Unit I

# 1

# MATRICES

**Unit Structure**

## 1.0 Objectives

After going through this chapter, students will able to learn

- Concept of adjoint of a matrix.
- Perform the matrix operations of addition, multiplication and express a system of simultaneous linear equations in matrix form.
- Determine whether or not a given matrix is invertible and if it is, find its inverse
- Rank of a matrix and methods finding these
- Solve a system of linear equations by row-reducing its augmented form
- Characteristics roots and characteristics vectors
- Reduction of matrix to a diagonal matrix

## 1.1 Introduction

A **matrix** is a rectangular arrangement of numbers into rows and columns. Matrices provide a method of organizing, storing, and working with mathematical information. We shall mostly be concerned with matrices having real numbers as entries. The horizontal arrays of a matrix are called its rows and the vertical arrays are called its columns. A matrix having m rows and n columns is said to have the order $m \times n$.

The numbers in a matrix can represent data, and they can also represent mathematical equations. Matrices have an abundance of applications and use in the real world. Matrices have wide applications in engineering, physics, economics, and statistics as well as in various branches of mathematics. In computer science, matrix mathematics lies behind animation of images in movies and video games. Matrices provide a useful tool for working with models based on systems of linear equations.

**Definitions:** A system of m x n numbers arranged in the form of an ordered set of m horizontal lines called rows & n vertical lines called columns is called an m x n matrix.

A matrix A of order m × n can be represented in the following form

$$\begin{bmatrix} a11 & a12 \dots\dots\dots\dots. & a1n \\ a21 & a22 \dots\dots\dots\dots. & a2n \\ \dots\dots & \dots\dots\dots\dots\dots.. & \dots\dots \\ \dots\dots & \dots\dots\dots\dots\dots\dots & \dots\dots. \\ am1 & am2 \dots\dots\dots.. & amn \end{bmatrix}$$

where $a_{ij}$ is the entry at the intersection of the $i^{th}$ row and $j^{th}$ column

Matrices are generally denoted by capital letters and the elements are generally denoted by corresponding small letters.

## 1.2 Types of Matrices

1. **Transpose of Matrix:** Let A be an (m x n) matrix. Then, the matrix obtained by interchanging the rows and columns of A is called the transpose of A, denoted by A' or $A^T$. Thus, if A = $[a_{ij}]_{m \times n}$ then A' = $[a_{ij}]_{n \times m}$

eg. If A = $\begin{bmatrix} 2 & -4 & 8 \\ -3 & 5 & 9 \end{bmatrix}$ then A' = $\begin{bmatrix} 2 & -3 \\ -4 & 5 \\ 8 & 9 \end{bmatrix}$

Note: 1. If A is any matrix, then (A')' = A

2. If A is any matrix and k is scalar, then (kA)' =k A'

3. If A and B are two matrices of same order then (A + B)' = A' + B'

2. **Determinant of a square matrix:** Corresponding to each square matrix

$$A = \begin{bmatrix} a11 & a12 & a13 \dots\dots\dots & a1n \\ a21 & a22 & a23 \dots\dots\dots & a2n \\ \dots\dots & \dots\dots & \dots\dots\dots\dots\dots & \dots\dots \\ an1 & an2 & a3n \dots\dots\dots & ann \end{bmatrix}$$

There is associated an expression, called the determinant of A, denoted by det A or $|A|$, written as

$$\det A = |A| = \begin{vmatrix} a11 & a12 & a13 \dots\dots\dots & a14 \\ a21 & a22 & a23 \dots\dots\dots & a24 \\ \dots\dots & \dots\dots & \dots\dots\dots\dots\dots & \dots\dots \\ an1 & an2 & an3 \dots\dots\dots & ann \end{vmatrix}$$

A matrix is an arrangement of numbers and so it has no fixed value, while each determinant has a fixed value. A determinant having n rows and n columns is known as a determinant of order n. The determinants of non-square matrices are not defined.

**Value of a determinant of order 1:** The value of a determinant of a (1 x 1) matrix [a] is defined as $|a|$=a.

**Value of a determinant of order 2:** $\begin{vmatrix} a11 & a12 \\ a21 & a22 \end{vmatrix}$ = (a11.a22 x a21.a12)

**Value of a determinant of order 3 or more:** For Finding the value of an order 3 or more, we need following definitions.

**Minor of $a_{ij}$ in $|A|$:** Minor of $a_{ij}$ in $|A|$defined as the value of the determinant obtained by deleting the ith row and jth column of $|A|$ is denoted by $M_{ij}$.

**Cofactor of $a_{ij}$ in $|A|$:** The cofactor $C_{ij}$ of an element $a_{ij}$ is defined as

$C_{ij} = (-1)^{i+j} . M_{ij}$

Eg. 1 Find the minor and cofactor of each element of A = $\begin{vmatrix} 1 & -3 & 2 \\ 4 & -1 & 2 \\ 3 & 5 & 2 \end{vmatrix}$

Sol: The minors of the elements of A are given by,

$M_{11} = \begin{vmatrix} -1 & 2 \\ 5 & 2 \end{vmatrix}$= -2-10= -12  $M_{12} = \begin{vmatrix} 4 & 2 \\ 3 & 2 \end{vmatrix}$= 8-6= 2

$M_{13} = \begin{vmatrix} 4 & -1 \\ 3 & 5 \end{vmatrix}$= 20 +3= 23  $M_{21} = \begin{vmatrix} -3 & 2 \\ 5 & 2 \end{vmatrix}$= -6-10= -16

$M_{22} = \begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix}$= 2- 6= -4  $M_{23} = \begin{vmatrix} 1 & -3 \\ 3 & 5 \end{vmatrix}$= 5+9= 14

$M_{31} = \begin{vmatrix} -3 & 2 \\ -1 & 2 \end{vmatrix}$= -6+2= -4  $M_{32} = \begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix}$= 2-8= -6

$M_{33} = \begin{vmatrix} 1 & -3 \\ 4 & -1 \end{vmatrix}$= -1+12= 11

SO, the cofactors of the corresponding elements of A are,

$C_{11} = (-1)^{1+1} .M_{11} = M_{11}$ = -12; $C_{12} = (-1)^{1+2} .M_{12} = -M_{12}$ = -2;

$C_{13} = (-1)^{1+3} .M_{13} = M_{13}$ = 23;  $C_{21} = (-1)^{2+1} .M_{21} = -M_{21}$ = 16;

$C_{22} = (-1)^{2+2} .M_{22} = M_{22}$ = -4; $C_{23} = (-1)^{2+3} .M_{23} = -M_{23}$ = -1;

$C_{31} = (-1)^{3+1} .M_{31} = M_{31}$ = 4; $C_{32} = (-1)^{3+2} .M_{32} = -M_{32}$ = 6;

$C_{33} = (-1)^{3+3} .M_{33} = M_{33}$ = 11;

**Value of Determinant:** The value of determinant is the sum of the products of elements of a row (or a column) with their corresponding cofactors.

We may expand a determinant by any arbitrarily chosen row or column.

Expansion of a Determinant: Expanding the given determinant by 1st row, we have

$$\begin{vmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \\ a31 & a32 & a33 \end{vmatrix} = a_{11}.\text{(its cofactor)} + a_{12}.\text{(its cofactor)} + a_{13}.\text{(its cofactor)}$$

$$= a_{11}.C_{11} + a_{12}.C_{12} + a_{13}.C_{13}$$

$$= a_{11}.M_{11} - a_{12}.M_{12} + a_{13}.M_{13} \qquad [\because C_{12} = -M_{12}]$$

Eg. Evaluate $A = \begin{vmatrix} 3 & 4 & 5 \\ -6 & 2 & -3 \\ 8 & 1 & 7 \end{vmatrix}$

Sol: Expanding the given determinant by 1st row, we get

$$A = 3.\begin{vmatrix} 2 & -3 \\ 1 & 7 \end{vmatrix} - 4\begin{vmatrix} -6 & -3 \\ 8 & 7 \end{vmatrix} + 5\begin{vmatrix} -6 & 2 \\ 8 & 1 \end{vmatrix}$$

$= 3\,(14 + 3) - 4.\,(-42+24) + 5\,(-6-16)$

$= 3\,(17) + 4(18) - 5(22) = 51 + 72 - 110 = 13$

3. **Adjoint of Matrix:** Let $A = [a_{ij}]$ be a square matrix of order n and let $A_{ij}$ denote the cofactor of $a_{ij}$ in $|A|$. Then, the adjoint of A, denoted by adj A, is defined as adj $A = [a_{ji}]_{n \times n}$

Thus, adj A is the transpose of the matrix of the corresponding cofactors of elements of $|A|$.

If $A = \begin{bmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \\ a31 & a32 & a33 \end{bmatrix}$ then Adj $A = \begin{bmatrix} A11 & A12 & A13 \\ A21 & A22 & A23 \\ A31 & A32 & A33 \end{bmatrix}'$

$= \begin{bmatrix} A11 & A21 & A31 \\ A12 & A22 & A32 \\ A13 & A23 & A33 \end{bmatrix}$, Where $A_{ij}$ denotes the cofactor of $a_{ij}$ in $|A|$.

Eg. 1. If $A = \begin{bmatrix} 1 & -2 & 4 \\ 0 & 2 & 1 \\ -4 & 5 & 3 \end{bmatrix}$ find adj A

Sol: $|A| = \begin{vmatrix} 1 & -2 & 4 \\ 0 & 2 & 1 \\ -4 & 5 & 3 \end{vmatrix}$

The cofactors of the elements of the $|A|$ are given by,

$A_{11} = \begin{vmatrix} 2 & 1 \\ 5 & 3 \end{vmatrix} = 1$; $\quad A_{12} = \begin{vmatrix} 0 & 1 \\ -4 & 3 \end{vmatrix} = -4$; $\quad A_{13} = \begin{vmatrix} 0 & 2 \\ -4 & 5 \end{vmatrix} = 8$;

$A_{21} = \begin{vmatrix} -2 & 4 \\ 5 & 3 \end{vmatrix} = -26$; $\quad A_{22} = \begin{vmatrix} 1 & 4 \\ -4 & 3 \end{vmatrix} = 19$; $\quad A_{23} = \begin{vmatrix} 1 & -2 \\ -4 & 5 \end{vmatrix} = 3$;

$A_{31} = \begin{vmatrix} -2 & 4 \\ 2 & 1 \end{vmatrix} = -10$; $\quad A_{32} = \left\|\begin{matrix} 1 & 4 \\ 0 & 1 \end{matrix}\right\| = -1$; $\quad A_{33} = \begin{vmatrix} 1 & -2 \\ 0 & 2 \end{vmatrix} = 2$

$$\therefore \text{adj } A = \begin{bmatrix} 1 & -4 & 8 \\ -26 & 19 & 3 \\ -10 & -1 & 2 \end{bmatrix}' = \begin{bmatrix} 1 & -26 & -10 \\ -4 & 19 & -1 \\ 8 & 3 & 2 \end{bmatrix}$$

## 1.3 Operations on Matrices:

1. **Addition of Matrices:** Let A and B be two comparable matrices, each of order (m x n). Then their sum (A + B) is a matrix of order (m x n), obtained by adding the corresponding elements of A and B.

Eg. Let $A = \begin{bmatrix} 6 & 1 & -7 \\ 5 & 4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & -3 & -6 \\ 1 & -3 & 0 \end{bmatrix}$

Here, Matrix A and Matrix B both are 2 x 3 matrices.

∴ A and B are comparable matrices. ∴ A + B is defined.

$$A + B = \begin{bmatrix} 6+5 & 1+(-3) & (-7)+(-6) \\ 5+1 & 4+(-3) & 2+0 \end{bmatrix} = \begin{bmatrix} 11 & -2 & -13 \\ 6 & 1 & 2 \end{bmatrix}$$

**Properties of Addition of Matrices:**

The basic properties of addition for real numbers also hold true for matrices. Let A, B and C be m x n matrices.

1. Matrix addition is commutative. i.e. A + B = B +A for all comparable matrices A and B.
2. Matrix addition is commutative. i.e. (A + B) + C = A + (B + C)
3. If O is an m x n null matrix, then A + O = O + A = A

**Students can solve proof of these properties as exercise.**

2. **Scalar Multiplication:** If A be a matrix and k be a number then the matrix obtained by multiplying each element of A by k is called the scalar multiple of A by k, denoted by kA.

If A is an (m X n) matrix then kA is also an (m X n) matrix.

If $A = \begin{bmatrix} 5 & 6 \\ 3 & -2 \\ -5 & 4 \end{bmatrix}$, Find i) 4A, ii) $\frac{1}{2}A$, iii) -3A

Sol: $4A = \begin{bmatrix} 20 & 24 \\ 12 & -8 \\ -20 & 16 \end{bmatrix}$, ii) $\frac{1}{2}A = \begin{bmatrix} \frac{5}{2} & 3 \\ \frac{3}{2} & -1 \\ \frac{-5}{2} & 2 \end{bmatrix}$, iii) $-3A = \begin{bmatrix} -15 & -18 \\ -9 & 6 \\ 15 & -12 \end{bmatrix}$

3. **Multiplication of Matrices:** For two given matrices A and B, multiplication of two matrices AB exists only when number of rows in A is equals the number of columns in B**.**

Let A = $[a_{ij}]_{m \times n}$ and B = $[b_{jk}]_{n \times p}$ be two matrices such that the number of columns in A equals the number of rows in B.

Then, AB exists and it is an (m x p) matrix, given by

AB = $[\ C_{ik}]_{m \times p}$ where $C_{ik} = (a_{i1}b_{1k} + a_{i2}b_{2k} + \ldots + a_{im}b_{mk}) = \sum_{j=1}^{n} aijbjk$

= sum of the products of corresponding elements of ith row of A and kth column of B.

**Properties of Matrix Multiplication:**

1. Matrix multiplication is not commutative in general.

Let A and B be two matrices.

a. If AB exists then it is quite possible that BA may not exist.
b. Similarly, if BA exists then AB may not exist.
c. If AB and BA both exist, they may not be comparable.

2. Associative Law: For any matrices A, B, C for which(AB)C and A(BC) both exist, we have (AB)C = A(BC)
3. Distributive laws of multiplication over addition:
   i) A. (B + C) = (AB + AC)
   ii) (A + B).C = (AC + BC)
4. The product of two non-zero matrices can be a zero matrix.
5. If A is a square matrix and I is an identity matrix of same order as A then we have A.I = I.A = A.
6. If A is a square matrix and 0 is an identity matrix of same order as A then we have A.0 = 0.A = 0.

Exercise:

Ex 1. If A = $\begin{bmatrix} 5 & 4 \\ 2 & 3 \end{bmatrix}$ and B = $\begin{bmatrix} 3 & 5 & 1 \\ 6 & 8 & 4 \end{bmatrix}$, find AB and BA whichever exists.

Ex 2. If A = $\begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & 0 \\ -2 & 0 & 1 \end{bmatrix}$, B = $\begin{bmatrix} 3 & 1 \\ 0 & 2 \\ -2 & 5 \end{bmatrix}$ and C = $\begin{bmatrix} 2 & 1 & -3 \\ 3 & 0 & -1 \end{bmatrix}$

Verify (AB)C = A(BC)

## 1.4 Elementary Transformation:

Following are three row operations and three column operations on a matrix, which are called Elementary operations or transformations.

**Equivalent Matrices:** Two matrices are said to be Equivalent if one is obtained from the other by one or more elementary operations and we, A~ B.

**Three Elementary Row Operations:**

i. **Interchange of any two rows:** The interchange of ith and jth rows is denoted by $R_i \leftrightarrow R_j$.

Eg. Let A = $\begin{bmatrix} 5 & 9 & 3 \\ -8 & 13 & 6 \\ -2 & 7 & 8 \end{bmatrix}$ Applying $R_1 \leftrightarrow R_2$, we get $\begin{bmatrix} -8 & 13 & 6 \\ 5 & 9 & 3 \\ -2 & 7 & 8 \end{bmatrix}$

ii. **Multiplication of the elements of a row by a nonzero number:** Suppose each element of ith row of a given matrix is multiplied by a nonzero number k. Then, we denote it by $R_i \longrightarrow kR_i$

Eg. Let A = $\begin{bmatrix} 5 & 9 & 3 \\ -8 & 13 & 6 \\ -2 & 7 & 8 \end{bmatrix}$ Applying $R_3 \longrightarrow 2R_3$, we get $\begin{bmatrix} 5 & 9 & 3 \\ -8 & 13 & 6 \\ -4 & 14 & 16 \end{bmatrix}$

iii. **Multiplying each element of a row by a nonzero number and then adding them to the corresponding elements of another row:** Suppose each element of jth row of a matrix A is multiplied by a nonzero number k and then added to the corresponding elements of ith row.

We denote it by $R_i \longrightarrow R_i + k\,R_j$

Eg. Let A = $\begin{bmatrix} 5 & 9 & 3 \\ -2 & 1 & 3 \\ -2 & 7 & 8 \end{bmatrix}$ Applying $R_1 \longrightarrow R_1 + 3R_2$, we get $\begin{bmatrix} -1 & 12 & 12 \\ -2 & 1 & 3 \\ -2 & 7 & 8 \end{bmatrix}$

**Three Elementary Column Operations:**

i. **Interchange of any two columns:** The interchange of ith and jth columns is denoted by $C_i \leftrightarrow C_j$.

Eg. Let A = $\begin{bmatrix} 4 & 9 & 3 \\ -6 & 1 & 6 \\ -2 & 7 & 9 \end{bmatrix}$ Applying $C_2 \leftrightarrow C_3$, we get $\begin{bmatrix} 4 & 3 & 9 \\ -6 & 6 & 1 \\ -2 & 9 & 7 \end{bmatrix}$

ii. **Multiplying each element of a column by a nonzero number:** Suppose each element of ith column of a given matrix is multiplied by a nonzero number k. Then, we denote it by $C_i \longrightarrow kC_i$

Eg. Let A = $\begin{bmatrix} 2 & 7 & 2 \\ -3 & 3 & 6 \\ 2 & 5 & 8 \end{bmatrix}$ Applying $C_2 \longrightarrow 2C_2$, we get $\begin{bmatrix} 2 & 14 & 2 \\ -3 & 6 & 6 \\ 2 & 10 & 8 \end{bmatrix}$

iii. **Multiplying each element of a column by a nonzero number and then adding them to the corresponding elements of another column:** Suppose each element of jth column of a matrix A is multiplied by a nonzero number k and then added to the corresponding elements of ith column.

We denote it by $C_i \longrightarrow C_i + k\, C_j$

Eg. Let A = $\begin{bmatrix} 5 & 2 & 3 \\ -2 & 1 & 3 \\ -2 & 4 & 8 \end{bmatrix}$ Applying $C_1 \longrightarrow C_1 + 2C_2$, we get $\begin{bmatrix} 9 & 2 & 3 \\ 0 & 1 & 3 \\ 6 & 4 & 8 \end{bmatrix}$

## 1.5 Inverse of Matrix:

**Invertible Matrices:** A square matrix A of order n is said to be invertible if there exists a square matrix B of order n such that AB = BA = I

Also, then B is called the inverse of A and we write, $A^{-1} = B$

Eg. Let A = $\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$ and B = $\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$ then

$$AB = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} . \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 6-5 & -15+15 \\ 2-2 & -5+6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$BA = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 6-5 & 10-10 \\ -3+3 & -5+6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$\therefore$ AB = BA = I Hence $A^{-1} = B$.

**Singular and Non-singular Matrices: A square** A is said to be singular if $|A| = 0$ and non-singular if $|A| \neq 0$.

Eg. Let A = $\begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix}$ then $|A| = \begin{vmatrix} 1 & 2 \\ 4 & 8 \end{vmatrix}$ = (8 – 8) = 0 $\therefore$ A is singular

Let B = $\begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}$ then $|A| = \begin{vmatrix} 1 & 2 \\ 4 & 8 \end{vmatrix}$ = (8 – 6) = 2 $\neq$ 0 $\therefore$ A is non-singular.

**Note 1: Uniqueness of Inverse:** Every invertible square matrix has a unique inverse.

**Note 2:** A square matrix A is invertible if and only if A is non-singular,

i.e. A is invertible $\Leftrightarrow |A| \neq 0$

### 1.5.1 Inverse of matrix by Elementary Row Operations:

Let A be a square matrix of order n.

We can write, A = I.A …………………. (i)

Now, let a sequence of elementary row operations reduce A on LHS of (i) to I and I on RHS of (i) to a matrix B.

Then, I =BA $\Rightarrow$ $I.A^{-1} = (BA)\,A^{-1} = B\,(A\,A^{-1}) = BI \Rightarrow A^{-1} = B$

We can write above method as given below.

1. Write A = I A
2. By using elementary row operations on A, transform it into a unit matrix.
3. In the same order we apply elementary operations on I to convert it into a matrix B.
4. Then, $A^{-1} = B$

Ex. 1. By using elementary row operations, find the inverse of the matrix

$$A=\begin{bmatrix}1 & 3 & -2\\ -3 & 0 & -5\\ 2 & 5 & 0\end{bmatrix}$$

Sol:
$$\begin{bmatrix}1 & 3 & -2\\ -3 & 0 & -5\\ 2 & 5 & 0\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}.A$$

$$R_2 \longrightarrow R_2 - 3R_1,\ R_3 \longrightarrow R_3 - 2R_1 \Rightarrow \begin{bmatrix}1 & 3 & -2\\ 0 & 9 & -11\\ 0 & -1 & 4\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 3 & 1 & 0\\ -2 & 0 & 1\end{bmatrix}.A$$

$$R_2 \leftrightarrow R_3 \Rightarrow \begin{bmatrix}1 & 3 & -2\\ 0 & -1 & 4\\ 0 & 9 & -11\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ -2 & 0 & 1\\ 3 & 1 & 0\end{bmatrix}.A$$

$$R_1 \longrightarrow R_1 + 3R_2,\ R_3 \longrightarrow R_3 + 9R_2 \Rightarrow \begin{bmatrix}1 & 0 & 10\\ 0 & -1 & 4\\ 0 & 9 & 25\end{bmatrix}=\begin{bmatrix}-5 & 0 & 3\\ -2 & 0 & 1\\ -15 & 1 & 9\end{bmatrix}.A$$

$$R_2 \longrightarrow (-1).\,R_2 \Rightarrow \begin{bmatrix}1 & 0 & 10\\ 0 & 1 & -4\\ 0 & 0 & 25\end{bmatrix}=\begin{bmatrix}-5 & 0 & 3\\ 2 & 0 & -1\\ -15 & 1 & 9\end{bmatrix}.A$$

$$R_3 \longrightarrow \left(\frac{1}{25}\right) R_3 \Rightarrow \begin{bmatrix}1 & 0 & 10\\ 0 & 1 & -4\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}-5 & 0 & 3\\ 2 & 0 & -1\\ \frac{-3}{5} & \frac{1}{25} & \frac{9}{25}\end{bmatrix}.A$$

$R_1 \longrightarrow R_1 - 10R_3,\ \ R_2 \longrightarrow R_2 + 4R_3 \Longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{-2}{5} & \frac{-3}{5} \\ \frac{-2}{5} & \frac{4}{25} & \frac{11}{25} \\ \frac{-3}{5} & \frac{1}{25} & \frac{9}{25} \end{bmatrix} . A$

Hence, $A^{-1} = \begin{bmatrix} 1 & \frac{-2}{5} & \frac{-3}{5} \\ \frac{-2}{5} & \frac{4}{25} & \frac{11}{25} \\ \frac{-3}{5} & \frac{1}{25} & \frac{9}{25} \end{bmatrix}$

### 1.5.2 Inverse of matrix by Formula:

**Formula for finding $A^{-1}$:**

Let A be a square matrix such that $|A| \neq 0$. Then, $\mathbf{A^{-1}} = \frac{1}{|A|}$. (adj A)

**Ex.1**. Find the inverse of the matrix $\begin{bmatrix} 3 & -10 & -1 \\ -2 & 8 & 2 \\ 2 & -4 & -2 \end{bmatrix}$

Sol: Let A = $\begin{bmatrix} 3 & -10 & -1 \\ -2 & 8 & 2 \\ 2 & -4 & -2 \end{bmatrix} \Longrightarrow |A| = \begin{vmatrix} 3 & -10 & -1 \\ -2 & 8 & 2 \\ 2 & -4 & -2 \end{vmatrix}$

$C_1 \longrightarrow C_1 + 3C_3$ and $C_2 \longrightarrow C_2 - 10C_3$

$|A| = \begin{vmatrix} 0 & 0 & -1 \\ 4 & -12 & 2 \\ -4 & 16 & -2 \end{vmatrix} = (-1).(64-48) = -16 \neq 0$

As $|A| \neq 0$ therefore $A^{-1}$ exists.

The cofactors of the elements of $|A|$ are given by,

$A_{11} = \begin{vmatrix} 8 & 2 \\ -4 & -2 \end{vmatrix} = -8;\ A_{12} = \begin{vmatrix} -2 & 2 \\ 2 & -2 \end{vmatrix} = 0;\ A_{13} = \begin{vmatrix} -2 & 8 \\ 2 & -4 \end{vmatrix} = -8$

$A_{21} = \begin{vmatrix} -10 & -1 \\ -4 & -2 \end{vmatrix} = -16;\ A_{22} = \begin{vmatrix} 3 & -1 \\ 2 & -2 \end{vmatrix} = -4\ \ A_{23} = \begin{vmatrix} 3 & -10 \\ 2 & -4 \end{vmatrix} = -8$

$A_{31} = \begin{vmatrix} -10 & -1 \\ 8 & 2 \end{vmatrix} = -12;\ A_{32} = \begin{vmatrix} 3 & -1 \\ -2 & 2 \end{vmatrix} = -4\ \ A_{33} = \begin{vmatrix} 3 & -10 \\ -2 & 8 \end{vmatrix} = 4$

$\therefore$ (Adj A) = $\begin{bmatrix} -8 & 0 & -8 \\ -16 & -4 & -8 \\ -12 & -4 & 4 \end{bmatrix}' = \begin{bmatrix} -8 & -16 & -12 \\ 0 & -4 & -4 \\ -8 & -8 & 4 \end{bmatrix}$

Hence $A^{-1} = \frac{1}{|A|}$. adj A

$$= \frac{1}{-16}\begin{bmatrix} -8 & -16 & -12 \\ 0 & -4 & -4 \\ -8 & -8 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 1 & \frac{3}{4} \\ 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{4} \end{bmatrix}$$

**Ex. 2** If A = $\begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$, B= $\begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}$, verify that $(AB)^{-1} = B^{-1}A^{-1}$

Sol. We have $|A| = \begin{vmatrix} 3 & 2 \\ 7 & 5 \end{vmatrix}$=15-14 = 1 $\neq$ 0

Cofactors of the elements of $|A|$ are

$A_{11} = 5$, $A_{12} = -7$, $A_{21} = -2$, $A_{22} = 3$

$\therefore$ adj A= $\begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}' = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$

Hence, $A^{-1} = \frac{1}{|A|}$ adj A = $\begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$ [ $\because |A|=1$]

$|B| = \begin{vmatrix} 6 & 7 \\ 8 & 9 \end{vmatrix}$=54-56 = -2 $\neq$ 0

Cofactors of the elements of $|B|$ are

$B_{11} = 9$, $B_{12} = -8$, $B_{21} = -7$, $A_{22} = 6$

$\therefore$ adj A= $\begin{bmatrix} 9 & -8 \\ -7 & 6 \end{bmatrix}' = \begin{bmatrix} 9 & -8 \\ -7 & 6 \end{bmatrix}$

Hence, $B^{-1} = \frac{1}{|B|}$ adj B = $-\frac{1}{2}\begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$ [ $\because |B| = -2$]

Now, $|AB| = |A||B|$ = 1 x -2 = -2 $\neq$ 0

adj AB = adj B. adj A

$= \begin{bmatrix} 9 & -8 \\ -7 & 6 \end{bmatrix}\begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 94 & -39 \\ -82 & 34 \end{bmatrix}$

$AB)^{-1} = \frac{1}{|AB|}$ adj AB = $-\frac{1}{2}\begin{bmatrix} 94 & -39 \\ -82 & 34 \end{bmatrix}$

$B^{-1}A^{-1} = -\frac{1}{2}\begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 94 & -39 \\ -82 & 34 \end{bmatrix}$

$(AB)^{-1} = B^{-1}A^{-1}$

**Exercise:**

1) Find the adjoint of given matrix verify A. (adj A)= (adj A)A = $|A|$.I

1) $\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$ 2) $\begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & -2 \\ 1 & 0 & 3 \end{bmatrix}$ 3) $\begin{bmatrix} 4 & 5 & 3 \\ 0 & 1 & 6 \\ 2 & 7 & 9 \end{bmatrix}$

[Ans: 1 . $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$, 2. $\begin{bmatrix} 3 & 3 & 0 \\ -11 & 1 & 8 \\ -1 & -1 & 4 \end{bmatrix}$, 3. $\begin{bmatrix} -42 & -24 & 30 \\ 3 & 30 & -21 \\ 7 & -18 & -5 \end{bmatrix}$]

2) If $A = \begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$, show that adj A = A

3) If $A = \begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$, show that adj A = 3A'

## 1.6 Rank of Matrix

The maximum number of its linearly independent rows (or columns) of a matrix A is called the rank of Matrix A. If we have a chance of solving a system of linear equations, when the rank is equals the number of variables, we may be able to find a unique solution. Rank of a matrix A is denoted by ρ (A) or R (A)

Note:

a. The rank of a matrix cannot exceed the number of its rows or columns.

b. The rank of a null matrix is zero.

c. Rank of a matrix $A_{m\times n}$ , $\rho(A_{m\times n}) \leq \text{Min}(m, n)$

d. $\rho(I_n) = n$ where $I_n$ = unit matrix of order n

e. If ρ (A) = m and ρ (B) = n then $\rho(AB) \leq \min(m, n)$

**1.6.1 Echelon or Normal Matrix:** a matrix is said to be echelon form if

a. There exists any zero row, they should be placed below the non- zero row

b. Number of zeros before a non – zero element in a row should increase according with row number.

Eg. $A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 5 & 4 \\ 0 & 0 & 1 \end{bmatrix}$ ∴ ρ(A)= 3 = number of non – zero row

$B = \begin{bmatrix} 1 & 6 & 5 & 4 \\ 0 & 5 & 4 & 6 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ ∴ ρ(B)= 3 = number of non – zero row

Note: To reduce a matrix into its echelon form only elementary row transformations are applied.

**Computing the Rank of a matrix:** A common approach for finding the rank of a matrix is to reduce it to a simpler form, generally row echelon form by elementary row operations. Row operations do not change the row rank

Ex 1. Find the rank of the matrix $A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & -1 & 3 \\ -2 & 8 & 2 \end{bmatrix}$

Sol: We have, $A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & -1 & 3 \\ -2 & 8 & 2 \end{bmatrix}$

To find the rank of a matrix, we will transform the matrix into its echelon form by row transformation. Then determine the rank by the number of non-zero rows

$R_2 = R_2 - 2R_1,\ R_3 = R_3 + 2R_1 \quad A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & -7 & -5 \\ 0 & 14 & 10 \end{bmatrix}$

$R_3 = R_3 + 2R_2 \quad A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & -7 & -5 \\ 0 & 0 & 0 \end{bmatrix}$

Number of non-zero rows in matrix A = 2 ∴ Rank of matrix A, $\rho(A) = 2$

Exercise:

Ex 1. Find the rank of the following matrices

1. $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 2 \end{bmatrix}$; 2. $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & 1 & 4 \end{bmatrix}$

## 1.7 Linear Equations

To find the solution to the system of equations is a matrix method. The steps to be followed are:

- All the variables in the equations should be written in the appropriate order.
- The variables, their coefficients and constants are to be written on the respective sides.

There are two types of system of equations.

1. **Consistent system of Equations:** A given system of equations is said to be consistent if it has one or more solutions.
2. **Inconsistent system of Equations:** A given system of equations is said to be inconsistent if it has no solution.

Consider the system of equations.

