1$, a contradiction. Thus λ_2 has to be positive. Then from (iv) and $x^* + y^* \leq 1$ we deduce $x^* + y^* = 1$. Using (v) we see that $x^* = 0$ and $y^* = 1$. Inserting these values for x^* and $y*$ into (i) and (ii) we find that $\lambda_2 = e^{-1}(4-e)$, which is positive. We conclude that the solution is: $x^* = 0$ and $y^* = 1$, with $\lambda_1 = 2$, $\lambda_2 = e^{-1}(4 - e)$.

EXAMPLE 2

A worker chooses both consumption c and labour supply l in order to maximize the utility function $\alpha \ln c + (1 - \alpha) \ln(1 - l)$ of consumption c and leisure $1 - l$, where $0 < \alpha < 1$. The worker's budget constraint is $c \leq wl + m$, where m is unearned income. In addition, the worker must choose $l \geq 0$ (otherwise there would be no work!). Solve the worker's problem.

Solution: The worker's constrained maximization problem is

$$\max \alpha \ln c + (1 - \alpha) \ln(1 - l) \text{ subject to } c \leq wl + m,\ l \geq 0$$

The Lagrangian is $\mathscr{L}(c, l) = \alpha \ln c + (1 - \alpha) \ln(1 - l) - \lambda(c - wl - m) - \mu l$, and the

Kuhn–Tucker conditions for (c^*, l^*) to solve the problem are

$$\mathscr{L}c = \alpha / c* - \lambda = 0 \qquad \text{(i)}$$

$$\mathscr{L}_l = -(1 - \alpha) / (1 - l^*) + \lambda w + \mu = 0 \qquad \text{(ii)}$$

$$\lambda \geq 0, \text{ and } \lambda = 0 \text{ if } c^* < wl^* + m \qquad \text{(iii)}$$

$$\mu \geq 0, \text{ and } \mu = 0 \text{ if } l^* > 0 \qquad \text{(iv)}$$

In the last two examples it was not too hard to find which constraints bind (i.e. hold with equality) at the optimum. But with more complicated nonlinear programming problems, this can be harder. A general method for finding all candidates for optimality in a nonlinear programming problem with two constraints can be formulated as follows: First, examine the case where both constraints bind. Next, examine the two cases where only one constraint binds. Finally, examine the case where neither constraint binds. In each case, find all vectors x, with associated values of the Lagrange multipliers that satisfy all the relevant conditions— if any do. Then calculate the value of the objective function for these values of x, and retain those x with the highest values. Except for perverse problems, this procedure will find the optimum. The next example illustrates how it works in practice.

EXAMPLE 3

Suppose your utility of consuming x1 units of good A and x2 units of good B is U(x1, x2) = ln x1 +ln x2, and that the prices per unit of A and B are 10 and 5, respectively. You have at most 350 to spend on the two goods. Suppose it takes 0.1 hours to consume one unit of A and 0.2 hours to consume one unit of B. You have at most 8 hours to spend on consuming the two goods. How much of each good should you buy in order to maximize your utility?

Solution: The problem is

max $U(x_1, x_2) = \ln x_1 + \ln x_2$ subject to $\begin{cases} 10x1 + 5x2 \le 350 \\ 0.1x1 + 0.2x2 \le 8 \end{cases}$

The Lagrangian is $\mathscr{L} = \ln x_1 + \ln x_2 - \lambda_1(10x_1 + 5x_2 - 350) - \lambda_2(0.1x_1 + 0.2x_2 - 8)$. Necessary conditions for (x^*_1, x^*_2) to solve the problem are that there exist numbers λ_1 and λ_2 such that

$$\mathscr{L}_1 = 1/x^*_1 - 10\lambda_1 - 0.1\lambda_2 = 0 \qquad \text{(i)}$$

$$\mathscr{L}_2 = 1/x^*_2 - 5\lambda_1 - 0.2\lambda_2 = 0 \qquad \text{(ii)}$$

$$\lambda_1 \ge 0, \text{ and } \lambda_1 = 0 \text{ if } 10x^*_1 + 5x^*_2 < 350 \qquad \text{(iii)}$$

$$\lambda_2 \ge 0, \text{ and } \lambda_2 = 0 \text{ if } 0.1x^*_1 + 0.2x^*_2 < 8 \qquad \text{(iv)}$$

We start the systematic procedure:

(I) Both constraints bind. Then

$$10x^*_1 + 5x^*_2 = 350 \qquad \text{(v)}$$

and $0.1x^*_1 + 0.2x^*_2 = 8$. The solution is $(x^*_1, x^*_2) = (20, 30)$. Inserting these values into (i) and (ii) yields the system $10\lambda_1 + 0.1\lambda_2 = 1/20$ and $5\lambda_1 + 0.2\lambda_2 = 1/30$, with solution $(\lambda_1, \lambda_2) = (1/225, 1/18)$. So we have found a candidate for optimality because all the Kuhn–Tucker conditions are satisfied. (Note that it is important to check that λ_1 and λ_2 are nonnegative.)

(II) Constraint 1 binds, 2 does not. Then (v) holds and $0.1x^*_1 + 0.2x^*_2 < 8$. From (iv) we obtain $\lambda_2 = 0$. Now (i) and (ii) give $x^*_2 = 2x^*_1$. Inserting this into (v), we get $x^*_1 = 17.5$ and then $x^*_2 = 2x^*_1 = 35$. But then $0.1x^*_1 + 0.2x^*_2 = 8.75$, which violates the second constraint. So no candidate arises in this case.

(III) Constraint 2 binds, 1 does not. Then $10x^*_1 + 5x^*_2 < 350$ and $0.1x^*_1 + 0.2x^*_2 = 8$. From (iii), $\lambda_1 = 0$, and (i) and (ii) yield $0.1x^*_1 = 0.2x^*_2$. Inserted into $0.1x^*_1 + 0.2x^*_2 = 8$ this yields $x^*_2 = 20$ and so $x^*_1 = 40$. But then $10x^*_1 + 5x^*_2 = 500$, violating the first constraint. So no candidate arises in this case either.

(IV) None of the constraints bind. Then $\lambda_1 = \lambda_2 = 0$ and (i) and (ii) make no sense.

Conclusion: There is only one candidate for optimality, (20, 30). Since the Lagrangian is easily seen to be concave, we have found the solution.

PROBLEMS

1. (a) Write down the Lagrangian and the necessary Kuhn–Tucker conditions for the problem max ½ x − y subject to x + e−x ≤ y, x ≥ 0

   (b) Find the solution to the problem.

2. Solve the following consumer demand problem where, in addition to the budget constraint, there is an upper limit ¯x which rations how much of the first good can be bought:

$$\max \alpha \ln x + (1 - \alpha) \ln y \text{ subject to } px + qy \le m,\ x \le \bar{x}$$

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6.4 NON-NEGATIVITY CONSTRAINTS

Consider the general nonlinear programming. Often, variables involved in economic optimization problems must be nonnegative by their very nature. If $x_1 \ge 0$, for example, this can be represented by the new constraint $h1(x1, \ldots, xn) = -x_1 \le 0$, and we introduce an additional Lagrange multiplier to go with it. But in order not to have too many Lagrange multipliers, the necessary conditions for the solution of nonlinear programming problems with non-negativity constraints are sometimes formulated in a slightly different way.

Consider first the problem

$$\max f(x, y) \text{ subject to } g(x, y) \le c,\ x \ge 0,\ y \ge 0 \qquad (1)$$

Here we introduce the functions $h_1(x, y) = -x$ and $h_2(x, y) = -y$, so that the constraints in problem (1) become $g(x, y) \le c$, $h_1(x, y) \le 0$, and $h_2(x, y) \le 0$. Applying the method for solving problem, we introduce the Lagrangian

$$\mathscr{L}(x, y) = f(x, y) - \lambda(g(x, y) - c) - \mu_1(-x) - \mu_2(-y)$$

The Kuhn–Tucker conditions are

$$\mathscr{L}_1 = f'_1(x, y) - \lambda g'_1(x, y) + \mu_1 = 0 \quad \text{(i)}$$

$$\mathscr{L}_2 = f'_2(x, y) - \lambda g'_2(x, y) + \mu_2 = 0 \quad \text{(ii)}$$

$$\lambda \geq 0, \text{ and } \lambda = 0 \text{ if } g(x, y) < c \quad \text{(iii)}$$

$$\mu_1 \geq 0, \text{ and } \mu_1 = 0 \text{ if } x > 0 \quad \text{(iv)}$$

$$\mu_2 \geq 0, \text{ and } \mu_2 = 0 \text{ if } y > 0 \quad \text{(v)}$$

From (i), we have $f'_1(x, y) - \lambda g'_1(x, y) = -\mu_1$. From (iv), we have $-\mu_1 \leq 0$ and $-\mu_1 = 0$ if $x > 0$. Thus, (i) and (iv) are together equivalent to

$$f'_1(x, y) - \lambda g'_1(x, y) \leq 0 \ (= 0 \text{ if } x > 0) \quad \text{(vi)}$$

In the same way, (ii) and (v) are together equivalent to

$$f'_2(x, y) - \lambda g'_2(x, y) \leq 0 \ (= 0 \text{ if } y > 0) \quad \text{(vii)}$$

So the new Kuhn–Tucker conditions are (vi), (vii), and (iii). Note that after replacing (i) and (iv) by (vi), as well as (ii) and (v) by (vii), only the multiplier λ associated with $g(x, y) \leq c$ remains.

The same idea can obviously be extended to the n-variable problem

$$\max f(x) \text{ subject to } \begin{cases} g1(x) \leq c1 \\ \dots\dots\dots\dots x1 \geq 0, \dots, xn \geq 0 \\ gm(x) \leq cm \end{cases} \quad (2)$$

Briefly formulated, the necessary conditions for the solution of (2) are that, for each $i = 1, \dots, n$,

$$\frac{\partial f(x)}{\partial xi} - \sum_{j=1}^{m} \lambda j \frac{\partial gj(x)}{\partial xi} \leq 0 \ (= 0 \text{ if } xi > 0) \quad (3)$$

$$\lambda_j \geq 0, \text{ with } \lambda_j = 0 \text{ if } g_j(x) < c_j, \quad j = 1, \dots, m \quad (4)$$

EXAMPLE 1

Consider the utility maximizing problem

maximize $x + \ln(1 + y)$ subject to $px + y \leq m$, $x \geq 0$, $y \geq 0$

(a) Write down the necessary Kuhn–Tucker conditions for a point $(x*, y*)$ to be a solution.

(b) Find the solution to the problem, for all positive values of p and m.

Solution:

(a) The Lagrangian is $L(x, y) = x + \ln(1 + y) - \lambda(px + y - m)$, and the Kuhn–Tucker conditions for $(x*, y*)$ to be a solution are that there exists a λ such that

$$\mathscr{L}_1(x^*, y^*) = 1 - p\lambda \leq 0, \text{ and } 1 - p\lambda = 0 \text{ if } x^* > 0 \quad \text{(i)}$$

$$\mathscr{L}_2(x^*, y^*) = 1 / (1 + y^*) - \lambda \leq 0, \text{ and } 1/ (1 + y^*) - \lambda = 0 \text{ if } y* > 0 \quad \text{(ii)}$$

$$\lambda \geq 0, \text{ and } \lambda = 0 \text{ if } px^* + y^* < m \quad \text{(iii)}$$

In addition $x^* \geq 0$, $y^* \geq 0$, and the budget constraint has to be satisfied, so $px^* + y^* \leq m$. (b) Note that the Lagrangian is concave, so a point that satisfies the Kuhn–Tucker conditions will be a maximum point. It is clear from (i) that λ cannot be 0. Therefore $\lambda > 0$, so (iii) and $px^* + y^* \leq m$ imply that

$$px^* + y^* = m \quad \text{(iv)}$$

There are four cases to consider:

I. Suppose $x^* = 0$, $y^* = 0$. Since $m > 0$, this is impossible because of (iv).

II. Suppose $x^* > 0$, $y^* = 0$. From (ii) and $y^* = 0$ we get $\lambda \geq 1$. Then (i) implies that $p = 1/\lambda \leq 1$. Equation (iv) gives $x^* = m/p$, so we get one candidate for a maximum point:

$$(x^*, y^*) = (m/p, 0),\ \lambda = 1/p, \text{ if } 0 < p \leq 1$$

III. Suppose $x^* = 0$, $y^* > 0$. By (iv) we have $y^* = m$. Then (ii) yields $\lambda = 1/(1 + y^*) = 1/(1 + m)$. From (i) we get $p \geq 1 / \lambda = m + 1$. This gives one more candidate:

$$(x^*, y^*) = (0, m),\ \lambda = 1/(1 + m), \text{ if } p \geq m + 1$$

IV. Suppose $x^* > 0$, $y^* > 0$. With equality in both (i) and (ii), $\lambda = 1/p = 1/(1 + y^*)$. It follows that $y^* = p - 1$, and then $p > 1$ because $y^* > 0$. Equation (iv) implies that $px^* = m - y^* = m - p + 1$, so $x^* = (m + 1 - p)/p$. Since $x^* > 0$, we must have $p < m+ 1$. Thus we get one last candidate

$$(x^*, y^*) = ((m + 1 - p) / p\ ,\ p - 1),\ \lambda = 1/p\ , \text{ if } 1 < p < m+ 1$$

Conclusion: Putting all this together, we see that the solution of the problem is

A. If $0 < p \leq 1$, then $(x^*, y^*) = (m/p, 0)$, with $\lambda = 1/p$. (Case II.)

B. If $1 < p < m+1$, then $(x^*, y^*) = ((m+1-p) / p, p -1)$, with $\lambda = 1/p$. (Case IV.)

C. If $p \geq m + 1$, then $(x^*, y^*) = (0, m)$, with $\lambda = 1/(m + 1)$. (Case III.)

Note that except in the intermediate case (B) when $1 < p < m+1$, it is optimal to spend everything on only the cheaper of the two goods — either x in case A, or y in case C. This makes economic sense.