$a_1x + b_1y + c_1z = d_1$; $a_2x + b_2y + c_2z = d_2$; $a_3x + b_3y + c_3z = d_3$

Let A = $\begin{bmatrix} a1 & b1 & c1 \\ a2 & b2 & c2 \\ c1 & c2 & c3 \end{bmatrix}$, X= $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and B = $\begin{bmatrix} d1 \\ d2 \\ d3 \end{bmatrix}$

Then the given system can be written as

$$\begin{bmatrix} a1 & b1 & c1 \\ a2 & b2 & c2 \\ a3 & b3 & c3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d1 \\ d2 \\ d3 \end{bmatrix}$$

∴ AX = B

**Case1:** when $|A| \neq 0$ , In this case, $A^{-1}$ exists.

∴ AX = B ⇒ $A^{-1}(AX) = A^{-1}B$ [multiplying both the sides by $A^{-1}$]

⇒ $(A^{-1}A)X = A^{-1}B$ [By associative law]

⇒ $I.X = A^{-1}B$ ⇒ $X = A^{-1}B$

Since $A^{-1}$ is unique, the given system has a unique solution.

Thus, when $|A| \neq 0$, then the given system is consistent and it has a unique solution.

**Case 2:** $|A| = 0$ and (adj A) B ≠ 0

In this case, the given system has no solution and hence it is inconsistent.

**Case 3:** $|A| = 0$ and (adj A) B = 0

In this case, the given system has infinitely many solutions.

**Ex.1** Use matrix method to show that the system of equations

2x + 5y = 7, 6x + 15y = 13 is inconsistent

**Sol:** The given equations are 2x + 5y = 7; 6x + 15y = 13

Let A = $\begin{bmatrix} 2 & 5 \\ 6 & 15 \end{bmatrix}$, X= $\begin{bmatrix} x \\ y \end{bmatrix}$ and B= $\begin{bmatrix} 7 \\ 13 \end{bmatrix}$

Then the given system in matrix form is AX = B

Now, $|A| = \begin{vmatrix} 2 & 5 \\ 6 & 15 \end{vmatrix} = 30 - 30 = 0$

The system will be inconsistent if (adj A) B ≠ 0

The minors of the elements of $|A|$ are $M_{11} = 15$, $M_{12} = 6$, $M_{21} = 5$, $M_{22} = 2$

The cofactors of the elements of $|A|$ are $A_{11} = 15$, $A_{12} = -6$, $A_{21} = -5$, $A_{22} = 2$

Adj A = $\begin{bmatrix} 15 & -6 \\ -5 & 2 \end{bmatrix}'$ = $\begin{bmatrix} 15 & -5 \\ -6 & 2 \end{bmatrix}$

$\Rightarrow$ (adj A) B = $\begin{bmatrix} 15 & -5 \\ -6 & 2 \end{bmatrix}\begin{bmatrix} 7 \\ 13 \end{bmatrix} = \begin{bmatrix} 105-65 \\ -42+26 \end{bmatrix} = \begin{bmatrix} 40 \\ -16 \end{bmatrix} \neq 0$

$|A|$=0, (adj A) B $\neq$ 0. Hence, the given system of equations is inconsistent.

Ex.2 Show that the following system of equations is consistent and solve it

2x + 5y = 1, 3x + 2y = 7

Sol: The given equations are

2x + 5y = 1; 3x + 2y = 7

Let A = $\begin{bmatrix} 2 & 5 \\ 3 & 2 \end{bmatrix}$, X= $\begin{bmatrix} x \\ y \end{bmatrix}$ and B= $\begin{bmatrix} 1 \\ 7 \end{bmatrix}$

Then the given system in matrix form is AX = B

Now, $|A| = \begin{bmatrix} 2 & 5 \\ 3 & 2 \end{bmatrix}$ = 4 - 15 = -11$\neq$ 0

Hence the given system has a unique solution.

The minors of the elements of $|A|$ are $M_{11}$ = 2, $M_{12}$ = 3, $M_{21}$= 5, $M_{22}$ = 2

The cofactors of the elements of $|A|$ are $A_{11}$ = 2, $A_{12}$ = -3, $A_{21}$= -5, $A_{22}$ = 2

Adj A = $\begin{bmatrix} 2 & -3 \\ -5 & 2 \end{bmatrix}'$ = $\begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix}$

$\Rightarrow A^{-1} = \frac{1}{|A|}$ adj A = $\frac{-1}{11}\begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} \frac{-2}{11} & \frac{5}{11} \\ \frac{3}{11} & \frac{-2}{11} \end{bmatrix}$

X = $A^{-1}$ B

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{-2}{11} & \frac{5}{11} \\ \frac{3}{11} & \frac{-2}{11} \end{bmatrix}\begin{bmatrix} 1 \\ 7 \end{bmatrix} = \begin{bmatrix} \frac{-2}{11} + \frac{35}{11} \\ \frac{3}{11} - \frac{14}{11} \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ $\Rightarrow$ x = 3 and y = -1

**Exercise:**

1) Use matrix method to solve the following system of equations

3x + 4y +2z = 8; 2y – 3z = 3; x-2y+6z = -2 [ Ans: x= -2, y = 3 and z = 1]

## 1.8 Linear dependence and linear independence of vectors

A collection of vectors is either linearly independent or linearly dependent. The vectors v1, v2 ……vk are linearly independent if the equation involving linear combination. In the theory of vector spaces, a set of vectors is said to be linearly dependent if there is a nontrivial linear combination of the vectors that equals the zero vector. If no such linear combination exists, then the vectors are said to be linearly independent.

A sequence of vectors $v_1, v_2, \ldots\ldots, v_k$ from a vector space V is said to be linearly dependent, if there exist scalars $a_1, a_2, \ldots, a_k$ not all zero, such that

$a_1v_1+a_2v_2+\ldots + a_kv_k = 0$, where 0 denotes the zero vector.

Ex. 1 State whether following set of vectors are linearly dependent or linearly independent. If dependent find the relation between them.

$X_1 = (1, 2, 3), X_2 = (3, -2, 1), X_3 = (1, -6, 5)$

Sol: Here, there are three vectors. For three vectors are take 3 scalars.

Let $\lambda_1$, $\lambda_2$ and $\lambda_3$ be three scalars.

Consider $\lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3 = 0$ ……………….. (1)

$\lambda_1(1, 2, 3) + \lambda_2 (3, -2, 1) + \lambda_3(1, -6, 5) = 0$

From these we make three simultaneous equations.

$\lambda_1 + 3\lambda_2 + \lambda_3 = 0;\ 2\lambda_1 - 2\lambda_2 - 6\lambda_3 = 0;\ 3\lambda_1 + \lambda_2 + 5\lambda_3 = 0$

Put them in matrix form

$$\begin{bmatrix}1 & 3 & 1\\ 2 & -2 & -6\\ 3 & 1 & 5\end{bmatrix}\begin{bmatrix}\lambda 1\\ \lambda 2\\ \lambda 3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$A\,\lambda = B$ ……………………………………… (2)

Now augmented matrix,

$$C = [A: B] \Rightarrow \quad = \begin{bmatrix}1 & 3 & 1 & 0\\ 2 & -2 & -6 & 0\\ 3 & 1 & 5 & 0\end{bmatrix}$$

Reduced this matrix in echelon matrix by row transformation

$$R_2 = R_2 - 2R_1; \quad R_3 = R_3 - 3R_1 \Longrightarrow C = \begin{bmatrix}1 & 3 & 1 & 0\\ 0 & -8 & -8 & 0\\ 0 & -8 & 2 & 0\end{bmatrix}$$

$$R_3 = R_3 - R_2,\ C = \begin{bmatrix}1 & 3 & 1 & 0\\ 0 & -8 & -8 & 0\\ 0 & 0 & 10 & 0\end{bmatrix}$$

Here we cannot further reduce.

$$\text{From (2)},\quad \begin{bmatrix}1 & 3 & 1\\ 0 & -8 & -8\\ 0 & 0 & 10\end{bmatrix}\begin{bmatrix}\lambda 1\\ \lambda 2\\ \lambda 3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

From matrix multiplication,

$\lambda_1 + 3\lambda_2 + \lambda_3 = 0$ ………………… (3)

$-8\lambda_2 - 8\lambda_3 = 0$ ………………………(4)

$\therefore \lambda_3 = 0$

Put $\lambda_3 = 0$ in (4) $\therefore \lambda_2 = 0$

Put $\lambda_2, \lambda_3$ (3) $\therefore \lambda_1 = 0$

$\because \lambda_1 = \lambda_2 = \lambda_3 = 0$ i.e all three scalars are 0.

$\therefore$ The given vectors are linearly independent and there exists no relationship.

Ex. 2 Test the linear dependency and find the relationship between if it exists for

$X_1 = (1, 1, 1, 3), X_2 = (1, 2, 3, 4), X_3 = (2, 3, 4, 7)$

Sol: Here, there are three vectors. For three vectors are take 3 scalars.

Let $\lambda_1$, $\lambda_2$ and $\lambda_3$ be three scalars.

Consider $\lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3 = 0$ ……………….. (1)

$\lambda_1(1, 1, 1, 3) + \lambda_2 (1, 2, 3, 4) + \lambda_3(2,3,4,7) = 0$

From these we make simultaneous equations.

$\lambda_1 + \lambda_2 + 2\lambda_3 = 0$; $\lambda_1 + 2\lambda_2 + 3\lambda_3 = 0$; $\lambda_1 + 3\lambda_2 + 4\lambda_3 = 0$; $3\lambda_1 + 4\lambda_2 + 7\lambda_3 = 0$

Put them in matrix form

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \\ 3 & 4 & 7 \end{bmatrix} \begin{bmatrix} \lambda 1 \\ \lambda 2 \\ \lambda 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$A\lambda = B$ ……………………………………… (2)

Now augmented matrix, C = [A: B]

$$= \begin{bmatrix} 1 & 1 & 2 & 0 \\ 1 & 2 & 3 & 0 \\ 1 & 3 & 4 & 0 \\ 3 & 4 & 7 & 0 \end{bmatrix}$$

Reduced this matrix in echelon (upper triangular) matrix by row transformation

$R_2 = R_2 - R_1$, $R_3 = R_3 - R_1$, $R_4 = R_4 - 3R_1$

$$C = \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$R_3 = R_3 - 2R_2, \quad R_4 = R_4 - R_2 \Longrightarrow \quad C = \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

from (2), $\begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda 1 \\ \lambda 2 \\ \lambda 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

From matrix multiplication,

$\lambda_1 + \lambda_2 + 2\lambda_3 = 0$ ………………… (3)

$\lambda_2 + \lambda_3 = 0 \Rightarrow \lambda_2 = -\lambda_3$ ………………………(4)

Consider $\lambda_3$ = k where k is non zero constant. $\therefore \lambda_2 = -k$

Put $\lambda_2, \lambda_3$ in equation (3) $\therefore \lambda_1 - k + 2k = 0 \Rightarrow \therefore \lambda_1 + k = 0 \quad \therefore \lambda_1 = -k$

All the scalars are non-zero.

∴ The given vectors are linearly dependent and there exists some relationship.

Now we find relationship between them.

We have, $\lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3 = 0 \quad \Rightarrow -k X_1 - k X_2 + k X_3 = 0$

Divide equation by -k , $X_1 + X_2 - X_3 = 0$

This is the required relationship.

## 1.9 Linear Transformation

Let U(F) and V(F) be two vector spaces.

A mapping f: U → V is called Linear Transformation of U into V if

i) f (x + y) = f(x) + f(y)

ii) f (ax) = a f(x) where x, y $\epsilon$ V, a $\epsilon$ F, f (x), f (y) $\epsilon$ V.

Sometimes linear transformation is also called vector space homomorphism.

Ex. 1 $V_3$ is a vector. A mapping is given as $T.V_3(R) \to V_2(R)$ by $T(x_1, x_2, x_3) = (x_1 - x_2), (x_1 + x_3)$. Check whether this is linear transformation.

Sol: Let $(x_1, x_2, x_3) = x \in V_3$ and let $(y_1, y_2, y_3) = y \in V_3(R)$

$T(x + y) = T[(x_1, x_2, x_3) + (y_1, y_2, y_3)] = T[(x_1 + y_1, x_2 + y_2, x_3 + y_3)]$

$= T[(x_1 + y_1 - x_2 - y_2, x_1 + y_1 + x_3 + y_3)]$

$= T[(x_1 - x_2 + y_1 - y_2, x_1 + x_3 + y_1 + y_3)]$

$= T[(x_1, x_2, x_3)] + T[(y_1, y_2, y_3)] = T(x) + T(y)$

$T(ax) = T[a(x_1, x_2, x_3)] = T[(ax_1, ax_2, ax_3)] = (ax_1 - ax_2, ax_1 + ax_3)$

$= a[(x_1 - x_2), (x_1 + x_2)]$

$T(ax) = aT(x) = T(x)$

Both the condition of linear transformation are satisfy.

∴ T is linear transformation.

### 1.9.1 Matrix representation of Linear Transformation:

Let U(F) and V(F) be two vector spaces over F.

T: U → V be a Linear Transformation

Let $B = \{u_1, u_2, u_3, \ldots\ldots\ldots\ldots, u_n\}$ and

$B' = \{v_1, v_2, v_3, \ldots\ldots\ldots\ldots, v_m\}$

Are two ordered bases for U and V respectively.

Now, if any $\alpha \in U \Rightarrow T(\alpha) \in V$

Also $T(\alpha)$ can be represented by B'

$T(u_1) = B_1 = a_{11}v_1 + a_{12}v_2 + a_{13}v_3 + \ldots\ldots\ldots + A_{1m}v_m$

$T(u_2) = B_2 = a_{21}v_1 + a_{22}v_2 + a_{23}v_3 + \ldots\ldots\ldots + A_{2m}v_m$

……………………………………………………

$T(u_n) = B_n = a_{n1}v_1 + a_{n2}v_2 + a_{n3}v_3 + \ldots\ldots\ldots + A_{nm}v_m$

$$\begin{bmatrix} T(u1) \\ T(u2) \\ . \\ . \\ T(un) \end{bmatrix} = \begin{bmatrix} a11 & a12 \ldots\ldots\ldots\ldots & a1m \\ a21 & a22 \ldots\ldots\ldots\ldots & a2m \\ \ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots \\ an1 & an2 \ldots\ldots\ldots & anm \end{bmatrix} \begin{bmatrix} v1 \\ v2 \\ . \\ . \\ vn \end{bmatrix}$$

$$\begin{bmatrix} a11 & a12 \ldots\ldots\ldots\ldots & a1m \\ a21 & a22 \ldots\ldots\ldots\ldots & a2m \\ \ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots \\ an1 & an2 \ldots\ldots\ldots & anm \end{bmatrix} = [T: B:B']$$

This is matrix of Linear Transformation.

If we have Linear Transformation T: U(F) → V(F)

then matrix form is [T: B], $[T]_B$

For any n dimensions vector spaces,

standard basis for $v_2(R)$= {(1, 0), (1, 0)}, $v_3(R)$= {(1, 0, 0), (0, 1, 0), (0, 0,1)}, $v_3(R)$= {(1, 0, 0, 0), (0, 1, 0, 0), (0, 0,1, 0), (0, 0, 0, 1)}

## 1.10 Characteristics roots and characteristics vectors

Characteristic vector or Eigen vector of a matrix A is a vector represented by a matrix X such that when X is multiplied with matrix A, then the direction of the resultant matrix remains same as vector X.

Let A be a square matrix of order n x n, then a number $\lambda$ is said to be eigen value of a matrix A if there exists a column matrix X of order n x 1 such that AX = $\lambda$X, where A is any arbitrary matrix, $\lambda$ are eigen values and X is an eigen vector corresponding to each eigen value.

$\Rightarrow AX - \lambda X = 0 \quad \Rightarrow (A- \lambda I)X = 0$ ……………………….(1)

Equation (1) is called characteristics equation of the matrix.

The roots of the characteristic equation are the eigen values of the matrix A.

**Ex.1** Find the eigen value (characteristics roots) and eigen vector (characteristics vector) for the matrix A = $\begin{bmatrix} 1 & -2 \\ -5 & 4 \end{bmatrix}$.

Sol: The characteristic equation for matrix A is,

$|A - \lambda I| = 0 \quad \Rightarrow \begin{vmatrix} 1-\lambda & -2 \\ -5 & 4-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda).(4-\lambda) - (-5).(-2) = 0 \quad \Rightarrow 4 - \lambda - 4\lambda + \lambda^2 - 10 = 0$

$\Rightarrow \lambda^2 - 5\lambda - 6 = 0 \quad \Rightarrow (\lambda-6).(\lambda+1) = 0 \Rightarrow \lambda = 6, \lambda = -1$

$\therefore$ Eigen value of A are 6 and -1.

Case I: $X_1 = \begin{bmatrix} X \\ Y \end{bmatrix}$ be the eigen vector of A corresponding to $\lambda = 6$

Then $(A - \lambda I)X_1 = 0$

i.e. $\begin{bmatrix} 1-\lambda & -2 \\ -5 & 4-\lambda \end{bmatrix}.\begin{bmatrix} X \\ Y \end{bmatrix} = 0, \quad \begin{bmatrix} 1-6 & -2 \\ -5 & 4-6 \end{bmatrix}.\begin{bmatrix} X \\ Y \end{bmatrix} = 0 \quad , \lambda = 6$

$\begin{bmatrix} -5 & -2 \\ -5 & -2 \end{bmatrix}.\begin{bmatrix} X \\ Y \end{bmatrix} = 0$

By row transformation,

$R_2 = R_2 - R_1 \quad \begin{bmatrix} -5 & -2 \\ 0 & 0 \end{bmatrix}.\begin{bmatrix} X \\ Y \end{bmatrix} = 0$

$-5X - 2Y = 0 \quad \Rightarrow -5X = 2Y$

$\frac{X}{2} = \frac{Y}{-5} = k$ (say)

X = 2k, Y = -5k for k=1

$\therefore$ Eigen vector $X_1 = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$

Case II: Let $X_2 = \begin{bmatrix} X \\ Y \end{bmatrix}$ be the eigen vector of A corresponding to λ = -1

Then $(A - \lambda I)X_2 = 0$

i.e. $\begin{bmatrix} 1-\lambda & -2 \\ -5 & 4-\lambda \end{bmatrix} . \begin{bmatrix} X \\ Y \end{bmatrix} = 0$

$\begin{bmatrix} 1-(-1) & -2 \\ -5 & 4-(-1) \end{bmatrix} . \begin{bmatrix} X \\ Y \end{bmatrix} = 0$ , λ = -1

$\begin{bmatrix} 2 & -2 \\ -5 & 5 \end{bmatrix} . \begin{bmatrix} X \\ Y \end{bmatrix} = 0$

By row transformation,

$R_2 = R_2 + \frac{5}{2} R_1 \implies \begin{bmatrix} 2 & -2 \\ 0 & 0 \end{bmatrix} . \begin{bmatrix} X \\ Y \end{bmatrix} = 0$

$2X - 2Y = 0 \implies 2X = 2Y$

$\frac{X}{1} = \frac{Y}{1} = k$ (say) $\implies X = k, Y = k$ for k=1

$\therefore$ Eigen vector $X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

## 1.11 Properties of characteristic vectors (eigen vector)

Following are the properties of Eigen vector:

1. Corresponding one eigen vector there exists one eigen value.

   Let $\lambda_1$ and $\lambda_2$ are two eigen values of A with one eigen vector $X \neq 0$.

   By condition of eigen values,

   $AX = \lambda_1 X$ and $AX = \lambda_2 X$

   $\implies \lambda_1 X = \lambda_2 X \implies (\lambda_1 - \lambda_2) X = 0$

   As $X \neq 0$, $(\lambda_1 - \lambda_2) = 0 \therefore \boldsymbol{\lambda_1 = \lambda_2}$

   So there exists one eigen value for one eigen vector

2. If $\lambda$ is eigen value of the matrix A of order n x n.

a) $\lambda^2$ is an eigen value of $A^2$

$AX = \lambda X$

Multiplying by A, $A^2X = \lambda AX = \lambda \lambda X = \lambda^2 X$

$\therefore \lambda^2$ is an eigen value of $A^2$

b) $\lambda^k$ is an eigen value of $A^k$, k is positive integer.

c) $f(\lambda) = a_0 \lambda + a_1 \lambda^2 + \ldots\ldots\ldots\ldots\ldots + a_n \lambda^n$

is an eigen value of $F(A) = a_0 I + a_1 A + \ldots\ldots\ldots\ldots\ldots + a_n A^n$

d) $e^{\lambda}$, $\log \lambda$, $\sin \lambda$ are eigen values of $e^A$, log A, $\sin A$ respectively.

e) $\frac{|A|}{\lambda}$ is an eigen value of adj A.

As $AX = \lambda X$

Let $\lambda_1$ is eigen value of adj A

adj $AX$ = adj $\lambda_1 X$

Multiplying by A, A adj $AX = \lambda_1 AX$

$|A|\, IX = \lambda_1 \lambda X$ $\qquad$ [ adj A = $|A|$ I]

$(|A| - \lambda_1 \lambda)\, X = 0$

As $X \neq 0$, $|A| - \lambda_1 \lambda = 0 \quad \Longrightarrow |A| = \lambda_1 \lambda \quad \Longrightarrow \lambda_1 = \frac{|A|}{\lambda}$

## 1.12 Caley Hamilton Theorem:

Consider $A - \lambda I = \begin{bmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{bmatrix}$ ←characteristic matrix, Where A is a square matrix

**Characteristic polynomial**: If we put characteristic matrix in determinant form and solved then we get polynomial that is called characteristic polynomial.

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix}$$

$|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

$(2-\lambda)\,[\,(1-\lambda)(2-\lambda) - 0] - 1\,[0] + 1[0 - 1 - \lambda] = 0$

$(2-\lambda)\,[\,(2-\lambda-2\lambda+\lambda^2)] - 1 + \lambda = 0$

$(2-\lambda)[(2-3\lambda+\lambda^2)] - 1 + \lambda = 0$

$4 - 6\lambda + 2\lambda^2 - 2\lambda + 3\lambda^2 - \lambda^3 - 1 + \lambda = 0$

$-\lambda^3 + 5\lambda^2 - 7\lambda + 3 = 0$

$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

This is called characteristic equation.

**Characteristic Roots (eigen values)** : Roots of characteristic equation is called characteristic roots.

**State and Prove Caley Hamilton Theorem**

**Statement:** Every square matrix A satisfy its own characteristic equation.

**Proof:** Let $A = [a_{ij}]_{n \times n}$ be any square matrix and $P(\lambda) = |A - \lambda I|$ be a characteristic equation where $\lambda$ be any constant, I is an identity matrix.

Show that $|A - \lambda I| = 0$

$$\begin{vmatrix} a11-\lambda & a12 & a13 & \dots\dots\dots & a1n \\ a21 & a21-\lambda & a23 & \dots\dots\dots & a2n \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ an1 & an2 & an3 & \dots\dots\dots & ann-\lambda \end{vmatrix} = 0$$

$a_0 + a_1\lambda + a_2\lambda^2 + a_3\lambda^3 \dots\dots\dots\dots\dots + a_n\lambda^n$

[ Matrix is n x n order therefore polynomial is of order n]

If we put matrix A in place of $\lambda$ then

$a_0 + a_1 A + a_2 A^2 + a_3 A^3 + \dots\dots\dots\dots + a_n A^n$

We know that, A adj A = $|A|$I

$(A - \lambda I)\ adj\ (A - \lambda I) = |A - \lambda I|.I \qquad [A = |A - \lambda I|]$

Now each element in $(A - \lambda I)$ is a polynomial of degree almost 1.

Hence adj $(A - \lambda I)$ has polynomial of degree n-1.

$adj\ (A - \lambda I) = B_0 + B_1\lambda + B_2\lambda^2 + B_3\lambda^3 \dots\dots\dots\dots\dots + B_{n-1}\lambda^{n-1}$

$(A - \lambda I)\ adj\ (A - \lambda I) = (A - \lambda I)[B_0 + B_1\lambda + B_2\lambda^2 + \dots\dots\dots\dots + B_{n-1}\lambda^{n-1}]$

$= AB_0 + AB_1\lambda + AB_2\lambda^2 + \dots\dots\dots\dots + AB_{n-1}\lambda^{n-1} -$

$B_0\lambda - B_1\lambda^2 - B_2\lambda^3 - \dots\dots\dots\dots - B_{n-1}\lambda^n$

$= AB_0 + (AB_1 - Bo)\lambda + (AB_2 - B_1)\lambda^2 + \dots\dots\dots - B_{n-1}\lambda^n$

Now $(A - \lambda I)\ adj\ (A - \lambda I) = |A - \lambda I|$

$AB_0 + (AB_1 - B_0)\lambda + (AB_2 - B_1)\lambda^2 +$

$\ldots\ldots - B_{n-1}\lambda^n = a_0 + a_1\lambda + a_2\lambda^2 + a_3\lambda^3 \ldots\ldots + a_n\lambda^n$

Compare coefficient of equal power of $\lambda$

$AB_0 = a_0$

$AB_1 - B_0 = a_1$

$AB_2 - B_1 = a_2$

$\ldots\ldots$

$AB_{n-1} - B_{n-2} = a_{n-1}$

$-B_{n-1} = a_n$

Multiplying with I, A, A2, ………, An and then add

$AB_0 = a_0 I$

$A^2B_1 - B_0 = a_1 A$

$A^3B_2 - B_1 = a_2 A^2$

$\ldots\ldots$

$A^nB_{n-1} - A^{n-1}B_{n-2} = a_{n-1}A^{n-1}$

$-A^nB_{n-1} = a_n A^n$

Hence $a_0 + a_1A + a_2A^2 + a_3A^3 + \ldots\ldots + a_n A^n = 0$

Hence proved.

Ex 1. Show that the matrix $A = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$

satisfies its characteristic equation and hence determine $A^{-1}$.

Sol: The characteristic matrix of A is,

$$\begin{vmatrix} 8-\lambda & -8 & -2 \\ 4 & -3-\lambda & -2 \\ 3 & -4 & 1-\lambda \end{vmatrix} = 0$$

Characteristic equation is given by $|A - \lambda I| = 0$

OR

If there is 2 x 2 matrix then $\lambda^2 - s_1\lambda + |A| = 0$

If there is 3 x 3 matrix then $\lambda^3 - s_1\lambda^2 + s_2\lambda - |A| = 0$

Where $s_1$= sum of diagonal element of matrix A

And $s_2$= sum of minors of diagonal element of matrix A

Here matrix A is 3 x 3

So Characteristic equation is given by

$\lambda^3$ - $s_1$ $\lambda^2$ + $s_2$ $\lambda$ - $|A|$ = 0

$s_1$ = sum of diagonal element of matrix A = 8 – 3 + 1 = 6

$s_2$ = sum of minors of diagonal element of matrix A

$= \begin{vmatrix} -3 & -2 \\ -4 & 1 \end{vmatrix} + \begin{vmatrix} 8 & -2 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 8 & -8 \\ 4 & -3 \end{vmatrix}$

= ( -3 - 8 ) + (8 + 6) + (-24 + 32) = (-11) + 14 + 8 = 11

Now $|A| = \begin{vmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{vmatrix}$

= 8 (-3 – 8) - (-8) (4 +6) + (-2) (-16+9) = 8(-11)+80 +14 = 6

$\lambda^3$ -6 $\lambda^2$ + 11 $\lambda$ – 6 = 0

Now, in LHS we replace $\lambda$ by A, we get

$A^3$ -6 $A^2$ + 11 $A$ – 6I

$$A^2 = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}\begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 26 & -32 & -2 \\ 14 & -15 & -4 \\ 11 & -16 & 3 \end{bmatrix}$$

$$A^3 = \text{A } A^2 = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}\begin{bmatrix} 26 & -32 & -2 \\ 14 & -15 & -4 \\ 11 & -16 & 3 \end{bmatrix} = \begin{bmatrix} 74 & -104 & 10 \\ 40 & -51 & -2 \\ 33 & -52 & 13 \end{bmatrix}$$

$A^3$ -6 $A^2$+ 11 $A$ – 6I =

$$\begin{bmatrix} 74 & -104 & 10 \\ 40 & -51 & -2 \\ 33 & -52 & 13 \end{bmatrix} - 6\begin{bmatrix} 26 & -32 & -2 \\ 14 & -15 & -4 \\ 11 & -16 & 3 \end{bmatrix} + 11\begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix} - 6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 74 & -104 & 10 \\ 40 & -51 & -2 \\ 33 & -52 & 13 \end{bmatrix} - \begin{bmatrix} 156 & -192 & -12 \\ 84 & -90 & -24 \\ 66 & -96 & 18 \end{bmatrix} +$$

$$\begin{bmatrix} 88 & -88 & -22 \\ 44 & -33 & -22 \\ 33 & -44 & 11 \end{bmatrix} - \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ $A^3$ -6 $A^2$+ 11 $A$ – 6I = 0 ………. (1)

Thus, A satisfy its characteristic equation.

To find $A^{-1}$, multiply equation (1) by $A^{-1}$

$A^3 A^{-1} - 6A^2 A^{-1} + 11 A A^{-1} - 6 I A^{-1} = 0$

$A^2 - 6A + 11I - 6A^{-1} = 0$ ………………… $[A A^{-1} = I, I = 1]$

$\Rightarrow 6A^{-1} = A^2 - 6A + 11I$

$$= \begin{bmatrix} 26 & -32 & -2 \\ 14 & -15 & -4 \\ 11 & -16 & 3 \end{bmatrix} - 6\begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix} + 11\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} -11 & 16 & 10 \\ -10 & 14 & 8 \\ -7 & 8 & 8 \end{bmatrix} A^{-1} = \frac{1}{6}\begin{bmatrix} -11 & 16 & 10 \\ -10 & 14 & 8 \\ -7 & 8 & 8 \end{bmatrix}$$

## 1.13 Similarity of matrices:

Matrix A and B of order n x n are said to be similar to each other if there exists an invertible n x n matrix P, such that AP = PB **i**.e. $B = P^{-1}AP$

For Similar matrices A, B, we have

i. $|A| = |B|$

Since A and B are similar, we have $B = P^{-1}AP$

$|B| = |P^{-1}AP|$ [ Taking determinant of both the side]

$= |P^{-1}||A|\,|P| \Rightarrow = |P^{-1}P||A| \Rightarrow = |I||A| \Rightarrow = |A|$ [ As $|I| = 1$]

$\therefore |A| = |B|$

ii. Characteristic equation for A and B are same.

If A and B are similar to each other then

$|A - \lambda I| = |B - \lambda I|$, for all real numbers $\lambda$.

$|B - \lambda I| = |P^{-1}AP - \lambda I|$ [As $B = P^{-1}AP$]

$= |P^{-1}AP - \lambda P^{-1}IP| = |P^{-1}(A - \lambda I)P| = |P^{-1}||(A - \lambda I)||P|$

$= |P^{-1}P||(A - \lambda I)| = |I||(A - \lambda I)| = |(A - \lambda I)|$

Since $|B - \lambda I| = |(A - \lambda I)|$, the similar matrices A and B have same characteristic equation.

## 1.14 Reduction of matrix to a diagonal matrix which has elements as characteristics values

If a square matrix A of order n has n linearly independent eigen vectors, then a matrix P can be found such that $P^{-1}AP$ is a diagonal matrix.

Proof: Let A be a square matrix of order 3.

Let $\lambda_1$ , $\lambda_2$ , $\lambda_3$ be its eigen values and

$X_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$, $X_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$, $X_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}$ be the corresponding eigen vectors.

Denoting the square matrix $[X_1, X_2, X_3]$ by P.

$$P = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_1 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$$

$AP = A\,[X_1, X_2, X_3] = [AX_1, AX_2, AX_3]$

We know that, $AX = \lambda X$

$\therefore AX_1 = \lambda_1 X_1,\ AX_2 = \lambda_2 X_2,\ AX_3 = \lambda_3 X_3$

$AP = [\lambda_1 X_1, \lambda_2 X_2, \lambda_3 X_3]$

$$= \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \lambda_3 x_3 \\ \lambda_1 y_1 & \lambda_2 y_1 & \lambda_3 y_3 \\ \lambda_1 z_1 & \lambda_2 z_2 & \lambda_3 z_3 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} \text{x} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

= P D, where D is diagonal matrix.