EXAMPLE 2

Peak Load Pricing: Consider a producer who generates electricity by burning a fuel such as coal or natural gas. The demand for electricity varies between peak periods, during which all the generating capacity is used and off-peak periods when there is spare capacity. We consider a certain time interval (say, a year) divided into n periods of equal length. Suppose the amounts of electric power sold in these n periods are $x_1, x_2, \ldots, x_n$. Assume that a regulatory authority fixes the corresponding prices at levels equal to $p_1, p_2, \ldots, p_n$. The total operating cost over all n periods is given by the function $C(x_1, \ldots, x_n)$, and k is the output capacity in each period. Let D(k) denote the cost of maintaining output capacity at level k. The producer's total profit is then

$$\pi(x_1, \ldots, x_n, k) = \sum_{i=0}^{n} pixi - C\,(x) - D\,(k)$$

Because the producer cannot exceed capacity k in any period, it faces the constraints

$$x_1 \leq k, \ldots, x_n \leq k \qquad \text{(I)}$$

We consider the problem of finding $x1 \geq 0 \ldots x_n \geq 0$ and $k \geq 0$ such that profit is maximized subject to the capacity constraints (I).

This is a nonlinear programming problem with n + 1 variables and n constraints. The Lagrangian $\mathscr{L}$ is

$$\mathscr{L}(x_1, \ldots, x_n, k) = \sum_{i=0}^{n} pixi - C(x_1\ldots\ldots, x_n) - D(k) - \sum_{i=0}^{n} \lambda i\,(x_i - k)$$

Following (4) and (5), the choice $(x^0_1, \ldots, x^0_n, k^0) \geq 0$ can solve the problem only if there exist Lagrange multipliers $\lambda 1 \geq 0, \ldots, \lambda n \geq 0$ such that

$$\frac{\partial \mathcal{L}}{\partial xi} = p_i - C'_i(x^0_1, \ldots, x^0_n) - \lambda_i \leq 0 \ (= 0 \text{ if } x^0_i > 0),\ i = 1, \ldots, n \qquad \text{(i)}$$

$$\frac{\partial \mathcal{L}}{\partial k} = p_i - D'_i(k^0) + \sum_{i=0}^{n} \lambda i \leq 0 \ (= 0 \text{ if } x^0 > 0) \qquad \text{(ii)}$$

$$\lambda_i \geq 0, \text{ and } \lambda_i = 0 \text{ if } x^0_i < k^0,\ i = 1, \ldots, n \qquad \text{(iii)}$$

Suppose that i is such that $x^0_i > 0$. Then (i) implies that

$$p_i = C'_i(x^0_1, \ldots, x^0_n) + \lambda_i \qquad \text{(iv)}$$

If period i is an off-peak period, then $x^0_i < k^0$ and so $\lambda_i = 0$ by (iii). From (iv) it follows that $p_i = C'_i(x^0_1, \ldots, x^0_n)$. Thus, we see that the profit-maximizing pattern of output $(x^0_1, \ldots, x^0_n)$ will bring about equality between the regulator's price in any off-peak period and the corresponding marginal operating cost.

On the other hand, λ_j might be positive in a peak period when $x^0_j = k^0$. If $k^0 > 0$, it follows from (ii) that $\sum_{i=0}^{n} \lambda i = D'(k^0)$. We conclude that the output pattern will be such that in peak periods the price set by the regulator will exceed the marginal operating cost by an additional amount λ_j, which is really the "shadow price" of the capacity constraint $x^0{}_j \leq k^0$. The sum of these shadow prices over all peak periods is equal to the marginal capacity cost.

PROBLEMS

1. (a) Consider the utility maximization problem

   maximize $x + \ln(1 + y)$ subject to $16x + y \leq 495$, $x \geq 0$, $y \geq 0$

   Write down the necessary K–T conditions (with non-negativity constraints) for a point (x^*, y^*) to be a solution.

   (b) Find the solution to the problem.

   (c) Estimate by how much utility will increase if income is increased from 495 to 500.

6.5 APPLICATIONS IN ECONOMICS

A resource allocation problem that a firm may encounter is how to decide on the product mix which will maximize profits when it has limited amounts of the various inputs required for the different products that it makes. The firm's objective is to maximize profit and so profit is what is known as the 'objective function'. It tries to optimize this function subject to the constraint of limited input availability. This is why it is known as a 'constrained optimization' problem.

When both the objective function and the constraints can be expressed in a linear form then the technique of linear programming can be used to try to find a solution. These are some example which we are taking from economic problems or economic applications.

Example 1 (Profit Maximisation)

A firm produces two goods A and B, which each contribute a net profit of £1 per unit sold. It uses two inputs K and L. The input requirements are:

3 units of K plus 2 units of L for each unit of A

2 units of K plus 3 units of L for each unit of B

If the firm has 600 units of K and 600 units of L at its disposal, how much of A and B should it produce to maximize profit?

Solution

Using the same method as in the previous example we can see that the constraints are:

for input K $3A + 2B \leq 600$ ……………………………………(1)

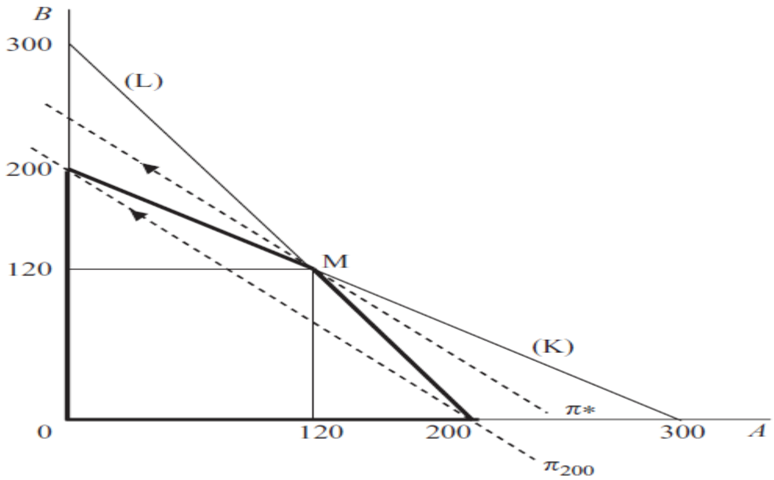
for input L $2A + 3B \leq 600$ ……………………………………(2)

Non-negativity $A \geq 0$ $B \geq 0$

The feasible area is therefore as marked out by the heavy black lines in Figure 5A.2.

As profit is £1 per unit for both A and B, the objective function is

$$\pi = A + B$$



If we suppose profit is £200, then

$$200 = A + B$$

This function corresponds to the line π 200 which can be used as a guideline for the slope of the objective function. The line parallel to π 200 that is furthest away from the origin but still within the feasible area will represent the maximum profit. This is the line π∗ through point M. The optimum values of A and B can thus be read off the graph as 120 of each. Alternatively, once we know that the optimum combination of A and B is at the intersection of the constraints (K) and (L), the values of A and B can be found from the simultaneous equations

$$3A + 2B = 600 \qquad (1)$$

$$2A + 3B = 600 \qquad (2)$$

From (1) $2B = 600 - 3A$

$$B = 300 - 1.5A \qquad (3)$$

Substituting (3) into (2)

$$2A + 3(300 - 1.5A) = 600$$

$$2A + 900 - 4.5A = 600$$

$$300 = 2.5A$$

$$120 = A$$

Substituting this value of *A* into (3)

$$B = 300 - 1.5(120) = 120$$

As both *A* and *B* equal 120 then

$$\pi* = 120 + 120 = £240$$

The optimum combination at M is where both constraints (K) and (L) bite. There is therefore no slack for either K or L.

It is possible that the objective function will have the same slope as one of the constraints. In this case there will not be one optimum combination of the inputs as all points along the section of this constraint that forms part of the boundary of the feasible area will correspond to the same value of the objective function.

Test Yourself,

1. A firm manufactures products A and B using the two inputs X and Y in the following quantities:

   1 tonne of A requires 80 units of X plus 148 units of Y

   1 tonne of B requires 200 units of X plus 120 units of Y

   The profit per unit of A is £20, and the per-unit profit of B is £30. If the firm has at its disposal 1,600 units of X and 1,800 units of Y, what combination of A and B should it manufacture in order to maximize profit? (Fractions of a tonne may be produced.)

Should the firm change its production mix if per-unit profits alter to (a) £25 each for both A and B, or (b) £30 for A and £20 for B?

2. A firm produces the goods A and B using the four inputs W, X, Y and Z in the following quantities:

   1 unit of A requires 9 units of W, 30 of X, 20 of Y and 20 of Z

   1 unit of B requires 13 units of W, 55 of X, 28 of Y and 20 of Z

   The firm has available 468 units of W, 1,980 units of X, 1,120 units of Y and 800 units of Z. What production mix will maximize its total profit if each unit of A adds £60 to profit and each unit of B adds £75?

Example 2 (Cost Minimisation)

A firm manufactures a medicinal product containing three ingredients X, Y and Z. Each unit produced must contain at least 100 g of X, 30 g of Y and 75 g of Z. The product is made by mixing the inputs A and B which come in containers costing respectively £3 and £6 each. These contain X, Y and Z in the following quantities:

1 container of A contains 50 g of X, 10 g of Y and 15 g of Z

1 container of B contains 20 g of X, 10 g of Y and 50 g of Z

What mix of A and B will minimize the cost per unit of the product subject to the above quality constraints? (It does not matter if these minimum requirements are exceeded and all other production costs can be ignored.)

Solution

Total usage of X will be 50 g for each container of A plus 20 g for each container of B. Total usage must be at least 100 g. This quality constraint for X can thus be written as

$$50A + 20B \geq 100$$

Note that the constraint has the $\geq$ sign instead of $\leq$. The quality constraints on Y and Z can also be written as

$$10A + 10B \geq 30$$

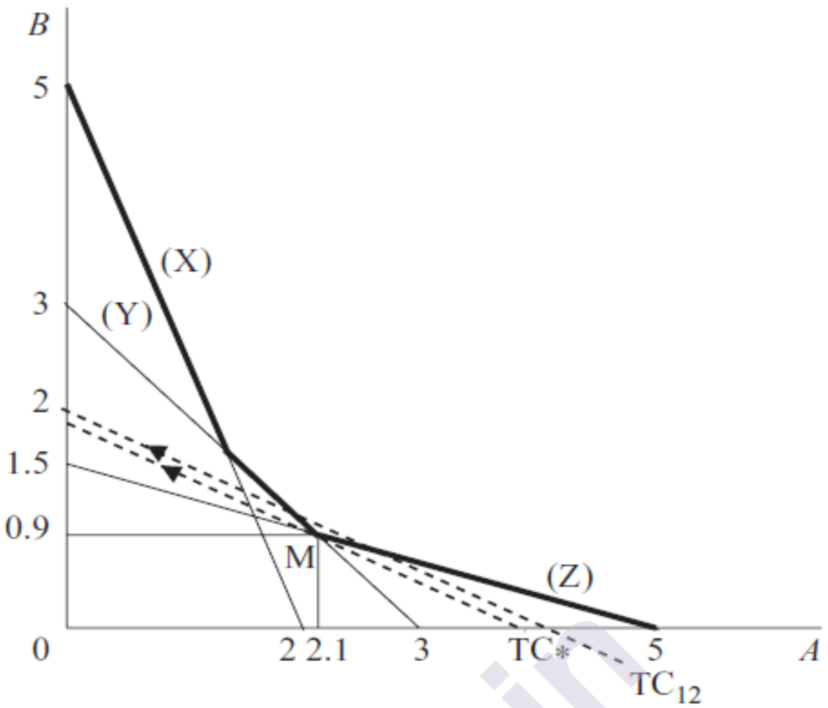
$$15A + 50B \geq 75$$

As negative amounts of the inputs A and Bare not feasible there are also the two non-negativity constraints

$$A \geq 0 \quad B \geq 0$$

If the quality constraint for X is only just met then

$$50A + 20B = 100 \ (X)$$



The line representing this function is drawn as (X) in Figure 5.A5. Any combination of A and B above this line will more than satisfy the quality constraint for X. Any combination of A and B below this line will not satisfy this constraint and will therefore not be feasible.

In a similar fashion the constraints for Y and Z are shown by the lines representing the functions

$$10A + 10B = 30 \qquad (Y)$$

and

$$15A + 50B = 75 \qquad (Z)$$

Taking all the constraints into account, the feasible area is marked out by the heavy black lines in Figure 5A.5, or at least its lower bounds are. As these are minimum constraints then theoretically there are no upper limits to the amounts of A and B that could be used to make a unit of the final product.

The objective function is total cost (TC) which the firm is seeking to minimize. Given the prices of A and B of £3 and £6 respectively, then

$$\text{TC} = 3A + 6B$$

To obtain a guideline for the slope of the TC function assumes any value for TC that is easily divisible by the two prices of £3 and £6. For example, if TC is assumed to be £12 then the line TC_{12} representing the function

$$12 = 3A + 6B$$

can be drawn, which has a slope of −0.5.

One now needs to ask the question ‘can a line with this slope be drawn closer to the origin (thus representing a smaller value for TC) but still going through the feasible area?’ In this case the answer is ‘yes’. The line

TC∗ through M represents the lowest cost method of combining A and B that still satisfies the three quality constraints. The optimum amounts of A and B can now be read off the graph at M as approximately 2.1 and 0.9 respectively.

More accurate answers can be obtained algebraically. The optimum combination M is where the quality constraints for Y and Z intersect. These correspond to the linear equations

$$10A + 10B = 30 \qquad (1)$$

$$15A + 50B = 75 \qquad (2)$$

Dividing (2) by 5 we get $\qquad 3A + 10B = 15$

Subtracting (1) $\qquad \underline{10A + 10B = 30}$

$$-7A = -15$$

$$A = \frac{15}{7}$$

Substituting this value for A into (1)

$$10\left(\frac{15}{7}\right) + 10\text{ B} = \frac{150}{7} + 10\text{ B} = 30 \qquad (3)$$

Multiplying (3) by 7

$$150 + 70B = 210$$

$$70B = 60$$

$$B = \frac{6}{7}$$

Thus the firm should use $\frac{15}{7}$ containers of A plus $\frac{6}{7}$ of a container of B for every unit of the final product it makes. As long as large quantities of the product are made, the firm does not have to worry about unused fractions of containers. It just needs to use containers A and B in the ratio $\frac{15}{7}$ to $\frac{6}{7}$ which is the same as the ratio 2.5 to 1.

The constraint on X does not bite and so there is some slack. In a minimization problem slack means overabundance. The total amount of X contained in a unit of the final product will be

$$50\text{A} + 20\text{B} = 50\left(\frac{15}{7}\right) + 20\left(\frac{6}{7}\right) = \frac{750+120}{7} = \frac{970}{7} = 38.57 \text{ mg.}$$

Test Yourself

1. Find the minimum value of the function $C = 40A+20B$ subject to the constraints

$$10A + 40B \geq 40 \qquad (x)$$

$$30A + 20B \geq 60 \qquad (y)$$

$$10A \geq 10 \qquad (z)$$

$$A \geq 0,\ B \geq 0$$

Will there be slack in any of the constraints at the optimum combination of A and B? If so, what is the excess capacity?

2. A firm manufactures a product that, per litre, must contain at least 18 g of chemical X and 10 g of chemical Y. The rest of the product is water whose costs can be ignored. The two inputs A and B contain X and Y in the following quantities:

1 unit of A contains 6 g of X and 5 g of Y

1 unit of B contains 9 g of X and 2 g of Y

The per-unit costs of A and B are £2 and £6 respectively. What combination of A and B will give the cheapest way of producing a litre of the final product?

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Example 3 (Mixed Constrained 'constrained optimization' in linear programming)

Minimize the objective function $C = 12A + 8B$ subject to the constraints

$$10A + 40B \geq 40 \quad (1)$$

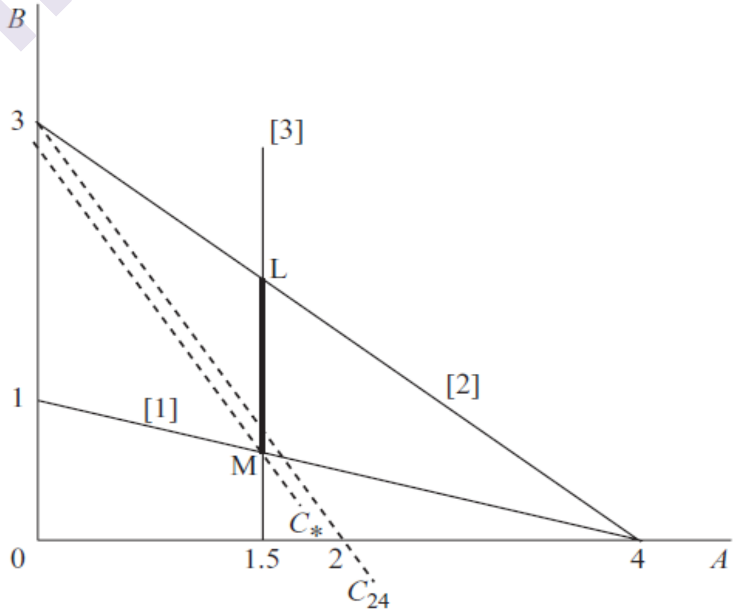
$$12A + 16B \leq 48 \quad (2)$$

$$A = 1.5 \quad (3)$$

Solution

The constraints are marked out in Figure 5.A7. Constraint (1) means that the feasible area must be above the line

$$10A + 40B = 40$$



Constraint (2) means that the feasible area must be below the line

$$12A + 16B = 48$$

Constraint (3) means that the feasible area must be along the vertical line through A = 1.5. The only section of the graph that satisfies all three of these constraints is the heavy black section LM of the vertical line through A = 1.5.

If C is assumed to be 24 then the line C_{24} representing the function

$$24 = 12A + 8B$$

can be drawn in and has a slope of −1.5. To minimize C, one needs to find the closest line to the origin that has this slope and also passes through the feasible area. This will be the line C_* through M.

The optimum value of A is therefore obviously 1.5.

The optimum value of B occurs at the intersection of the two lines

$$A = 1.5$$

$$\text{and } 10A + 40B = 40$$

$$\text{Thus } 10(1.5) + 40B = 40$$

$$15 + 40B = 40$$

$$40B = 25$$

$$B = 0.625$$

Test Yourself

1. A firm makes two goods A and B using the three inputs X, Y and Z in the following quantities:

20 units of X, 8 units of Y and 20 units of Z per unit of A

20 units of X, 20 units of Y and 14 units of Z per unit of B

The per-unit profit of A is £1,500, and for B the figure is £1,000. Input availability is restricted to 60 units of X, 40 units of Y and 70 units of Z. The firm has already committed itself to a contract to supply one customer with 1 unit of B. What combination of A and B should it produce to maximize total profit?

Example 4 (Maximisation of output)

Firm faces the production function $Q = 12K^{0.4}\ L^{0.4}$ and can buy the inputs K and L at prices per unit of £40 and £5 respectively. If it has a budget of

£800 what combination of K and L should it use in order to produce the maximum possible output?

Solution

The problem is to maximize the function $Q = 12K^{0.4}L^{0.4}$ subject to the budget constraint

$$40K + 5L = 800 \qquad (1)$$

The theory of the firm tells us that a firm is optimally allocating a fixed budget if the last £1 spent on each input adds the same amount to output, i.e. marginal product over price should be equal for all inputs. This optimization condition can be written as

$$MP_K / P_K = MP_L / P_L \qquad (2)$$

The marginal products can be determined by partial differentiation:

$$MP_K = \frac{\partial Q}{\partial K} = 4.8\ K^{-0.6}\ L^{0.4} \qquad (3)$$

$$MP_L = \frac{\partial Q}{\partial L} = 4.8\ K^{0.4}\ L^{-0.6} \qquad (4)$$

Substituting (3) and (4) and the given prices for P_K and P_L into (2)

$$\frac{4.8\ K^{-0.6}\ L^{0.4}}{40} = \frac{4.8\ K^{0.4}\ L^{-0.6}}{5}$$

Dividing both sides by 4.8 and multiplying by 40 gives

$$K^{-0.6}\ L^{0.4} = 8\ K^{0.4}\ L^{-0.6}$$

Multiplying both sides by $K^{0.6}\ L^{0.6}$ gives

$$L = 8K \qquad (5)$$

Substituting (5) for L into the budget constraint (1) gives

$$40K + 5(8K) = 800$$

$$40K + 40K = 800$$

$$80K = 800$$

Thus the optimal value of K is

$$K = 10$$

and, from (5), the optimal value of L is

$$L = 80$$

Note that although this method allows us to derive optimum values of K and L that satisfy condition (2) above, it does not provide a check on whether this is a unique solution, i.e. there is no second-order condition check. However, it may be assumed that in all the problems in this section the objective function is maximized (or minimized depending on the question) when the basic economic rules for an optimum are satisfied.

The above method is not the only way of tackling this problem by substitution. An alternative approach, explained below, is to encapsulate the constraint within the function to be maximized, and then maximize this new objective function.

From the budget constraint

$$40K + 5L = 800$$

$$5L = 800 - 40K \qquad (1)$$

$$L = 160 - 8K \qquad (2)$$

Substituting (2) into the objective function $Q = 12K^{0.4}L^{0.4}$ gives

$$Q = 12K^{0.4}(160 - 8K)^{0.4} \qquad (3)$$

We are now faced with the unconstrained optimization problem of finding the value of K that maximizes the function (3) which has the budget constraint (1) 'built in' to it by substitution. This requires us to set d Q/dK = 0. However, it is not straight forward to differentiate the function in (3), and we must wait until further topics in calculus have been covered before proceeding with this solution. To make sure that you understand the basic substitution method, we shall use it to tackle another constrained maximization problem.

Example 5 (Budget Constraint)

A firm faces the production function $Q = 20K^{0.4}L^{0.6}$. It can buy inputs K and L for £400 a unit and £200 a unit respectively. What combination of L and K should be used to maximize output if its input budget is constrained to £6,000?

Solution

$$MP_L = \frac{\partial Q}{\partial L} = 12K^{0.4}L^{-0.4} \qquad MP_K = \frac{\partial Q}{\partial K} = 8K^{-0.6}L^{0.6}$$

Optimal input mix requires

$$MP_L / P_L = MP_K / P_K$$

Therefore

$$\frac{12K^{0.4}L^{-0.4}}{200} = \frac{8K^{-0.6}L^{0.6}}{400}$$

Cross multiplying gives

$$4{,}800K = 1{,}600L$$

$$3K = L$$

Substituting this result into the budget constraint

$$200L + 400K = 6{,}000$$

Gives

$$200(3K) + 400K = 6{,}000$$

$$600K + 400K = 6{,}000$$

$$1{,}000K = 6{,}000$$

$$K = 6$$

Therefore

$$L = 3K = 18$$

The examples of constrained optimization considered so far have only involved output maximization when a firm faces a Cobb–Douglas production function, but the same technique can also be applied to other forms of production functions.

Example 6 (Production function)

A firm faces the production function

$$Q = 120L + 200K - L^2 - 2K^2$$

For positive values of Q It can buy L at £5 a unit and K at £8 a unit and has a budget of £70. What is the maximum output it can produce?

Solution

$$MP_L = \frac{\partial Q}{\partial L} = 120 - 2L \qquad MP_K = \frac{\partial Q}{\partial K} = 200 - 4K$$

For optimal input combination

$$MP_L / P_L = MP_K / P_K$$

Therefore, substituting MP_K and MP_L and the given input prices

$$\frac{120 - 2L}{5} = \frac{200 - 4K}{8}$$

$$8(120 - 2L) = 5(200 - 4K)$$

$$960 - 16L = 1{,}000 - 20K$$

$$20K = 40 + 16L$$

$$K = 2 + 0.8L \qquad (1)$$

Substituting (1) into the budget constraint

$$5L + 8K = 70$$

Gives

$$5L + 8(2 + 0.8L) = 70$$

$$5L + 16 + 6.4L = 70$$

$$11.4L = 54$$

$$L = 4.74 \text{ (to 2 dp)}$$

Substituting this result into (1)

$$K = 2 + 0.8(4.74) = 5.79$$

Therefore maximum output is

$$Q = 120L + 200K - L^2 - 2K^2$$

$$= 120(4.74) + 200(5.79) - (4.74)^2 - 2(5.79)^2$$

$$= 1{,}637.28$$

This technique can also be applied to consumer theory, where utility is maximized subject to a budget constraint.

Example 7 (Utility function)

The utility a consumer derives from consuming the two goods A and B can be assumed to be determined by the utility function $U = 40A^{0.25}B^{0.5}$. If A costs £4 a unit and B costs £10 a unit and the consumer's income is £600, what combination of A and B will maximize utility?

Solution

$MU_A = \frac{\partial U}{\partial A} = 10A^{-0.75}B^{0.5}$ $\qquad$ $MP_B = \frac{\partial U}{\partial B} = 20A^{0.25}B^{-0.5}$

Consumer theory tells us that total utility will be maximized when the utility derived from the last pound spent on each good is equal to the utility derived from the last pound spent on any other good. This optimization rule can be expressed as

$$MU_A / P_A = MU_B / P_B$$

Therefore, substituting the above MU functions and the given prices of £4 and £10, this condition becomes

$$\frac{10A^{-0.75}B^{0.5}}{4} = \frac{20A^{0.25}B^{-0.5}}{10}$$

$$100B = 80A$$

$$B = 0.8A \qquad (1)$$

Substituting (1) for *B* in the budget constraint

$$4A + 10B = 600$$

Gives

$$A + 10(0.8A) = 600$$

$$4A + 8A = 600$$

$$12A = 600$$

$$A = 50$$

Thus from (1)

$$B = 0.8(50) = 40$$

The substitution method can also be used for **constrained minimization** problems. If output is given and a firm is required to minimize the cost of this output, then one variable can be eliminated from the production function before it is substituted into the cost function which is to be minimized.

Test Yourself

1. If a firm has a budget of £378 what combination of *K* and *L* will maximize output given the production function $Q = 40K^{0.6}L^{0.3}$ and prices for K and L of £20 per unit and £6 per unit respectively?

2. A firm faces the production function $Q = 6K^{0.4}L^{0.5}$. If it can buy input K at £32 a unit and input L at £8 a unit, what combination of L and K should it use to maximize production if it is constrained by a fixed budget of £36,000?

6.6 SUMMARY

1. Simple inequality constraints are those requiring certain variables to be nonnegative. The bounds on resource availability are expressed as inequalities rather than equalities.
2. Introduce the complementary slackness condition $\lambda \geq 0$, and $\lambda = 0$ if $g(x, y) < c$ and require (x, y) to satisfy the constraint $g(x, y) \leq c$ are called Kuhn–Tucker conditions.
3. The firm's objective is to maximize profit and so profit is what is known as the 'objective function'. It tries to optimize this function subject to the constraint of limited input availability. This is why it is known as a 'constrained optimization' problem.

6.7 QUESTIONS

1. Explain Constrained Optimisation Problem with inequality constraint with suitable example.
2. Explain multiple inequality constraints with suitable example.
3. Explain non-negativity constraint problem with suitable example.
4. Suppose a firm earns revenue $R(Q) = aQ - bQ2$ and incurs cost $C(Q) = \alpha Q + \beta Q2$ as functions of output $Q \geq 0$, where a, b, α, and β are positive parameters. The firm maximizes profit $\pi(Q) = R(Q)-C(Q)$ subject to the constraint $Q \geq 0$. Solve this one-variable problem by the Kuhn–Tucker method, and find conditions for the constraint to bind at the optimum.
5. Consider the problem

$$\max xz + yz \text{ subject to } x^2 + y^2 + z^2 \leq 1$$

(a) Write down the Kuhn–Tucker conditions.

(b) Solve the problem.

6. A firm uses three inputs X, Y and Z to manufacture two goods A and B. The requirements per tonne are as follows.

   A: 5 loads of X, 4 containers of Y and 6 hours of Z

   B: 5 loads of X, 6 containers of Y and 2 hours of Z

   Each tonne of A brings in £400 profit and each tonne of B brings in £300. What combination of A and B should the firm produce to maximize profit if it has at its disposal 150 loads of X, 240 containers of Y and 150 hours of Z?

7. A firm makes the two food products A and B and the contribution to profit is £2 per unit of A and £3 per unit of B. There are three stages in the production process: cleaning, mixing and tinning. The number of hours of each process required for each product and the total number of hours available for each process are given in Table 5.A1. Given these constraints what combination of A and B should the firm produce to maximize profit?