$\therefore P^{-1}AP = P^{-1}PD \Rightarrow$ **$P^{-1}AP = D$**

P constitute eigen vectors of A and is called Modal matrix of A.

D has eigen values as its diagonal elements and is called special matrix of A.

Ex. 1 Reduce the matrix $A = \begin{bmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{bmatrix}$ into a diagonal matrix.

Sol: We know that, $D = P^{-1}AP$

The characteristic equation of A is $|A - \lambda I| = 0$

$$\begin{bmatrix} 11-\lambda & -4 & -7 \\ 7 & -2-\lambda & -5 \\ 10 & -4 & -6-\lambda \end{bmatrix} = 0$$

Here matrix A is 3 x 3

So Characteristic equation is given by $\lambda^3 - s_1\lambda^2 + s_2\lambda - |A| = 0$

After solving for $s_1$, $s_2$ and $|A|$ , we get characteristic equation as,

$\lambda^3 - 3\lambda^2 + 2\lambda = 0 \Rightarrow \lambda(\lambda - 1)(\lambda - 2) = 0$

$\Rightarrow \lambda = 0,\ \lambda = 1$ and $\lambda = 2$ are the eigen values.

Now consider $[A - \lambda I]\,[X] = [0]$

$$\begin{bmatrix} 11-\lambda & -4 & -7 \\ 7 & -2-\lambda & -5 \\ 10 & -4 & -6-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$(11-\lambda)x - 4y - 7z = 0;\ 7x + (-2-\lambda)y - 5z = 0;\ 10x - 4y + (-6-\lambda)z = 0$

Case i: $\lambda_1 = 0$ in above equations

$11x - 4y - 7z = 0 \Rightarrow 7x - 2y - 5z = 0 \Rightarrow 10x - 4y - 6z = 0$

Now take any two equations. By rule of cross multiplication,

$$\frac{x}{20-14} = \frac{-y}{-55+49} = \frac{z}{-22+28} \Rightarrow \frac{x}{6} = \frac{-y}{-6} = \frac{z}{6} \Rightarrow \frac{x}{1} = \frac{-y}{-1} = \frac{z}{1}$$

$X_1 = (1, 1, 1)'$ eigen vector corresponding to $\lambda_1 = 0$

Case ii: $\lambda_2 = 1$ in main equations

$10x - 4y - 7z = 0 \Rightarrow 7x - 3y - 5z = 0\ \ 10x - 4y - 7z = 0$

Now take any two equations. By rule of cross multiplication,

$$\frac{x}{20-21} = \frac{-y}{-50+49} = \frac{z}{-30+28} \Rightarrow \frac{x}{-1} = \frac{-y}{-1} = \frac{z}{-2}$$

Divide by -1 , $X_2 = (1, -1, 2)'$ eigen vector corresponding to $\lambda_2 = 1$

Case iii: $\lambda_3 = 2$ in main equations

$9x - 4y - 7z = 0 \Rightarrow 7x - 4y - 5z = 0 \Rightarrow \quad 10x - 4y - 8z = 0$

Now take any two equations. By rule of cross multiplication,

$$\frac{x}{-8} = \frac{-y}{4} = \frac{z}{-8}$$

Divide by -4, $\quad \Rightarrow \frac{x}{2} = \frac{y}{1} = \frac{z}{2}$

Take -1 common

$X_3 = (2, 1, 2)'$ eigen vector corresponding to $\lambda_3 = 2$

$$P = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

Now for $P^{-1}$

We know that, $\quad P^{-1} = \frac{1}{|P|}\text{ adj } P$

We know how to calculate $P^{-1}$ and adj P.

After calculation we get $P^{-1}$ = 1 and adj P = $\begin{bmatrix} -4 & 2 & 3 \\ -1 & 0 & 1 \\ 3 & -1 & -2 \end{bmatrix}$

$$P^{-1} = \frac{1}{1}\begin{bmatrix} -4 & 2 & 3 \\ -1 & 0 & 1 \\ 3 & -1 & -2 \end{bmatrix}$$

$$D = P^{-1}AP$$

$$= \begin{bmatrix} -4 & 2 & 3 \\ -1 & 0 & 1 \\ 3 & -1 & -2 \end{bmatrix}\begin{bmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{bmatrix}\begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 2 & 3 \\ -1 & 0 & 1 \\ 3 & -1 & -2 \end{bmatrix}\begin{bmatrix} 0 & 1 & 4 \\ 0 & -1 & 2 \\ 0 & 2 & 4 \end{bmatrix} \Rightarrow D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Whatever the eigen values is appear in the diagonal matrix.

## 1.15Summary

In this chapter, we learned about types of matrices, matrix operations and a system of simultaneous linear equations in matrix form. We now understand what is adjoint of a matrix, invertible matrix and rank of a matrix and methods finding these. Students can solve a system of linear equations by row-reducing its augmented form. Students differentiated between Characteristics roots and characteristics vectors also able to reduce a matrix to a diagonal matrix.

## 1.16 References

1. Applied Mathematics II by P. N. Wartikar and J. N. Wartikar
2. Higher Engineering Mathematics by Dr. B. S. Grewal
3. Fundamentals of Matrix Computation by David S. Watkins

## 1.17 Exercise

Ex 1. If A = $\begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}$ Show that $A^2 - 5A - 14I = 0$

Ex 2. A = $\begin{bmatrix} 4 & -1 & -4 \\ 3 & 0 & -4 \\ 3 & -1 & -3 \end{bmatrix}$, show that $A^2 = I$

Ex 3. Show that AB ≠ BA in each of the following cases.

a. $A = \begin{bmatrix} 5 & -1 \\ 6 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$ b. $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 3 & 4 \end{bmatrix}$

Ex 4. Find the inverse of given matrices

1) $\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$ 2) $\begin{bmatrix} 1 & 2 & 5 \\ 1 & -1 & -1 \\ 2 & 3 & -1 \end{bmatrix}$ 3) $\begin{bmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix}$

Ex 5. If $A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$, show that $A^{-1} = \frac{1}{19} A$

Ex 6. If $A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$, show that $A^2 + 3A + I = 0$ and hence find $A^{-1}$.

Ex 7. Find the rank of the following matrices

1. $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & 1 & 4 \end{bmatrix}$; 2. $A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix}$

Ex 8. If $A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & 3 & -4 \end{bmatrix}$, find $A^{-1}$ and hence solve the system of linear equations:

$x + 2y - 3z = -4$; $2x + 3y + 2z = 2$; $3x - 3y - 4z = 1$ [ Ans: $x = 3, y = -2, z = 1$]

Ex 9. Use matrix method to show that the following system of equations is inconsistent: $3x - y + 2z = 3$:$2x + y + 3z = 5$; $x - 2y - z = 1$

Ex 10. Show that the matrix $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

satisfies its characteristic equation and hence determine $A^{-1}$.

Ex 11. Show that the matrix $A = \begin{bmatrix} 4 & -3 & -3 \\ 3 & -2 & -3 \\ -1 & 1 & 2 \end{bmatrix}$

satisfies its characteristic equation and hence determine $A^{-1}$.

Ex 12. Reduce the matrix $A = \begin{bmatrix} -1 & 3 \\ -2 & 4 \end{bmatrix}$ into a diagonal matrix. [ Ans: $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ ]

Ex 13. Reduce matrix $A = \begin{bmatrix} -19 & 7 \\ -42 & 16 \end{bmatrix}$ into a diagonal matrix.[Ans: $D = \begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix}$]

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Unit I

# 2

# COMPLEX NUMBERS

**Unit structure**

## 2.0 Objective

After going through this chapter, students will able to

- Compute sums, products, quotients, conjugate, modulus and argument of complex numbers.
- Understand the graphical representation of complex numbers
- Write the complex numbers in polar form, exponential form
- Learn about circular, hyperbolic function, inverse hyperbolic function
- Obtain relations between circular and hyperbolic functions
- Learn about graphs of the hyperbolic functions and logarithms of complex quality

## 2.1 Introduction:

This chapter is concerned with the representation and manipulation of complex numbers. It has some introductory ideas associated with complex numbers, their algebra and geometry, algebraic properties of complex numbers, Argand plane and polar representation of complex numbers, exponential form of complex numbers, mathematical operation with complex numbers and their representation on Argand's diagram, circular functions of complex angles, hyperbolic functions, relations between circular and hyperbolic functions, Inverse hyperbolic functions, graphs of the hyperbolic functions. This includes how complex numbers add and multiply, and how they can be represented graphically. Finally, we look the logarithms of complex quality and application of complex number in electrical circuit.

## 2.2 Complex number:

**Imaginary Numbers:** If the square of a given number is negative then such a number is called an imaginary number.

Eg. $\sqrt{-1}$, $\sqrt{-2}$ are imaginary numbers.

We denote $\sqrt{-1}$ as i.

Thus, $\sqrt{-4} = 2i$, $\sqrt{-9} = 3i$ and $\sqrt{-5} = i\sqrt{5}$

Powers of i:

$i^0 = 1$, $i^1 = i$, $i^2 = -1$, $i^3 = i^2 \times i = (-1) \times i = -i$, $i^4 = i^2 \times i^2 = (-1) \times (-1) = 1$

Thus, $i^0 = 1, i^1 = i, i^2 = -1, i^3 = -i, i^4=1$

**Complex Numbers:** The numbers of the form (a + ib), where a and b are real numbers and i = $\sqrt{-1}$, are known as complex numbers. The set of all complex numbers is denoted by C.

$\therefore$ C = {(a + ib):a , b $\in R$}

Eg. Each of the numbers (5 + 6i), (-4 + $\sqrt{3}$ i), and ($\frac{3}{4}$ - $\frac{5}{7}$ i) is a complex number.

For a complex number, z = (a + ib),

a = real part of z, written as Re (z) and b = imaginary part of z, written as Im (z).

If z = (5 + 6i) then Re(z) = 5 and Im(z) = 9.

**Purely Real and Purely Imaginary Numbers:**

A complex numbers z is said to be

i. Purely real, if Im(z) = 0

ii. Purely imaginary, if Re(z) = 0

Thus, each of the numbers 2, -8, $\sqrt{4}$ is purely real and 3i, ($\sqrt{5}$ i), - $\frac{5}{7}$ i is purely imaginary.

**Conjugate of a Complex Number:**

Conjugate of a complex number z = (a + ib) is defined as, $\bar{z}$ = (a - ib).

Eg, $\overline{(3 + 7i)}$ = (3 - 7i)

**Modulus of Complex Number:**

Modulus of complex number Z = (a + ib), denoted by $|z| = \sqrt{a^2 + b^2}$.

Eg. If z = (2 + 3i) then $|z| = \sqrt{2^2 + 3^2} = \sqrt{13}$

If z = (-5 - 4i) then $|z| = \sqrt{(-5)^2 + (-4)^2} = \sqrt{41}$

## 2.3 Equality of Complex Number:

If $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ then $z_1 = z_2 \Leftrightarrow a_1 = a_2$ and $b_1 = b_2$.

Ex. If 2y + (3x - y) i = (5 - 2i), find the values of x and y.

Sol: Equating the real and imaginary parts, we get

$2y + (3x - y)\,i = (5 - 2i) \Leftrightarrow 2y = 5$ and $3x - y = -2$

$\Leftrightarrow y = \frac{5}{2}$ and $3x - \frac{5}{2} = -2 \Leftrightarrow y = \frac{5}{2}$ and $x = \frac{1}{6}$

Hence $x = \frac{1}{6}$ and $y = \frac{5}{2}$

**Sum and Difference of Complex Number:**

If $z_1 = (a_1 + ib_1)$ and $z_2 = (a_2 + ib_2)$ then

i. $z_1 + z_2 = (a_1 + a_2) + i\,(b_1 + b_2)$ ii. $z_1 - z_2 = (a_1 - a_2) + i\,(b_1 - b_2)$

Ex. i. $z_1 = (3 + 5i)$ and $z_2 = (-5 + 2i)$ then

$z_1 + z_2 = \{(3 + (-5)\} + i\,(5 + 2) = (-2 + 7i)$

$z_1 - z_2 = \{(3 - (-5)\} + i\,(5 - 2) = (8 + 3i)$

**Properties of Addition of Complex Numbers:**

**i. Closure Property:** The sum of two complex numbers is always a complex number.

**ii. Commutative Law:** Addition of two complex numbers is commutative.

For any two complex numbers $z_1$ and $z_2$, $z_1 + z_2 = z_2 + z_1$, for all $z_1, z_2 \in C$

**iii. Associative Law:** Addition of three complex numbers is associative.

For any complex numbers $z_1$, $z_2$ and $z_3$,

$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ for all $z_1, z_2, z_3 \in C$

**iv. Existence of Additive Identity:** For any complex numbers z,

**$z + 0 = 0 + z = z$**, 0 is the additive identity for complex number.

**v. Existence of Additive Identity:** For any complex numbers z,

$z + (-z) = (-z) + z = 0$

Thus, every complex number z has (-z) as its additive inverse.

**Multiplication of Complex Numbers:**

Let $z_1 = (a_1 + ib_1)$ and $z_2 = (a_2 + ib_2)$ then $z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2)$

$= (a_1 a_2 - b_1 b_2) + i\,(a_1 b_2 + b_1 a_2)$

$\therefore z_1 z_2 = \{Re(z_1).\,Re(z_2) - Im(z_1).\,Im(z_2)\} + i\{Re(z_1).$

$Im(z_2) - Im(z_1).\,Re(z_2)\}$

Ex. 1. Let $z_1 = (4 + 2i)$ and $z_2 = (6 + 3i)$ then

$z_1\, z_2 = (4.6 - 2.3) + i\,(4.3 + 6.2) = (24 - 6) + i\,(12 + 12) = 18 + 24i$

**Properties of Multiplication of Complex Numbers:**

**a.** **Closure Property:** The product of two complex numbers is always a complex number.

**b.** **Commutative Law:** Multiplication of two complex numbers is commutative.

For any two complex numbers $z_1$ and $z_2$,

$z_1.z_2 = z_2.z_1$, for all $z_1, z_2 \in C$

**c.** **Associative Law:** Multiplication of three complex numbers is associative.

For any complex numbers $z_1$, $z_2$ and $z_3$,

$(z_1.z_2).\, z_3 = z_1.\,(z_2.z_3)$ for all $z_1, z_2, z_3 \in C$

d. **Existence of Multiplicative Identity:** the complex number

(1 + i0) is multiplicative identity in C.

Let z = (a + ib) then

$z \times 1 = (a + ib).\,(1 + i0) = \{(a.1 - b.0) + i(a.0 + b.1)\} = (a + ib) = z$

Similarly, $z \times 1 = 1 \times z = z$ for all $z \in C$

Hence, the complex number 1 = (1 + i0) is the multiplicative identity.

e. **Existence of multiplicative Identity:**

Let z = (a + ib) then

$$z^{-1} = \frac{1}{z} = \frac{1}{(a+ib)} = \frac{1}{(a+ib)} \times \frac{(a-ib)}{(a-ib)} = \frac{(a-ib)}{a^2+b^2}$$

Clearly, $z \times z^{-1} = z^{-1} \times z = 1$

Thus, every z = (a + ib) has its multiplicative inverse, given by,

$$z^{-1} = \frac{1}{z} = \frac{(a-ib)}{a^2+b^2} = \frac{\bar{z}}{|z|^2} \quad \therefore zz^{-1} = |z|^2$$

Points to remember:

1. $z = (a + ib) \Rightarrow \bar{z} = (a - ib)$ and $|z|^2 = (a^2 + b^2)$

2. $z = (a + ib) \Rightarrow z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{(a-ib)}{a^2+b^2}$

**f.** **Distributive Laws:** For any complex numbers $z_1$, $z_2$ and $z_3$,

$z_1.\,(z_2 + z_3) = z_1z_2 + z_1z_3$

$(z_1 + z_2).z_3 = z_1z_3 + z_2z_3$ for all $z_1, z_2, z_3 \in C$

**Division of Complex Numbers:**

Let $z_1$ and $z_2$ be complex numbers such that $z_2 \neq 0$ then

$\frac{z1}{z2} = z_1.\frac{1}{z2} = z_1.\, z_2^{-1}$.

Eg. Find $\frac{z1}{z2}$ when $z_1 = (6+3i)$ and $z_2 = (3 - i)$

Sol: We have $\frac{z1}{z2} = z_1.\, z_2^{-1}$.

$$z_2^{-1} = \frac{\overline{z2}}{|z2|^2} = \frac{\overline{(3-\iota)}}{[3^2+(-i)^2\,]} = \frac{(3+\text{i})}{10}$$

$$\frac{z1}{z2} = z_1.\, z_2^{-1}$$

$$= (6+3i).\frac{(3+\text{i})}{10} = \frac{(6+3i).(3+i).}{10} = \frac{(6.3-3.1)+i(6.1+3.3).}{10} = \frac{(15+15i).}{10} = \frac{15(1+i).}{10} = \frac{3(1+i).}{2}$$

**Some Identities on Complex Numbers:**

For any complex numbers $z_1$ and $z_2$,

i. $(z_1 + z_2)^2 = z_1^2 + z_2^2 + 2\, z_1 z_2$

ii. $(z_1 - z_2)^2 = z_1^2 + z_2^2 - 2\, z_1 z_2$

iii. $(z_1^2 - z_2^2) = (z_1 + z_2)\,(z_1 - z_2)$

iv. $(z_1 + z_2)^3 = z_1^3 + z_2^3 + 3\, z_1 z_2(z_1 + z_2)$

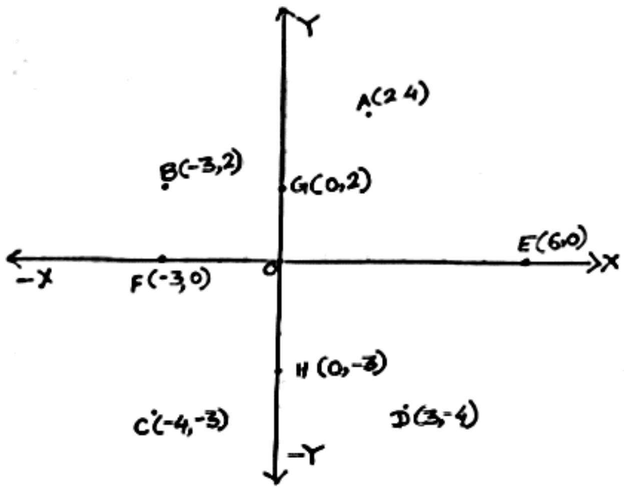
v. $(z_1 + z_2)^3 = z_1^3 - z_2^3 - 3\, z_1 z_2(z_1 - z_2)$

Students can solve these identities as exercise**.**

## 2.4 Graphical representation of Complex Number (Argand's Diagram):

**Complex Plane or Argand Plane:**

Let X'OX and YOY' be the mutually perpendicular lines, known as the x axis and the y axis respectively. The complex number (x + iy) corresponds to the ordered pair (x, y) and it can be represented by the point P(x, y) in the x-y plane. The x-y plane is known as the complex plane or Argand plane. X axis is called the real axis and y axis is called the imaginary axis.




Note that every number on the x axis is a real number, while each on the y axis is an imaginary number.

The complex numbers represented geometrically in the above diagram are

(2 + 4i), (-3 + 2i), (-4 -3i), (3 - 4i), (5 + 0i), (-4 + 0i), (0 + 3i), (0 - 3i)

Represented by the points, A (2,4), B (-3, 2), C (-4, -3), D (3, -4), E (6,0),

F (-3, 0), G (0, 2) and H (0, -3) respectively.

## 2.5 Polar form of a Complex Number:

Let the complex number z = x + iy be represented by the point P (x, y) in the complex plane. Let $\angle$ XOP = $\theta$ and $|OP|$= r > 0.

Then, P (r, $\theta$) are called the polar coordinates of P.

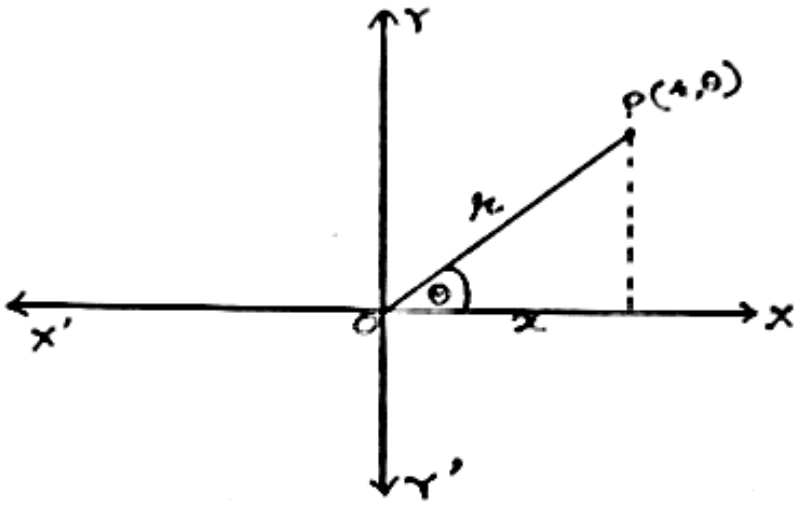
We call the origin O as pole.

Clearly, x = r cos$\theta$ and y = r sin$\theta$

We have, z = x + iy = r cos$\theta$ + i r sin$\theta$

= r (cos$\theta$ + i sin$\theta$).

This is called the polar form, or trigonometric form, or modulus-amplitude form, of z.




Here, r = $\sqrt{x^2 + y^2} = |z|$ is called the modulus of z.

And $\theta$ is called the argument, or amplitude of z, written as arg (z), or amp (z).

The value of $\theta$ such that -$\pi < \theta \leq \pi$ is called the principal argument of z.

**2.5.1 Polar form of x + iy for different signs of x, y: -**

**Method for finding the Principal Argument of a Complex Number**

**Case I** When z = (x +iy) lies on one of the axes:

I. When z is purely real. In this case, z lies on the x axis.

   i. If z lies on positive side of the x axis, then $\theta = 0$.

   ii. If z lies on negative side of the x axis, then $\theta = \pi$.

II. When z is purely imaginary. In this case, z lies on the y axis.

   i. If z lies on the y axis and above the x axis then $\theta = \frac{\pi}{2}$.

   ii. If z lies on the y axis and above the x axis then $\theta = \frac{\pi}{2}$.

**Case II** When z = (x +iy) does not lies on any axes:

Step 1. Find the acute angle $\alpha$ by tan $\alpha = \left|\frac{Im\ (z)}{Re\ (z)}\right|$.

Step 2. Find the quadrant in which P (x, y) lies.

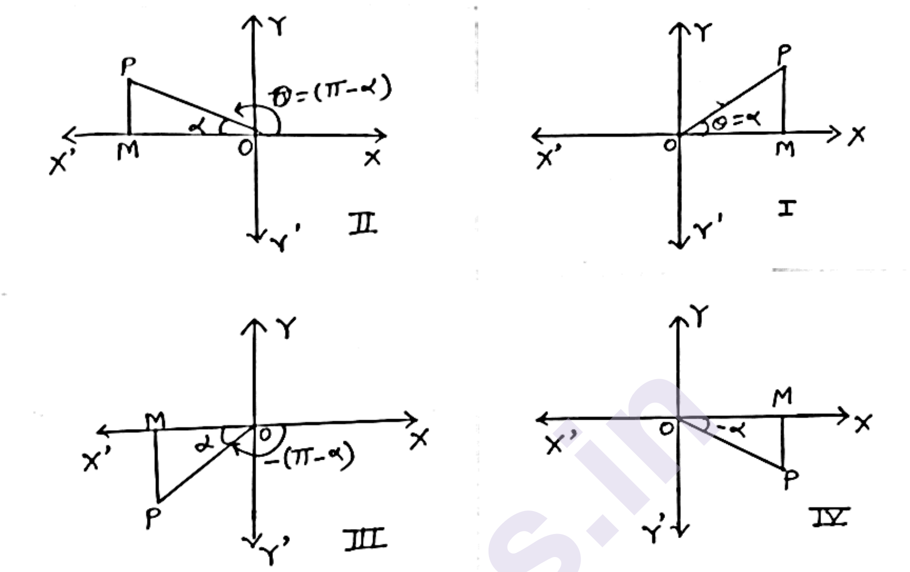
Then, $\theta$ = arg (z) may be obtained as under.

   i. When z lies in quad I; Then, $\theta = \alpha \Rightarrow$ arg (z) = $\alpha$

ii. When z lies in quad II; Then, $\theta = (\pi - \alpha) \Rightarrow \arg(z) = (\pi - \alpha)$

iii. When z lies in quad III; Then, $\theta = (\alpha - \pi) or (\pi + \alpha) \Rightarrow \arg(z) = (\alpha - \pi)\ or\ (\pi + \alpha)$

iv. When z lies in quad IV; Then, $\theta = -\alpha \Rightarrow \arg(z) = -\alpha$




Ex. 1. For following complex numbers find the polar form.

i. $z = (1+i\sqrt{3})$ ii. $z = (-1- i\sqrt{3})$

Sol. i. Let $z = (1+i\sqrt{3})$ i.e. x=1 and $y = \sqrt{3}$

We know that, Polar form = r $(\cos\theta + i\sin\theta)$

We have to find e and $\theta$

We know that, $r = \sqrt{x^2 + y^2} = = \sqrt{1+3} = 2$

$\therefore r = 2$

Let $\tan\alpha = \left|\frac{Im\ (z)}{Re\ (z)}\right| = \left|\frac{y}{x}\right| = \left|\frac{\sqrt{3}}{1}\right| = \sqrt{3}$

$\tan\alpha = \tan\frac{\pi}{3}$ $[\because \tan 60 = \sqrt{3}, \tan 60 = \tan\frac{\pi}{3}. \pi = 180]$

$\alpha = \frac{\pi}{3}$

$\because$ points $(1, \sqrt{3})$ lies in I quad, $\therefore \theta = \alpha$ $\therefore \theta = \frac{\pi}{3}$

$\therefore$ Polar form of $z = 2\ (\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$

ii. Let $z = (-1- i\sqrt{3})$ i.e. x= - 1 and $y = -\sqrt{3}$

Sol: We know that, Polar form = r $(\cos\theta + i\sin\theta)$

We have to find e and $\theta$

We know that, r = $\sqrt{x^2+y^2}$ = = $\sqrt{1+3}$ = 2 $\quad\therefore$ r = 2

Let tan $\alpha = \left|\frac{Im\,(z)}{Re\,(z)}\right| = \left|\frac{y}{x}\right| = \left|\frac{-\sqrt{3}}{-1}\right| = \sqrt{3}$

$\tan\alpha = \tan\frac{\pi}{3}$ $\qquad$ [$\because$ tan 60 = $\sqrt{3}$, tan 60 = $\tan\frac{\pi}{3}.\pi = 180$]

$\alpha = \frac{\pi}{3}$

$\because$ points (- 1, - $\sqrt{3}$) lies in III quad, $\therefore \theta = \alpha - \pi$

$\therefore \theta = \frac{\pi}{3} - \pi = \frac{\pi - 3\pi}{3} = \frac{-2\pi}{3}$

$\therefore$ Polar form of z = 2 (cos $\frac{-2\pi}{3}$ + i sin $\frac{-2\pi}{3}$ )

**Exercise:**

Ex 1. If $z$ is a non-zero complex number, such that $2iz2 = \bar{z}$ the find $|z|$

[Ans: $|z|$=1/2]

Ex. 2 If | z | = 1, then find the value of $\frac{1+z}{1+\bar{z}}$.

[Ans: z]

## 2.6 Exponential form of Complex Numbers:

We know that if x is a real number, then

$e^x$= 1 + x + $\frac{x^2}{2!}$ + $\frac{x^3}{3!}$ + $\frac{x^4}{4!}$ + …………………… (1)

Assuming this is true for all values of x (real or complex)

Let substitute i$\theta$ for x in equation (1)

$e^{i\theta}$= 1 + i$\theta$ + $\frac{i^2\theta^2}{2!}$ + $\frac{i^3\theta^3}{3!}$ + $\frac{i^4\theta^4}{4!}$+ ……………………

Put $i^2$ = -1

$e^{i\theta}$= 1 + i$\theta$ - $\frac{\theta^2}{2!}$ - $\frac{i\theta^3}{3!}$ + $\frac{\theta^4}{4!}$+ ……………………

$e^{i\theta}$ = (1 - $\frac{\theta^2}{2!}$ + $\frac{\theta^4}{4!}$ - $\frac{\theta^6}{6!}$……) + i ( $\theta$ - $\frac{i\theta^3}{3!}$ + $\frac{\theta^5}{5!}$ - $\frac{\theta^5}{7!}$………)

We know that,

sin $\theta$ = $\theta$ - $\frac{i\theta^3}{3!}$ + $\frac{\theta^5}{5!}$ - $\frac{\theta^5}{7!}$…… and cos $\theta$ = 1 - $\frac{\theta^2}{2!}$ + $\frac{\theta^4}{4!}$ - $\frac{\theta^6}{6!}$…………

$\therefore e^{i\theta}$ = (cos$\theta$ + i sin$\theta$)

$\therefore$ Complex number $z = x + iy$ (cartesian form)

$= r(\cos\theta + i\sin\theta)$ (Polar form)

$= r\,e^{i\theta}$ (Exponential form)

**Exponential form of x + iy = r $e^{i\theta}$**

$e^{i\theta} = (\cos\theta + i\sin\theta)$ and $e^{-i\theta} = (\cos\theta - i\sin\theta)$

## 2.7 Mathematical operation with complex numbers and their representation on Argand's Diagram:

**1. Addition of Complex Numbers:**

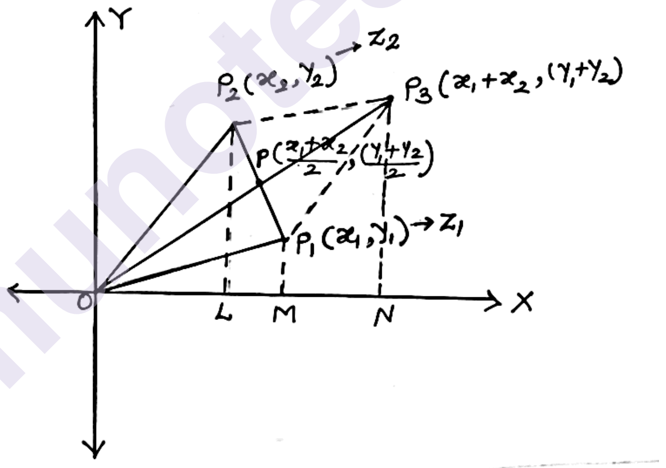
Let $z_1$ and $z_2$ be two complex numbers.

$z_1 = (x_1 + iy_1)$ and $z_2 = (x_2 + iy_2)$

$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$

Graphical representation (Argand's diagram):

Represent the complex numbers $z_1$ and $z_2$ by vectors $\overrightarrow{OP1}$ and $\overrightarrow{OP2}$ respectively.




Now complete the parallelogram $OP_1P_3P_2$.

By properties of parallelograms, opposite sides of parallelogram are equal and diagonals of parallelogram bisect each other.

$\therefore$ 0 (0,0) and P $\left(\frac{x1+x2}{2}, \frac{y1+y2}{2}\right)$

We can calculate coordinates of $P_3$.

Let consider $P_3(X, Y)$

$\therefore$ coordinates of $P_3$,

$\frac{X+0}{2} = \frac{x1+x2}{2} \Rightarrow X = (x_1 + x_2)$

$\frac{Y+0}{2} = \frac{y1+y2}{2} \Rightarrow Y = (y_1 + y_2)$

∴ coordinates of $P_3(x_1 + x_2, y_1 + y_2)$

If we represent $P_3$ in complex number as $z_3$

$z_3 = (x_1 + x_2) + i\,(y_1 + y_2)$

$= x_1 + iy_1 + x_2 + iy_2$

$\mathbf{z_3 = z_1 + z_2}$ [∵ $z_1 = (x_1 + iy_1)$ and $z_2 = (x_2 + iy_2)$]

**2. Subtraction of Complex Numbers:**

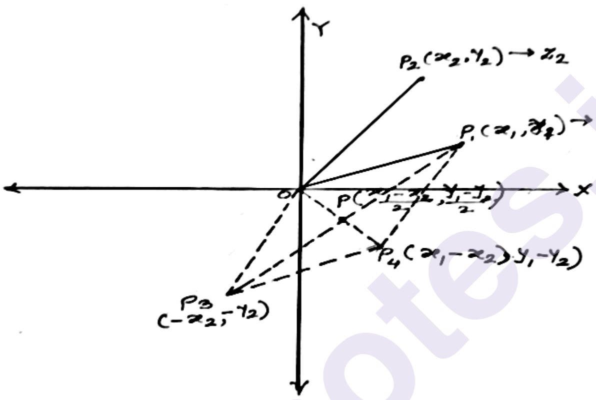
Let $z_1$ and $z_2$ be two complex numbers.