8. firm manufactures two goods A and B which require the two inputs K and L in the following amounts:

   1 unit of A requires 6 units of K and 4 of L

   1 unit of B requires 8 units of K and 10 of L

   The firm has at its disposal 96 units of K and 100 of L. The per-unit profit of A is £600 and for B the figure is £300. The firm is under contract to produce a minimum of 6 units of B. How many units of A should it make to maximize profit?

9. A consumer spends all her income of £120 on the two goods A and B. Good A costs £10 a unit and good B costs £15. What combination of A and B will she purchase if her utility function is $U = 4A^{0.5}B^{0.5}$?

10. If a firm faces the production function $Q = 4K^{0.5}L^{0.5}$, what is the maximum output it can produce for a budget of £200? The prices of K and L are given as £4 per unit and £2 per unit respectively.

11. A firm faces the production function $Q = 2K^{0.2}L^{0.6}$ and can buy L at £240 a unit and K at £4 a unit.

    (a) If it has a budget of £16,000 what combination of K and L should it use to maximize output?

    (b) If it is given a target output of 40 units of Q what combination of K and L should it use to minimize the cost of this output?

12. A firm has a budget of £1,140 and can buy inputs K and L at £3 and £8 respectively a unit. Its output is determined by the production function

$$Q = 6K + 20L - 0.025K^2 - 0.05L^2$$

for positive values of Q. What is the maximum output it can produce?

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7

INTRODUCTION TO MATRICES

Unit Structure

7.0 OBJECTIVES

After going to this unit you will be able to

- Understand the meaning of matrix.
- Understand the various types of matrices.
- Calculate the addition of matrices.
- Calculate the subtraction of matrices
- Calculate the multiplication of matrices.
- Understand and derive the transpose of matrix

7.1 INTRODUCTION AND MEANING OF MATRIX

The English mathematician Cayley invented the concept of matrices in 1858. Matrix is an array of numbers in rectangular brackets. In another words, the collection of vectors is called as matrix. Numbers in matrices are written in square or rectangular brackets [ ] or parentheses () or pairs of double bars ||. Matrix is denoted by capital letters as A, B, C, D, E, F, --- ----, Z.

For example,

$$A=\begin{bmatrix} 9 & 8 \\ 7 & 6 \end{bmatrix}, \qquad B = \begin{bmatrix} 7 & 8 & 5 \\ 4 & 1 & 3 \\ 7 & 8 & 3 \end{bmatrix}$$

In the matrices there are horizontal and vertical lines. Horizontal lines are known as rows and vertical lines are called as columns.

General Form of Matrix

A matrix is an rectangular array of elements or numbers arranged in rows and columns.

Suppose, there are 2 girls whose are Friends: Anjali (A) and B (Bhagyashri).

A has a set of 3 kurtis, 2 pants

B has a set of 4 kurtis, 2 pants.

Now we will arrange this data in the form of matrix.

$$\begin{array}{ccc} & \begin{matrix} \text{Kurtis} & \text{Pants} \\ (K) & (P) \end{matrix} & \\ \begin{matrix} \text{Anjali } (A) \rightarrow \\ \text{Bhagyashri } (B) \rightarrow \end{matrix} & \begin{bmatrix} 3 & 2 \\ 4 & 2 \end{bmatrix} & \begin{matrix} \rightarrow 1^{st} \text{ Row} \\ \rightarrow 2^{nd} \text{ Row} \end{matrix} \\ & \begin{matrix} \downarrow & \downarrow \\ 1^{st} \text{ Column} & 2^{nd} \text{ Column} \end{matrix} & \end{array}$$

If we denote to this matrix by C, then we can write as below –

$$C = \begin{bmatrix} 3 & 2 \\ 4 & 2 \end{bmatrix}$$

The general form of Matrix is :

$$A = \begin{bmatrix} a_{11} & a_{12} & .. & .. & a_{1n} \\ a_{21} & a_{22} & .. & .. & a_{2n} \\ \vdots & \vdots & .. & .. & \vdots \\ \vdots & \vdots & .. & .. & \vdots \\ a_{m1} & a_{m2} & .. & .. & a_{mn} \end{bmatrix}$$

It is a matrix of m rows and n columns.

Order of Matrix :

Order of matrix is the number of the rows and columns of matrix. Order of matrix is also known as dimension of matrix or size of matrix.

If, there is 2 rows and 3 columns in a matrix, the order of matrix is 2x3.

The number of rows is written before the number of columns while discussing the order of matrix.

For example,

i. $A = \begin{bmatrix} 5 & 4 & 6 \\ 7 & 3 & 9 \end{bmatrix}$, then find the order of A.

In the above matrix, rows are 2 and columns are 3. So, the order of matrix is 2x3 which is written as –

$A = \begin{bmatrix} 5 & 4 & 6 \\ 7 & 3 & 9 \end{bmatrix}_{2x3}$

ii. $B = \begin{bmatrix} 2 & 3 & 4 & 1 & 4 \end{bmatrix}$ find the order of B.

Order of matrix B is 1x5

$\therefore B = \begin{bmatrix} 2 & 3 & 4 & 1 & 4 \end{bmatrix}_{1x5}$

Check your progress :

Find the order of following matrices –

i. $A=\begin{bmatrix} 2 & 4 & 5 & 6 \\ 4 & 3 & 2 & 1 \end{bmatrix}$

ii. $B=\begin{bmatrix} 5 \\ 3 \\ 2 \\ 1 \end{bmatrix}$

iii. $C=\begin{bmatrix} 5 & 3 & 2 \\ 1 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}$

7.2 TYPES OF MATRICES

The several types of matrices have been given in the below figure.

Figure No. 7.1

Types of Matrices

The explanation of each matrix in detail is as follows-

1. Row Matrix:

If a matrix having only one row and n columns, it is called as ‘Row Matrix’ or ‘Row Vector’. The row of the matrix is a ‘vector’.

For example,

- $A = \begin{bmatrix} 5 & 2 & 3 & 4 \end{bmatrix}_{1x4}$
- $B = \begin{bmatrix} 7 & 3 & 2 & 1 & 5 & 6 & 7 \end{bmatrix}_{1x7}$

2. Column Matrix :

If à matrix having only one column and n number of rows, it is known as 'Column Matrix' or 'Column Vector'. Column of the matrix is a 'Vector'.

For example,

$$A = \begin{bmatrix} 5 \\ 6 \\ 3 \\ 2 \\ 1 \end{bmatrix}_{5x1}, B = \begin{bmatrix} 5 \\ 4 \\ 2 \end{bmatrix}_{3x1}, C = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 4 \end{bmatrix}_{4x1}$$

3. Zero or Null Matrix :

If all the numbers or elements of a matrix are zero, this matrix is known as zero matrix or null matrix. Zero matrix or null matrix is denoted by 0.

For example,

i. $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3x3}$

ii. $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{2x3}$

iii. $C = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}_{1x3}$

iv. $D = \begin{bmatrix} 0 \end{bmatrix}_{1x1}$

4. Sub-Matrix :

Suppose matrix A is given matrix. If we delete a few rows and columns, we get a new matrix which is different from previous or given matrix A, it is knows as the 'sub-matrix of A'

For example,

$A=\begin{bmatrix} 7 & 8 \\ 8 & 5 \end{bmatrix}$

Then the sub-matrices of A are –

i) $B=\begin{bmatrix} 7 & 8 \end{bmatrix}$ ii) $C=\begin{bmatrix} 8 & 5 \end{bmatrix}$

iii) $D=\begin{bmatrix} 7 \\ 8 \end{bmatrix}$ iv) $E=\begin{bmatrix} 8 \\ 5 \end{bmatrix}$

5. Equal Matrices:

Suppose, Matrix A and Matrix B are two matrices. If the all elements and order of matrix are same or equal for these two matrices, then we can called that Matrix A and Matrix B are equal matrices (A= B).

For example,

1) If $A = \begin{bmatrix} 5 & 3 & 1 \\ 4 & 2 & 0 \\ 2 & 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 3 & 1 \\ 4 & 2 & 0 \\ 2 & 1 & 2 \end{bmatrix}$,

then A = B means matrix A and matrix B are equal matrices; because matrix A and matrix B have same order (3x3) and all equal elements.

2) If $A = \begin{bmatrix} 4 & 8 \\ 3 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 \\ 4 & 2 \end{bmatrix}$,

then $A \neq B$ means matrix A and matrix B are unequal matrices; corresponding elements are not equal in these matrices.

6. Rectangular Matrix:

If a matrix having the number of rows is greater than number of its columns or the number of columns is greater than the number of rows, then the matrix is known as a 'Rectangular Matrix.' It means m>n or m<n. (m= number of rows & n = number of columns)

For Example,

1. $A= \begin{bmatrix} 5 & 4 & 3 & 2 \\ 8 & 3 & 4 & 5 \end{bmatrix}$

2. $B= \begin{bmatrix} 5 & 4 \\ 3 & 2 \\ 2 & 1 \end{bmatrix}$

3. $C= \begin{bmatrix} 1 & 2 \end{bmatrix}$

   Note: All non-square matrices are rectangular matrices. It means that square matrix is never rectangular matrix.

7. Square Matrix:

When a matrix having the number of rows is equal to the number of columns, then the matrix is known as a 'Square Matrix.' It means, m=n / number of rows = number of columns.

For Example,

1. $A= \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}$, it is 2x2 square matrix

2. $B= \begin{bmatrix} 5 & 4 & 3 \\ 8 & 7 & 6 \\ 7 & 5 & 2 \end{bmatrix}$, it is 3x3 square matrix

3. C= $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{bmatrix}$, it is 4x4 square matrix

There are several sub types of square matrix as follows-

7-A) Identity or Unit Matrix:

If a square matrix having all the diagonal elements are one and non-diagonal elements are zero, It is known as an 'Identity Matrix' or 'Unit Matrix'. The Identity Matrix Is denoted by 'I'.

For Example,

1. A= $[1]$, it is 1x1 unit matrix

2. B= $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, it is 2x2 unit matrix

3. C= $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, it is 3x3 unit matrix

7-B) Diagonal Matrix:

A square matrix in which all non-diagonal elements are zero and diagonal elements are any number, it is known as diagonal matrix. In other words, in the diagonal matrix all elements are zero excluding the main diagonal matrix.

For Example,

A= $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

7-C) Scalar Matrix::

Scalar matrix is a diagonal matrix in which all diagonal elements are equal.

For example,

1. A= $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

2. B= $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Note: Each identity or unit matrix is a scalar matrix, but scalar matrix is never the identity or unit matrix.

7-D) Symmetric Matrix:

Suppose A is a square matrix. If the A^T(Transpose of A) is equal to A, then matrix A is called as 'Symmetric Matrix'.

Symbolically,

$A^T=A \Rightarrow A$ is symmetric matrix.

For Example,

1. $A=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$

2. $B=\begin{bmatrix}5 & 0 & 0\\0 & 3 & 0\\0 & 0 & 2\end{bmatrix}$

3. $C=\begin{bmatrix}3 & 0 & 0\\0 & 3 & 0\\0 & 0 & 3\end{bmatrix}$

Note: Identity or Unit, Diagonal and Scalar Matrices are Symmetric Matrices.

7-E) Skew Symmetric Matrix:

Skew symmetric matrix is always square matrix. Suppose, A is a square matrix. If the A^T(Transpose of A) is equal to -A, then A is known as 'Skew Symmetric Matrix'.

Symbolically,

$A^T=-A \Rightarrow A$ is skew symmetric matrix.

It means, in this case the transpose of a square matrix is equal to the negative of the original matrix and diagonal elements are zero.

For example,

$A=\begin{bmatrix}0 & 3\\-3 & 0\end{bmatrix}$

7-F) Idempotent Matrix:

Idempotent Matrix is a symmetric matrix for which transpose of A is equal to A ($A^T=A$) and square of A is equal to A ($A^2=A$)

For Example,

$A=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

i. $A^2=A \Rightarrow \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

ii. $A^T=A \Rightarrow \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

Note : Identity or Unit Matrix is Idempotent matrix.

7-G) Triangular Matrix:

A Triangular Matrix is a symmetric matrix which having all the elements above or below the principle diagonal are zero.

There are mainly two subtypes of Triangular Matrix as below:

- **Upper Triangular Matrix :**
  The Triangular Matrix in which all the elements below the principle diagonal (Lower Diagonal elements) are zero, it is known as Upper Triangular Matrix.

For Example,

A= $\begin{bmatrix} 5 & 3 \\ 0 & 1 \end{bmatrix}$

B= $\begin{bmatrix} 5 & 2 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

- **Lower Triangular Matrix :**
  The Triangular Matrix in which all the elements above the principle diagonal (Upper Diagonal elements) are zero, it is known as Lower Triangular Matrix.

For Example,

A= $\begin{bmatrix} 5 & 0 \\ 3 & 1 \end{bmatrix}$

B= $\begin{bmatrix} 5 & 0 & 0 \\ 2 & 3 & 0 \\ 1 & 2 & 3 \end{bmatrix}$

7-H) Singular Matrix:

If the determinant of a square matrix is equal to zero, that matrix is known as singular matrix.

Symbolically,

|A| = 0 => Matrix A is a singular matrix

For Example,

If, A= $\begin{bmatrix} 5 & 5 \\ 2 & 2 \end{bmatrix}$, then

|A| = Determinant of A = (5x2) – (5x2)

= (10) – (10)

= 10 – 10

= 0

∴ |A| = 0

7-H) Non-Singular Matrix:

If the determinant of a square matrix is equal to non-zero, it is called as non-singular matrix.

Symbolically,

$|A| \neq 0$ => Matrix A is a non-singular matrix

For Example,

If, A= $\begin{bmatrix} 5 & 4 \\ 2 & 3 \end{bmatrix}$, then

$|A| = (5x3) – (4x2)$

$= 15 – 8$

$= 7$

$\therefore |A| \neq 0$

7-J) Hermitian Matrix:

If transpose of a square matrix A is equal to matrix A (A^T=A), then matrix A is known as 'Hermitian Matrix'.

7-K) Skew-Hermitian Matrix:

If transpose of a square matrix A is not equal to matrix A ($A^T \neq A$), then matrix A is known as 'Skew-Hermitian Matrix'.