$z_1 = (x_1 + iy_1)$ and $z_2 = (x_2 + iy_2)$

$z_1 - z_2 = (x_1 - x_2) + i\,(y_1 - y_2)$

Graphical representation (Argand's diagram):

Represent the complex numbers $z_1$ and $z_2$ by vectors $\overrightarrow{OP1}$ and $\overrightarrow{OP2}$ respectively.






Take negative of complex number of $z_2$

Now complete the parallelogram $OP_3P_4P_1$.

By properties of parallelograms, opposite sides of parallelogram are equal and diagonals of parallelogram bisect each other.

∴ o (0,0) and P $\left(\frac{x1-x2}{2}, \frac{y1-y2}{2}\right)$

We can calculate coordinates of $P_4$.

Let consider $P_4(X, Y)$

∴ coordinates of $P_4$,

$\frac{X+0}{2} = \frac{x1-x2}{2} \Rightarrow X = (x_1 - x_2)$

$\frac{Y+0}{2} = \frac{y1-y2}{2} \Rightarrow Y = (y_1 - y_2)$

∴ coordinates of $P_3(x_1 - x_2, y_1 - y_2)$

If we represent $P_4$ in complex number as $z_4$

$Z_4 = (x_1 - x_2) + i\,(y_1 - y_2)$

$= (x_1 + iy_1) - (x_2 + iy_2)$

$\mathbf{Z_4 = z_1 - z_2}$ [∵ $z_1 = (x_1 + iy_1)$ and $z_2 = (x_2 + iy_2)$]

## 3. Multiplication of Complex Numbers:

Let $z_1$ and $z_2$ be two complex numbers.

$z_1 = (x_1 + iy_1)$ and $z_2 = (x_2 + iy_2)$

$z_1.z_2 = (x_1 - x_2).\ i\ (y_1 - y_2)$

$= x_1x_2 + ix_1y_2 + ix_2y_1 - y_1y_2$ [as i2 = -1]

$= (x_1x_2 - y_1y_2) + i\ (x_1y_2 + x_2y_1)$

Let consider the complex numbers in polar form.

Let $z_1 = (x_1 + iy_1) = r_1(\cos\theta_1 + i\sin\theta_1) = r_1\, e^{i\theta 1}$

$z_2 = (x_2 + iy_2) = r_2(\cos\theta_2 + i\sin\theta_2) = r_2\, e^{i\theta 2}$

Then $z_1.\ z_2 = r_1\, e^{i\theta 1}.\ r_2\, e^{i\theta 2}$

$= r_1r_2 e^{i(\theta 1 + \theta 2)}$

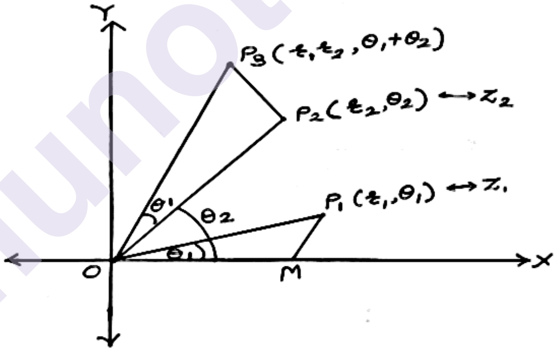
$= r_1r_2\, [\cos(\theta 1 + \theta 2) + i\sin(\theta 1 + \theta 2)]$

The product of the complex numbers is a complex number whose modulus is the product of their moduli and whose amplitude is the sum of their amplitudes.

**Graphical representation (Argand's diagram):**

Let $P_1$ represent $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$,

$P_2$ represent $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ and OM = 1 unit




We get Δ $OP_1M$.

Construct the Δ $OP_3P_2$ similar to Δ $OP_1M$.

For modulus, $\frac{OP3}{OP1} = \frac{OP2}{OM} \Rightarrow \frac{OP3}{r1} = \frac{r2}{1} \Rightarrow OP_3 = r_1r_2$

To calculate argument,

$\angle XOP_3 = \angle XOP_2 + \angle P_2OP_3 = \theta_2 + \theta_1 = \theta_1 + \theta_2$

$P_1(r_1, \theta_1)$ represents the complex number $r_1(\cos\theta_1 + i\sin\theta_1)$ and

$P_2(r_2, \theta_2)$ represents the complex number $r_2(\cos\theta_2 + i\sin\theta_2)$.

Similarly, $P_3(r_1r_2, \theta_1 + \theta_2)$ represents $r_1r_2\, [\cos(\theta 1 + \theta 2) + i\sin(\theta 1 + \theta 2)]$ whose modulus is the product of their moduli and whose amplitude is the sum of their amplitudes.

Hence $z_1 . z_2$ giving simple graphical construction for a product.

**4. Quotient of Complex Numbers:**

The product of two conjugate complex numbers is a real number i.e $(x + iy)(x - iy) = x^2 + y^2$ leads to the following method of division, where the denomination is always expressed as a real number.

Let $z_1$ and $z_2$ be two complex numbers.

$z_1 = (x_1 + iy_1)$ and $z_2 = (x_2 + iy_2)$

$$\text{Thus, } \frac{x1 + iy1}{x2 + iy2} = \frac{x1 + iy1}{x2 + iy2} \cdot \frac{x2 - iy2}{x2 - iy2}$$

$$= \frac{(x1.x2 + y1.y2) + i(x2.y1 - x1.y2)}{x2^2 + y2^2}$$

$$= \frac{(x1.x2 + y1.y2)}{x2^2 + y2^2} + \frac{i(x2.y1 - x1.y2)}{x2^2 + y2^2}$$

But it is more convenient to divide the complex numbers in their polar forms or better in exponential form.

$$\frac{x1 + iy1}{x2 + iy2} = \frac{r1\, e^{i\theta 1}}{r2\, e^{i\theta 2}} = \frac{r1}{r2}\, e^{i(\theta 1 - \theta 2)}$$

$$\frac{x1 + iy1}{x2 + iy2} = \frac{r1\, e^{i\theta 1}}{r2\, e^{i\theta 2}} = \frac{r1}{r2} \{\cos(\theta 1 - \theta 2) - i\ \sin(\theta 1 - \theta 2)\}$$

∴ The modulus of the quotient of two complex numbers is the quotient of their moduli and amplitude of the quotient is the difference of their amplitudes.

**Graphical representation (Argand's diagram):**

Let $\theta_1 > \theta_2$

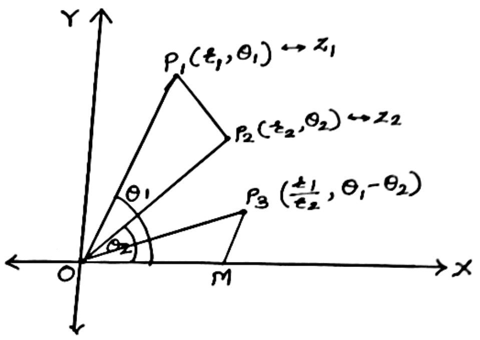
Let $P_1$ represent $z_1 = r_1(\cos\theta_1 + i \sin\theta_1)$,

$P_2$ represent $z_2 = r_2(\cos\theta_2 + i \sin\theta_2)$ and OM = 1 unit along X axis.

Construct $\Delta\ OP_1 P_2$ similar to $\Delta\ OP_3M$.

$$\frac{OP1}{OP3} = \frac{OP2}{OM} \Rightarrow \frac{r1}{OP3} = \frac{r2}{1} \Rightarrow OP_3 = \frac{r1}{r2}$$

$\angle XOP_3 = \angle XOP_1 - \angle XOP_2 = \theta_1 - \theta_2$



$P_1(r_1, \theta_1)$ represents the complex number $r_1(\cos\theta_1 + i \sin\theta_1)$ and

$P_2(r_2, \theta_2)$ represents the complex number $r_2(\cos\theta_2 + i \sin\theta_2)$.

We get, $P_3 = (\frac{r1}{r2}, (\theta_1 - \theta_2)$ gives complex number

$\frac{r1}{r2}$ $[\cos(\theta_1 - \theta_2) - i \sin(\theta 1 - \theta 2)]$ which is equal to $\frac{z1}{z2}$.

$\therefore \left|\frac{z1}{z2}\right| = \frac{|z1|}{|z2|}$ and $\arg\frac{z1}{z2} = \arg z_1 - \arg z_2$

## 5. Powers of Complex Numbers DeMoivre's Theorem:

**Statement:** If n is any real number, one of the values of $(\cos\theta + i \sin\theta)^n$ is $\cos n\theta + i \sin n\theta$.

**Proof:** Here we consider three cases.

i. n is positive integer ii. n is negative integer and iii. n is a fraction

**i. Let n is positive integer:**

$(\cos\theta + i \sin\theta)^n = (\cos\theta + i \sin\theta)(\cos\theta + i \sin\theta)$........n times

$= \cos[\theta + \theta + \ldots\ldots \text{ n times}] + i \sin[\theta + \theta + \ldots\ldots \text{ n times}]$

$= \cos n\theta + i \sin n\theta$

**ii. Let n is negative integer:**

Let n = -m, where m is a positive integer

$(\cos\theta + i \sin\theta)^n = (\cos\theta + i \sin\theta)^{-m}$

$$= \frac{1}{(\cos\theta + i \sin\theta)^m} \qquad [\because a^{-m} = \frac{1}{a^m}]$$

$$= \frac{1}{\cos m\theta + i \sin m\theta} \qquad [\text{ from ( i)}]$$

$$= \frac{1}{\cos m\theta + i \sin m\theta} \cdot \frac{\cos m\theta - i \sin m\theta}{\cos m\theta - i \sin m\theta}$$

$$= \frac{\cos m\theta - i \sin m\theta}{\cos^2 m\theta + \sin^2 m\theta} \qquad [\because i^2 = -1]$$

$$= \cos m\theta - i \sin m\theta \qquad [\because \cos^2 m\theta + \sin^2 m\theta = 1]$$

$$= \cos(-m)\theta + i \sin(-m)\theta$$

$$= \cos n\theta + i \sin n\theta$$

**iii. Let n be a fraction:**

**$n = \frac{p}{q}$, where p and q are + ve or -ve integer.**

from (i) and (ii) we have,

$(\cos\frac{\theta}{q} + i \sin\frac{\theta}{q})^q = \cos\theta + i \sin\theta$

$\because (\cos\theta + i \sin\theta)^{\frac{1}{q}} = \cos\frac{\theta}{q} + i \sin\frac{\theta}{q}$

$(\cos\theta + i \sin\theta)^n = (\cos\theta + i \sin\theta)^{\frac{p}{q}} \qquad [\because n = \frac{p}{q}]$

$= [\ (\cos\theta + \ i\ \sin\theta)^{\frac{1}{q}}]^p = [\cos\frac{\theta}{q} + i\ \sin\frac{\theta}{q}]^p$

$= \cos\frac{p}{q}\ \theta + i\ \sin\frac{p}{q}\ \theta$

$(\cos\theta + \ i\ \sin\theta)^n = \cos n\theta + i\ \sin n\theta$

Ex 1. Express sin 3θ and cos 3θ in terms of powers of cos θ and sin θ.

Sol: Using de Moivre's theorem,

$\cos 3\theta + i \sin 3\theta = (\cos\theta + i\sin\theta)^3$

$= (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\sin\theta\cos^2\theta - \sin^3\theta)$

We can equate the real and imaginary coefficients separately,

i.e. $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta = 4\cos^3\theta - 3\cos\theta$

$\sin 3\theta = 3\sin\theta\cos^2\theta - \sin^3\theta = 3\sin\theta - 4\sin^3\theta$

Ex 2. If $z = (\cos\theta + i\sin\theta)$, show that $z^n + \frac{1}{z^n} = 2\cos n\theta$ and $z^n - \frac{1}{z^n} = 2i\sin n\theta$

Sol: Let $z = (\cos\theta + i\sin\theta)$

By de Moivre's theorem,

$z^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

$\frac{1}{z^n} = z^{-n} = \cos n\theta - i\sin n\theta$

$z^n + \frac{1}{z^n} = (\cos n\theta + i\sin n\theta) + (\cos n\theta - i\sin n\theta) = 2\cos n\theta$

Also, $z^n - \frac{1}{z^n} = (\cos n\theta + i\sin n\theta) - (\cos n\theta - i\sin n\theta) = 2\sin n\theta$

Ex 3 Simplify $\left(\frac{1+\cos 2\theta + i\sin 2\theta}{1+\cos 2\theta - i\sin 2\theta}\right)^{30}$

**Sol:** Let $Z = \cos 2\theta + i\sin 2\theta$

As $|z| = |z|^2 = z\bar{z} = 1$, we get $\bar{z} = \frac{1}{z}\cos 2\theta - i\sin 2\theta$

$\therefore \frac{1+\cos 2\theta + i\sin 2\theta}{1+\cos 2\theta - i\sin 2\theta} = \frac{1+z}{1+\frac{1}{z}} = \frac{(1+z)z}{z+1} = z$

$\therefore \left(\frac{1+\cos 2\theta + i\sin 2\theta}{1+\cos 2\theta - i\sin 2\theta}\right)^{30} = Z^{30} = (\cos 2\theta + i\sin 2\theta)^{30} = \cos 60\theta + i\sin 60\theta$

Ex 4. Simplify $(1+i)^{18}$

Sol: Let $1+i = r(\cos\theta + i\sin\theta)$ then we get

$r = \sqrt{1^2+2^2} = \sqrt{2}$ ; $\alpha = \tan^{-1} = (\frac{1}{1}) = \frac{\pi}{4}$

$\theta = \alpha = \frac{\pi}{4}$ [$\because$ 1 + i lies in the first quadrant]

$\therefore (1+i) = \sqrt{2}\,(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$

Raising to power 18 on both sides

$(1+i)^{18} = [\sqrt{2}\,(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})]^{18} = \sqrt{2}^{\,18}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})^{18}$

By de Moivre's theorem,

$(1+i)^{18} = 2^9\,(\cos\frac{18\pi}{4} + i\sin\frac{18\,\pi}{4}) = 2^9\,(\cos\frac{9\pi}{2} + i\sin\frac{9\,\pi}{2})$

$= 2^9\,(\cos[4\pi + \frac{\pi}{2}] + i\sin[4\pi + \frac{\pi}{2}]) = 2^9\,(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}) = 2^9 i =$

ol: we have to find $1^{1/3}$

et $z = 1^{1/3}$ i.e. $z^3 = 1$

ı polar form, $z^3 = 1$ can be written as

$= \cos(0 + 2k\pi) + i \sin(0 + 2k\pi) = e^{i2k\pi}$, k = 0, 1, 2,...

$= (\cos\frac{2k\pi}{3} + i \sin\frac{2k\pi}{3}) = e^{i\frac{2k\pi}{3}}$, k = 0, 1, 2

aking k = 0, 1, 2 we get,

= 0, $z = (\cos 0 + i \sin 0) = 1$

= 1, $z = (\cos\frac{2\pi}{3} + i \sin\frac{2\pi}{3}) = (\cos(\pi - \frac{\pi}{3}) + i \sin(\pi - \frac{\pi}{3}))$

$- \cos\frac{\pi}{3} + i \sin\frac{\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

= 2, $z = (\cos\frac{4\pi}{3} + i \sin\frac{4\pi}{3}) = (\cos(\pi + \frac{\pi}{3}) + i \sin(\pi + \frac{\pi}{3}))$

$= -\cos\frac{\pi}{3} - i \sin\frac{\pi}{3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

The cube root of unity are 1, $\frac{-1+i\sqrt{3}}{2}$, $\frac{-1-i\sqrt{3}}{2}$

**ːe:**

implify $(-\sqrt{3} + 3i)$ [Ans: $2\sqrt{3}(\cos\frac{2\pi}{3} + i \sin\frac{2\pi}{3})$

implify $\left(\sin\frac{\pi}{6} + i\cos\frac{\pi}{6}\right)^{18}$ [ Ans: 1]

## rcular functions of complex angles:

ıw that,

$\cos x + i \sin x)$ and $e^{-ix} = (\cos x - i \sin x)$

$e^{-ix} = (\cos x + i \sin x) + (\cos x - i \sin x) = 2\cos x$

$- e^{-ix} = (\cos x + i \sin x) - (\cos x - i \sin x) = 2i \sin x$

$= \frac{e^{ix} + e^{-ix}}{2}$ **and** $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$

re known as exponential values of the sine and cosine.

non-real quantity z, where the geometrical definitions of sin z, cos z no ıave a meaning, we may regard them as defined as above so that,

$\frac{e^{iz} - e^{-iz}}{2i}$; $\cos z = \frac{e^{iz} + e^{-iz}}{2}$

$\frac{\sin z}{\cos z} = \frac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})}$; $\operatorname{cosec} z = \frac{1}{\sin z} = \frac{2i}{e^{iz} - e^{-iz}}$

$\sec z = \frac{1}{\cos z} = \frac{2}{e^{iz} + e^{-iz}}$; $\cot z = \frac{1}{\tan z} = \frac{i(e^{iz} + e^{-iz})}{e^{iz} - e^{-iz}}$

## 2.9 Definition of Hyperbolic Function:

**Hyperbolic Functions:** The hyperbolic functions are the complex analogues of the trigonometric functions. The analogy may not be immediately apparent and their definitions may appear at first to be somewhat arbitrary. However, careful examination of their properties reveals the purpose of the definitions. For example, their close relationship with the trigonometric functions, both in their identities and their calculus, means that many of the familiar properties of trigonometric functions can also be applied to the hyperbolic functions.

Definitions: The two fundamental hyperbolic functions are cosh x and sinh x, which, as their names suggest, are the hyperbolic equivalents of cos x and sin x. They are defined by the following relations.

Hyperbolic cosine of x, $\cosh x = \frac{e^x + e^{-x}}{2}$

Hyperbolic sine of x, $\sinh x = \frac{e^x - e^{-x}}{2}$

$\cosh x$ is an even function and $\sinh x$ is an odd function. By analogy with the trigonometric functions, the remaining hyperbolic functions are,

$\tanh x = \frac{\sinh x}{\cosh x} = \frac{(e^x - e^{-x})}{(e^x + e^{-x})}$; $\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{(e^x + e^{-x})}$

$\operatorname{cosech} x = \frac{1}{\sinh x} = \frac{2}{(e^x - e^{-x})}$; $\coth x = \frac{1}{\tanh x} = \frac{(e^x + e^{-x})}{(e^x - e^{-x})}$

Identities of Hyperbolic function:

1. $\sinh(-x) = -\sinh x$ 2. $\cosh(-x) = \cosh x$

3. $\tanh(-x) = -\tanh x$ 4. $1 - \tanh^2 x = \operatorname{sech}^2 x$

5. $\cosh^2 x - \sinh^2 x = 1$ 6. $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$

7. $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$

Now, we prove identity 5, rest of the identities can solve by students as exercise.

Prove that $\cosh^2 x - \sinh^2 x = 1$

Proof: L. H. S. = $\cosh^2 x - \sinh^2 x$

$= [\frac{e^x + e^{-x}}{2}]^2 - [\frac{e^x - e^{-x}}{2}]^2$

$= \frac{e^{2x} + 2.e^x e^{-x} + e^{-2x}}{4} - \frac{e^{2x} - 2.e^x e^{-x} + e^{-2x}}{4}$

$$= \frac{1}{4}\left[e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}\right]$$

$$= \frac{1}{4}[4] = 1 = \text{R.H.S}$$

Prove that 1 - $\tanh^2 x = \operatorname{sech}^2 x$

Proof: Just now we proved, $\cosh^2 x - \sinh^2 x = 1$

Divide by $\cosh^2 x$

$$\frac{\cosh^2 x}{\cosh^2 x} - \frac{\sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}$$

$1 - \tanh^2 x = \operatorname{sech}^2 x$

## 2.10 Relations between Circular and Hyperbolic Functions:

By definitions of sin z and cos z,

$\sin z = \frac{e^{iz} - e^{-iz}}{2\,i}$ and $\cos z = \frac{e^{iz} + e^{-iz}}{2}$

Put z = ix

$$\sin(\mathrm{ix}) = \frac{e^{\mathrm{i(ix)}} - e^{-\mathrm{i(ix)}}}{2\,i}$$

$$= \frac{e^{-\mathrm{x}} - e^{\mathrm{x}}}{2\,i} \qquad [\because i^2 = -1]$$

$$= \frac{-1}{i}\left[\frac{e^{\mathrm{x}} - e^{-\mathrm{x}}}{2}\right] = \frac{i^2}{i}\left[\frac{e^{\mathrm{x}} - e^{-\mathrm{x}}}{2}\right] = \mathrm{i}\sinh x$$

$$\cos(\mathrm{ix}) = \frac{e^{\mathrm{i(ix)}} + e^{-\mathrm{i(ix)}}}{2} = \frac{e^{-\mathrm{x}} + e^{\mathrm{x}}}{2} = \cosh x$$

Thus, we have,

$\sin(\mathrm{ix}) = \mathrm{i}\sinh x$; $\cos(\mathrm{ix}) = \cosh x$; $\tan(\mathrm{ix}) = \mathrm{i}\tanh x$

These definitions enable us to deduce the properties of hyperbolic functions from those of circular functions.

I. $\cos^2 z + \sin^2 z = 1$.

$$\cos^2 z + \sin^2 z = \left(\frac{e^{iz} + e^{-iz}}{2}\right)^2 + \left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2$$

$$= \left(\frac{e^{2iz} + 2 + e^{-2iz}}{4}\right) - \left(\frac{e^{2iz} - 2 + e^{-2iz}}{4}\right) = \frac{4}{4} = 1 \quad [\because i^2 = -1]$$

II. $\cosh^2 x - \sinh^2 x = 1$.

Put z = ix in I

$\cos^2(\mathrm{ix}) + \sin^2(\mathrm{ix}) = 1 \Rightarrow \cosh^2 x + (\mathrm{i}\,sinh\,x)^2 = 1$

$\Rightarrow \cosh^2 x + i^2\sinh^2 x = 1 \Rightarrow \cosh^2 x - \sinh^2 x = 1 \qquad [\because i^2 = -1]$

III. $\sin(z_1 \pm z_2) = \sin z_1.\cos z_2 \pm \cos z_1.\sin z_2$

Put $z_1$ = ix and $z_2$ = iy

sin i (x ± y) = sin (ix). cos (iy) ± cos (ix). sin (iy)

sinh (x ± y) = sinh x. cosh y ± cosh x. sinh y

Similarly, from the expansion of cos ($z_1 \pm z_2$), we get,

cosh (x ± y) = cosh x. cosh y ± sinh x. sinh y

We have following formulae for hyperbolic function which can be deduced from those of circular functions by similar methods as illustrated above.

a. tanh (x ± y) = $\frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$

b. sinh x + sinh y = 2 sinh $\frac{x+y}{2}$ . cosh $\frac{x+y}{2}$

sinh x - sinh y = 2 sinh $\frac{x-y}{2}$ . cosh $\frac{x-y}{2}$

cosh x + cosh y = 2 cosh $\frac{x+y}{2}$ . cosh $\frac{x+y}{2}$

cosh x - cosh y = 2 sinh $\frac{x+y}{2}$ . sinh $\frac{x+y}{2}$

c. $\cosh^2 x = \frac{1}{2}$ (1 + cosh 2x) and $\sinh^2 x = \frac{1}{2}$ (cosh 2x -1)

## 2.11 Inverse Hyperbolic Functions:

Let x and y be two complex numbers.

If $\sinh y$ = x then y is called the inverse hyperbolic sin of x and is written as y = $\sinh^{-1} x$.

$\sinh^{-1} x$, $\cosh^{-1} x$, $\tanh^{-1} x$ etc are called inverse hyperbolic function.

1. Prove that $\sinh^{-1} x$ = log (x + $\sqrt{x^2+1}$ )

Proof: Let sinh y = x then y = $\sinh^{-1} x$

sinh y = x ……………….. (1)

$\sinh^2 y = x^2$ ………………. (squaring both the sides)

$\sinh^2 y + 1 = x^2 + 1$ ……….. (adding 1 toboth the sides)

$\cosh^2 y = x^2 + 1$ …………..... ( ∵$\cosh^2\theta$ – $\text{sinnh}^2\,\theta$=1)

cosh y = $\sqrt{x^2+1}$………..(2) (Take square root )

Add (1) and (2)

sinh y + cosh y = x + $\sqrt{x^2+1}$

If x is real, we have, $\cosh y = \frac{e^y + e^{-y}}{2}$ and $\sinh y = \frac{e^y - e^{-y}}{2}$

$\therefore \frac{e^y - e^{-y}}{2} + \frac{e^y + e^{-y}}{2} = x + \sqrt{x^2+1} \implies \frac{e^y - e^{-y} + e^y + e^{-y}}{2} = x + \sqrt{x^2+1}$

$\frac{2\,e^y}{2} = x + \sqrt{x^2+1} \implies e^y = (x + \sqrt{x^2+1}\,) = \; y = \log\,(x + \sqrt{x^2+1}\,)$

$\sinh^{-1} x = \log\,(x + \sqrt{x^2+1}\,)$ ……… ($\because y = \sinh^{-1} x$ )

2. Prove that $\cosh^{-1} x = \log\,(x + \sqrt{x^2-1}\,)$

Proof: Let cosh y = x then $y = \cosh^{-1} x$

cosh y = x ……………….. (1)

$\cosh^2 y = x^2$ ………………. (squaring both the sides)

$\cosh^2 y - 1 = x^2 - 1$ ……….. (subtracting 1 from both the sides)

$\sinh^2 y = x^2 - 1$ …………..  ( $\because \cosh^2\theta - \text{sinnh}^2\,\theta = 1$)

$\sinh y = \sqrt{x^2-1}$………..(2) (Take square root )

Add (1) and (2)

$\cosh y + \sinh y = x + \sqrt{x^2-1}$

If x is real, we have, $\cosh y = \frac{e^y + e^{-y}}{2}$ and $\sinh y = \frac{e^y - e^{-y}}{2}$

$\therefore \frac{e^y + e^{-y}}{2} + \frac{e^y - e^{-y}}{2} = x + \sqrt{x^2-1}$

$\frac{e^y + e^{-y} + e^y - e^{-y}}{2} = x + \sqrt{x^2-1}$

$\frac{2\,e^y}{2} = x + \sqrt{x^2-1} \implies e^y = (x + \sqrt{x^2-1}\,)$

$y = \log\,(x + \sqrt{x^2-1}\,)$

$\cosh^{-1} x = \log\,(x + \sqrt{x^2-1}\,)$ ……… ($\because y = \cosh^{-1} x$ )

3. Prove that $\tanh^{-1} x = \frac{1}{2} \log \frac{1+x}{1-x}$

Proof: Let tanh y = x then $y = \tanh^{-1} x$

tanh y = x

$\frac{e^y - e^{-y}}{e^y + e^{-y}} = x$ ………………. ( $\tan x = \frac{sinx}{\cos x}$ )

$\frac{e^y + e^{-y}}{e^y - e^{-y}} = \frac{1}{x}$

$\frac{e^y + e^{-y} + e^y - e^{-y}}{e^y + e^{-y} - e^y - e^{-y}} = \frac{1+x}{1-y}$……. (if $\frac{x}{y} = \frac{a}{b}$ then $\frac{x+y}{x-y} = \frac{a+b}{a-b}$)

$$\frac{e^{y}+e^{y}}{e^{-y}+e^{-y}} = \frac{1+x}{1-y}$$

$$\frac{2e^{y}}{2e^{-y}} = \frac{1+x}{1-y}$$

$$e^{y}.\, e^{y} = \frac{1+x}{1-y}$$

$$e^{2y} = \frac{1+x}{1-y}$$

$$2y = \log \frac{1+x}{1-y}$$

$$y = \frac{1}{2} \log \frac{1+x}{1-y}$$

$$\tanh^{-1} x = \frac{1}{2} \log \frac{1+x}{1-y} \;\ldots\ldots\ldots\ldots\; (y = \tanh^{-1} x)$$

## 2.12 Differentiation and Integration:

a. $y = cosh\, x,\quad \frac{dy}{dx} = sinh\, x,\qquad \therefore \int sinh\, x\, dx = cosh\, x$

b. $y = sinh\, x,\quad \frac{dy}{dx} = cosh\, x,\qquad \therefore \int cosh\, x\, dx = sinh\, x$

c. $y = tanh\, x,\quad \frac{dy}{dx} = \text{sech}^2\, x,\qquad \therefore \int \text{sech}^2\, x\; dx = tanh\, x$

d. $y = \sinh^{-1} \frac{x}{a},\quad \frac{dy}{dx} \frac{1}{\sqrt{a^2+x^2}},\qquad \therefore \int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1} \frac{x}{a}$

e. $y = \cosh^{-1} \frac{x}{a},\quad \frac{dy}{dx} \frac{1}{\sqrt{x^2+a^2}},\qquad \therefore \int \frac{dx}{\sqrt{x^2+a^2}} = \cosh^{-1} \frac{x}{a}$

f. $y = \tanh^{-1} \frac{x}{a},\quad \frac{dy}{dx} \frac{a}{a^2-x^2},\qquad \therefore \int \frac{dx}{a^2-x^2} = \frac{1}{a} \tanh^{-1} \frac{x}{a}$

g. $y = \text{cosech}^{-1} \frac{x}{a},\quad \frac{dy}{dx} \frac{-a}{x\sqrt{a^2+x^2}},\qquad \therefore \int \frac{dx}{x\sqrt{a^2+x^2}} = -\frac{1}{a} \text{cosech}^{-1} \frac{x}{a}$

h. $y = \text{sech}^{-1} \frac{x}{a},\quad \frac{dy}{dx} \frac{-a}{x\sqrt{a^2+x^2}},\qquad \therefore \int \frac{dx}{x\sqrt{a^2+x^2}} = -\frac{1}{a} \text{sech}^{-1} \frac{x}{a}$

Series for cosh x and sinh x :-

$$e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\ldots\ldots\ldots\ldots\ldots$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots\ldots\ldots\ldots\ldots\ldots$$

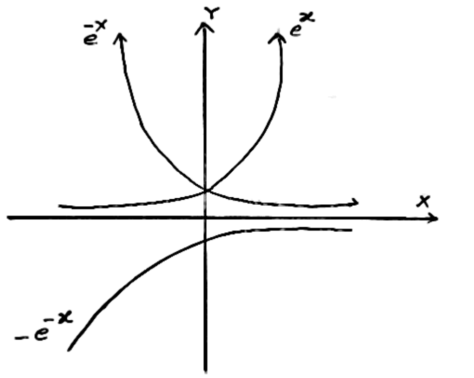
$$\cosh x = \frac{1}{2}(e^{x} + e^{-x}) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\ldots\ldots\ldots\ldots\ldots$$

$$\sinh x = \frac{1}{2}(e^{x} - e^{-x}) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\ldots\ldots\ldots\ldots\ldots$$

$$\tanh x = \frac{sinh\, x}{cosh\, x} = x - \frac{1}{3!} x^3 + \frac{2}{15} x^5 + \ldots\ldots\ldots\ldots\ldots\ldots$$

## 2.13 Graphs of the hyperbolic functions:

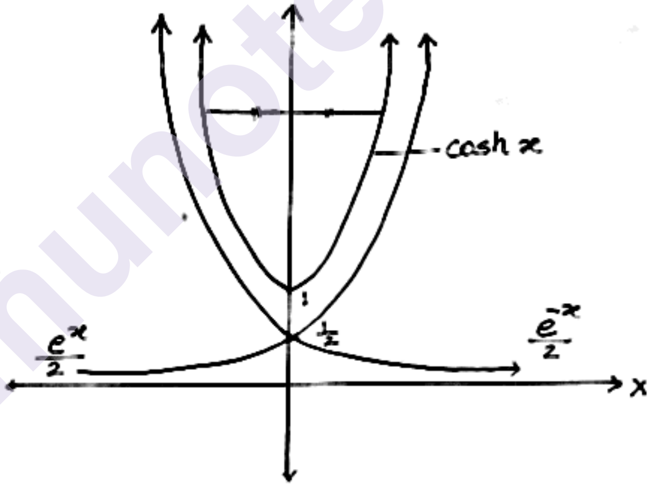
First, we draw the graphs of $e^x, e^{-x}$ and - $e^{-x}$




We know that,

$\cosh x = \frac{e^x + e^{-x}}{2}$, $\sinh x = \frac{e^x - e^{-x}}{2}$ and $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

For $\cosh x$, $\quad \cosh x = \frac{e^x + e^{-x}}{2} = \frac{e^x}{2} + \frac{e^{-x}}{2}$






Note: $\cosh x$ is an EVEN function. It is symmetric about Y axis and cosh $(-x)$ = $\cosh x$

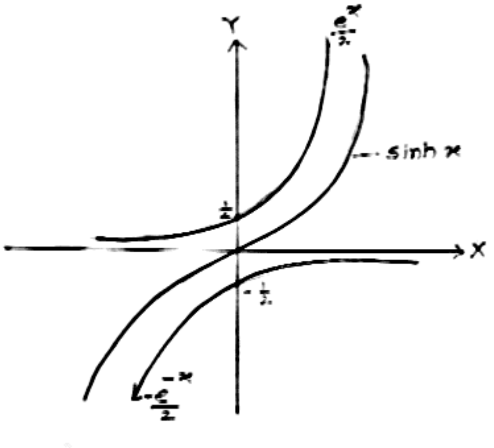
Domain: $\{ x \in R\}$ and Range: $\{y \in R \,/\, y \geq 1\}$

$x \to -\infty$ then $\cosh x \to \infty$ and

$x \to \infty$ then $\cosh x \to \infty$

For $\sinh x$,

$\sinh x = \frac{e^x - e^{-x}}{2} = \frac{e^x}{2} - \frac{e^{-x}}{2}$




Note: $\sinh x$ is an ODD function and sinh (-$x$) = $-\sinh x$

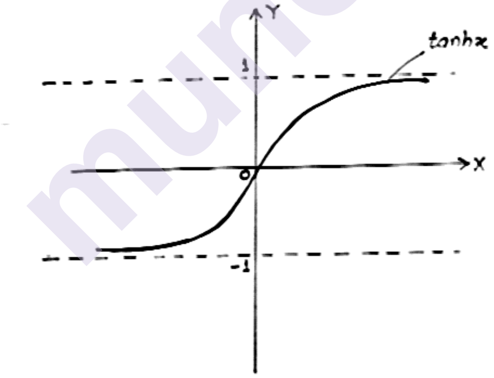
Domain: { x $\epsilon$ $R$} and Range: {y $\epsilon$ $R$}

x→ - ∞ $then$ $\sinh x$ → - ∞ and

x→ ∞ then $\sinh x$ → ∞

For $\tanh x$,

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{\sinh x}{\cosh x}$$




Note: $\tanh x$ is an ODD function. It is symmetric about origin and tanh (-$x$) = $-$ tanhh $x$

Domain: { x $\epsilon$ $R$} and Range: {y $\epsilon$ R / -1 < y < 1}

x→ - ∞ $then$ $\tanh x$ → - 1and x→ ∞ then $\tanh x$ → 1

The values of sinh x, cosh x and tanh x for x = - ∞, 0 and + ∞ from definition are as follows

| x | sinh x | cosh x | tanh x |
|---|---|---|---|
| $-\infty$ | $-\infty$ | $+\infty$ | -1 |
| 0 | 0 | 1 | 0 |
| $+\infty$ | $+\infty$ | $+\infty$ | 1 |

## 2.14 Logarithms of complex quality:

Let z = x + iy

Expressing the complex number in general polar form,

$z = r(\cos\theta + \sin\theta)$

$x + iy = r(\cos\theta + \sin\theta)$………….. (A)

Equating real and imaginary parts,

$x = r\cos\theta$………………. (1)

$y = r\sin\theta$……………….. (2)

Eq $(1)^2$+ Eq$(2)^2$

$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta$

$x^2 + y^2 = r^2$ …………………… $[\cos^2\theta + \sin^2\theta = 1]$

$\therefore r = \sqrt{x^2 + y^2}$

Eq (2) / Eq (1)

$\frac{r\sin\theta}{r\cos\theta} = \frac{y}{x}$

$\tan\theta = \frac{y}{x}$

$\theta = \tan^{-1}\frac{y}{x}$

Take a log of Eq (A)

$\log(x + iy) = \log r(\cos\theta + \sin\theta) = \log r\, e^{i\theta} = \log r + \log e^{i\theta}$

$= \log r + i\theta$ $[\log e = 1]$

$$\mathbf{\log(x + iy) = \log\sqrt{x^2 + y^2} + i\tan^{-1}\frac{y}{x}}$$

1. Prove that $\log(x + iy) = \log\sqrt{x^2 + y^2} + i\tan^{-1}\frac{y}{x} + 2n\pi i$

Proof: Let z = x + iy

Expressing the complex number in general polar form,

z = r (cos$\theta$ + sin $\theta$)

x + iy = r (cos$\theta$ + sin $\theta$)………….. (A)

Equating real and imaginary parts,

x = r cos$\theta$………………. (1)

y = r sin $\theta$……………….. (2)

Eq $(1)^2$+ Eq$(2)^2$

$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta$

$x^2 + y^2 = r^2$ …………………..        ($\cos^2\theta + \sin^2\theta = 1$)

$\therefore r = \sqrt{x^2 + y^2}$

Eq (2) / Eq (1)

$\frac{r\sin\theta}{r\cos\theta} = \frac{y}{x}$

$\tan\theta = \frac{y}{x}$

$\theta = \tan^{-1}\frac{y}{x}$

Take a Log of Eq (A)

Log (x + iy) = Log r (cos$\theta$ + sin $\theta$)

(Take general value of Log)

= log r {cos $(2n\pi + \theta)$ + i sin $(2n\pi + \theta)$}

= log r $e^{i(2n\pi+\theta)}$

= log r + log $e^{i(2n\pi+\theta)}$ [∵ log mn=log m + log n]

= log $\sqrt{x^2 + y^2}$ + i$(2n\pi + \theta)$ log e        [∵log $m^n$=n log m]

Log (x + iy) = log $\sqrt{x^2 + y^2}$ + i $(2n\pi + \tan^{-1}\frac{y}{x})$        [∵ log e=1]

This shows that for different value of n, the logarithm of a complex quantity x + iy is multivalued

**Ex. 1.** Prove that Log (1 + i) = $\frac{1}{2}$ log 2 + i$(2n\pi + \frac{\pi}{4})$

Sol: we know that,

Log (x + iy) = log $\sqrt{x^2 + y^2}$ + i$(2n\pi + \tan^{-1}\frac{y}{x})$

L. H. S. = Log (1 + i)

$= \log \sqrt{\mathbf{1^2 + 1^2}} + i(2n\pi + \tan^{-1}\frac{1}{1})$

$= \log \sqrt{\mathbf{2}} + i(2n\pi + \frac{\pi}{4})$ $\quad (\because \tan^{-1}1 = \frac{\pi}{4})$

$= \log (2^{\frac{1}{2}}) + i(2n\pi + \frac{\pi}{4})$

$= \frac{1}{2} \log (2) + i(2n\pi + \frac{\pi}{4})$

= R. H. S.

**Ex. 2.** Prove that Log (-5) = $\log 5 + i(2n\pi + \pi)$

Sol: we know that,

$\text{Log}(x + iy) = \log \sqrt{x^2 + y^2} + i(2n\pi + \tan^{-1}\frac{y}{x})$

L. H. S. = Log (-5)

$= \log (-5) + 2n\pi\, i$

$= \log 5(-1) + 2n\pi\, i$

$= \log 5 + \log (-1) + 2n\pi\, i$

$= \log 5 + \log (\cos \pi + i \sin \pi) + 2n\pi\, i \; (\because \cos\pi + i\sin \pi = 1)$

$= \log 5 + \log e^{i\pi} + 2n\pi\, i$

$= \log 5 + i\,\pi \log e + 2n\pi\, i$

$= \log 5 + i\,(2n\pi + \pi)$

= R.H.S

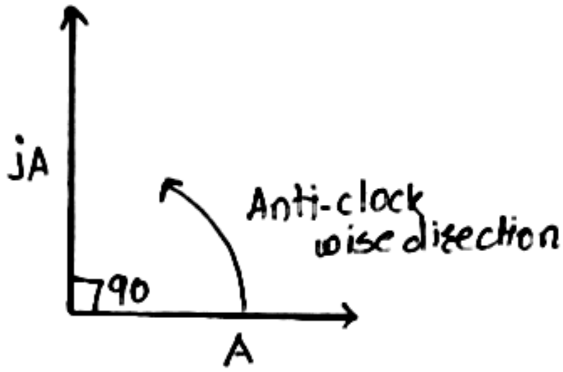
Exercise:

Ex1. Prove that $\log \left(\frac{a+ib}{a-ib}\right) = 2i \tan^{-1}\left(\frac{x}{y}\right)$

Ex. 2 Show that $\log (1 + e^{i\theta}) = \log (2 \cos \frac{\theta}{2}) + \frac{1}{2} i\theta$, if $-\pi < \theta < \pi$

## 2.15 j(=i) as an operator (Electrical circuits)

j operator is a mathematical operator which when multiplied with any vector, rotate that vector by $90^0$ in anti-clock wise direction.




j operator has assigned a value of $\sqrt{-1}$. Thus, it is an imaginary number.

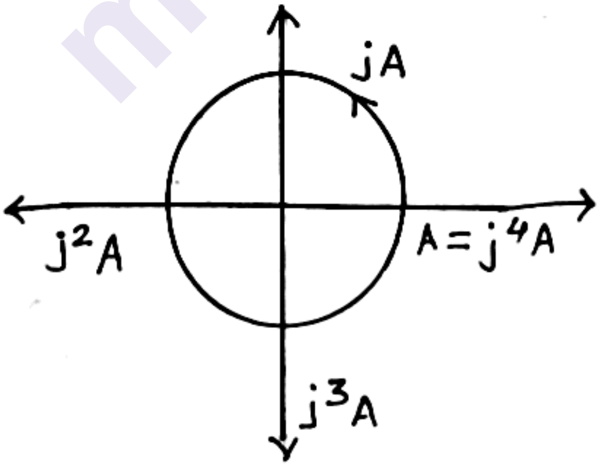
When operator j is operated on vector A, will get new vector jA. This new vector is displaced from displaced the original vector by $90^0$ in anti-clockwise direction. the magnitude of vector is remains unchanged when the vector is operated by j.

If the j is applied on the vector jA, the new vector $j^2A$ will be the $180^0$ in anti-clockwise direction. The new vector $j^2A$ is in opposite to the original vector A. Hence $j^2A = -A$.

Similarly, when $j^2A$ is operated with j, the new vector so produced $j^3A$ will $270^0$ ahead of the A. Hence, $j^3A = -jA$. In the same way $j^4A = A$

From above, we can say that,

$j^2 = -1$; $j^3 = j^2.j = -j$; $j^2 = j^2\, j^2 = 1$; $(1/j) = -j$




**We know that from Euler's Formula,**

$e^{ix} = (\cos x + i \sin x)$

Substitute $x = \frac{\pi}{2}$ since $\cos\frac{\pi}{2} = 0$, $\sin\frac{\pi}{2} = 1$, we get, $e^{i\frac{\pi}{2}} = i$

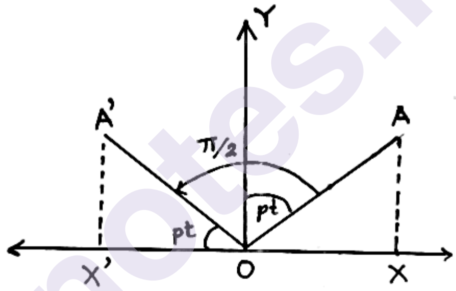
If we take a radius vector of length 'a' along a horizontal line then

$ai = a\,e^{i\frac{\pi}{2}} = ia$; $\quad ai^2 = a\,e^{i\pi} = -a$; $\quad ai^3 = a\,e^{i\frac{3\pi}{2}} = -ia$; $ai^4 = a\,e^{i2\pi} = a$

Thus, if we take a radius vector of length 'a' along a horizontal line the effect of raising i to a power n is equivalent to turning this radius vector through an angle $n\frac{\pi}{2}$.

i. Operation of j (=i) on a sin pt:

A sin pt is the projection of vector $\overrightarrow{OA}$ (= a) on the horizontal line, where pt is an angle made by it with vertical, as shown in the fig.




Then j (a sin pt) represents the projection of $\overrightarrow{OA'}$ (= a) on the horizontal line, when $\overrightarrow{OA}$ is turned through $\frac{\pi}{2}$.

∴ j (a sin pt) = Projection of $OA'$ on XOX' = a cos pt

**∴ j (a sin pt) = a cos pt**

ii. Operation of (a + jb) on a sin pt:

(a + jb) sin pt = a sin pt + jb sin pt

= a sin pt + b cos pt [from (i)]

**∴ (a + jb) sin pt = $\sqrt{a^2 + b^2}$ (sin pt + $\alpha$),** Where $\tan\alpha = \frac{b}{a}$

Operation of (a - jb) on a sin pt:

(a - jb) sin pt = a sin pt - jb sin pt

= a sin pt - b cos pt [from (i)]

$\therefore$ **(a - jb) sin pt = $\sqrt{a^2 + b^2}$ (sin pt - $\alpha$),** Where $\tan \alpha = \frac{b}{a}$

iii. Operation of $\frac{1}{a + jb}$ on a sin pt:

$\frac{1}{a + jb}$ sin pt = $\frac{a - jb}{a^2+b^2}$ sin pt

= $\frac{1}{a^2+b^2}\sqrt{a^2 + b^2}$ (sin pt - $\alpha$), Where $\tan \alpha = \frac{b}{a}$ [from (ii)]

$\therefore \frac{1}{a + jb}$ sin pt = $\frac{1}{\sqrt{a^2+b^2}}$ (sin pt - $\alpha$)

Similarly,

$\therefore \frac{1}{a - jb}$ sin pt = $\frac{1}{\sqrt{a^2+b^2}}$ (sin pt + $\alpha$)

In electrical engineering, j operator has a great significance and application. You will encounter this operator often in electrical machine, power system, AC Network etc.

We know that impedance of a circuit is a complex quantity i.e. it is having real part and imaginary part. Real part signifies resistive portion whereas imaginary part denotes reactance part of the impedance.

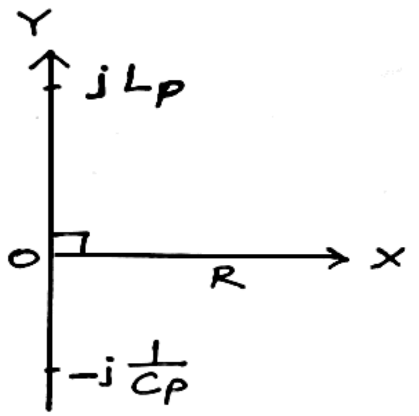
In an electric circuit containing resistance R, inductance L and capacity C in series. We know that, if current I flow through the circuit at any time due to applied hormonic E. M. F. $E_0$ sin pt, we have,

$E_R$ = RI in phase with I

$E_L$ = LpI in quadrature with I (leading)

$E_C = \frac{I}{Cp}$ in quaduture with I (lagging)

Where $E_R$, $E_L$ and $E_C$ are voltage drops across R, L and C respectively.



As current through reactance either lags or lead the voltage by $90^{0.}$ Therefore, this reactance is represented by using j operator. The current through resistance remain in phase with the voltage, hence resistance is taken as reference and reactance (say E) is rotated with respect to this reference when operated with j operator.

The total impendence which impedes the circuit in AC circuit given by addition of these vectors.

Hence impendence Z is written as Z = (R $\pm$ jE). It may be noted that the capacitive and inductive reactance are (-j/$Cp$) and jLp.

$$\therefore z \quad = R + jLp + \left(\frac{-j}{Cp}\right)$$

$$= R + j\left(Lp - \frac{1}{Cp}\right)$$

If $E_0$ sin pt be applied voltage, the current I in the circuit is given by,

$$\frac{E0 \sin pt}{I} = z$$

$$\therefore I = \frac{E0 \sin pt}{z}$$

$$\therefore I = \frac{E0}{R + j\left(Lp - \frac{1}{Cp}\right)} \sin pt$$

$$= \frac{E0}{\sqrt{R^2 + \left(Lp - \frac{1}{Cp}\right)^2}} \sin(pt - \alpha), \text{ where } \alpha = \tan^{-1} \frac{\left(Lp - \frac{1}{Cp}\right)}{R}$$

$$\left[\because \frac{1}{a + jb} \sin pt = \frac{1}{\sqrt{a^2 + b^2}} (\sin pt - \alpha)\right]$$

## 2.16 Summary:

Complex Numbers can be presented in rectangular, polar or exponential form with the conversion between each complex number algebra form including addition, subtracting, multiplication and division. We learned about introductory ideas associated with complex numbers, their algebra and geometry, algebraic properties of complex numbers, Argand plane and polar representation of complex numbers, mathematical operation with complex numbers and their representation on Argand's Diagram, circular functions of complex angles, hyperbolic functions, relations between circular and hyperbolic functions, inverse hyperbolic functions, graphs of the hyperbolic functions. Finally, we looked the Logarithms of complex quality and application of complex number in electrical circuit.

## 2.17 References:


1. Applied Mathematics II by P. N. Wartikar and J. N. Wartikar
2. Higher Engineering Mathematics by Dr. B. S. Grewal
3. Complex numbers from A to Z by Titu Andreescu and Dorin Andrica


## 2.17 Exercise:

Ex. 1 If $|z1| = 1$, $|z2| = 2$, $|z3| = 3$ and $|9z1\ z2 + 4z1\ z3 + z2\ z3| = 12$, then find the value of $|z1 + z2 + z3|$. [Ans: $|z1 + z2 + z3| = 2$]

Ex.2 $z_1$, $z_2$, and $z_3$ are complex numbers such that $z_1 + z_2 + z_3 = 0$ and $|z1| = |z2| = |z3| = 1$ then find $z_1^2 + z_2^2 + z_3^3$

[Ans: 0]

Ex. 3 Find the fourth roots of unity [Ans: 1, i, -1, -i]

Ex. 4 Find all cube root of $(\sqrt{3} + i)$

[Ans: $2^{1/3}$ $(\cos\frac{\pi}{18} + i\sin\frac{\pi}{18})$, $2^{1/3}$ $(\cos\frac{13\pi}{18} + i\sin\frac{7\pi}{8})$, $2^{1/3}$ $(\cos\frac{25\pi}{18} + i\sin\frac{25\pi}{18})$]

Ex. 5 Simplify $\left(\frac{1+\sqrt{3}i}{1-\sqrt{3}i}\right)^{10}$ [Ans: $(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3})$]

Ex 6 Prove that $\text{Log}\ i = \log i + 2n\pi i$

Ex 7 Prove that $i\log\left(\frac{x-i}{x+i}\right) = \pi - 2\tan^{-1}x$

Ex 8 Show that $\tan\left(i\log\frac{a+ib}{a-ib}\right) = \frac{2ab}{a^2 + b^2}$

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Unit 2

# 3

# DIFFERENTIAL EQUATION

## EQUATION OF THE FIRST ORDER AND OF THE FIRST DEGREE

Unit Structure



## 3.1 Objectives

- recognize and solve problems in ordinary differential equations
- Understand the application of differential equation in physics and engineering branches such as electronics, electrical, mechatronics etc.
- Evaluate first order differential equations including separable, homogeneous, non-homogeneous exact, and linear and partial

- Identify research problems where differential equations can be used to model the system
- Analyze mathematical models to solve application problems such as circuits, population modeling, orthogonal trajectories, and slopes

## 3.2 Introduction

In an equation constituting of dependent and independent variable, when the derivatives of the former can be represented with respect to one or more independent variables such equations are called Differential Equation. Some of the differential equations that can be solved by standard procedures are as follows:

- Differential equation in which variables are separable
- Homogeneous differential equations
- Non homogeneous differential equations which can be reduced homogeneous differential equations
- Linear differential equations
- Bernoulli's differential equations that are nonlinear and can be reduced to linear form.
- Exact differential equations

A first order differential equation is an equation that can be represented in the form

F (t, y, dy/dt) = 0 or in other words $F(t, y, y')$
Equation 1

where $y'$ is the first order derivative of y

This equation can also be represented as

$F(t, f(t), f'(t)) = 0$ for every value of t
Equation 2

and is function of three variables $(t, y, y')$.

A differential equation's order is determined by the highest-order derivative whereas the degree is the highest power to which a variable is raised within an equation. The higher the order of the differential equation, the more arbitrary constants need to be added to the general solution. Below are a few examples that depict different scenarios

Examples $y''' + y'' + y' + c = 0$ Equation 3

$y' - y = 4$ Equation 4

## 3.3. Ordinary Differential Equation

Ordinary Differential Equation (ODE) is described as the relation having an independent variable x,a dependent variable y and associated derivatives of y. The order of the ordinary differential equation is the order of the highest derivative in that equation. Few examples of ordinary differential equation are as follows:

| Equation | Order | Degree | |
|---|---|---|---|
| $y^3 + x^3 dy/dx = 0$ | 1 | 3 | Equation 5 |
| $y^3 + x^3 d^2y/dx^2 = 0$ | 2 | 4 | Equation 6 |

**Example 1**

$\partial y/\partial x = 4y - 2$ $\partial y/4y\text{-}2 = \partial x$

$\int \partial y/(4y-2) = \partial x$

$1/4 \log |4y-2| = x + c$

$\log |4y-2| = 4x + 4c$

$4y - 2 = (+-)e^{4x+4c}$

$4y = (+-)e^{4x+4c} + 2$

$y = 1/4(+-)e^{4x+4c} + 1/2$

$y = (+-)1/4\exp(4(x+c)) + 1/2$

Let $C = 1/4\exp(4c)$

$y(x) = Ce^{4x} + 1/2$

$\frac{\partial y}{\partial x} = 4Ce^{4x}$

$4y - 2 = 4Ce^{4x}$

$4y = 4Ce^{4x} + 2$

Substituting for y in the above

$4(Ce^{4x} + 1/2) = 4Ce^{4x} + 2$

The two equations are proved equal.

With y(2) where x = 2 and y(2) = 4 the proof is as follows :

y(2) = 1 then

$Ce^8 + 1/2 = 1$

$Ce^8 = 1/2$ or $C = 1/2(e^{-8})$

**$4(1/2e^{-8} .e^8 + 2) = 4(1/2e^{-8}. e^8) + 2 = 4$ Ans**

**Example 2**

$= \partial y / \partial x = 7y^2x^3$ and $y(2) = 3$

$= \partial y / 7y^2 = x^3 dx$

$= 1/7 \int \partial y / y^2 = \int x^3 \, dx$

$= -\frac{1}{7} y^{-2+1} = \frac{x^4}{4} + c$

$= (-1)y^{-1} = 7x^4 + c$

$y = -1/(7/4(x^4) + c) = \frac{-1}{\frac{7}{4}x^4 + c}$

Putting x =2

$3 = (-1)/(7/4(16) + c)$

$3 = -1/28 + c$

$c = -85/3$

$y = \frac{(-1)}{(\frac{7}{4})x^4 - \frac{85}{3}}$ **Ans**

$\frac{\partial y}{\partial x} = \frac{\partial}{\partial x}\left(\frac{-1}{\frac{7}{4}x^4 + c}\right) = \frac{7x^3}{(\frac{7}{4}x^4 + c)^2} = 7x^3 \times y^2$ where $y = \frac{-1}{\frac{7}{4}x^4 + c}$

## 3.4 Separable Variables - Differential Equation

Variables are said to be separable when all the similar terms are on the same side i.e. x and dx on one side and y and dy on the other side. The general representation of the equation is as follows :

$f(x)dx = g(y)dy$ (or) $f(x)dx + g(y)dy = 0$ Equation 7

Consider an example as follows:

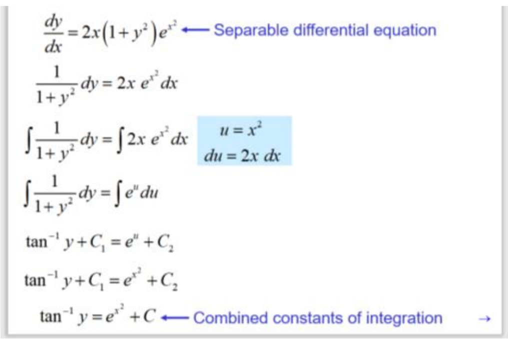
$(y^2+1)\partial y + (x^2 + 3)\partial x = 0$

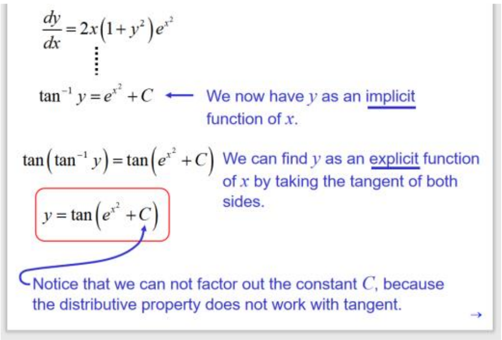
$\int (y^2 + 1)\partial y + \int (x^2 + 3)\partial x = 0$

$\frac{y^3}{3} + y + \frac{x^3}{3} + 3x = c$

## Examples

**a)**




**b)** $\partial y/\partial x = e^{x-y} + x^2e^{-y} = e^{-y}(e^x + x^2)$

$= \partial y/e^{-y} = (e^x + x^2)\,\partial x$

$= \int e^y\,\partial y = \int e^x \partial x + \int x^2 \partial x$

$= e^y = e^x + \frac{x^3}{3} + c$ **Ans**

**c)** y - x dy/dx = a($y^2$ +dy/dx)

= y - xdy/dx **= ay2** + ady/dx

= y - a$y^2$ = dy/dx(x+a) = dx/(x+a) = dy/y-a$y^2$

$\int \mathbf{dx/(x+a)} = \int \boldsymbol{dy/y - ay^2}$

Let $\frac{1}{y(1-ay)} = \frac{A}{y} + \frac{B}{1-ay}$

1 = A (1-ay) +By

1 = A- a(Ay) + By

1-A = -y(aA -B)

1-A = -y or y = A-1

B-aA= 1-A if A= 1 then B- a = 0 or B=a

$\int dx/(x+a) = \int dy/y - ay^2$

1/y(1-ay) = A/y +B/1-ay

Upon integrating it is

log y +a(-1/a)log(1-ay) = log y - log(1-ay) = log(y/1-ay) +c

log (x+a) = log(y/1-ay) +c

log(x+a) -log(y) +log(1-ay) =log c

log(x+a)(1-ay)/log y = log c

**(x+a)(1-ay) = cy Ans**

**Example**

$y(1+x)dx + x(1+y)dy = 0$

(1+x)dx = -x(1+y)dy

(1+x)dx/x = -(1+y)dy

$\int \frac{dx}{x} + \int dx = -\int \frac{dy}{y} - \int dy$

$= \log x + x = -\log y - y$

= log |x |+log |y| +x+y = c

$= \log|xy| + x + y = c$

## 3.5 Equations reducible to homogeneous forms

A function f(x,y) is called Homogeneous of degree n if

$$f(x,y) = t^n f(x,y)$$

Equation 8

and where t is a nonzero real number. Thus

$$\sqrt{xy}, \frac{x^{10}+y^{10}}{x^2+y^2} and \ldots \sin\left(\frac{x}{y}\right)$$

Equation 9 are homogeneous function of degree 1, 8 and 0 respectively

A first order differential equation of the form $\frac{dy}{dx} = f(x,y)$ is said to be homogeneous if the function f depends only on ratio of (y/x).Thus first order homogeneous equation are of the form

$$\frac{dy}{dx} = g\left(\frac{y}{x}\right)$$

Equation 10

and is transformed into an equation that is separable by substituting y = vx and

$$\frac{dy}{dx} = v + x\left(\frac{dv}{dx}\right)$$

Equation 11

and

**g(v) = v +x(dv/dx) and** $\int$ **dv/(v-g(v) =** $-\int dx/x$

Equation 12

**a)** $(x-y)\frac{\partial y}{\partial x} = x+3y$

$\frac{x+3y}{x-y} = \frac{\partial y}{\partial x}$ let y = hx; $\frac{\partial y}{\partial x} = h + x\frac{\partial h}{\partial x} = \frac{x+3vx}{x-vx} = \frac{1+3h}{1-h}$

$= x\frac{\partial h}{\partial x} = \frac{1+3h}{1-h} - h = \frac{1+3h-h+h^2}{1-h} = \frac{1+2h+h^2}{1-h}$

$= \frac{\partial x}{x} = \frac{(1-h)}{(1+h)^2} = \log|x| + c = \int \frac{1-h}{(1+h)^2}\partial h$

Putting m = 1+h

$= \int \frac{2-m}{m^2}dm = 2\int \frac{\partial m}{m^2} - \int \frac{\partial m}{m} = -\log|m| - \frac{2}{m} = \frac{-2}{(1+h)}log|1+h|$

$= \frac{-2}{(1+h)}log|1+h|$

$= \log|x| + c = \frac{-2}{1+\frac{y}{x}} - \log|1+\frac{y}{x}| = \log|x| + c + \frac{2x}{x+y} + \log|\frac{x+y}{x}| = 0$

$= \log|x+y| + \frac{2x}{x+y} = c$ **Ans**

**b)** Solve : $(x + 9y - 7)dx = (2x + 3y - 6)dy$

$$\frac{dy}{dx} = \frac{(x + 9y - 7)}{(2x + 3y - 6)}$$

$= \frac{dy}{x+9y-7} = \frac{dx}{2x+3y-6}$

Let $x = X + h$, $y = Y + k$ here h and k can be solved for their values

Equations to be considered are as follows:

$(h + 9k - 7)$ and $(2h + 3k - 6)$ that are solved to get $h = \frac{11}{5}$ and $k = \frac{8}{15}$

$\frac{\partial y}{\partial x} = \frac{X+9Y+(h+9k-7)}{2X+3Y+(2h+3k-6)}$ $\frac{\partial y}{\partial x}$ = Let $Y = hX$ then $\frac{\partial y}{\partial x} = h + X\frac{\partial h}{\partial x}$ i.e.equal to

$\frac{X+9Y}{2X+3Y} = h + X\frac{\partial v}{\partial x} = \frac{X+9hX}{2X+3hX} = \frac{1+9h}{2+3h}$ i.e. $X\frac{\partial v}{\partial x} = \frac{1+9h}{2+3h} - h = \frac{1+7v-3v^2}{2+3v} = X\frac{\partial v}{\partial x}$

$$\frac{2+3v}{1+7v-3v^2}\partial v = \frac{\partial X}{X}$$

$$\int \frac{2+3v}{1+7v-3v^2}\partial v = log|X| + c$$

$2\int \frac{\partial v}{1+7v-3v^2} + 3\int \frac{v\partial v}{1+7v-3v^2} = log|X| + c$

$2\log(1+7v-3v^2) + 3v\log(1+7v-3v^2) + \frac{3}{(1+7v-3v^2)}\frac{v^2}{(2)} = log|X| + c$

**$2\log(1+7\frac{Y}{X} - 3(\frac{Y}{X})^2 + 3(\frac{Y}{X})\log(1+7(\frac{Y}{X}) - 3(\frac{Y}{X})^2 + \frac{3(\frac{y}{x})^2}{(1+7(\frac{y}{x})-3(\frac{y}{x})^2)(2)} = \log|X| + c$ Ans**

**Example**

$$sec^2 x \tan y\, \partial x + \sec^2 y \tan x \partial y = 0$$

$$sec^2 x\, \partial x = -\frac{\sec^2 y \tan x\, \partial y}{tany} = -\frac{tanx\, \partial y}{sinycosy} = \frac{sec^2 x \partial x}{tanx} = -\frac{\partial x}{sinxcosx} = \frac{\partial y}{sinycosy}$$

Upon integrating

$$\int \frac{\partial x}{sinxcosx} = -\int \frac{\partial y}{sinycosy} + \log|c| = \int \frac{(sin^2\, x + cos^2\, x)\partial x}{sinxcosx}$$

$$= -\int \frac{(sin^2 y + cos^2 y)\partial y}{sinycosy} + c = 0$$

$$\int \frac{sinx}{cosx}\,\partial x + \int \frac{cosx}{sinx}\,\partial x + \int \frac{siny}{cosy}\,\partial y + \int \frac{cosy}{siny}\,\partial y = c$$

= log (secx) + log (sinx) +log (secy)+log(sin y) = log(c)

=log (tanx)+log(tany) = log(c)

=log (tanxtany) = log(c)

= **tan x tan y = c Ans**

**Example**

The cost of producing x socks is $6 + 10x - 6x^2$ . The total cost of producing a pair is INR 100. Find the function representing total and average cost.