7-L) Orthogonal Matrix:

If A is a square matrix,

If, $A^TA = AA^T = I$, it is known as 'Orthogonal Matrix'.

7-M) Commute:

If A and B are square matrices and AB = +BA, then A and B are called 'commutative' or 'commute'.

7-N) Anti- Commute:

If A and B are square matrices and AB = -BA, then A and B are called as 'Anti-commute'.

7.3 ALGEBRA OF MATRICES OR OPERATIONS WITH MATRICES

In this case, addition of matrices, subtraction of Matrices and multiplication of matrices have been discussed.

7.3.1 Addition of Matrices:

If A and B are two matrices and we have to take sum or addition of these two matrices, then for that first condition is that these two matrices have same order i.e. the same number of rows and columns. It means the number of rows of matrix A is equals to the number of rows of matrix B and number of columns of matrix A is equal to the number of columns of matrix B. This condition is necessary for taking sum or addition of two or more than two matrices.

The sum of same order two matrices is obtained by adding corresponding elements of the two matrices.

For Example,

1. A= $\begin{bmatrix} 5 & 8 \\ 7 & 9 \end{bmatrix}_{2x2}$ and B = $\begin{bmatrix} 7 & 4 \\ 9 & 3 \end{bmatrix}_{2x2}$

Find A + B.

Solution :

A + B= $\begin{bmatrix} 5+7 & 8+4 \\ 7+9 & 9+3 \end{bmatrix} = \begin{bmatrix} 12 & 12 \\ 16 & 12 \end{bmatrix}_{2x2}$

2. A= $\begin{bmatrix} -3 & 4 \\ 5 & 7 \end{bmatrix}_{2x2}$ and B = $\begin{bmatrix} -4 & 11 \\ 3 & 13 \end{bmatrix}_{2x2}$

Find A + B.

Solution :

A + B $= \begin{bmatrix} (-3)+(-4) & 4+11 \\ 5+3 & 7+13 \end{bmatrix}$

$= \begin{bmatrix} -7 & 15 \\ 8 & 20 \end{bmatrix}_{2x2}$

3. A= $\begin{bmatrix} 10 & 12 & 13 \\ 25 & 13 & 11 \\ 15 & 14 & 16 \end{bmatrix}$ and B = $\begin{bmatrix} 20 & 5 & 7 \\ 8 & 3 & 2 \\ 9 & 5 & 4 \end{bmatrix}$

Find A + B.

Solution :

A + B= $\begin{bmatrix} 10+20 & 12+5 & 13+7 \\ 25+8 & 13+3 & 11+2 \\ 15+9 & 14+5 & 16+4 \end{bmatrix}$

$= \begin{bmatrix} 30 & 17 & 20 \\ 33 & 16 & 13 \\ 24 & 19 & 20 \end{bmatrix}$

Check your progress:

1. A= $\begin{bmatrix} -15 & -18 \\ 18 & 20 \end{bmatrix}$ and B = $\begin{bmatrix} 17 & 18 \\ 14 & 13 \end{bmatrix}$, Find A + B.

2. A= $\begin{bmatrix} 30 & -50 & 40 \\ 20 & 30 & -40 \\ 10 & 50 & -70 \end{bmatrix}$ and B = $\begin{bmatrix} 5 & 7 & 8 \\ 9 & 3 & 2 \\ 1 & 4 & 5 \end{bmatrix}$

Find A + B.

3. A= $\begin{bmatrix} 5 & 2 \\ 9 & 3 \end{bmatrix}$, B = $\begin{bmatrix} 11 & 15 \\ 17 & 19 \end{bmatrix}$ and C = $\begin{bmatrix} 18 & 15 \\ 13 & 14 \end{bmatrix}$

Find A + B + C, A + C, B + C and A + B.

Laws or Properties of Matrix Addition:

a) Associative Law of Addition:
A + B = B + A

b) Commutative Law of Addition:
A + (B + C) = (A + B) + C

c) Existence of Identity:
A + 0 = 0 + A = A

d) Existence of Inverse:
If A+X = 0, then X = -A

7.3.2 Subtraction of Matrices:

If the matrices A and B are of the same order (m x n), then these matrices can be subtracted.

If, A = $[a_{ij}]$ & B = $[b_{ij}]$, then

A – B = $[a_{ij} - b_{ij}]$

In other words, while subtracting two matrices, corresponding elements are subtracted. After subtraction when we get new matrix, it will be of the same order m x n.

For Ex.

A= $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2x2}$ and B = $\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}_{2x2}$

A – B = $\begin{bmatrix} a_{11} - b_{11} & a_{12} - b_{12} \\ a_{21} - b_{21} & a_{22} - b_{22} \end{bmatrix}_{2x2}$

Matrix (A-B) is of the same order 2 x 2

Prob.1 If A= $\begin{bmatrix} 5 & 3 & 2 \\ 7 & 8 & 9 \end{bmatrix}$ and B = $\begin{bmatrix} 8 & 7 & 3 \\ 5 & 4 & 1 \end{bmatrix}$,

Find A – B & B – A

Solution :

i) $A - B = \begin{bmatrix} 5 & 3 & 2 \\ 7 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 8 & 7 & 3 \\ 5 & 4 & 1 \end{bmatrix}$

$A - B = \begin{bmatrix} 5-8 & 3-7 & 2-3 \\ 7-5 & 8-4 & 9-1 \end{bmatrix}$

$A - B = \begin{bmatrix} -3 & -4 & -1 \\ 2 & 4 & 8 \end{bmatrix}$

ii) $B - A = \begin{bmatrix} 8 & 7 & 3 \\ 5 & 4 & 1 \end{bmatrix} - \begin{bmatrix} 5 & 3 & 2 \\ 7 & 8 & 9 \end{bmatrix}$

$B - A = \begin{bmatrix} 8-5 & 7-3 & 3-2 \\ 5-7 & 4-8 & 1-9 \end{bmatrix}$

$B - A = \begin{bmatrix} 3 & 4 & 1 \\ -2 & -4 & -8 \end{bmatrix}$

Prob.2 If $A= \begin{bmatrix} 4 & 5 \\ 6 & 7 \end{bmatrix}$ *and* $B = \begin{bmatrix} 8 & 9 \\ 7 & 8 \end{bmatrix}$,

Find A – B & B – A.

By subtracting correspondence elements of matrix B from A

i) $A - B = \begin{bmatrix} 4-8 & 5-9 \\ 6-7 & 7-8 \end{bmatrix}$

$A - B = \begin{bmatrix} -4 & -4 \\ -1 & -1 \end{bmatrix}$

ii) $B - A = \begin{bmatrix} 8 & 9 \\ 7 & 8 \end{bmatrix} - \begin{bmatrix} 4 & 5 \\ 6 & 7 \end{bmatrix}$

By subtracting the corresponding elements of matrix A from B.

$B - A = \begin{bmatrix} 8-4 & 9-5 \\ 7-6 & 8-7 \end{bmatrix}$

$B - A = \begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix}$

Exercise –

Prob 1. If $A= \begin{bmatrix} 11 & 23 & 4 & 8 \\ 3 & 12 & 9 & 7 \end{bmatrix}$ *and* $B = \begin{bmatrix} 9 & 7 & 5 & 3 \\ 5 & 3 & 4 & 1 \end{bmatrix}$

Find A – B & B – A.

Prob 2. If $A= \begin{bmatrix} 7 \\ 8 \end{bmatrix}$ *and* $B = \begin{bmatrix} 11 \\ 13 \end{bmatrix}$, then

Find A – B & B – A.

Prob 3. If $A= \begin{bmatrix} 3 & 4 & 8 & 9 \end{bmatrix}$ *and* $B = \begin{bmatrix} 7 & 11 & 9 & 4 \end{bmatrix}$

Then find A – B & B – A.

Note : Two matrices A and B can be added or subtracted if and only if they those matrices have same order or dimension i.e., the number of column of Matrix A is equal to the number of column of Matrix B and the number of Rows of Matrix A is equal to the number of Rows of Matrix B. In other words, two matrices of the same order are said to be conformable for subtraction or addition.

7.3.3 Multiplication of Matrices :

i. Scalar Multiplication

ii. Multiplication of two vectors.

iii. Multiplication of two matrices.

i. Scalar Multiplication

In this case, to multiply, a Matrix A of order m x n by a scalar or a number 'K', we have to multiply every element of matrix A by the scalar K. The order of new matrix KA (after multiplication) is also m x n as the Matrix A.

So,

KA = K[a_{ij}] = [Ka_{ij}]

For Example,

Prob 1. A= $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and scalar is K, then find KA.

Solution -

KA= K $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

By multiplying the scalar to each element of Matrix A

KA= $\begin{bmatrix} Ka & Kb \\ Kc & Kd \end{bmatrix}$

Prob 2. A= $\begin{bmatrix} 5 & 3 & 2 \\ 7 & 8 & 9 \end{bmatrix}$ and K=3, then find KA.

Solution -

KA= 3 x $\begin{bmatrix} 5 & 3 & 2 \\ 7 & 8 & 9 \end{bmatrix}$

KA= $\begin{bmatrix} 5x3 & 3x3 & 2x3 \\ 7x3 & 8x3 & 9x3 \end{bmatrix}$

KA= $\begin{bmatrix} 15 & 9 & 6 \\ 21 & 24 & 27 \end{bmatrix}$

Exercise :

Prob 1. A= $\begin{bmatrix} 3 & 4 & 5 & 6 & 7 \end{bmatrix}$ and K=4, then find KA.

Prob 2. A= $\begin{bmatrix} 5 & 7 \\ 3 & 6 \\ 2 & 5 \end{bmatrix}$ and K=9, then find KA.

Prob 3. A= $\begin{bmatrix} 3 & 4 & 5 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$ and K=7, then find KA.

ii. **Multiplication of Two Vectors**

In this case, two matrices A and B can be multiplied it and only if the number of columns of the first Matrix (Matrix A) must be equal to the number of rows of the second Matrix (Matrix B). In another words, multiplication of two vectors which are A and B (i.e. Column and Row) is possible only when A and B are 'conformable'.
when, number of columns of the first Matrix is equal to the number of rows of the second matrix, these two matrices (First & Second namely A and B.) are said to be conformable for multiplication. Multiplication of two vectors means multiplication or product of a Row Matrix and a column matrix.

Example :

1. If A= $\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}_{1x3}$ and B = $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$, find AB.

Solution:

Multiplication of Matrix A and Matrix B (i.e. AB) is possible or defined, because number of columns of first matrix (Matrix A) is equal to number of rows of second matrix (Matrix B). In other words, Matrix A and Matrix B are conformable.

Steps :

i. Firstly, we have to confirm that the multiplication of two vectors is possible or not.

'The number of columns of the first matrix should be the same to number of rows of second matrix' for multiplication of two vectors.

ii. Then multiply the elements of first matrix (row matrix) and the elements of second matrix as below-

If A= $\begin{bmatrix} 3 & 2 & 1 \end{bmatrix}_{1x3}$ and B = $\begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix}_{3x1}$, find AB.

AB= $\begin{bmatrix} 3 & 2 & 1 \end{bmatrix}_{1x3}$ X $\begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix}_{3x1}$

= [(3x5) + (2x4) + (1x3)]

= [15 + 8 + 3]

= $[26]_{1x1}$

iii. After multiplication of two vectors you will get new matrix which will be order of number of rows of first matrix x number of columns of second matrix.

Prob 1. If A= $\begin{bmatrix} -5 & 7 & -8 \end{bmatrix}$ and B = $\begin{bmatrix} 11 \\ -12 \\ 25 \end{bmatrix}$, then find AB.

Solution : Matrix A and Matrix B are conformable matrices. So solution is possible.

AB= $\begin{bmatrix} -5 & 7 & -8 \end{bmatrix}$ X $\begin{bmatrix} 11 \\ -12 \\ 25 \end{bmatrix}$

=[(-5 x 11) + (7 x (-12)) + (-8 x 25)]

=[(-55) + (-84) + (-200)]

=[-55 -84 -200]

=[-339]

Prob 2. If A = $\begin{bmatrix} 8 \\ 3 \end{bmatrix}_{2x1}$ $and\ B = \begin{bmatrix} 15 & 9 \end{bmatrix}_{1x2}$, find AB

Solution : Matrix A and Matrix B are conformable because number of columns of first matrix (Matrix A) is equal to the number of rows of second matrix (Matrix B). So, solution is possible.

AB $=\begin{bmatrix} 8 \\ 3 \end{bmatrix}_{2x1} X \begin{bmatrix} 15 & 9 \end{bmatrix}_{1x2}$

Now multiply by row elements of Matrix A to the column elements of Matrix B as follow

AB $=\begin{bmatrix} 8x15 & 8x9 \\ 3x15 & 3x9 \end{bmatrix}$

$=\begin{bmatrix} 120 & 72 \\ 45 & 27 \end{bmatrix}_{2x2}$

Exercise :

Prob 1. If A = $\begin{bmatrix} 13 \\ -9 \\ 7 \end{bmatrix}_{3x1}$ $and\ B = \begin{bmatrix} 8 & 3 & 4 & 5 \end{bmatrix}_{1x4}$, find AB

Prob 2. If A = $\begin{bmatrix} 13 & 11 & 9 \end{bmatrix}_{1x3}\ and\ B = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}_{3x1}$, find AB

Prob 3. If A = $[15 \quad 2]_{1x2}$ $and\ B = \begin{bmatrix} 8 \\ 3 \end{bmatrix}_{2x1}$, find AB

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iii. Multiplication of Two Matrices

In the case of matrix A and matrix B when, the number of columns of the first matrix (i.e. Matrix A) must be equal to the number of rows of second matrix (i.e. Matrix B), two matrices A and B can be multiplied.

In another words, the multiplication of two Matrices A and B is possible when Matrix A and Matrix B are conformable.

Examples :

Prob.1- If, A = $\begin{bmatrix} 5 & 4 \\ 3 & 2 \\ 7 & 6 \end{bmatrix}_{3x2}$ $and\ B = \begin{bmatrix} 7 & 8 \\ 9 & 6 \end{bmatrix}_{2x2}$, then find AB.