Cost = $6 + 10x - 6x^2$

$$\frac{\partial c}{\partial x} = 6 + 10x - 6x^2 = \int \partial c = \int (6 + 10x - 6x^2)\partial x + k$$

$$C = 6x + 10\frac{x^2}{2} - 6\frac{x^3}{3} + k = C = 6x + 5x^2 - 2x^3 + k$$

When x = 2 and C = 100 then K = 84

Hence Average Cost if there are x units of socks is $\frac{6x+5x2 - 2x3 + 84}{x}$ **Ans**

**Example**

A curve passes through points (1, 2) and lines to the curve pass through the point (1,0). Formulate the equation of the curve using differential equation

Slope of a line given by y = mx + c

$$\partial y/dx = m = \frac{y2 - y1}{x2 - x1} = \frac{y-0}{x-1} = \frac{\partial y}{y} = \frac{\partial x}{x-1}$$

Slope of a normal at any given point P(x, y) is represented by $-\frac{\partial x}{\partial y} = \frac{y-0}{x-1}$

$= \int y\partial y = -\int (x-1)\partial x + c = -\frac{x^2}{2} + x + c$

$= \frac{y^2}{2} = -\frac{x^2}{2} + x + c$

Passing through points (1, 2) we have $c = \frac{5}{2} - 1 = \frac{3}{2}$

Putting $c = \frac{3}{2} = \frac{y^2}{2} + \frac{x^2}{2} - x - \frac{3}{2} = y^2 + x^2 - 2x - 3 = y^2$ = **$2x - x^2 + 3$ Ans**

**Example**

A sum of INR 4,000 is compounded at a 10% per annum rate of interest. In how many years will the amount be double the original principal? ($\log_e 2 = 0.69$)

Principal = P, Rate of Interest = 10 percent per annum, Sum = P+ P*($\frac{10}{100}$)

$\frac{\partial P}{\partial t} = \frac{10}{100}P = \int \frac{\partial P}{P} = \frac{1}{10}\int \partial t + c = log_e|P| = (0.1\, t) + c = P = e^{.1t} e^{c} =$ c’ $e^{.1t}$ = 4000 = c’ when t = 0 and e = 1, = 8000 = $4000e^{.1t}$, 2 = $e^{.1t}$, t/10 = log 2

.69 = t/10, t **= 6.9 years. Ans**

## 3.6 Existence of a solution for a differential equation

The general solution of the equation dy/dx = h(x, y) and has the form f(x, y, C) = 0, C being a constant. Below is the theorem that presents the scenario :

A general solution of dy/dx = h(x, y) exists over a region S of points (x, y) based on certain conditions

a) h(x, y) is continuous and single-valued over S

b) $\partial g/\partial y$ exists and is continuous at all points of S

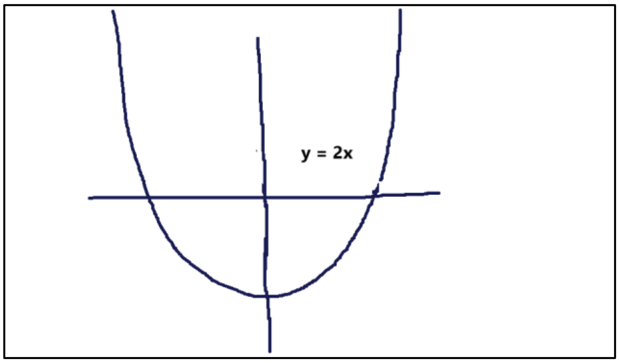
The general solution f(x, y, C) = 0 of a differential equation dy/dx = h(x, y) over some region S consists of set of curves, where each curve represents a particular solution, such that through each point in S there passes one and only one curve for different values of C.

The differential equation associates with each point ($x_0$, $y_0$) in the region S a direction that is given by

$m = \frac{\partial y}{\partial x}|_{x,y} = h(x,y)$

The direction at each point of S is the tangent to that curve of the family f(x, y, C) = 0 that passes through the point.

A region S in which a direction is associated with each point is called a **direction field**. For an equation such as $y = x^2 + c$ the direction would be 2x. The curves or parabolas can be represented as shown in the diagram




## 3.7 Homogeneous polynomial

A polynomial whose terms sum to the same degree with respect to all the variables taken together. Thus

| | |
|---|---|
| $m^2 + 2mn - 2n^2$ | degree 2 homogeneity |
| $2m^3n + 3m^2n^2 + 5n^4$ | degree 4 homogeneity |
| $2m + 5n$ | degree 1 homogeneity |

### 3.7.1 Homogeneous function

A function is said to be homogeneous when we can take a function: f(x, y) and multiply each of the variable so that the function is of the form n: f (nx,ny) and represent it in the form zn f(x, y).Thus

| | |
|---|---|
| $2m^2 \ln(\frac{m}{n}) + 4n^2$ | is homogeneous of degree 2 |
| $m^2 n + n^3 \sin(\frac{m}{n})$ | is homogeneous of degree 3 |

### 3.7.2 Homogeneous Differential Equation

A homogeneous equation is a differential equation of the form

$$M(p, q)\, dp + N(p, q)\, dq = 0$$

Equation 13

where M(p, q) and N(p, q) are homogeneous functions of the same degree.Here variables can be separated by substitution by introducing a new variable p = sq (or q = sp), where s is a new variable.

**Note.** Differentiating p = sq gives dp = s dq + q ds, a quantity that must be substituted for dp when sq is substituted for p.

**Example**

Solve the equation

$$(x^2 - y^2)dx + 2xy\, dy = 0$$

**Solution** Separation of variables though not possible the can be represented as homogeneous function as follows. Substituting

$$y = vx \quad \text{and} \quad dy = v\, dx + x\, dv$$

we get

$$(x^2 - v^2x^2)dx + 2x(vx)(vdx+xdv) = 0$$

$$x^2dx - v^2x^2dx + 2v^2x^2dx + 2x^3vdv = 0$$

$$x^2dx + v^2x^2dx + 2x^3vdv = 0$$

$$(1+v^2)x^2dx = -2x^3vdv$$

$$\mathbf{dx/x = -2v\, dv/(1+v^2)}$$

**Upon integrating**

$\int \frac{dx}{x} = -2\int \frac{vdv}{(1+v^2)}$

= -log (x)+ log C = log(1+$v^2$)

=x(1+$v^2$) = C

Since y = vx

=x(1+$(\frac{y}{x})^2$) = C

=x($\frac{(x^2+y^2}{x^2}$) = C

= $\mathbf{x^2 + y^2 = C}$ **Ans**

### 3.7.3 Non Homogeneous Differential Equation

These can be represented in the form as follows:

$\frac{\partial y}{\partial x} = \frac{px+qy+r}{p'x+q'y+r'}$ Equation 14

We can now replace x = X + h and y = Y + k

$$\frac{p(X+h)+q(Y+k)+r}{p'(X+h)+q'(Y+k)+r'} = \frac{p(X)+p(h)+q(Y)+q(k)+r}{p'(X)+p'(h)+q'(Y)+q'(k)+r'} = \frac{p(X)+q(Y)+ph+qk+r}{p'(X)+q'(Y)+p'h+q'k+r'}$$

$ph + qk + r = 0$; $p'h + q'k + r' = 0$; $ph + qk = -r$; $p'h + q'k = -r'$;

**h** $= \frac{r'q-rq'}{pq'-p'q}$ **and** **k** $= \frac{p'r-pr'}{pq'-p'q}$ and subject to the condition that the term pq' - p'q $\neq$ 0 the equation takes the form as follows:

$\frac{\partial Y}{\partial X} = \frac{pX+qY}{p'X+q'Y}$ that is transformed into a homogeneous equation.

If $pq' - p'q = 0$

then the values of h and k are infinite and the method is not suitable to find th

of a non homogeneous differential equation. If $pq' - p'q = 0$ then

$\frac{p}{p'} = \frac{q}{q'} = \frac{1}{n}$ i.e. p' = np and q' = nq and the differential equation becomes

$$\frac{\partial y}{\partial x} = \frac{px+qy+r}{p'x+q'y+r'} = \frac{\partial y}{\partial x} = \frac{px+qy+r}{npx+nqy+r'} = \frac{px+qy+r}{n(px+qy)+r'}$$

$$v = px + qy = \frac{\partial v}{\partial x} = p + q\frac{\partial y}{\partial x} = \frac{\partial y}{\partial x} = \frac{(\frac{\partial v}{\partial x} - p)}{q} = \frac{v+c}{nv+c'} = F(v) = \partial x = \frac{\partial v}{p+q(F(v))}$$

that which can be integrated.

**Example**

$$(6x - 4y + 1)\frac{\partial y}{\partial x} = (3x - 2y + 1) = \frac{\partial y}{\partial x} = \frac{3x-2y+1}{2(3x-2y)+1}$$

Following the above transformation let $v = 3x - 2y$, $\frac{\partial v}{\partial x} = 3 - 2\frac{\partial y}{\partial x} = \frac{\partial y}{\partial x}$

$$= \frac{3-\frac{\partial v}{\partial x}}{2} = \frac{v+1}{2v+1} = 3 - \frac{\partial v}{\partial x} = \frac{2v+2}{2v+1} = 3 - \frac{2v+2}{2v+1} = \frac{6v+3-2v+1}{2v+1} = \frac{\partial v}{\partial x} = \frac{4v+4}{2v+1}$$

$$= \partial x = \frac{(2v+1)}{4v+4}\partial v$$

$$= \frac{1}{2}\{v + \frac{1}{4}\log(4v+1)\} = x + c'$$

$$= \{v + \frac{1}{2}\log(4v+1)\} = 2x + 2c$$

$= \{v + \frac{1}{2}\log(4v+1)\} = 2x + c'$ where $c' = 2c$, $v + \frac{1}{2}\log(4v+1) = 2x + c'$

= 3x - 2y = v and substituting for v in terms of x and y we get value of c'

$$= 3x - 2y + \frac{1}{2}\log\{4(3x-2y)+1\} = 2x + c'$$

$$\mathbf{= x - 2y + \frac{1}{2}\log\{4(3x-2y)+1\} = c'} \text{ **Ans**}$$

## 3.8 Exact Differential Equation

The total differential of a function u(x, y) is, by definition

$$du(x,y) = \frac{\partial u(x,y)}{\partial x}dx + \frac{\partial u(x,y)}{\partial y}dy \qquad \text{Equation 15}$$

The exact differential is given as follows:

$$du(x,y) = \frac{\partial u(x,y)}{\partial x}dx + \frac{\partial u(x,y)}{\partial y}dy = 0 \qquad \text{Equation 16}$$

or

**M (x, y) dx + N (x, y) dy = 0**

For example to see if this equation is exact or not

$(3x^2y - y)dx + \boxed{(x^3 - x + 2y)dy} = 0$

**(M)** **(N)**

$\frac{\partial M}{\partial y}$ **= 3x² - 1** $\frac{\partial N}{\partial x}$ **= 3x² - 1 Ans**

In the above the left hand side is an exact differential of the right side of the equation hence the differential is said to be an exact differential or in other words a relevant factor u(x, y) known as integrating factor has been appended to the given differential equation.

$\int \boxed{(x^3 - x + 2y)dy} = x^3 y - xy + y^2$ upon differentiating with respect to x gives $3x^2 y - y$ which is the left side of the equation . Here the integrating factor is y.

Similarly considering another example

2ydx + xdy = 0 This cannot be considered as an exact differential equation but if it multiplied by x then it gets transformed into an exact equation

$= (2xy)\, dx + (x^2)\, dy = 0 =$ **M (x, y) dx + N (x, y) dy = 0**

$= \frac{\partial M}{\partial y} = 2x$ and $\frac{\partial N}{\partial x} = 2x$ and also

$\int x^2\, \partial y = x^2 y +$
c whose differential is the equation in the left half i.e. $(2xy)dx$ ,here "y" is again an integrating factor.

Few more examples to showcase whether the differential equations are exact or not $= (2xy - 3x^2)\,\partial x + (x^2 - 2y)\,\partial y$ With respect to y and x= $\frac{\partial M}{\partial y} = 2x$ and $\frac{\partial N}{\partial x} = 2x$ they are exact .

Now there exists there is a function u(x, y)of which the left hand side is exactly the total differential. To find this function we integrate as follows without terms in x $\int N\,\partial y$and that is [-$y^2$]. $\int Mdx = \int (2xy - 3x^2)dx$

The final result is as follows : **$xy^2 - x^3 - y^2 = c$ the general solution**

**Example**

$= (xy^2 + x)\,\partial x + (yx^2)\,dy = 0$

$= M = (xy^2 + x), N = yx^2,$

$\frac{\partial M}{\partial y} = 2xy, \frac{\partial N}{\partial y} = 2xy$ Hence they are exact

Integrating M $\partial x$ and N $\partial y$ we get

$=\int M\,\partial x = y^2\,x^2/2 + \frac{x^2}{2} + c(y)$ ,

Differentiating with respect to y this previous equation

$f(x,y) = x^2y + dc/dy$ and $dc/dy = 0$

Hence the generalized equation becomes **$x^2\,y^2 + x^2 = c$** is the general solution.

**Example**

$\frac{(y\cos x + \sin y + y)}{(\sin x + x\cos y + x)} = \frac{\partial y}{\partial x}$

$M = y\cos x + \sin y + y$, $\frac{\partial M}{\partial y} = \cos x + \cos y + 1$

$N = \sin x + x\cos y + x$, $\frac{\partial N}{\partial x} = \cos x + \cos y + 1$ hence the equations are exact

$= \int (y\cos x + \sin y + y)\,\partial x + \int (\text{terms of N not containing x})\,\partial y = c$

= **$y\sin x + (\sin y + y)x = c$ Ans**

**Example**

$= (1 + 2xy\cos x^2 - 2xy)\,dx + (\sin x^2 - x^2)dy = 0$

$= \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 2x\cos x^2 - 2x$ I.e. equation is exact

$=\int (\mathbf{1 + 2xy\cos x^2 - 2y})\mathbf{)\partial x} + \int (\textbf{terms of N not containing x}) = \textbf{c Ans}$

**Example**

$2xydx + (x^2 + 3y^2)dy = 0$

$M = 2xy,\ N = x^2 + 3y^2, \frac{\partial M}{\partial y} = 2x$ and $\frac{\partial N}{\partial x} = 2x$

Hence these equations are exact

Now to find function u(x, y) we have $\int M\partial x = x^2y + c(y)$ and

Substituting the expression for u(x, y) in the 2nd equation

We have $u(x, y) = \frac{\partial}{\partial y}(x^2y + c(y))$ i.e. $x^2 + c'(y) = x^2 + 3y^2$ .

Hence $c'(y) = 3y^2$

Hence the integral of last equation in the above is given as $\int 3y^2\partial y = y^3$

The final form is $\boldsymbol{x^2y + y^3 = C}$ **Ans**

**Example**

$(6x^2 - y + 3)\, dx + (3y^2 - x - 2)\partial x$

$=\frac{\partial M}{\partial y} = -1$ and $\frac{\partial N}{\partial x} = -1$ Hence the equations are exact

Now to find function u(x, y) we have $\int \mathbf{M\partial x} = \mathbf{2x^3 - xy + 3x^2 + k(y)}$ **.**

Now $\mathbf{\partial f/\partial y = -k(y) + (-x) = 3y^2 - x - 2}$

So the final equation becomes**:** $\mathbf{2x^3 - xy + 3x + 3y^2 - 2 = c}$

**Example**

$\mathbf{(3x^2 + 4xy)dx + (2x^2 + 2y)dy = Mdx + Ndy = 0}$

$dM/dy = 4x,\ dN/dx = 4x$ and hence the equations are exact

**Integrating M(x,y)dx =** $\int (3x^2 + 4xy)\partial x = x^3 + 2x^2y + k(y) = f(x,y)$

Differentiating with respect to y

$df/dy = 2x^2 + \partial/\partial y(k(y)) = 2x^2 + \partial k/\partial y = 2x^2 + 2y$

So $\partial k/\partial y = 2y$ Upon integrating $k(y) = y^2 + c'$

$f(x,y) = x^3 + 2x^2y + y^2 + c' = c'' =$ $\mathbf{x^3 + 2x^2y + y^2 = c}$ **is the general solution.**

## 3.9 Integrating Factors

The equation $P\partial x + Q\partial y$ cannot be represented as an exact differential equation, then there exists a multiplying factor μ that is a function of x and y that makes the equation exact. This factor is otherwise known as the **Integrating Factor.**

A given differential equation assumes the form as follows;

$$\frac{\partial y}{\partial x} + M(x)y = N(x)$$

Equation 17

then the integrating factor μ is defined as follows:

$$\mu = e^{\int M(x)\partial x}$$

Equation 18

Where M(x) (the function of x) is a multiple of y and μ denotes integrating factor.

OR

$$\frac{\partial y}{\partial x} + P(y) = Q$$

$$\frac{\partial y}{\partial x} e^{\int P\partial x} + y(e^{\int P\partial x}P) = Qe^{\int P\partial x}$$

**Upon integration**

$$ye^{\int P\partial x} = \int Qe^{\int P\partial x}\,dx + c$$

Equation 19

For example consider the function

$(x - y)\partial x + x\partial y = 0$ or $x\partial y = -(x - y)\partial x$ **or** $\frac{\partial y}{\partial x} = \frac{y-x}{x}$ **where** $\frac{y-x}{x}$ is considered as M.

The steps for the integrating factor are as follows :The differential equation is represented in the above form and the value of M(x) is found out. The integration factor needs to be calculated i.e. μ.The equation at the next step needs to be represented as follows:

$$\mu\frac{\partial y}{\partial x} + \mu M(x)y = \mu N(x)$$

On the left-hand side of the equation, a particular differential is obtained as follows:

$$\frac{\partial}{\partial x}(\mu, y) = \mu N(x)$$

In the end, integration of the expression needs to follow and the required solution to the given equation is represented as: **μ y = ∫μN(x) + C**.

Now considering this equation $(x - y)\,\partial x + x\,\partial y = 0$ here **M = x-y and N = x** hence $\frac{\partial M}{\partial y}$ = **-1** and $\frac{\partial N}{\partial x}$ = **1** and the equations are not exact.

In order to make the equations exact the ***μ*** should be such that $\frac{\partial M}{\partial y}$ should be equal to $\frac{\partial N}{\partial x}$ .

Hence if we multiply M and N with $\frac{1}{x^2}$ then $\frac{\partial M}{\partial y} = \frac{x-y}{x^2}$ and $\frac{\partial N}{\partial x} = \frac{1}{x}$ then differentiating M and N gives us the results as follows: $\frac{\partial M}{\partial y} = -\frac{1}{x^2}$ and $\frac{\partial N}{\partial x} = -\frac{1}{x^2}$ hence $\mu = -\frac{1}{x^2}$ and this becomes the integrating factor.

**Example**

Solve the equation: $(xy^2 - 2y^3)dx + (3 - 2xy^2)dy = 0.$

The given equation is not exact, because

$\frac{\partial M}{\partial y} = 2xy - 6y^2, \frac{\partial N}{\partial x} = 2y^2$ and the equations are not exact

We try to find the general solution of the equation using an integrating factor. Calculate the difference i.e. $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}$ = **$2xy - 4y^2$**

$\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{2xy - 4y^2}{xy^2 - 2y^3} = \frac{2}{y}$

and the integrating factor is μ that is dependent on y.

$\mu = \frac{2}{y}, \frac{\partial \mu}{\partial y} = -2y^2, \frac{1}{\mu}\left(\frac{\partial \mu}{\partial y}\right) = -\frac{2}{y}$

Upon integrating

$-2\int \frac{\partial y}{y} = \ln|y| = \mu = (+-)\frac{1}{y^2}$

Now the exact equation is got i.e.

$(xy^2 - 2y^3)/y^2\,\partial x + (3 - 2xy^2)/y^2\,\partial y = (x - 2y)\,\partial x + \left(\frac{3}{y^2} - 2x\right) = 0$

$\frac{\partial M}{\partial y} = -2, \frac{\partial N}{\partial x} = -2.$ Now to find u from the above

$\frac{\partial u}{\partial x} = x - 2y$ and $\frac{\partial u}{\partial y} = \frac{3}{y^2} - 2x$, $u(x,y) = \int (x - 2y)dx = x^2 - 2yx$ + (this is from the first equation and from the second equation

$\frac{\partial}{\partial y}(x^2 - 2yx + \emptyset) = \frac{3}{y^2} - 2x = -2x + \emptyset'(y) = \frac{3}{y^2} - 2x$ and $\emptyset'(y) = \frac{3}{y^2}$, $\emptyset(y) = \int \frac{3}{y^2}\,\partial y = -\frac{3}{y}$

Hence the final equation becomes **$x^2 - 2yx - \frac{3}{y} = c$ with y = 0 Ans**

**Example**

y(log y )∂x + (x-log y)∂y = 0

∂x/∂y +x/(y log y) = 1/y which is a leibnitz's equation in x

Integrating Factor = $e^{\int \frac{1}{y\log y}\partial y} = e^{log(log\, y)}$ = log y

Thus the solution is as follows: x (I.F.) = $\int \frac{1}{y}$(Integrating Factor)∂y + c

= x log y = $\int \frac{1}{y}$(Integrating Factor)∂y + c = (1/2) (log y)$^2$ + c

**= x = (1/2) log y + c/log y Ans**

**Example**

**Solve (x + 1)$\frac{\partial y}{\partial x}$ - ye$^{3x}$(x+1)$^2$**

$\frac{\partial y}{\partial x}$ - y/(x+1) = e$^{3x}$(x+1) here P = - 1/(x+1) and $\int P\partial x$

$= -\int \frac{\partial x}{x+1}$ = -log (x+1)

Here integrating factor is as follows:$e^{\int P\partial x} = e^{log(x+1)1^{-1}}$ = 1/x+1

As per the above equation **y.** (**Integrating Factor**) =

$\int e^{3x}$(x+1)(**Integrating factor**).∂x + c

**y(1/(x+1) = $\int e^{3x}$ ∂x +c or y = (x+1)(c+1/3$e^{3x}$)**

**Integrating Factor found by Inspection**

**Example**

y (2xy +e$^x$)dy +2xy$^2$ dx = 0

Dividing by 1/y$^2$ that is the Integrating Factor then equation becomes

$\int$ M ∂x + $\int$(terms of N not containing x) = c

**= $\frac{e^x}{y}$ + x$^2$ = c Ans**

## 3.10 Integrating Factor of a homogeneous equation

If Mdx +Ndy =0 be a homogeneous equation then its integrating factor is 1/(Mx+Ny) and Mx+Ny ≠ 0

**Example**

$(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$ .

It is in the homogeneous form Integrating Factor = $1/x^2y^2$

Multiplying the equation with IF, the equation becomes exact in the form $(1/y-2/x)dx - (x/y^2 - 3/y)dy = 0$ and is exact

Applying $\int M\,\partial x + \int (\text{terms of N not containing x}) = c$

**= x/y -2logx +3logy = c Ans**

## 3.11 Linear Equation and equation reducible to homogeneous form

A differential equation is said to be linear if its differential coefficient occur in the first degree and is not multiplied together and is represented as follows:

$\frac{\partial y}{\partial x}$ **+ P(y) = Q where P and Q are functions of x.**

Here when Q = 0 then $\frac{\partial y}{y} + P\,\partial x = 0$

Upon integration $\int \frac{\partial y}{y} + \mathbf{P} \int \partial x = \log y + \mathbf{P}\int \partial x = \log c$ **or y/c = - P**$\int \partial x$ **= y/c =** $e^{-\int P\partial x}$ and the rest is the same as the liebnitz equation.

Bernoulli Equation can be represented as follows:

$\frac{\partial y}{\partial x}$ **+ P(y) = Q $y^n$ where P and Q are functions of x and upon solving gives**

$\frac{\partial z}{\partial x}$ **+(1-n)$P_z$ = (1-n)Q**

Another equation that can be linear in the form is $f'(y)\partial y/\partial x + Pf(y) = Q$

Then **dz/dx + P(z )= Q where f(y) = z**

**Example**

Solve

$$(y^2 + 2xy) + x^2 dy = 0$$

$$\frac{dy}{dx} = -\frac{y^2 + 2xy}{x^2}$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$ for y =vx

$$v + x\frac{dv}{dx} = -\frac{v^2x^2 + 2xvx}{x^2} = -(v^2 + 2v)$$

$$x\frac{dv}{dx} = -(v^2+3v) \Rightarrow \int \frac{dv}{v(v+3)} = -\int \frac{dx}{x}$$

$$\Rightarrow \frac{1}{3}\int\left(\frac{1}{v} - \frac{1}{v+3}\right)dv = -\int \frac{dx}{x}$$

$$\frac{1}{3}\log v - \frac{1}{3}\log(v+3) = -\log x + \log c$$

$$\Rightarrow \log\frac{v}{v+3} = 3\log\frac{c_1}{x}$$

$$\Rightarrow \log\left[\frac{v}{v+3}\right] = \log\left[\frac{c_1}{x}\right]^3$$

$$\Rightarrow \left[\frac{v}{v+3}\right] = \frac{c^3{}_1}{x^3} = \frac{c}{x^3}$$

$$\Rightarrow \left[\frac{\frac{y}{x}}{\frac{y}{x}+3}\right] = \frac{c}{x^3}$$

$$\Rightarrow x^3 y = c(y+3) \qquad \textbf{Ans}$$

**Example**

Solve (2x-5y )dx +(4x-y)dy = 0

$dy/dx = \frac{2x-5y}{4x-y}$

Put $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$

So eq(1) becomes

$$v + x\frac{dv}{dx} = -\frac{(2x-5vx)}{(4x-vx)} = -\frac{(2-5v)}{(4-v)}$$

$$x\frac{dv}{dx} = -\frac{(2-5v)}{(4-v)} - v \quad \Rightarrow \int \frac{(4-v)dv}{(v-1)(v+2)} = \int \frac{dx}{x}$$

$$\Rightarrow \int \frac{dv}{v-1} - 2\int \frac{dv}{v+2} = \int \frac{dx}{x}$$

$$\log(v-1) - 2\log(v+2) = \log x + \log c$$

$$\Rightarrow \log\frac{(v-1)}{(v+2)^2} = \log cx \quad \Rightarrow \frac{(v-1)}{(v+2)^2} = cx$$

$$\Rightarrow [\frac{y}{x} - 1] = cx[\frac{y}{x} + 2]^2 \quad \Rightarrow [y-x] = c[y+2x]^2$$

$$\Rightarrow [4-1] = c[4+2]^2 \Rightarrow [4-1] = c[4+2]^2$$

$$\Rightarrow c = \frac{1}{12} \qquad \Rightarrow 12[y-x] = [y+2x]^2$$

**Example**

Solve $x\frac{\partial y}{\partial x} + y = x^3y^6$

= Dividing by $xy^6$ we have $\frac{\partial y}{\partial x}y^{-6} + y^{-5} = x^2$

Let $y^{-5} = z = =5y^{-6} = dz/dx$ or $^{-}dz/dx -(5/x)z = -5x^2$ which is linear in z

Applying Integration Factor

i.e. $\boldsymbol{e^{\int \frac{-5}{x}\partial x} = e^{-5\log x} = x^{-5}}$

= z * (Integrating Factor) = c +Q(Integrating factor ) dx

= z* $\mathbf{x^{-5}} = \int(-5x^2\mathbf{x^{-5}}dx + c$

= $\mathbf{y^{-5}\ x^{-5} = -5x^{-2}/(-2) + c}$ **Ans**

**Example**

**tan y(∂y/∂x) +tan x = cos y cos²x**

Dividing by cos y it gives

secy tan y(∂y/∂x)+secy tanx = $\cos^2 x$

Let sec y = z then ∂z/ ∂x = secy tan y ∂y/∂x

$\partial z/\partial x + z \tan x = \cos^2 x$

It is in the linear form hence the integrating factor I.F

$e^{\int tanx\partial x} = e^{\log \sec x}$ = sec x

So z * (Integrating Factor) = c +Q (Integrating factor ) dx

= z*(sec x) = c+ ($\cos^2 x\sec x$) dx

=z*(sec x) = c+ Integration of (cosx) dx

secy secx = c+ sin x

**secy = (c+sinx)cosx Ans**

The DE is not homogeneous.

$$(a_1x + b_1y + c_1)dx + (a_2x + b_2y + c_2)dy = 0$$

It can be reduced to homogeneous form

**Type-1** If $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

then the transformation is as follows: x = X + h, y = Y + k

$$(a_1X + b_1Y + a_1h + b_1k + c_1)dX$$
$$+ (a_2X + b_2Y + a_2h + b_2k + c_2)dY = 0$$

**Type 2**

If $\frac{a_1}{a_2} = \frac{b_1}{b_2}$

then put $z = a_1 x + b_1 y$ and the given equation will reduce to a separable equation.

**Example**

**Solve dy/dx = (2x+y+1)/(x-2y+3)**

$$\frac{dy}{dx} = \frac{2(X+h)+Y+k+1}{X+h-2(Y+k)+3}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2X+Y+2h+k+1}{X-2Y+h-2k+3}$$

Now

$$\Rightarrow \begin{cases} 2h+k+1=0 \\ h-2k+3=0 \end{cases} \Rightarrow \begin{matrix} 5h+5=0 \\ h=-1 \\ k=1 \end{matrix}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2X+Y-2+1+1}{X-2Y-1-2+3}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2X+Y}{X-2Y} \Rightarrow \frac{dy}{dx} = \frac{2+\frac{Y}{X}}{1-2\left(\frac{Y}{X}\right)}$$

Put Y = vX

$$\Rightarrow v + X\frac{dv}{dX} = \frac{2+v}{1-2v}$$

$$\Rightarrow X\frac{dv}{dX} = \frac{2+v}{1-2v} - v$$

$$\Rightarrow X\frac{dv}{dX} = \frac{2+v-v+2v^2}{1-2v} = \frac{2(1+v^2)}{1-2v}$$

$$\Rightarrow \int \frac{(1-2v)}{1+v^2}dv = 2\int \frac{dX}{X}$$

$$\Rightarrow \int \frac{dv}{1+v^2} - \int \frac{2v\,dv}{1+v^2} = 2\int \frac{dX}{X}$$

$$\Rightarrow \tan^{-1} v - \ln(1+v^2) = 2\ln X + \ln c$$

$$\Rightarrow \tan^{-1} v = \ln(1+v^2) + \ln X^2 + \ln c$$

$$\Rightarrow \tan^{-1} v = \ln c(1+v^2)X^2$$

$$\Rightarrow \tan^{-1}\frac{Y}{X} = \ln c\left(1+\frac{Y^2}{X^2}\right)X^2$$

$$\Rightarrow \tan^{-1}\frac{Y}{X} = \ln c(X^2+Y^2)$$

$$\Rightarrow \tan^{-1}\left(\frac{(y-1)}{(x+1)}\right) = \ln c[(x+1)^2$$

$$+(y-1)^2]$$

**Ans**

**Example**

**Solve dy/dx = 3x-4y-2/3x-4y-3**

$$\frac{dz}{dx} = 3-4\frac{dy}{dx} \quad \Rightarrow \frac{dy}{dx} = \frac{3}{4}-(\frac{1}{4})\frac{dz}{dx}$$

$$\Rightarrow \frac{3}{4}-(\frac{1}{4})\frac{dz}{dx} = \frac{z-2}{z-3}$$

$$\Rightarrow (\frac{1}{4})\frac{dz}{dx} = \frac{3}{4}-\frac{z-2}{z-3}$$

$$\Rightarrow (\frac{1}{4})\frac{dz}{dx} = -\frac{(z+1)}{4(z-3)}$$

$$\Rightarrow \frac{dz}{dx} = -\frac{(z+1)}{(z-3)}$$

$$\Rightarrow \int \frac{(z-3)dz}{(z+1)} = -\int dx$$

$$\Rightarrow \int dz - 4\int \frac{dz}{(z+1)} = -\int dx$$

$$\Rightarrow z+x+c_1 = 4\ln(z+1)$$

Put $z = 3x-4y$

$$\Rightarrow 3x-4y+x+c_1 = 4\ln(3x-4y+1)$$

$$\Rightarrow x-y+\frac{c_1}{4} = \ln(3x-4y+1)$$

$$\Rightarrow x-y+c = \ln(3x-4y+1)$$

## 3.12 Partial Differential Equation - An overview

A differential equation that constitutes of partial derivatives is known as a partial differential equation. The differential equation presented below is a partial differential equation since a derivative can result with respect to both x and y.

**Example** Consider an equation of the form F(x,y).A partial differential equation that can be represented is as follows :

$d/dx(F(x,y)$ with respect to x otherwise written as $F_x$(x,y) or $\partial f/\partial x$ where x is allowed to vary.

Upon finding the derivative of the same function with respect to y the representation is as follows:

$F_{xy}$(x,y) i.e. $\partial/\partial y(\partial f/\partial x)$ which is equivalent to $\partial^2 f/\partial y\,\partial x$.Few examples of the partial differential equation are as follows for ready reference and a basic understanding:

$\partial/\partial x(\partial u/\partial x) + \partial/\partial y(\partial u/\partial y) = 0$ Equation 20

$\partial/\partial x(\partial u/\partial x) + \partial/\partial y(\partial u/\partial y) + (\partial/\partial x(\partial u/\partial y) = x^2 + y^2$
Equation 21

## 3.13 Summary

This chapter discusses on the concepts of differential equation and their solving methodologies, as differential equation formulation and representation with respect to heat conduction, oscillation in mechanical and electrical systems and circuitry take a centre stage in all modern scientific and engineering studies. In applied mathematics generally, the study of differential equation constitutes of modelling the equation, solving the equation using different criterion and conduction as rules of separation, reduction, multiplication by a certain integration factor to make it exact. Here even mechanisms to find certain integrating factors by inspection or of a homogeneous equation or represented in a complex format to find a general solution to the real world problems. This chapter introduces the students to the fundamental problem solving in the segment of first order and first degree equations that are moderately complex to model and solve.

## 3.14 References

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2. Differential Equation, Shepley L Ross Wiley Publications, 3rd Edition
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## 3.15 Questions

1. Given the differential equation $dp/dq=p^4-q^4(p^2+q^2)pq$ the degree of differential equation.

2. Solve $(m^2 + n^2 + m)\, dm + mndn = 0$.

3. Solve the following equations by the method of inspection

   a) $y(3yx + e^x)\, dx - e^x\, dy = 0$

   b) ydx-xdy+lnxdx=O fr all x,y>O.

   c) $(xy - 2y^2)\, dx - (x^2 - 3xy)dy = 0$.

4. Solve the homogeneous equation : Solve $(x^2y - 2xy^2)\, dx - (x3 - 3x^2y)\, dy = 0$ using Integrating Factor.

5. Solve $(p^4+y^4)dp - py^3dy = 0$. (Hint When bp - ay # 0 and the different equation a(p,y) dy + b(p,y) dp = 0 can be written in the form qf,(p y)dp + pf,(p,y)dy = 0 with I as an integrating factor).

6. Check for exactness of the equation :

   Solve $y(x^2y^2 + 2)\, dx + x(2 - 2x^2\, y^2)\, dy = 0$

7. Solve for exactness and find the integrating factor

   (3x2y4+2xy) dx + (2xY3-x2) dy = 0.

❖❖❖❖❖❖❖

Unit 2

# 4

# DIFFERENTIAL EQUATION OF THE FIRST ORDER OF A DEGREE HIGHER THAN THE FIRST

**Unit Structure**



**Please note two conventions of differentials have been used ($\frac{\partial y}{\partial x}$,dy/dx)**



## 4.0 Objectives

Here nonlinear equations are considered where the derivatives are of first order and of higher degree. The equations are not solvable by any structured methodology. Here, some typical types of equations are considered to describe the techniques of solution of such equations. One will able to solve differential equation of first order and higher degree solvable for solvable for x, solvable for y and the Clairaut's form of the equation. Also obtain the solution of the differential equations in which x or y is absent

## 4.1 Introduction

Isaac Newton (1642-1727), the English mathematician and scientist, classified differential equations of the first order then known as fluxional equation which was published in 1736. Then Count Jacopo Riccati (1676-1754), an Italian mathematician, contributed towards advancing the theory of differential equations with reduction of an equation of the second order in y to an equation of first order in p. In 1723, he exhibited the solution of an equation to which the name of Riccati is attached. Later the French mathematician Alexis Claude Clairaut (1713-1765) pioneered the idea of differentiating a given differential equations in a specific form to solve them.

These equations are described as equations constituting of dependent and independent variable, that are solvable using the following : equations that are solvable for p,y,x and the Clairaut's form, the techniques and methodologies of which are described in the succeeding section.

**Equations that are solvable for p**

**( $p = \frac{\partial y}{\partial x}$ )** and for **y : y = f (x, p)** and for **x = f (y,p )**

For Clauriat's form of equation it is a follows : **y = p(x) + f (p )**

The equations that are solvable for p of the first order and the $n^{th}$ degree is represented as follows:

$$= p^n + f_1(x,y)p^{n-1} + f_2(x,y)p^{n-2} + f_3(x,y)p^{n-3} + \ldots\ldots + f_{n-1}(x,y)p + f_n(x,y) = 0$$

Now the left hand side of the above equation is split up into n linear representative equations as follows :

**$[p - \theta_1(x,y)], [p - \theta_2(x,y)] \ldots\ldots\ldots [p - \theta_n(x,y)]$** and these are of first order and first degree. Each individual solution to the above can be represented in the form as follows :**$f_1(x_1,y_1,c) = f_2(x_2,y_2,c) \ldots . f_n(x_n,y_n,c) = 0$** and these together form the solution for the above equation as follows.

**Example**

**Solve** $m^2 + m\left(e^x + \frac{1}{e^x}\right) + 1 = 0$

$= m(m+e^x) + \frac{1}{e^x}(m + e^x) = 0$

$= (m+e^x)(m+\frac{1}{e^x}) = 0$

$= y + e^x + k' = 0, \; y+e^{-x} + k'' = 0$ Ans

Here k' and k" can be replaced by k and the final equation constitutes of first degree and first order representation.

**$=(y + e^x + k)(y+e^{-x} + k) = 0$ Ans**

**Example**

**Solve** $\frac{dp}{dx} - \frac{dx}{dp} = \frac{p}{x} - \frac{x}{p}$

$= q - \frac{1}{q} = \frac{p}{x} - \frac{x}{p}$

$= q^2 - 1 = q(\frac{p}{x} - \frac{x}{p})$

$= q^2 - 1 - q(\frac{p}{x} - \frac{x}{p}) = 0$

$= q(q - \frac{p}{x}) + \frac{x}{p}\ (q - \frac{p}{x}) = 0$

$= (q - \frac{x}{p})(q - \frac{p}{x}) = 0$

$= \mathbf{q = -\frac{x}{p}, \frac{p}{x}}$

$= q = \frac{dp}{dx} = -x/p$ , Upon integrating $\int pdp = \int -xdx$

$= \frac{p^2}{2} + \frac{x^2}{2}$ **= c i.e.** $\mathbf{p^2 + x^2 = c}$ **is** the first solution

When $q = \frac{p}{x}$

Then $\frac{dp}{dx} - \frac{p}{x} = 0$

$= \frac{dp}{p} - \frac{dx}{x} = 0$

$= \ln(p) - \ln(x) = 0$

**=ln(p/x) = ln (c), p = xc** is the required solution

**Example**

**Solve** $\mathbf{p^2 + 2\,py\cot x = y^2}$

The square root of p will be equal to

**=(-b(+-)square root of ((b² -4ac))/2a**

$=(1/2)(-2y\cot x\ (+-)\sqrt{4y^2\cot^2 x - 4y^2}$

$= -y\cot x \pm y\,\mathrm{cosec}\,x$

$= \frac{\partial y}{\partial x} = -y\cot x + y\,\mathrm{cosec}\,x$

$= \frac{\partial y}{\partial x} = y(\mathrm{cosec}\,x - \cot x)$

$= \frac{\partial y}{y} = (\mathrm{cosec}\,x - \cot x)\partial x$

$=\int \frac{\partial y}{y} = \int (\text{cosecx} - \text{cotx})\partial x$

$= \log y = \log \tan x(\frac{x}{2}) - \log(\sin x) + \log(c)$

$\mathbf{= y\ (1+cosx) = c}$

Similarly for the equation

$\frac{\partial y}{\partial x} = -y\cot x - y\,\text{cosec}x$

$\mathbf{= y\ (1 - cos\ x) = c}$

**= y (1(+-) cosx) = c Ans**

**Example**

**Solve** $\mathbf{xyp^3 + +(x^2 - 2y^2)p - 2xyp = 0}$

$= p\,[xyp^2 + (x^2 - 2y^2)p - 2xy] = 0$

$= p\,(xp - 2y)(yp + x) = 0$

$= (p = 0,\ y - c = 0),(xp - 2y = 0),\ (yp + x = 0)$

$= (xp - 2y) = 0$, let $p = \frac{dy}{dx} = x\frac{dy}{dx} = 2y$, or $\frac{dy}{y} = 2\frac{dx}{x}$, $y = cx^2$

$= yp + x = 0$, with $p = \frac{dy}{dx}$, $ydy + xdx = 0$, $x^2 + y^2 - 2c = 0$

So the final equation becomes $\mathbf{(y-c)(y-cx^2)(x^2 + y^2 - 2c) = 0}$ **Ans**

### 4.1.1 Equations solvable for x

Let there be equation of the form x= f(y,p)

Differentiating with respect to y it can be represented as follows:

$= 1/p = dx/dy = \theta(y,p,dp/dx)$

The solution that can be deduced is as follows: F(y,p,c) = 0 that can be shown through the following example.

**Example**

**Solve** $\mathbf{x = 4(p+p^2)}$

$dx/dy = 1/p =$ $pdx = dy$ Differentiating with respect to y

$= 1/p = 4(1 + 2p)dp/dy$

$= dy = 4p(1 + 2p)dp$

Integrating we have

$\int dy = \int 4p(1 + 2p)dp$

$= \mathbf{y = 2p^2 + (8/3)p^3 + c}$ **Ans**

**Example**

**Solve $y = 2px + y^2p^3$**

$= y - y^2p^3 = 2px$

$= \frac{y - y^2p^3}{2p} = x$

$= y/2p - y^2p^2/2 = x$

Differentiating the above with respect to y

The first component is $\frac{1}{p} = \frac{2p - 2y\frac{\partial p}{\partial y}}{4p^2}$

and the second component is $\frac{2yp^2 + y^2 2p\frac{dp}{dy}}{2}$

$= 0 = \frac{2p - 2y\frac{\partial p}{\partial y}}{4p^2} - \frac{2yp^2 + y^2 2p\frac{dp}{dy}}{2} - \frac{1}{p}$

$=0 = (y\frac{\partial p}{\partial y} + p)(yp + \frac{1}{2p^2}) = 0$

$= (y\frac{\partial p}{\partial y} + p) = c = \log(py) = \log c$

**$= py = c$**

Eliminating p from the main equation

**We have $y^2 = 2cx + c^3$** as the solution

**Example**

**Solve: $y^2p^2 - 3xp + y = 0$.**

The differential equation is of the form $x = f(y,p)$,

where $f(y,p) = (1/3)(y/p + y^2p)$.

Differentiating with respect to y we get

$\frac{dx}{dy}(3) = 3(1/p)$

$= (1/p) - (y/p^2)(dp/dy) + 2yp + y^2(dp/dy)$

Simplifying we get

$2p + y(dp/dy) = 0$ so $(dp/p) + 2(dy/y) = 0$

$= p = \frac{c}{y^2}$

Hence **$y^3 - 2cx + c^2 = 0$** x then becomes **$p + 1/p$**

$(dx/dy) = (dp/dy) - (1/p^2)(dp/dy)$

Integrating

$\int (p - 1/p)dp = \int (p - 1/p) + c$

**$= y = (p^2/2) - \log p + c$ and $x = p + 1/p$ Ans**

**Example**

**Solve $y^2p^2$ - 3xp + y = 0**

The equation can be represented in the form

**x = f(y,p) = p = $\tan^{-1}(p + \frac{p}{1+p^2})$**

$= \frac{\partial x}{\partial y} = \frac{1}{p} = (\frac{1}{1+p2}) \frac{\partial p}{\partial y} + \frac{1+p^2-2p^2}{(1+p^2)1^2}$

$= \frac{\partial y}{\partial p} = \frac{2p}{(1+p^2)1^2}$

Upon integration

**y = c (1 +$p^2$) - 1** from where y cannot be removed. **Ans**

### 4.1.2 Equations solvable for y

A differential equation of first order and higher degree takes the form y = f (x, p). Differentiating the equation w.r.t x , we have p = $\frac{dy}{dx}$, = θ(y,p.dp/dx). The solution for the same will be in the form of : F(x,p,c) = 0. Now taking into consideration y = f(x, p) and solution being F(x, p, c) the (x, y) variables in the equation can be represented as x = $F_1$(p, c) and y = $F_2$(p, c) respectively as the solution.

**Example**

**Solve y = p x +a p(1-p)**

We differentiate the above with respect to x.

dy/dx = p + x dp /dx + a(dp/dx) - a(2p)dp/dx

dy/dx = p + dp/dx[x +a -2ap]

p = p + dp/dx[x +a -2ap]

0 = dp/dx[x +a -2ap]

Here p is a constant hence the equation becomes p = 1/2a(x +a)

**y = (1/2a)(x + a)[x + a(1/2a(x + a))(1-(1/2a)(x + a))**

**Example**

**Solve x+ 2(xp - y) + $p^2$**

The equation to be represented as y = f(x, p) and hence is solvable for y

Representing in the form with y on the left hand side the equation takes the form = y= $\frac{x}{2}$ + xp + $\frac{p^2}{2}$

$= \frac{dy}{dx} = p = \frac{1}{2} + p + x\frac{dp}{dx} + p\,\frac{dp}{dx}$

$= (x + p)\frac{dp}{dx} + \frac{1}{2} = 0$

Let $(x + p) = u$ , $1 + \frac{dp}{dx} = \frac{du}{dx}, \frac{2u}{2u-1}\,\partial u = \partial x$

$= \int (1 + \frac{1}{2u-1})\,\partial u = \int \partial x + c = 0$

$= u + \frac{1}{2}\log(2u - 1) = x + c$

Replacing p with x + u we have

$= 2p + \frac{1}{2}\log(2(x+p) - 1) = x + c$

$= 2x + 2p - 1$

$= e^{2p - c} =$ **$x = (\frac{1}{2})e^{2p-c} + 1 - p$ and $y = \frac{x}{2} + xp + \frac{p^2}{2}$ Ans**

**Example**

**Solve p2 - py +x**

$= y = (x + p^2)/p$

$= x/p + p$

$= xp^{-1} + p$

$= dp/dx + 1/p - xp^{-2}(dp/dx) = p = dy/dx$

Solving this equation

$= \frac{dx}{dp} + \frac{x}{(p+1)p(p-1)} = \frac{p}{p^2-1}$

The integrating factor is

$e^{\int \frac{pdp}{p^2-1}} = e^{\int \left[\frac{1}{2(p-1)} + \frac{1}{2(p+1)} - \frac{1}{p}\right]dp}$

$= e^{\int \frac{\ln[(p+1)(p-1)^{\frac{1}{2}}}{p}} = \frac{(p^2-1)^{\frac{1}{2}}}{p}$

Hence the final solution is

$x(\frac{(p^2-1)^{\frac{1}{2}}}{p}) = \int \frac{p}{p^2-1}\frac{(p^2-1)^{\frac{1}{2}}}{p}dp = \int \frac{dp}{\sqrt{p^2-1}} = c + \cosh^{-1} p$

**$= x = p(c + \cosh^{-1}p)\,(p^2 - 1)^{1/2}$ Ans**

**Example**

**Solve $y = 2px + p^n$**

Differentiating with respect to x

$p = 2p + 2x(dp/dx) + np^{n-1}(dp/dx)$

$0 = p + 2x(dp/dx) + np^{n-1}(dp/dx)$

$0 = p + 2x(dp/dx) + (np/p)(dp/dx)$

$-p = (dp/dx)(2x) + np^{n-1}(dp/dx)$

$= -\frac{dx}{dp}[p] = 2x + np^{n-1}$

$0 = \frac{\partial x}{\partial p}[p] + [2x] + [\frac{p^n}{p^2}]$

Integrating factor $e^{2logp} = p^2$

Solution is

$xp^2 = -\int np^n \partial p + c$

$= xp^2 = -n\frac{p^{n+1}}{n+1} + c$

$= x = -n\, p^{n+1-2} + cp^{-2}$

$= \mathbf{x = -np^{n-1} + cp^2}$

Then substitute for y in the given equation

**$y = 2px + p^n$ and the solution is as follows :**

$\mathbf{y = \frac{2c}{p} + \frac{1+n}{1-n}p^n}$ **Ans**

## 4.2 Equations not containing dependent/independent variable

Sometimes the equations do not contain dependent/independent variable and either it contains y or x and not both such equations can be represented in the form as follows: f(x,p) = 0 or f(y,p) = 0. For example $y = \frac{1}{1+p^2}$ is one of form of equation where the x is missing as an independent variable from the equation.

**Type I**

In the former scenario equations do not contain independent variable

This equation can be represented as follows after differentiating with respect to x as follows :

**$p = dy/dx = \theta(y)$**

In order to seek clarity lets consider the below example

**$y = 3p + 6p^2$** This equation is already in the form y = f(p)

$p = 3\left(\frac{dp}{dx}\right) + 12p\frac{dp}{dx}$

$p = (\frac{dp}{dx})$ (3 + 12 p)

dx = (3 + 12p)/p (dp)

=x = 3 ln(p) +12p +c  and y =  3p +6p$^2$  **Ans**

**Example**

**Solve  $y^2 = a^2 (1 + p^2)$**

The above equation is in y and p only.It can be written as follows :

**$= p^2 = \frac{y^2}{a^2} - 1$**

$= p = (+-)\sqrt{\frac{y^2}{a^2}} - 1$

$= \frac{\partial y}{\partial x} = \frac{\sqrt{y^2 - a^2}}{a}$

= a ln | y $+ \sqrt{y^2}$ -a$^2$| = x +c

= a ln  | y $+ \sqrt{y^2}$ -a$^2$| (-+)x-c  Ans

**Type II**

**Equations not containing a "y" as the dependent variable**

Let the equation be as follows:

$x= \frac{1}{1+ p^2}$

= 1/p = (1 +p$^2$)$^{-1}$

$^{=}$ 1/p = (1+p)$^{-2(}$-2p)(dp/dy)

$= dy = \frac{-2p}{(1+p)^2}$

$= \int \partial y = \int \frac{-2p^2}{(1+p^2)^2} dp$

$= \int \partial y = \int 2[\frac{-1}{(1+p^2)} + \int \frac{1}{(1+p^2)^2}]$ dp

= y = tan$^{-1}$ p + 2

We will use the substitution x=tanθ, implying that dx=sec$^2$θdθ:

I=∫sec$^2$θ dθ(1+tan$^2$θ)$^2$

Note that 1+tan$^2$θ=sec$^2$θ:

**Integrating Factor**=∫$\sec^2\theta d\theta \sec^4\theta$=∫$d\theta \sec^2\theta$=∫$\cos^2\theta d\theta$

Recall that $\cos 2\theta=2\cos^2\theta-1$, so $\cos^2\theta=1/2\cos^2\theta+1/2$.

**Integrating Factor**=$1/2\int\cos^2\theta d\theta+\int 1/2(d\theta)$

**Integrating Factor**=$1/4\sin^2\theta+1/2\theta+C$

From $x=\tan\theta$ we see that $\theta=\arctan x$.

We see that $1/4\sin^2\theta=(1/2)\sin\theta\cos\theta$.

Also, since $\tan\theta=x$, for a right angle triangle with the side opposite $\theta$ being x, the adjacent side being 1, and the hypotenuse $\sqrt{1+x^2}$.
Thus, $\sin\theta=x/\sqrt{1+x^2}$ and $\cos\theta=1/\sqrt{1+x^2}$:

**Integrating Factor** =$1/2(\sin\theta\cos\theta)+1/2(\arctan x)+C$

**Integrating Factor**=$1/2(x/\sqrt{1+x^2})(1/\sqrt{1+x^2})+\text{arc}\tan x/2+c$

**Integrating Factor** =$x/2(1+x^2)+\arctan x/2+c$ Ans

## 4.3 Clairaut's Form of the Equation

When an equation is of first degree in x and y, it is solvable for both independent and dependent x and y variables both and hence it can be put in the following forms:

$\mathbf{y = xf_1(p) + f_2(p)}$ or

$\mathbf{x = yg_1(p) + g_2(p)}$ and these can be solved normally.

But if $f_1(p) = p$ then it takes the Clairaut's form as follows **: y = xp + f(p)** and these equations can be non linear in nature. Here f(p) is a known function that does not contain an x or y.

Instances , like $y = px + p^2$ and $y = x + e^q$ are examples of Clairaut's equation whereas equations $y = xy^2 + p$ or $y - x^2p^2 + yp^2$ are not of the Clairaut's form.

Let there be an equation of the form $y = px + f(p)$ where y is the dependent variable and (p,x) are the independent variable.

$= \frac{dy}{dx} = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$

$= p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$

$= 0 = x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$

Now with $\frac{dp}{dx} = 0$ and $p = c$ we have $y = cx + f(c)$ which is the general solution of Clairaut's equation.

**Example**

**Solve** $y = mx + \frac{a}{m}$ Since the said equation is exactly in the form of a Clairaut's representation hence there is no need to solve it further.

**Example**

**Solve $q = \log(qx - y)$**

$e^q = qx - y$

$y = qx - e^q$ Replacing q with c the equation becomes

$y = cx - e^c$ and this equation is in the required Clairaut's form.

**Example**

**Solve $y = xy' + (y')^2$**

let $y' = p$

$y = x(p) + (p)^2$

$p(dx) = x(dp) + p(dx) + 2pdp$

$0 = x(dp) + 2pdp$

$= dp(x + 2p) = 0$

$= dp = 0;\ p = c;\ x = -2p;\ p = c$

**$= x = -2p;\ y = xp + p^2$**

**$= p = -\frac{x}{2} = -\frac{x^2}{4}$ (Eliminating p )Ans**

**Example**

**Solve $y = xy' + \sqrt{(y'^2) + 1}$**

Let $y' = q$

$y = xq + \sqrt{(q^2) + 1}$

$dy = xdq + qdx + \frac{qdq}{\sqrt{q^2+1}}$

$0 = xdq + \frac{qdq}{\sqrt{q^2+1}}$

Now $dq = 0$ and $p = c$

**So $y = cx + \sqrt{(c^2) + 1}$**

The other equation is as follows :

$$\mathbf{x} = \frac{-q}{\sqrt{q^2+1}} \text{ and } \mathbf{y} = \left(\frac{-q}{\sqrt{q^2+1}}\right)\mathbf{x} + \sqrt{(\mathbf{q^2})+\mathbf{1}}$$

$$\mathbf{y} = \frac{1}{\sqrt{q^2+1}}$$

Elimination of p happens by putting the equation in the form of $x^2 + y^2 = 1$ **Ans**

**Example**

**Solve:** $e^{4x}(p-1)+e^{2y}p^2 = 0$.

The differential equation is not in the Clairaut's form, but by taking $e^{2x} = u$ and $e^{2y} = v$ and can be converted it into Clairaut's form.

$v = u\, dv/du + (dv/du)^2$ and now this is in the Clairaut's form

$dv/du = c \Longrightarrow v = uc + c^2 \Longrightarrow e^{2y} = ce^{2x} + c^2$ is the general solution.

### 4.2.1 Equations reducible to Clairaut's form

Many differential equations of the first order but of the higher degree can be reduced to Clairaut's form with substitutions.

**Example**

Transform and solve the following equation i.e. $x^2(y-px) = p^2y$ is transformed into Clairaut's form

Here $x^2$ and $y^2$ can be considered as u and v respectively i.e. $2xdx = du, 2ydy = dv$

$$\frac{dv}{du} = \left(\frac{dv}{dy}/\frac{du}{dx}\right)/\left(\frac{dy}{dx}\right).$$

$$\text{Let } p = \frac{dy}{dx} = \frac{x}{y}\frac{dv}{du} = \left(\frac{u}{v}\right)^{1/2}\frac{dv}{du}$$

$$= (u)(v^{1/2} - \left(\frac{u}{v}\right)^{1/2}(u)^{1/2}\frac{du}{dv}) = \frac{u}{v}\left(\frac{dv}{du}\right)^2(v)^{1/2}$$

$$= \mathbf{v} = \mathbf{u}\frac{dv}{du} + \left(\frac{dv}{du}\right)^2 \quad \textbf{Ans}$$

**Example**

$$\left(\frac{dy}{dx}\right) + 4x\left(\frac{dy}{dx}\right) - 4y = 0$$

Let $\frac{dy}{dx} = p$ then $p + 4xp - 4y = 0$

Or $y = (p + 4xp)/4$ and this is in the Clairaut's Equation

Differentiating with respect to x

$p = p + p'(x) + (p/2)p'$

$= 0 = p'(x) + (p/2)(p')$

Assuming $p = c$ , $y = cx + (c^2/4)$

Eliminating p we have $y(x) = -x^2$ as it satisfies $y = (p + 4xp)/4$

## 4.4 Summary

There are equations where the left-hand side of the equation can be resolved into rational factors of the first degree and also there are equations where the left-hand side of the equations cannot be factorized. Equations that cannot be factorized in addition to exact and homogeneous are summarized below.Differential equations of the first order but of a higher degree can be solved by one or more of the following four methods :

- Equations solvable for p , i.e. $p = \frac{dy}{dx}$ where the general solution can be represented as $p - f_i(x,y) = 0$ and $F_i(x,y,c) = 0$
- Equations solvable for y i.e. y = f ( x,p ), solution for the same can be represented as f(x,p,c) and the elimination of p if not possible then $x = f_1(p,c)$ and $y = f_2(p,c)$ are combined to form the solutions.
- Equations solvable for x i .e. x = f ( y,p ), solution for the same can be represented as f(y,p,c) and the elimination of p if not possible then $x = f'_1(p,c)$ and $y = f'_2(p,c)$ are combined to form the solutions.

  Clairaut's equation takes the form y = px + f(p) .The general solution for the same is obtained by replacing p by c . Some complex differential equations can be reduced to Clairaut's form with the help of appropriate substitutions.

## 4.5 References

1. Differential equations with Application and Programs. S. Balachandra Rao and H. R. Anuradha, University Press (India) Limited 1996.
2. Lecture notes on Differential Equation by Dr. B. Patel,Department of Mathematics,Gujarat University

## 4.6 Problems

Find for the below problems whether they are solvable for x,y and p

1. $py^2 - 2pyx(\tan^2\theta) + (y^2\sec^2\theta - x^2\tan^2\theta) = 0$
2. Given $p^3 - 4xyp + 8y^2 =$ where p = dy/dx
3. Given y = p tan p + log(cos p)
4. Given $y = px + (1 - p)^{1/2}$

5. $xp^2 - 2yp + x + 2y = 0$

6. $y = x + c\tan^{-1}q$

7. $x = \tan^{-1}q + q/(1+q^2)$

8. $yq^2 + (x-y)q - x$ = Hint [(($x-y+c)(x^2 + y^2 +$constant) = 0) the r.h.s of the equation is the answer Ans]

9. Represent in the Clairaut's form and solve the following :

a) $y = 2px + 6y^2 p^2$ ( $y = v^3$)

b) $\sin qx \cos y = \cos qx \sin y + q$

c) $e^{4x}(p-1) + e^{2y}p^2 = 0$

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Unit 2

# 5

# LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

**Unit Structure**



## 5.0 Objectives

After going through this chapter, students will able to learn

- The Differential Operator
- Properties of operators
- Linear Differential Equation f(D) y = 0 and solution Of f(D) y = 0
- Different cases depending on the nature of the root of the equation f(D) = 0
- Linear differential equation f(D) y = X
- The complimentary Function
- The inverse operator 1/f(D)
- Particular Integral

## 5.1 Introduction

A linear equation or polynomial, with more than one term, constituting of the derivatives of the dependent variable with regard to one or more than one independent variable is known as a linear differential equation.

A differential equation which comprises of the differential coefficients and the dependent variable in the first degree, that does not include the product of a derivative with another derivative or with dependent variable, and in which the coefficients are as constants is called a linear differential equation with constant coefficients.

The general form of such a differential equation of order "n" is

$b_0\frac{d^n y}{dx^n} + b_1\frac{d^{n-1} y}{dx^{n-1}} + b_2\frac{d^{n-2} y}{dx^{n-2}} + \dots\dots\dots + b_{n-1}\frac{dy}{dx} + b_2 y = X$ ………………… (Equation)

Here $b_0$, $b_1$, $b_2$ … are constants. Above equation is a nth order linear differential equation with constant coefficients.

E.g. when n = 3 is put in the equation we get

$$b_0\frac{d^3 y}{dx^3} + b_1\frac{d^2 y}{dx^2} + b_2\frac{dy}{dx} + b_3 y = X$$

which is a 3rd order linear differential equation with constant coefficients.

Using the differential operator D as $\frac{d}{dx}$ i.e. $Dy = \frac{dy}{dx}$; $D^2 y = \frac{d^2 y}{dx^2}$, …… $D^n y = \frac{d^n y}{dx^n}$,

the above equation will take the form

$b_0 D^n y + b_1 D^{n-1} y + b_2 D^{n-2} y + \dots + b_{n-1} Dy + b_n y = X$

OR

$(b_0 D^n + b_1 D^{n-1} + b_2 D^{n-2} + \dots + b_{n-1} D + b_n)y = X$ …………………………..(Equation)

in which each term in the parenthesis is multiplied to y and the results are added to form the equation.

Let $f(D) \equiv b_0 D^n + b_1 D^{n-1} + b_2 D^{n-2} + \dots + b_{n-1} D + b_n$

f (D) is called as nth order polynomial in D.

∴ Then the above equation can be written as f(D) y = f(x) … (Equation)

If in equation (1), if $b_0$, $b_1$, $b_2$ … … $b_n$ are functions of x then it is called $n^{th}$ order linear differential equation.

## 5.2 The Differential Operator D

It is appropriate to present the symbol D to denote the operation of differentiation with respect to x.

$D^2$ designate differentiation twice.

$D^3$ designate differentiation three times.

In general, let $D^k$ designate differentiation k times.

i.e. $D \equiv \frac{d}{dx}$, so that

$\frac{dy}{dx} = Dy$; $\frac{d^2y}{dx^2} = D^2y$; $\frac{d^3y}{dx^3} = D^3y$; ……; $\frac{d^ny}{dx^n} = D^n y$

and $\frac{dy}{dx} + ay = (D + a) y$

The differential operator D or ($D^n$) correlates to the algebraic laws.