The given above two matrices are confirmable. Therefore, AB is possible. For multiplication of two matrices, each element of the row is multiplied into the corresponding element of the column and then the products are summed.

$$A = \begin{bmatrix} 5x7 + 4x9 & 5x8 + 4x6 \\ 3x7 + 2x9 & 3x8 + 2x6 \\ 7x7 + 6x9 & 7x8 + 6x6 \end{bmatrix}$$

$$= \begin{bmatrix} 35 + 36 & 40 + 24 \\ 21 + 18 & 21 + 12 \\ 49 + 54 & 56 + 36 \end{bmatrix} = \begin{bmatrix} 71 & 64 \\ 39 & 33 \\ 103 & 92 \end{bmatrix}_{3x2}$$

Prob.2- If, A = $\begin{bmatrix} 5 & 6 & 2 \\ 7 & 8 & 2 \end{bmatrix}_{2x3}$ $and\ B = \begin{bmatrix} 2 & 0 \\ 1 & 5 \\ 3 & 1 \end{bmatrix}_{3x2}$, then find AB.

Solu.- The given two matrices A and B are conformable. Hence, AB (Multiplication of Matrix A and B) is possible or defined. And after that the order of matrix is 2x2.

$$AB = \begin{bmatrix} 5x2 + 6x1 + 2x3 & 5x0 + 6x5 + 2x1 \\ 7x2 + 8x1 + 2x3 & 7x0 + 8x5 + 2x1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 10 + 6 + 6 & 0 + 30 + 2 \\ 14 + 8 + 6 & 0 + 40 + 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 22 & 32 \\ 28 & 42 \end{bmatrix}_{2x2}$$

Prob.3- If, $A = \begin{bmatrix} 7 & 8 & 4 \end{bmatrix}_{1x3}$ $and\ B = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}_{3x1}$, then find AB.

Solu.- The given two matrices A and B are conformable, because the number of columns of Matrix A (First Matrix) are equal to the number of rows of Matrix B (Second Matrix). Therefore, AB is possible. Hence, the order of answer matrix is 1x1.

AB=$[7x5 + 8x7 + 4x9]$

AB=$[35 + 56 + 36]$

AB=$[127]_{1x1}$

Prob.4- If, $A = \begin{bmatrix} 5 & 7 \\ 4 & 4 \end{bmatrix}_{2x2}$ $and\ B = \begin{bmatrix} 7 \\ 3 \end{bmatrix}_{2x1}$, then find AB.

Solu.- Matrix A and Matrix B are conformable. So AB is possible. Hence, the order of answer is 2x1.

$AB = \begin{bmatrix} 5x7 + 7x3 \\ 4x7 + 4x3 \end{bmatrix}$

$AB = \begin{bmatrix} 35 + 21 \\ 28 + 12 \end{bmatrix}$

$AB = \begin{bmatrix} 56 \\ 40 \end{bmatrix}_{2x1}$

Prob.5- If, $A = \begin{bmatrix} 4 & 7 & 8 \end{bmatrix}_{1x3}$ $and\ B = \begin{bmatrix} 4 & 7 & 3 \\ 8 & 4 & 5 \end{bmatrix}_{2x3}$, then find AB.

Solu.- AB is not possible, because the number of columns of Matrix A is not equal to number of rows in Matrix B. In other words, the given matrices are not conformable.

Therefore, the multiplication of these two matrices are not possible.

Exercise:

Prob.1- If, $A = \begin{bmatrix} 7 & 8 & 3 \\ 5 & 4 & 6 \\ 8 & 9 & 2 \end{bmatrix}_{3x3}$ $and\ B = \begin{bmatrix} 7 \\ 8 \\ 4 \end{bmatrix}_{3x1}$, then find AB.

Prob.2- If, $A = \begin{bmatrix} 4 & 5 & 8 \\ 3 & 2 & 1 \end{bmatrix}_{2x3}$ $and\ B = \begin{bmatrix} 5 & 4 \\ 3 & 2 \\ 1 & 3 \end{bmatrix}_{3x2}$, then find AB.

Prob.3- If, $A = \begin{bmatrix} 4 & 3 \\ 7 & 1 \end{bmatrix}_{2x2}$ $and\ B = \begin{bmatrix} 7 & 8 & 3 & 4 \\ 4 & 2 & 4 & 5 \\ 3 & 9 & 8 & 4 \end{bmatrix}_{3x4}$, then find AB.

7.4 TRANSPOSE OF A MATRIX.

Transpose of a Matrix A is a new matrix in which the rows and columns of Matrix A have been interchanged. Rows should be transferred or converted into columns and columns should be converted or transferred into rows. Transpose of Matrix is denoted by A' or A^T.

Therefore,

If A = $[a_{ij}]$
A' or A^T = $[a_{ji}]$

Examples:

Prob.1 If, $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2x2}$, find A^T.

Solu. $A^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}_{2x2}$

Prob.2 If, $A = \begin{bmatrix} 7 & 8 & 4 \\ 9 & 5 & 3 \end{bmatrix}_{2x3}$, find A^T.

Solu. $A^T = \begin{bmatrix} 7 & 9 \\ 8 & 5 \\ 4 & 3 \end{bmatrix}_{3x2}$

Exercise –

Prob.1 If, $A = \begin{bmatrix} 7 & 8 & 9 & 5 \end{bmatrix}$, find A^T.

Prob.2 If, $A = \begin{bmatrix} 7 & 8 \\ 4 & 3 \end{bmatrix}$, find A^T.

Prob.2 If, $A = \begin{bmatrix} 8 & 9 & 7 \\ 5 & 4 & 3 \end{bmatrix}$, find A^T.

7.5 SUMMARY

The important concepts regarding the matrices are as below-

- **Trace -**
  Sum of the diagonal elements of a square matrix A is called the 'Trace of A'. The trace of 'A' is denoted by Tr (A).

  Example=

  If $A = \begin{bmatrix} 4 & 5 & 3 \\ 7 & 8 & 9 \\ 1 & 2 & 3 \end{bmatrix}$, then

  Tr (A) = 4+8+3 = 15.

- **Minor :**
  If we delete the row and the column containing the element a_{ij}, we obtain a square matrix of order n-1 and the determinant of this square matrix is called the 'Minor' of the element a_{ij} and is denoted by M_{ij}.
  In other words, the determinant, obtained by omitting the row and column containing a particular element is called as the 'Minor' of that element.

- **Cofactor:**
  The minor(M_{ij}) multiplied by $(-1)^{i+j}$ is called as the 'Cofactor of the element a_{ij}.
  Cofactor is denoted by A_{ij}. In another words, a cofactor matrix in which every element of a matrix, a_{ij}, is replaced by its cofactor A_{ij}.
  Therefore, $A_{ij} = (-1)^{i+j} M_{ij}$. Hence, the signed Minors are called 'cofactors'.

- **Determinant:**
  A determinant is a single number which is associated with a square matrix. It is denoted by |A|.

  If A=[3], then |A| = 3

  If $A = \begin{bmatrix} 4 & 5 \\ 3 & 2 \end{bmatrix}$, then

  |A|=(4x2) – (5x3)

= 8-15 =-7

- **Transpose of a Matrix:**
  In case of Matrix A, transpose of Matrix is denoted by A' or A^T.
  For getting the transpose of a matrix, rows should be converted into columns and columns should be converted into rows.

  If, $A = \begin{bmatrix} 5 & 4 & 3 \\ 9 & 8 & 7 \end{bmatrix}_{2x3}$, then

  $A^T = \begin{bmatrix} 5 & 9 \\ 4 & 8 \\ 3 & 7 \end{bmatrix}_{3x2}$

- **Adjoint of the Matrix :**
  Transpose of a cofactor matrix (A_{ij}) is called as the 'Adjoint of the Matrix A' or the 'Adjugate of the Matrix A'.

  It is denoted by Adj. A of matrix A.
  To get the Adjoint of a Matrix, rows of the cofactor matrix should be converted into columns and columns of the cofactor matrix should be converted into rows.

- **Inverse of a Matrix :**
  Inverse of a Matrix is denoted by A^{-1}. A^{-1} is said to be the 'Inverse of a square matrix A', if it satisfies the following property -

$AB = BA = I$ (Identity Matrix)

$AA^{-1} = A^{-1}A = I$

In other words, one matrix is the inverse of the another, if and only if their product is the Identity Matrix.

Inverse of a Matrix is also called a Reciprocal Matrix.

Only square matrix posses inverses.

The necessary and sufficient condition for a square Matrix 'A' to possess an inverse is that,

$|A| \neq 0$ i.e., A is non-singular matrix.

7.6 QUESTIONS

Q.1 Write a short note on -

i. Trace

ii. Transpose of a Matrix

iii. Rank of a Matrix

iv. Inverse of a Matrix

Q.2 Explain the various types of matrices.

Q3. $A = \begin{bmatrix} 5 & 7 \\ 8 & 3 \end{bmatrix}_{2x2}$ and $B = \begin{bmatrix} 8 & 4 \\ 5 & 1 \end{bmatrix}_{2x2}$,

Find A + B and A – B.

❄❄❄❄❄❄❄

8

INVERSE MATRIX AND SOLVING LINEAR EQUATIONS WITH MATRICES

Unit Structure

8.0 OBJECTIVES

After going to this unit you will be able to -

- Understand the process to calculate the inverse of matrix.
- Understand the solving methods of linear equations with matrices.

8.1 INTRODUCTION

In this unit, we study the calculation of inverse matrix and the various methods of solving the linear equations with matrices as matrix inversion m technique, Cramer's rule and Guass elimination method etc.

Inverse of a matrix is called as "Reciprocal Matrix'. Only square matrices possess inverses. Inverse of a square matrix is denoted by A^{-1}. One matrix is the inverse of the another if and only if their product is the identity matrix.

$AB = BA = I$

8.2 INVERSE MATRIX

Inverse matrix is denoted by A^{-1}. Only square matrices possess inverses. There is a necessary and sufficient condition for a square matrisx A to possess an inverse is that,

$|A| \neq 0$

where,

$|A|$ = determinant of matrix A.

$|A| \neq 0$ i.e., A is a non-singular matrix.

Formula of Inverse Matrix (A^{-1}):

$A^{-1} = \frac{Adj.of\ A}{|A|}$

Steps to calculate Inverse Matrix :

- First, check that the given matrix is a square matrix.
- Find the determinant ($|A|$) of the given matrix.

If, $|A| \neq 0$, then go to the step 3. If, $|A| = 0$, then there is no need to calculate inverse. Because we cannot find inverse.

- Compute minors (Mij)
- Compute cofactor matrix (Aij).
- Compute Adjoint of matrix (Adj.A) means transpose of the cofactor matrix.
- Compute inverse of matrix (A^{-1}) using below formula

  $A^{-1} = \frac{Adj.of\ A}{|A|}$

- Check the inverse by using

  $A^{-1} A = I$

Examples :

1. If $A = \begin{bmatrix} 4 & 3 \\ 5 & 6 \end{bmatrix}_{2x2}$ Find the inverse of the matrix.

Solution –

i) First, find the determinant of Matrix A.

$|A| = \begin{bmatrix} 4 & 3 \\ 5 & 6 \end{bmatrix} = (4 \text{ x } 6) – (3 \text{ x } 5)$

$|A| = 24 – 15 = 9.$

$\therefore |A| \neq 0$, then follow the next step.

ii) Find the minors of A

Minors of $A = \begin{bmatrix} 6 & 5 \\ 3 & 4 \end{bmatrix}$

iii) Find the cofactor matrix

Cofactor of A = $\begin{bmatrix} 6 & -5 \\ -3 & 4 \end{bmatrix}$

iv) Find adjoint of A i.e., transpose of cofactor matrix

Adj. of A = $\begin{bmatrix} 6 & -3 \\ -5 & 4 \end{bmatrix}$

v) Find inverse of matrix (A^{-1})

$A^{-1} = \frac{Adj. of\ A}{|A|}$

$A^{-1} = \frac{\begin{bmatrix} 6 & -3 \\ -5 & 4 \end{bmatrix}}{9}$

$A^{-1} = \begin{bmatrix} \frac{6}{9} & \frac{-3}{9} \\ \frac{-5}{9} & \frac{4}{9} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{-1}{3} \\ \frac{-5}{9} & \frac{4}{9} \end{bmatrix}$

vi) Check the inverse (answer matrix) by using

$A^{-1} A = I$

$= \begin{bmatrix} \frac{2}{3} & \frac{-1}{3} \\ \frac{-5}{9} & \frac{4}{9} \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 5 & 6 \end{bmatrix}$

$= \begin{bmatrix} \left(\frac{2}{3} x\, 4\right) + \left(\frac{-1}{3} x\, 5\right) & \left(\frac{2}{3} x\, 3\right) + \left(\frac{-1}{3} x\, 6\right) \\ \left(\frac{-5}{9} x\, 4\right) + \left(\frac{4}{9} x\, 5\right) & \left(\frac{-5}{9} x\, 3\right) + \left(\frac{4}{9} x\, 6\right) \end{bmatrix}$

$. = \begin{bmatrix} \left(\frac{8}{3} - \frac{5}{3}\right) & \left(\frac{6}{3} - \frac{6}{3}\right) \\ \left(\frac{-20}{9} + \frac{20}{9}\right) & \left(\frac{-15}{9} + \frac{24}{9}\right) \end{bmatrix}$

$= \begin{bmatrix} \left(\frac{8-5}{3}\right) & \left(\frac{6-6}{3}\right) \\ \left(\frac{-20+20}{9}\right) & \left(\frac{-15+24}{9}\right) \end{bmatrix} = \begin{bmatrix} \frac{3}{3} & \frac{0}{3} \\ \frac{0}{9} & \frac{9}{9} \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

$A^{-1} A = I$

Therefore, answer matrix is correct

2. Find A^{-1}, if A= $\begin{bmatrix} 5 & 3 & 1 \\ 4 & 8 & 2 \\ 6 & 5 & 7 \end{bmatrix}$

Solution –

Find determinant of A

$$|A| = 5\begin{bmatrix}8 & 2\\5 & 7\end{bmatrix} -3\begin{bmatrix}4 & 2\\6 & 7\end{bmatrix} +1\begin{bmatrix}4 & 8\\6 & 5\end{bmatrix}$$

= 5(56 – 10) -3(28 – 12) +1(20 – 48)

=5(46) -3(16) +1(-28)

= 230 -48 -28

|A| = 154

ii) Compute minor of A

$$\text{Minor of A} = \begin{bmatrix}\begin{bmatrix}8 & 2\\5 & 7\end{bmatrix} & \begin{bmatrix}4 & 2\\6 & 7\end{bmatrix} & \begin{bmatrix}4 & 8\\6 & 5\end{bmatrix}\\ \begin{bmatrix}3 & 1\\5 & 7\end{bmatrix} & \begin{bmatrix}5 & 1\\6 & 7\end{bmatrix} & \begin{bmatrix}5 & 3\\6 & 5\end{bmatrix}\\ \begin{bmatrix}3 & 1\\8 & 2\end{bmatrix} & \begin{bmatrix}5 & 1\\4 & 2\end{bmatrix} & \begin{bmatrix}5 & 3\\4 & 8\end{bmatrix}\end{bmatrix}$$

$$= \begin{bmatrix}(56-10) & (28-12) & (20-48)\\(21-5) & (35-6) & (25-18)\\(6-8) & (10-4) & (40-12)\end{bmatrix}$$

$$= \begin{bmatrix}46 & 16 & -28\\16 & 29 & 7\\-2 & 6 & 28\end{bmatrix}$$

iii) Find the cofactor of A

$$\text{cofactor of A} = \begin{bmatrix}46 & -16 & -28\\-16 & 29 & -7\\-2 & -6 & 28\end{bmatrix}$$

iv) Find the Adjoint of A

$$\text{Adj of A} = \begin{bmatrix}46 & -16 & -2\\-16 & 29 & -6\\-28 & -7 & 28\end{bmatrix}$$

v) Find inverse of matrix (A^{-1})

$$A^{-1} = \frac{Adj.of\ A}{|A|}$$

$$A^{-1} = \frac{\begin{bmatrix}46 & -16 & -2\\-16 & 29 & -6\\-28 & -7 & 28\end{bmatrix}}{154}$$

$$= \begin{bmatrix}\frac{46}{154} & \frac{-16}{154} & \frac{-2}{154}\\ \frac{-16}{154} & \frac{29}{154} & \frac{-6}{154}\\ \frac{-28}{154} & \frac{-7}{154} & \frac{28}{154}\end{bmatrix}$$

vi) Check the inverse

$A^{-1} A = I$

$$= \begin{bmatrix} \frac{46}{154} & \frac{-16}{154} & \frac{-2}{154} \\ \frac{-16}{154} & \frac{29}{154} & \frac{-6}{154} \\ \frac{-28}{154} & \frac{-7}{154} & \frac{28}{154} \end{bmatrix} \begin{bmatrix} 5 & 3 & 1 \\ 4 & 8 & 2 \\ 6 & 5 & 7 \end{bmatrix}$$

$$a_{11} = \left(\frac{46}{154} \; x \; 5\right) + \left(\frac{-16}{154} \; x \; 4\right) + \left(\frac{-2}{154} \; x \; 6\right)$$

$$= \frac{230}{154} - \frac{64}{154} - \frac{12}{154}$$

$$= \frac{230 - 64 - 12}{154} = \frac{154}{154} = 1$$

$$a_{12} = \left(\frac{46}{154} \; x \; 3\right) + \left(\frac{-16}{154} \; x \; 8\right) + \left(\frac{-2}{154} \; x \; 5\right)$$

$$= \frac{138}{154} - \frac{128}{154} - \frac{10}{154} = \frac{138 - 128 - 10}{154}$$

$$= \frac{0}{154} = 0$$

$$a_{13} = \left(\frac{46}{154} \; x \; 1\right) + \left(\frac{-16}{154} \; x \; 2\right) + \left(\frac{-2}{154} \; x \; 7\right)$$

$$= \frac{46}{154} - \frac{32}{154} - \frac{14}{154} = \frac{46 - 32 - 14}{154}$$

$$= \frac{0}{154} = 0$$

$$a_{21} = \left(\frac{-16}{154} \; x \; 5\right) + \left(\frac{29}{154} \; x \; 4\right) + \left(\frac{-6}{154} \; x \; 6\right)$$

$$= \frac{-80}{154} + \frac{116}{154} - \frac{36}{154} = \frac{-18 + 116 - 36}{154}$$

$$= \frac{0}{154} = 0$$

$$a_{22} = \left(\frac{-16}{154} \; x \; 3\right) + \left(\frac{29}{154} \; x \; 8\right) + \left(\frac{-6}{154} \; x \; 5\right)$$

$$= \frac{-48}{154} + \frac{232}{154} - \frac{30}{154} = \frac{-48 + 232 - 30}{154}$$

$$= \frac{154}{154} = 1$$

$$a_{23} = \left(\frac{-16}{154} \; x \; 1\right) + \left(\frac{29}{154} \; x \; 2\right) + \left(\frac{-6}{154} \; x \; 7\right)$$

$$= \frac{-16}{154} + \frac{58}{154} + \left(\frac{-42}{154}\right) = \frac{-16 + 58 - 42}{154}$$

$$= \frac{0}{154} = 0$$

$a_{31} = \left(\frac{-28}{154} \, x \, 5\right) + \left(\frac{-7}{154} \, x \, 4\right) + \left(\frac{28}{154} \, x \, 6\right)$

$= \frac{-140}{154} - \frac{28}{154} + \frac{168}{154} = \frac{-140 - 28 + 168}{154}$

$= \frac{0}{154} = 0$

$a_{32} = \left(\frac{-28}{154} \, x \, 3\right) + \left(\frac{-7}{154} \, x \, 8\right) + \left(\frac{28}{154} \, x \, 5\right)$

$= \frac{-84}{154} - \frac{56}{154} + \frac{140}{154} = \frac{-84 - 56 + 140}{154}$

$= \frac{0}{154} = 0$

$a_{33} = \left(\frac{-28}{154} \, x \, 1\right) + \left(\frac{-7}{154} \, x \, 2\right) + \left(\frac{28}{154} \, x \, 7\right)$

$= \frac{-28}{154} - \frac{14}{154} + \frac{196}{154} = \frac{-28 - 14 + 196}{154}$

$= \frac{154}{154} = 1$

$A^{-1} A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

Therefore, answer matrix (Inverse Matrix) is correct.

Exercise :

Q1. If $A = \begin{bmatrix} 5 & 10 \\ 8 & 7 \end{bmatrix}$, find A^{-1}.

Q2. If $A = \begin{bmatrix} -3 & 4 \\ -8 & 6 \end{bmatrix}$, find A^{-1}.

Q3. If $A = \begin{bmatrix} 5 & 4 & 1 \\ 6 & 7 & 3 \\ 8 & 9 & 4 \end{bmatrix}$, find A^{-1}.

8.3 SOLVING LINEAR EQUATIONS WITH MATRICES :

Mainly there are three methods or techniques for solving the linear equations with the help of matrices as Matrix Inversion Techniques, Cramer's Rule and Guass Elimination method etc.

8.3.1 Matrix Inversion Techniques :

With the help of matrix inversion techniques linear equations are solved in the two case as two simultaneous equations in two unknowns and three simultaneous equations in three unknowns.

1) Two simultaneous Equations in Two Unknowns :

Suppose, AX = B, is the given equation

Premultiplying the both sides of above equation by A^{-1},

$A^{-1} AX = A^{-1}B$

$(A^{-1} A)X = A^{-1}B$

$IX = A^{-1}B$ (Since $A^{-1}A = I$)

$X = A^{-1}B$

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Steps to solve the Linear Equations:

i) Write the equations in Matrix form.

ii) Compute the determinant of the Coefficient matrix.

If $|A| \neq 0$, go to step 3.

If $|A| \neq 0$, don't go to step 3, because we cannot find inverse.

iii) Find minors (Mij)

iv) Compute cofactor Matrix (Aij)

v) Compute Adjoint matrix i.e., transpose of the cofactor matrix (adj. A)

vi) Find inverse of matrix (A^{-1})

vii) Finally multiply the Inverse Matrix (A^{-1}) by the constant vector (B) i.e. $A^{-1}B$. Then the values of variables will be get.

viii) For proving the values of variable are correct , please keep the values of variables in the any simultaneous equation. If the both sides of equation will be same, the answer is correct.

Examples :

1. Solve the following linear simultaneous equations.

$5x_1 + 2x_2 = 10$ ------------- (1)

$4x_1 + 8x_2 = 12$ ------------- (2)

Solution :

Step 1 – Write the above equations in the matrix form.

$$\begin{bmatrix} 5 & 2 \\ 4 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 12 \end{bmatrix}$$

(A) (X) (B)

Where, A = Coefficient Matrix

X = Variable Vector

B = Constant Vector

Step 2 – Find the determinant of the Coefficient Matrix

$$A = \begin{bmatrix} 5 & 2 \\ 4 & 8 \end{bmatrix}$$

|A| = (5 x 8) – (2 x 4)

= 40 – 8

= 32

$\therefore\ |A| \neq 0,$ So follow step 3.

Step 3 – Find the Minors

$$\text{Mij} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = \begin{bmatrix} M_{22} & M_{21} \\ M_{12} & M_{11} \end{bmatrix}$$

$$\text{Mij} = \begin{bmatrix} 8 & 4 \\ 2 & 5 \end{bmatrix}$$

Step 4 – Find the Cofactor Matrix

$$\text{Aij} = \begin{bmatrix} 8 & -4 \\ -2 & 5 \end{bmatrix}$$

Step 5 – Find the Adjoint Matrix (Adj-A) means transpose of the cofactor matrix.

$$\text{Adj. of A} = \begin{bmatrix} 8 & -2 \\ -4 & 5 \end{bmatrix}$$

Step 6 – Calculate inverse Matrix (A^{-1})

$$A^{-1} = \frac{Adj.of\ A}{|A|}$$

$$A^{-1} = \frac{\begin{bmatrix} 8 & -2 \\ -4 & 5 \end{bmatrix}}{32} = \begin{bmatrix} \frac{8}{32} & \frac{-2}{32} \\ \frac{-4}{32} & \frac{5}{32} \end{bmatrix}$$

Step 7 – Multiply the inverse matrix (A^{-1}) by the constant vector (B)

$$X = A^{-1}B$$

$$= \begin{bmatrix} \frac{8}{32} & \frac{-2}{32} \\ \frac{-4}{32} & \frac{5}{32} \end{bmatrix} \begin{bmatrix} 10 \\ 12 \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{8}{32} x\ 10\right) + \left(\frac{-2}{32} x\ 12\right) \\ \left(\frac{-4}{32} x\ 10\right) + \left(\frac{5}{32} x\ 12\right) \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{80}{32}\right) + \left(\frac{-24}{32}\right) \\ \left(\frac{-40}{32}\right) + \left(\frac{60}{32}\right) \end{bmatrix} = \begin{bmatrix} \frac{80-24}{32} \\ \frac{-40+60}{32} \end{bmatrix}$$

$$X = \begin{bmatrix} \frac{56}{32} \\ \frac{20}{32} \end{bmatrix}$$

$$X = \begin{bmatrix} \frac{7}{4} \\ \frac{5}{8} \end{bmatrix}$$

$\therefore\ x_1 = \frac{7}{4}$ & $x_1 = \frac{5}{8}$

Proof –

Substitute the value of x_1 and x_2 in the equation (1)

$5x_1 + 2x_2 = 10$

$5 \text{ x } \frac{7}{4} + 2 \text{ x } \frac{5}{8} = 10$

$\frac{35}{4} + \frac{10}{8} = 10$

$\frac{(35\ x\ 8)+(10\ x\ 4)}{(4x8)} = 10$

$\frac{240 + 40}{32} = 10$

$\frac{320}{32} = 10$

$10 = 10$

Hence, answer is correct.

2. Solve the following set of Linear simultaneous equations.

$2x_1 + 4x_2 - x_3 = 15$ ---------- (1)

$x_1 - 3x_2 + 2x_3 = -5$ ---------- (2)

$6x_1 + 5x_2 + x_3 = 28$ ---------- (3)

Solution –

Step 1 – Write these three equations in the matrix form.

$$\begin{bmatrix} 2 & 4 & -1 \\ 1 & -3 & 2 \\ 6 & 5 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 15 \\ -5 \\ 28 \end{bmatrix}$$

(A) (X) (B)

Step 2 - Find the determinant of the coefficient matrix.

$$A = \begin{bmatrix} 2 & 4 & -1 \\ 1 & -3 & 2 \\ 6 & 5 & 1 \end{bmatrix}$$

$$|A| = 2 \begin{vmatrix} -3 & 2 \\ 5 & 1 \end{vmatrix} - 4 \begin{vmatrix} 1 & 2 \\ 6 & 1 \end{vmatrix} + (-1) \begin{vmatrix} 1 & -3 \\ 6 & 5 \end{vmatrix}$$

$=2(-3 -10) -4(1 - 12) -1(5 + 18)$

$=2(-13) -4(-11) -1(23)$

$=-26 + 44-23$

$= -5 \neq 0$

$\therefore\ |A| \neq 0$

Step 3 – Find Minors

$$\text{Minor of A} = \begin{bmatrix} -13 & -11 & 23 \\ \begin{vmatrix} 4 & -1 \\ 5 & 1 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ 6 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 4 \\ 6 & 5 \end{vmatrix} \\ \begin{vmatrix} 4 & -1 \\ -3 & 2 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 4 \\ 1 & -3 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} -13 & -11 & 23 \\ (4+5) & (2+6) & (10-24) \\ (8-3) & (4+1) & (-6-4) \end{bmatrix}$$

$$= \begin{bmatrix} -13 & -11 & 23 \\ 9 & 8 & -14 \\ 5 & 5 & -10 \end{bmatrix}$$

Step 4 – Find cofactor matrix

$$\text{Aij} = \begin{bmatrix} -13 & -11 & 23 \\ -9 & 8 & 14 \\ 5 & -5 & -10 \end{bmatrix}$$

Step 5 – Find the Adjoint Matrix means transpose of the cofactor matrix.

$$\text{Adj. of A} = \begin{bmatrix} -13 & -9 & 5 \\ 11 & 8 & -5 \\ 23 & 14 & -10 \end{bmatrix}$$

Step 6 – Find the inverse of A

$$A^{-1} = \frac{Adj.of\ A}{|A|}$$

$$= \frac{\begin{bmatrix} -13 & -9 & 5 \\ 11 & 8 & -5 \\ 23 & 14 & -10 \end{bmatrix}}{-5}$$

$$= \begin{bmatrix} \frac{-13}{-5} & \frac{-9}{-5} & \frac{5}{-5} \\ \frac{11}{-5} & \frac{8}{-5} & \frac{-5}{-5} \\ \frac{23}{-5} & \frac{14}{-5} & \frac{-10}{-5} \end{bmatrix}$$

Step 7 – Multiply Inverse Matrix by Constant Vector (B) i.e.

$$X = A^{-1} B$$

$$= \begin{bmatrix} \frac{-13}{-5} & \frac{-9}{-5} & \frac{5}{-5} \\ \frac{11}{-5} & \frac{8}{-5} & \frac{-5}{-5} \\ \frac{23}{-5} & \frac{14}{-5} & \frac{-10}{-5} \end{bmatrix} \begin{bmatrix} 15 \\ -5 \\ 28 \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{-13}{-5}\ x\ 15\right) & \left(\frac{-9}{-5} x\ (-5)\right) & \left(\frac{5}{-5}\ x\ 28\right) \\ \left(\frac{11}{-5} x\ 15\right) & \left(\frac{8}{-5} x\ (-5)\right) & \left(\frac{-5}{-5}\ x\ 28\right) \\ \left(\frac{23}{-5} x\ 15\right) & \left(\frac{14}{-5} x\ (-5)\right) & \left(\frac{-10}{-5}\ x\ 28\right) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-195}{-5} + \frac{45}{-5} + \frac{140}{-5} \\ \frac{165}{-5} + \frac{-40}{-5} + \frac{-140}{-5} \\ \frac{345}{-5} + \frac{-70}{-5} + \frac{-280}{-5} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-195 + 45 + 140}{-5} \\ \frac{165 - 40 - 140}{-5} \\ \frac{345 - 70 - 280}{-5} \end{bmatrix} = \begin{bmatrix} \frac{-10}{-5} \\ \frac{-15}{-5} \\ \frac{-5}{-5} \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 15 \\ -5 \\ 28 \end{bmatrix}$$

Therefore, $x_1 = 2, x_2 = 3\ and\ x_3 = 1$

Step 8 – Substitute the values of $x_1, x_2\ and\ x_3$ in the equation (1)

$2x_1 + 4x_2 - x_3 = 15$

$2(2) + 4(3) - 1(1) = 15$

$4 + 12 - 1 = 15$

$15 = 15$

So, Answer is correct.

Exercise :

Q1. Solve the following simultaneous equations

$2x + 5y = 8$

$3x - 8y = 10$

Q2. Solve the following simultaneous equations

$2x - 3y + 4z = 5$

$x + 2y - 3z = 8$

$x - y - z = 1$

8.3.2 Cramer's Rule :

Cramer's rule is a rule using determinant to solve the simultaneous equations.

If $|A| \neq 0$, then the solutions of the simultaneous equations are given by,

$$X = \frac{|\Delta X|}{|\Delta|},\ Y = \frac{|\Delta Y|}{|\Delta|} \text{ and } Z = \frac{|\Delta Z|}{|\Delta|}$$

where,

Δ= coefficient matrix

ΔX = Matrix obtained from Δ by replacing the coefficient of X (i.e. first column of the coefficient matrix) viz a_{11}, a_{21} and a_{31} by the constant vectors d_1, d_2 and d_3 respectively.

ΔY= Matrix which is obtained from Δ by replacing the coefficient of Y (i.e. second column of the coefficient matrix) viz a_{12}, a_{22} and a_{32} by the constant vectors d_1, d_2 and d_3 respectively.

ΔZ = Matrix which is obtained from Δ by replacing the coefficient of Z (i.e. third column of the coefficient matrix) viz a_{13}, a_{23} and a_{33} by the constant vectors d_1, d_2 and d_3 respectively.

I. Two Simultaneous Linear Equations with Two Variable :

Steps

i) First, write down the simultaneous equations in the matrix form.

ii) Calculate the determinant of the coefficient matrix. If it is non-zero, follow the next step because solution is exist. If the determinant is zero, then no need to follow next step. Because in this case solution is not exist.

iii) Find the value of X_1 or Y_1 or Z_1 by following formula.

$$X_1 = \frac{|\Delta x_1|}{|\Delta|}$$

iv) Find the value of X_2 or Y_2 or Z_2 by following formula.

$$X_2 = \frac{|\Delta x_2|}{|\Delta|}$$

v) Check the answers by putting values of X_1 and X_2 in any simultaneous equation.

Examples :

10) Solve the following two simultaneous linear equation by using Cramer's Rule

$3x_1 + 5x_2 = 13$ ---------------- (1)

$x_1 + 6x_2 = 14$ ---------------- (2)

Solution –

Step 1 – Write down the equation in the matrix form

$$\begin{bmatrix} 3 & 5 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 13 \\ 14 \end{bmatrix}$$

Step 2 – Find the determinant of the coefficient matrix

$|\Delta| = \begin{vmatrix} 3 & 5 \\ 1 & 6 \end{vmatrix}$ = (3 x 6) –(5 x 1)

= 18 – 5

= 13

$\therefore |\Delta| \neq 0$, follow next step, because solution exists.

Step 3 – find the value of x_1

$x_1 = \frac{|\Delta x_1|}{|\Delta|}$

$|\Delta x_1| = \begin{vmatrix} 13 & 5 \\ 14 & 6 \end{vmatrix} = (13 \text{ x } 6) – (5 \text{ x } 14)$

$= 78 - 70$

$|\Delta x_1| = 8$

$x_1 = \frac{|\Delta x_1|}{|\Delta|} = \frac{8}{13}$

Step 4 – find the value of x_2

$x_2 = \frac{|\Delta x_2|}{|\Delta|}$

$|\Delta x_2| = \begin{vmatrix} 3 & 13 \\ 1 & 14 \end{vmatrix} = (3 \text{ x } 14) – (13 \text{ x } 11)$

$= 42 - 13$

$|\Delta x_2| = 29$

$x_2 = \frac{29}{13}$

$\therefore x_1 = \frac{8}{13}$ and $x_2 = \frac{29}{13}$

Step 5 – Put the values of x_1 and x_2 in equation (1)

$3x_1 + 5x_2 = 13$

$\left(3 \text{ x } \frac{8}{13}\right) + \left(5 \text{ x } \frac{29}{13}\right) = 13$

$\frac{24}{13} + \frac{145}{13} = 13$

$\frac{24 + 145}{13} = 13$

$\frac{169}{13} = 13$

$13 = 13$

Therefore, values of x_1 and x_2 are correct.

II. Three Simultaneous Linear Equations With Three Variables

Example :

1. **Solve the following simultaneous equations by using Cramer's Rule.**

$2x_1 + 3x_2 - x_3 = 9$ ---------- (1)

$x_1 + x_2 + x_3 = 9$ ---------- (2)

$3x_1 - x_2 - x_3 = -1$ ---------- (3)

Solution –

Step 1 – Write down the above equation in the matrix form

$$\begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 3 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 9 \\ -1 \end{bmatrix}$$

Step 2 – Compute the determinant of the coefficient matrix

$$|\Delta| = \begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 3 & -1 & -1 \end{vmatrix}$$

= 2(-1 +1) -3(-1 -3) -1(-1 -3)

= 2(0) -3(-4) -1(-4) = 0 + 12 +4

= 16

$$|\Delta| = 16 \neq 0$$

Step 3 – find the value of x_1

$$x_1 = \frac{|\Delta x_1|}{|\Delta|}$$

$$|\Delta x_1| = \begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 3 & -1 & -1 \end{vmatrix}$$

$|\Delta x_1|$= 9(-1 +1) -3(-9+1) -1(-9+1)

= 9(0) -3(-8) -1(-8)

=0 + 24 +8

= 32

$$x_1 = \frac{|\Delta x_1|}{|\Delta|} = \frac{32}{16} = 2$$

Step 4 – find the value of x_2

$$x_2 = \frac{|\Delta x_2|}{|\Delta|}$$

$$|\Delta x_2| = \begin{vmatrix} 2 & 9 & -1 \\ 1 & 9 & 1 \\ 3 & -1 & -1 \end{vmatrix}$$

Δx_2|= 2(-9 +1) -1(-1-3) -1(-1 -27)

=2(-8) -9(-4) -1(-28)

=-16 +36 +28

= 48

$$x_2 = \frac{|\Delta x_2|}{|\Delta|} = \frac{48}{16} = 3$$

Step 5 – find the value of x_3

$x_3 = \frac{|\Delta x_3|}{|\Delta|}$

$|\Delta x_3| = \begin{vmatrix} 2 & 3 & 9 \\ 1 & 1 & 9 \\ 3 & -1 & -1 \end{vmatrix}$

$\Delta x_3|$= 2(-1 +9) -3(-1 -27) +9(-1-3)

=2(8) -3(-28) +9(-4)

=-16 +84 -36

= 100 – 36

= 64

$x_3 = \frac{|\Delta x_3|}{|\Delta|} = \frac{64}{16} = 4$

Therefore, x_1=2 , x_2=3 , x_3=4

Exercise :

Solve the following simultaneous equations by using Cramer's Rule

i) $3x_1 - 4x_2 = 5$

$6x_1 + 3x_2 = 7$

ii) $5x_1 - 3x_2 - x_3 = 5$

$x_1 + x_2 + x_3 = 7$

$2x_1 - 2x_2 - x_3 = 9$

8.3.3 Guass Elimination Method

Examples :

1. Solve the following simultaneous equations by using Cramer's Rule.

$2x_1 + 3x_2 - x_3 = 9$ ---------- (1)

$x_1 + x_2 + x_3 = 9$ ---------- (2)

$3x_1 - x_2 - x_3 = -1$ ---------- (3)

Solution –

Step 1 – Write down the above equation in the matrix form

$$\begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 3 & -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 9 \\ -1 \end{bmatrix}$$

Step 2 – Write the augmented matrix which is

$$\begin{bmatrix} 2 & 3 & -1 & 9 \\ 1 & 1 & 1 & 9 \\ 3 & -1 & -1 & -1 \end{bmatrix}$$

By Elementary row operations, we gave augmented

$$\begin{bmatrix} 1 & 1 & 1 & 9 \\ 2 & 3 & -1 & 9 \\ 3 & -1 & -1 & -1 \end{bmatrix} \quad (R_1 \leftrightarrow R_2)$$

$$\begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & 1 & -3 & -9 \\ 0 & -4 & -4 & -28 \end{bmatrix} \begin{pmatrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{pmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 9 \\ 0 & 1 & -3 & -9 \\ 0 & -4 & -4 & -28 \end{bmatrix} R_3 \to R_3 + 4R_2$$

Therefore, the given system of simultaneous equation is equivalent to

$x_1 + x_2 + x_3 = 9$ ---------- (1)

$x_2 - 3x_3 = -9$ ---------- (2)

$-16x_3 = -64$ ---------- (3)

From equation (3) we calculate the value of x_3

$-16x_3 = -64$

$x_3 = \frac{-64}{-16}$

$x_3 = 4$

Now let us substitute the value of x_3 (4) in the equation (2), we get

$x_2 - 3x_3 = -9$

$x_2 - 3 \times 4 = -9$

$x_2 = -9 + 12$

$x_2 = 3$

Now we substitute the values of x_2 and x_3 (x_2=3 and x_3 = 4) in the equation (1), we get

$x_1 + x_2 + x_3 = 9$

$x_1 + 3 + 4 = 9$

$x_1 = 9 - 7$

$x_1 = 2$

Therefore,

$x_1 = 2$, $x_2 = 3$, $x_3 = 4$

Exercise :

Q1. Solve the following simultaneous equations by using Guass elimination method.

$3x_1 - 2x_2 + x_3 = 7$

$9x_1 + 5x_2 + 7x_3 = 9$

$x_1 - x_2 - x_3 = -5$

Q2. Solve the following simultaneous equations by using Guass elimination method.

$x_1 + x_2 + x_3 = 5$

$-x_1 - 2x_2 + 6x_3 = 7$

$3x_1 + 2x_2 + x_3 = 5$

8.4 SUMMARY

This unit has been delivered the complete knowledge about the process to compute inverse of matrix in 2x2 and 3x3 matrix forms and the technique to prove the correct answer i.e. $A^{-1}A = I$ · And the three methods to solve the simultaneous equations in two simultaneous equations and three simultaneous equations case. Remember following points.

- Simultaneous equations can be solved by three techniques as matrix inversion technique, Cramer's Rule and Guass elimination method.
- $A^{-1}A = I$ i.e. the multiplication or product of inverse matrix (A^{-1})= and given matrix (A) is always Identity matrix.
- For getting the inverse of matrix or for solving the simultaneous equations determinant should be non-zero If the determinant will be zero, no answer exist in this case.

8.5 QUESTIONS

Q.1 Find the inverses of following matrices.

i) $A = \begin{bmatrix} 5 & 10 \\ 15 & -9 \end{bmatrix}$

ii) $A = \begin{bmatrix} -5 & -3 \\ 4 & 5 \end{bmatrix}$

iii) $A = \begin{bmatrix} 8 & 4 & 2 \\ 7 & 5 & 3 \\ 9 & 6 & 1 \end{bmatrix}$

Q.2 Solve the following simultaneous equations by matrix inversion method.

i) $5x_1 + 3x_2 = 15$

$7x_1 - x_2 = 20$

ii) $3x_1 - 4x_2 + x_3 = 8$

$4x_1 + x_2 - 2x_3 = 7$

$x_1 - 6x_2 - 5x_3 = 5$

Q.3 Solve the following simultaneous equations by using Crammer's Rule.

i) $3x_1 + 5x_2 = 25$

$-5x_1 - 30x_2 = 30$

ii) $4x_1 - 6x_2 - 7x_3 = 8$

$7x_1 + 3x_2 - 5x_3 = -9$

$x_1 + x_2 + x_3 = 10$

Q.3 Solve the following simultaneous equations by using Guass elimination method.

$5x_1 + 10x_2 - 3x_3 = 9$

$4x_1 - 3x_2 + x_3 = 5$

$14x_1 + 13x_2 - 11x_3 = -11$

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