**Properties of the operator D**

Suppose $y_1$ and $y_2$ are differentiable functions of x and "b" is a constant and p, q are positive integer then the following holds true

**a.** $D^p (D^q) y = D^q (D^p) y = D^{p+q} y$

**b.** $(D - p_1)(D - p_2) y = (D - p_2)(D - p_1) y$

**c.** $(D - p_1)(D - p_2) y = [D^2 - (p_1 + p_2) D + p_1 p_2] y$

**d.** $D (bu) = b \cdot D(u)$; $D^n (bu) = b \cdot D^n (u)$

**e.** $D (y_1 + y_2) = D (y_1) + D (y_2)$; $Dn (y_1 + y_2) = D^n (y_1) + D^n (y_2)$.

## 5.3 Linear Differential Equation f(D) y = 0

Consider $f(D) y = 0$ …………………… (Equation)

where, $f(D) = b_0 D^n + b_1 D^{n-1} + b_2 D^{n-2} + \dots + b_{n-1} D + \dots\dots\dots b_n$

is nth order polynomial in D and D obeys the laws of algebra, f(D) can be factorized into n linear factors as follows :

$f(D) = (D - p_1)(D - p_2)(D - p_3) \dots (D - p_n)$ where $p_1, p_2, p_3, \dots p_n$ are the roots of the algebraic equation $f(D) = 0$

Therefor the equation can be written as follows:

$f(D) y = (D - p_1)(D - p_2)(D - p_3) \dots (D - p_n) y = 0$ … (Equation)

The equation $f(D) = 0$ is called as an auxiliary equation for the above equations.

e.g. $\frac{d^2y}{dx^2}$ $+D$ + 12y = 0

By using operator D for $\frac{d}{dx}$,

we have ($D^2$ + D = 12) y = 0

∴ f (D) = $D^2$ +(4D -3D) -12 = 0 is an auxiliary equation.

∴ (D2 + D - 12) y = (D + 4) (D -3) y =0

**5.3.1 Solution of f(D) y = 0**

Being $n^{th}$ order Differential Equation, the above equations will have exactly n constants in its general solution.

The equation (5) will be satisfied by the solution of the equation

$(D - p_n)$ y = 0

i.e. $\frac{dy}{dx} - p_n$ y = 0

On solving this first order and first degree differential equation by separating variables, we get $y = c_n e^{p_n x}$, where, $c_n$ is an arbitrary constant.

Similarly, since the factors in equation can be taken in any order, the equation will be satisfied by independently solving each of these equations $(D - p_1)$ y = 0, $(D - p_2)$ y = 0 … etc., that is by $y = c_1 e^{p_1 x}$, $y = c_2 e^{p_2 x}$ ………… etc.

It can, therefore, easily be proved that the sum of these individual solutions is the sum of n arbitrary constants, i.e. $y = c_1 e^{p_1 x} + c_2 e^{p_2 x} + \ldots + c_n e^{p_n x}$ … where the original equation is of terms containing till the nth order and so also are the constants for the above said equations.

∴ The general solution of the equation f (D) y = 0 is,

$$\mathbf{y = c_1 e^{p_1 x} + c_2 e^{p_2 x} + \ldots + c_n e^{p_n x}}$$

where $p_1$, $p_2$, … $p_n$ are the roots of the auxiliary equation f(D) = 0.

**Example :**

**Solve** $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$

**Solution:** Let D stand for $\frac{d}{dx}$ and the given equation can be written as

$(D^3 - 6D^2 + 11D - 6)$ y = 0.

Here auxiliary equation is $D^3 - 6D^2 + 11D - 6 = 0$

i.e. (D – 1) (D – 2) (D – 3) = 0

⇒ $p_1 = 1$, $p_2 = 2$, $p_3 = 3$, are roots of auxiliary equation.

∴ The general solution is $\mathbf{y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}}$

## 5.4 Different cases depending on the nature of the root of the equation f(D) = 0

**a. The Case of Real and Different Roots**

If roots of f (D) = 0 be $p_1, p_2, p_3 \ldots\ldots p_n$, all are real and different, then the solution of f (D) y = 0 will be

$$y = c_1 e^{p_1 x} + c_2 e^{p_2 x} + \ldots + c_n e^{p_n x}$$

**b. The Case of Real and Repeated Roots**

Let $p_1, p_2, p_3, p_4 \ldots p_n$ be the roots of f(D) = 0, then the part of solution corresponding to $p_1$ and $p_2$ will look like $c_1 e^{p_1 x} + c_2 e^{p_1 x}$ ($p_1 = p_2$)

$= (c_1 + c_2) e^{p_1 x} = c' e^{p_1 x}$

But this means that number of arbitrary constants now in the solution will be n – 1 if 2 p's are the same. Hence it is no longer the general solution. The rectification of the anomaly is done as follows:

Pertaining to $p_1 = p_2$, the part of the equation will be $(D - p_1)(D - p_1) y = 0$

Put $(D - p_1) y = t$, then we have

$(D - p_1) t = 0$

$\therefore t = c_1 e^{p_1 x}$

Hence putting value of t in $(D - p_1) y = t$,

we have $(D - p_1) y = c_1 e^{p_1 x}$

or $\left(\frac{dy}{dx} - p_1\right) y = c_1 e^{p_1 x}$ which is a linear differential equation.

Its I.F. $= e^{-\int p_1 dx} = e^{-p_1 x}$ and hence solution is

$y (e^{-p_1 x}) = \int c_1 e^{p_1 x} \cdot e^{-p_1 x} dx + c_2 = c_1 x + c_2$

$\therefore y = (c1\ x + c2) e^{p_1 x}$

If p1 = p2 are real, and the remaining roots $p_3, p_4, p_5, \ldots, p_n$ are real and different then solution of f(D) y = 0 is

$y = (c_1 x + c_2) e^{p_1 x} + c_3 e^{p_3 x} + c_4 e^{p_4 x} + \ldots + c_n e^{p_n x}$

Similarly, when three roots are repeated.

i.e. if $p_1 = p_2 = p_3$ are real, and the remaining roots $p_4, p_5, \ldots p_n$ are real and different then solution of f(D) y = 0 is

$y = (c_1 x^2 + c_2 x + c_3) e^{p_1 x} + c_4 e^{p_4 x} + \ldots + c_n e^{p_n x}$

If $p_1 = p_2 = p_3 = \ldots = p_n$

i.e. n roots are real and equal then solution of f(D) y = 0 is

$y = (c_1 x^{n-1} + c_2 x^{n-2} + \ldots + c_{n-1} x + c_n) e^{p_1 x}$

**Example**

For $(D^2 - 6D + 9) y = 0$

Auxiliary Equation = $(D - 3)^2 y = 0$

and solution takes the form $(c_1 x + c_2)e^{3x}$

and the final representation is as follows:

$\mathbf{y = (c_1 x + c_2) e^{3x}}$ **Ans**

**Example**

For (D – 1)3 (D + 1) y = 0

solution is $y = (c_1 x^2 + c_2 x + c_3) e^x + c_4 e^{-x}$ where $p_1 = p_2 = p_3$

**Example**

For (D – 1)2 (D + 1)2 y = 0 where $p_1 = p_2 = p_3 = p_4$

solution is $\mathbf{y = (c_1 x + c_2) e^x + (c_3 x + c_4) e^{-x}}$.

**c. The Case of Imaginary or the Complex Roots**

The coefficients of the auxiliary equation that are real will have the imaginary roots that will occur in conjugate pairs.

Let $\alpha \pm i\beta$ be one such pair.

$\therefore p_1 = \alpha + i\beta,\ p_2 = \alpha - i\beta$

Then the solution of the equation f(D) y = 0 takes the form as follows :

$y = Pe^{(\alpha + i\beta) x} + Q e^{(\alpha - i\beta) x}$

$= e^{\alpha x} [P e^{i\beta x} + Qe^{-i\beta x}]$

$= e^{\alpha x} [P (\cos \beta x + i \sin \beta x) + Q(\cos \beta x - i \sin \beta x)]$

$= e^{\alpha x} [(P + Q) \cos \beta x + i (P - Q) \sin \beta x]$

$\mathbf{y = e^{\alpha x} [c_1 \cos \beta x + c_2 \sin \beta x]}$

where, $c_1 = P + Q$ and $c_2 = i (P - Q)$ are arbitrary constants.

$\mathbf{y = C e^{\alpha x} \cos (\beta x + \theta)}$ where C, θ are arbitrary constants,

using $c_1 = C \cos \theta,\ c_2 = -\sin \theta$

**Example:**

**Solve** $(D^2 + 2D + 5)\, y = 0$.

**Solution:** The auxiliary equation is $D^2 + 2D + 5 = 0$

whose roots are $D = -1 \pm 2i$ which are both imaginary.

Here $\alpha = -1, \beta = 2$.

Hence the solution is $y = e^{-x}\, [P \cos 2x + Q \sin 2x]$

**Example:**

**Solve** $\frac{d^4y}{dx^4} - 5\frac{d^2y}{dx^2} + 12\frac{dy}{dx} + 28y = 0$

**Solution:** The auxiliary equation is $D^4 - 5D^2 + 12D + 28 = 0$

Roots are $D = -2, -2, 2 \pm \sqrt{3}\, i$.

(Here $\alpha = 2, \beta = 3$). Hence the solution is

$y = (c_1 x + c_2)\, e^{-2x} + e^{2x}\, [P \cos \sqrt{3}\, x + Q \sin \sqrt{3}\, x]$

**Example:**

**Solve** For $(D^2 + 4)\, y = 0$, $D = 0 \pm 2i$ (Here $\alpha = 0, \beta = 2$)

$\Rightarrow y = P \cos 2x + Q \sin 2x$.

**d. The Case of Repeated Imaginary Roots**

If the imaginary roots $p_1 = \alpha + i\beta$ and $p_2 = \alpha - i\beta$ occur twice, then the part of solution of $f(D)\, y = 0$ will be

$y = (P x + Q)\, e^{p_1 x} + (Rx + S)\, e^{p_2 x}$ … (by using case 2)

$= (P x + Q)\, e^{(\alpha + i\beta)x} + (Rx + S)\, e^{(\alpha - i\beta)x}$

$= e^{\alpha x}\, [(P x + Q)\, e^{i\beta x} + (Rx + S)\, e^{-i\beta x}]$

$= e^{\alpha x}\, [(P x + Q)\, \{\cos \beta x + i \sin \beta x\} + (Rx + S)\, \{\cos \beta x - i \sin \beta x\}]$

$= e^{\alpha x}\, [(P x + Q + Rx + S) \cos \beta x + i\, (Px + Q - Rx - S) \sin \beta x]$

**$y = e^{\alpha x}\, [(c_1 x + c_2) \cos \beta x + (c_3 x + c_4) \sin \beta x]$ with**

constants as $c_1$, $c_2$, $c_3$ and $c_4$.

**Example:**

**Solve** $\frac{d^6y}{dx^6} + 6\frac{d^4y}{dx^4} + 9\frac{d^2y}{dx^2} = 0$

**Solution :** The auxiliary equation $D^6 + 6D^4 + 9D^2 = 0$ has roots

$D = 0, 0, \pm i\sqrt{3}, \pm i\sqrt{3}$ where the imaginary roots $\pm i\sqrt{3}$ are seen to occur in a recurrent manner.

Hence the solution is

$y = c_1 x + c_2 + (c_3 x + c_4) \cos \sqrt{3} x + (c_5 x + c_6) \sin \sqrt{3} x$

**Example:**

**Solve** $(D^4 + 2D^2 + 1) y = 0$.

**Solution:** The auxiliary equation $D^4 + 2D^2 + 1 = 0$ has roots

$D = \pm i, \pm i$, recurring imaginary roots. Hence the solution is

$y = (c_1 x + c_2) \cos x + (c_3 x + c_4) \sin x$.

**Summary of four cases**

**Case 1: Real & Distinct Roots:**

**Auxiliary Equation** $\Rightarrow (D - p_1)(D - p_2)(D - p_3) \ldots (D - p_n) = 0$

$\therefore$ Solution is $y = c_1 e^{p_1 x} + c_2 e^{p_2 x} + c_3 e^{p_3 x} + \ldots + c_n e^{p_n x}$

**Case 2: Repeated Real Roots**

For p1 = p2 $\Rightarrow$ Auxiliary Equation $\Rightarrow$

$(D - p_1)(D - p_2)(D - p_3) \ldots (D - p_n) = 0$

Solution is $y = (c_1 x + c_2) e^{p_1 x} + c3\, e^{p_3 x} + \ldots + c_n e^{p_n x}$

For $p_1 = p_2 = p_3 \Rightarrow$ A.E. $\Rightarrow (D - p_1)(D - p_1)(D - p_1)(D - p_4) \ldots (D - p_n) = 0$

Solution is $y = (c_1 x^2 + c_2 x + c_3) e^{p_1 x} + c4\, e^{p_4 x} + \ldots + c_n e^{p_n x}$

**Case 3: Imaginary Roots**

For $D = \alpha \pm i\beta$

Solution is $y = e^{\alpha x} [c_1 \cos \beta x + c_2 \sin \beta x]$

**Case 4: Repeated Imaginary Roots**

For $D = \alpha \pm i\beta$ be repeated twice

Solution is $y = e^{\alpha x} [(c_1 x + c_2) \cos \beta x + (c_3 x + c_4) \sin \beta x]$

## 5.5 Linear differential equation f (D) y = X

The general solution of the equation f (D)y = X can be represented as

$y = Y_c + Y_p$

i.e. General solution = Complementary function + Particular integral

$Y_c$ is the solution of the given equation with X = 0 that is of equation f (D)y = 0 and is called the complementary function. It involves n arbitrary constants and is denoted by Complementary function (C.F).

By definition of $Y_c$, $f(D)\, Y_c = 0$.

$Y_p$ is any function of s, which satisfies the equation f(D)y = X, so that $f(D)\, Y_p = X$. $Y_p$ is called the particular integral and is denoted by particular integral(P.I). It does not contain any arbitrary constants.

Thus, on substituting $y = Y_c + Y_p$ in f (D) y, we have

$$f(D)\,[Y_c + Y_p] = f(D)\,Y_c + f(D)\,Y_p$$

$$= 0 + X \quad \ldots\ldots\ldots\ldots\ldots \text{[by definition of } Y_c \text{ and } Y_p\text{]}$$

$$= X$$

∴ $y = Y_c + Y_p$ satisfies the equation f (D)y = X and it contains n arbitrary constants, is the general (or complete) solution of the equation.

## 5.6 The Complimentary Function

The solution where the order of the differential equation matches the number of arbitrary constants is called the complementary function (C.F.) of a Differential equation.

**Method of Finding Complementary Function (C.F)**

Step I: Find auxiliary equation (Auxiliary .Equation.)

Step II: Find the roots of the equation. i.e. values of p. Let the roots are $p_1$, $p_2, \ldots\ldots, p_n$ .

Step III: Required C.F. is obtained as per the roots stated below.

**Rules of finding C.F**

If all roots $p_1, p_2, \ldots\ldots, p_n$ are real and distinct of auxiliary equation then complementary function will be $c_1 e^{p_1 x} + c_2 e^{p_2 x} + \ldots + c_n e^{p_n x}$ .

If $p_1 = p_2$, but other roots are real and distinct then complementary function will be $(c_1 x + c2)\, c_1 e^{p_1 x} + c_3 e^{p_3 x} + c_4 e^{p_4 x} + \ldots + c_n e^{p_n x}$.

If roots are imaginary $(\alpha \pm i\, \beta)$ then complementary function will be $e^{\alpha x}\,[c_1 \cos \beta x + c_2 \sin \beta x]$.

If roots are imaginary and repeated twice then complementary function will be $e^{\alpha x}$ [($c_1$ x + $c_2$) cos βx + ($c_3$ x + $c_4$) sin βx]

**Example**

**Solve** $(D^2 - 3D - 4)\, y = 0$.

**Solution:** Here Auxiliary equation is $(D^2 - 3D - 4) = 0$.

$D^2 - 3D - 4 = 0$

$(D - 4).(D + 1) = 0$

$D = 4, -1$

Hence roots are 4 and -1, real and different

∴ Complementary Function is $y = c_1 e^{4x} + c_2 e^{-x}$

**Example**

**Solve** (D3 - 8) y = 0.

**Solution:** Here Auxiliary equation is $(D^3 - 8) = 0$.

$D^3 - 8 = 0$.

(D - 2). (D2 + 2D + 4) = 0

$D = 2,\ D = -1 \pm i\sqrt{3}$

Hence roots are 2, and $-1 \pm i\sqrt{3}$,

one is real and the rest is a pair of imaginary roots.

∴ Complementary Function is,

$y = c_1 e^{2x} + e^{-x} (c_2 \cos \sqrt{3}x + c_3 \sin\sqrt{3}x)$

## 5.7 The inverse operator 1/f(D) and the symbolic expiration for the particular integral 1/f(D) X

To find the Particular Integral, it is essential to specify the inverse operator $\frac{1}{f(D)}$.So If X is any function of x, then $\frac{1}{f(D)}$ X is that function of x that is free from arbitrary constant which when operated by f (D) gives the function X.

The order of operator f (D) and $\frac{1}{f(D)}$ can be interchanged.

$f(D) \{ \frac{1}{f(D)} X\} = \frac{1}{f(D)} f(D) X = X$

General Method of finding the Particular Integral

**Factor Method**

To evaluate $\frac{1}{f(D)}$ X, where X is a function of x, resolve f (D) into factors of the type (D - a), then operate on X successively by the reciprocal of these factor in any order using the formula

$\frac{1}{(D-a)}$ X = $e^{ax} \int Xe^{-ax}$dx

If X = $e^{ax}$

$\frac{1}{f(D)} e^{ax} = \frac{e^{ax}}{f(a)}$ ; f(a) ≠ 0

= $\frac{x\,e^{ax}}{f'(a)}$ ; f '(a) ≠ 0 = $\frac{x^2 e^{ax}}{f''(a)}$ ; f ''(a) ≠ 0 and so on

Same formula is applicable for sin(ax +b) and cos(ax +b)

Similarly when there are functions like $\frac{1}{f(D)} e^{ax} \sqrt{x} = e^{ax} \frac{1}{f(D+a)} \sqrt{x}$ we can use the above methodology

**Method of partial fractions**

Resolve $\frac{1}{f(D)}$ into partial fractions and then operate on X by each of these fractions.

To find the value of $\frac{1}{f(D)} x^p$, p is any positive integer, then

$\frac{1}{f(D)} x^p = [f(D)]^{-1} x^p$

Since D is an operator, which can be manipulated as expanding $[f(D)]^{-1}$ by the Binomial theorem in ascending power of D as far as the result of expanding $D^{p+1}$ on $x^p$ is 0.Then operating upon $x^p$ with each term of the expansion.

**Examples**

**Example :**

**Solve** $\frac{d^2y}{dx^2}$ - y = 3 + 6x

**Solution:** Auxiliary equation is $D^2 - 1 = 0$

∴ Roots are 1 and -1 and

∴ C.F. is,

$Y_c = C_1 e^{x} + C_2 e^{-x}$

The P. I of the equation is given by,

$\frac{1}{(D^2-1)}[3+6x] = \frac{1}{2}\left\{ \frac{1}{D-1} - \frac{1}{D+1} \right\}[3+6x]$

$= \frac{1}{2}$ { $\frac{1}{D-1}$ [3 +6x] - $\frac{1}{D+1}$ [3+6x]}

$= \frac{1}{2}$ {$e^{x} \int e^{-x}$ [3 +6x] dx - $e^{-x} \int e^{x}$ [3 +6x] dx}

$= \frac{1}{2}$ [- 6 – 12x]

= -3 – 6x =-3[1-2x].

∴ P. I. is, $Y_p$ = - 3[1– 2x]

The complete solution of the equation is

$Y = Y_c + Y_p$

$Y = C_1 e^{x} + C_2 e^{-x}$ - 3 – 6

## 5.8 Particular integral

Short method for finding Particular integral (P.I.):

If X ≠ 0, in equation $a_0\frac{d^n y}{dx^n} + a_1\frac{d^{n-1} y}{dx^{n-1}} + a_2\frac{d^{n-2} y}{dx^{n-2}} + \ldots\ldots\ldots + a_{n-1}\frac{dy}{dx} + a_2 y = X$

then P.I. $= \frac{1}{f(D)}$ X

Following are the methods for finding particular integral

Rules for finding Particular Integral :

| Types of function | What to do | Corresponding P.I. |
|---|---|---|
| $X = e^{ax}$ | Put D = a in f(D) | $\frac{1}{f(D)}$ $e^{ax}$, provided f(a) ≠ 0. If f(a) = 0 then (D-a) is one of the factor of f(D).This factor is solved by using the formula $\frac{1}{(D-a)}$ X $= e^{ax} \int e^{-ax} X dx$. And rest is solved by the above method given here. |
| $X = x^m$ | Put [f(D)]- 1$x^m$ | Expand [f(D)]-1 using binomial expansions and if (D-a) remains in the denominator then take rationalization of denominator and place D in the numerator as derivative |

| | | |
|---|---|---|
| | | of the corresponding function. |
| X = = $e^{ax}$v | First operate on $e^{ax}$ on $\frac{1}{f(D)}$ then operate | $e^{ax}\frac{1}{f(D+a)}$v, then solve for v by above method |
| X = sin ax (or cos ax) | Put D2= -a2 in f(D) | $\frac{1}{f(-a^2)}$ sin ax (or cos ax) , provided $\frac{1}{f(-a^2)}$ ≠ 0 or otherwise use following formula: $\frac{1}{D^2+a^2}$ sin ax = -$\frac{1}{2a}$ cos ax or $\frac{1}{D^2+a^2}$ cos ax = $\frac{x}{2a}$ sin ax |

**Example:**

**Solve** $(D^2+4D+3)y = e^{-2x}$

**Solution:** Here auxiliary equation is $(D^2+4D+3) = 0$

$D^2+4D+3 = 0$

$(D+3)(D+1) = 0$

$D = -3,-1$

Hence roots are -3 and -1, real and different.

Therefore C.F. is

$C.F = C_1e^{-3x}\ C_2e^{-x}$

Now to find P.I:

$P.I = \frac{1}{f(D)}\ X$

$= \frac{1}{f(D)}\ X$

$= \frac{1}{D^2+4D+3}\ e^{-2x}$

Here X = $e^{ax}$ therefore put D = a = -2

$= \frac{1}{(-2)^2+4(-2)+3}\ e^{-2x}$

P. I. $= -\ e^{-2x}$

Hence the general solution is y = C.F.+ P.I.

$Y = C_1e^{-3x}\ C_2e^{-x} - e^{-2x}$

## 5.9 Particular Integral: Other methods

### Method of Variation by Parameters

The method of Variation of Parameters is a generalized method that can be used in many more cases. However, there are two disadvantages to the method. First, the complementary solution is required to solve the problem. Secondly, in order to complete the method a couple of integrals need to be solved.

In some cases we may not be able to actually find the solutions if the integrals are too difficult or if we are unable to find the complementary solution.

**Example:**

**Solve** by Method of Variation by Parameters $[D^2 + 4] = \tan 2x$

Solution: The Auxiliary Equation is $p^2 + 4 = 0$

$p^2 = -4$

$p = \pm 2i$

Complimentary Function is represented as follows :

$Y_c = C_1 \cos 2x + C_2 \sin 2x$

Particular Integral = $- y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X}{W} dx$

$y_1 = \cos 2x$, $y_2 = \sin 2x$; $X = \tan 2x$

and for W, by Wronskian determinant,

$$W = \begin{vmatrix} y1 & y2 \\ y1' & y2' \end{vmatrix}$$

$$= \begin{vmatrix} cos2x & sin2x \\ -2\,sin2x & 2cos2x \end{vmatrix}$$

$= 2\cos^2 2x + 2 \sin^2 2x$

$= 2 [\cos^2 2x + \sin^2 2x]$

$= 2$

Particular Integral $= - \cos 2x \int \frac{sin2x.tan2x}{2} dx + \sin 2x \int \frac{cos2x.tan2x}{2} dx$

$= - \cos 2x \int \frac{sin^2 2x}{2cos2x} dx + \sin 2x \int \frac{sin2x}{2} dx$ $\quad [\ \tan x = \frac{sinx.}{\cos x2}\ ]$

$= - \cos 2x \int \frac{1-cos^2 2x}{2cos2x} dx + \frac{sin2x}{2}\ [\frac{-cos2x}{2}\ ]$

$= - \frac{cos2x}{2} \{\int \frac{1}{cos2x} dx - \int \frac{cos^2 2x}{cos2x} dx\} + [\ \frac{sin2x}{2}\ \frac{-cos2x}{2}\ ]$

$= -\frac{cos2x}{2}\{\int sec2x\, dx - \int cos2x\ dx\} - \frac{sin2x\ cos2x}{4}$

$= -\frac{cos2x}{2}\{\frac{\log[sec2x+\tan 2x]}{2} - \frac{sin2x}{2}\} - \frac{sin2x\ cos2x}{4}$

$= -\frac{cos2x}{4}\log[sec2x + \tan 2x] + \frac{cos2x\ sin2x}{4} - \frac{sin2x\ cos2x}{4}$

Particular I ntegral= $-\frac{cos2x}{4}\log[sec2x + \tan 2x]$

The complete solution of the equation is

$Y = Y_c + Y_p$

$Y = C_1 \cos 2x + C_2 \sin 2x - \frac{cos2x}{4}\log[sec2x + \tan 2x]$

By the method of variation of parameters,

solve the following differential equation:

$\frac{\partial^2 y}{\partial x^2} + 4y = 4\tan 2x$

$= y'' + 4y = 0$

$= p^2 + 4 = 0$

$= p\ (+-)2 = 0$

$= p_1 = -2\ ;\ p_2 = 2$

$= y_c = c_1\cos 2x + c_2\sin 2x$

Now let $y_1 = \cos 2x$ and $y_2 = \sin 2x$

$y_1' = -2\sin 2x$ and $y_2' = 2\cos 2x$

$W = y_1 * y_2' - y_2 * y_1' = 2[\cos^2 x + \sin^2 x] = 2$

$A' = \frac{-y2 * 4tan2x}{W}$

$B' = \frac{y1 * 4tan2x}{W}$

$A = \int \frac{-2sin^2 2x}{cos2x}\partial x$

$B = \int 2\sin 2x\, \partial x$

**$= A = -\log(\sec 2x + \tan 2x) + \sin 2x + c_1, B = -\cos 2x + c_2$**

$= y = A\cos 2x + B\sin 2x$ , we put A and B in this equation

and get the final result.

## 5.10 Differential equations reducible to the linear differential equations with constant coefficients

Linear differential Equation:

X Linear:

$\frac{dx}{dy} + P(y).x = Q(y)$

Integrating Factor (I. F.) = $e^{\int p(y)dy}$

x. IF = $\int$ IF Q(y)dy + c

Y Linear:

$\frac{dy}{dx} + P(x).y = Q(x)$

Integrating Factor (I. F.) = $e^{\int p(x)dx}$

y. IF = $\int$ IF Q(x)dx + c

Example:

Ex 1: $(1 + y^2) + (x - \tan^{-1} y)\frac{dy}{dx} = 0$

Sol: Multiply by $\frac{dx}{dy}$

$(1 + y^2)\frac{dx}{dy} + x - \tan^{-1} y = 0$

$(1 + y^2)\frac{dx}{dy} + x = \tan^{-1} y$

$\frac{dx}{dy} + \frac{x}{(1 + y^2)} = \frac{\tan^{-1} y}{(1 + y^2)}$ ,

$P = \frac{1}{(1 + y^2)}$ and $Q = \frac{\tan^{-1} y}{(1 + y^2)}$

$$\begin{aligned} IF &= e^{\int p(y)dy} \\ &= e^{\int \frac{1}{(1 + y^2)}dy} \\ &= e^{\tan^{-1} y} \end{aligned}$$

x. IF = $\int$ IF Q(y)dy + c

$x.\ e^{\tan^{-1} y} = \int e^{\tan^{-1} y}\frac{\tan^{-1} y}{(1 + y^2)}\ dy + c$

Let $\tan^{-1} y = t$

$\frac{1}{(1 + y^2)}dy = dt$

$x.\ e^t = \int e^t\, t\, dt\ + c$

$= \int e^t t\, dt \ + c$

$= t\, e^t - e^t + c$

Put $t = \tan^{-1} y$

$x.\, e^{\tan^{-1} y} = \tan^{-1} y\, e^{\tan^{-1} y} - e^{\tan^{-1} y} + c$

Reducible to Linear differential Equation

$\frac{dy}{dx} + P(x).\, y = Q(x)\, y\, n$

Dividing by y n

$\frac{1}{y^n} \frac{dy}{dx} + P(x) \frac{1}{y^{n-1}} = Q(x)$ ……….. (1)

Let $\frac{1}{y^{n-1}} = t$

Differentiating. with respect to y

$(-n+1) \frac{1}{y^n} \frac{dy}{dx} = \frac{dt}{dx}$

$\frac{1}{y^n} \frac{dy}{dx} = \frac{1}{1-n} \frac{dt}{dx}$

Eq (1) becomes

$\frac{1}{1-n} \frac{dt}{dx} + P(x)\, t = Q(x)$

This is a linear equation in t.

Example:

Ex1. $\frac{dy}{dx} - y \tan x = - y\, 2 \sec x$

Sol: $\frac{dy}{dx} - y \tan x = - y\, 2 \sec x$

Dividing by y2

$\frac{1}{y^2} \frac{dy}{dx} - \frac{1}{y} \tan x = - \sec x$

Let $\frac{1}{y} = t$

$- \frac{1}{y^2} \frac{dy}{dx} = \frac{dt}{dx}$

$- \frac{dt}{dx} - t \tan x = - \sec x$

$\frac{dt}{dx} + t \tan x = \sec x$

This is a linear equation in t

P = tan x and Q = sec x

IF $= e^{\int \tan x \, dx}$

$= e^{\log \sec x}$

= sec x

t. IF $= \int \text{IF } Q(x) dx + c$

t. sec x $= \int \sec^2 x dx + c$

Ans:

$\frac{1}{y} \sec x = \tan x + c$

## 5.11 Summary

This chapter provides the students with an understanding of linear differential equation of higher order and degree with constant coefficients and goes on to explain the concepts of complimentary functions and integral values and their usage in solving the problems that constitute the above. Students are made to use the concept of inverse operator and the case of real, repeated and imaginary roots to solve complex differential equations of the higher order and higher degree. The techniques of using substitution methods to solve the differential equations by using the concept of reduction is also dealt with in this chapter.

## 5.12 References

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b. Applied Mathematics II by P. N. Wartikar and J. N. Wartikar.

c. https://www.library.gscgandhinagar.in/assets/admin/images/MAT-102(UNIT1,2).pdf

d. http://www.math.utah.edu/~zwick/Classes/Fall2013_2280/Lectures/Lecture6_with_Examples.pdf

e. http://www.rahulandmaths.com/bsc-students/differential-equations

## 5.13 Questions

Solve $\frac{d^2y}{dx^2} + 4y = 0$

Solve $\frac{d^4y}{dx^4} - 16y = 0$

Solve (D2- 3D - 4) y = 0.

Solve (D3 - 8) y = 0

Solve (D2-3D+2)y=$e^5$x

Solve(D3-3D2+4)y=$e^3$x

3. $\frac{d^3 y}{dx^3} - y = e^{2x}$

4. (D2-2D+1)y = $e^3$x

5. (D2-2D+1)y = $e^x$

6. Assuming that the rate of growth of any organism is directly proportional to N(t) present at time t, so to find the value of N(t) given that N(0) = 100 and after (t+1 with t = 0), the size of the organism has grown to 200.

**Solution :**

In this case t = 0, N(0) = 100. The solution of the problem is given by

N(t) = 100 exp (kt), t >=0

Determine m from the additional condition

N(l) = 200 (N(1) = size of I at time t = 1).

Hence 200 = 100 exp (k) ,k = 1n2

Hence the solution is

N(t) = 100 exp (t ln2) = 100 exp ($\ln 2^t$) or N(t) = (100) $2^t$ . So the equation can be represented as shown here.

7. **Applications - Electrical circuits**

$E= L\frac{di}{dt} + R_i$

**a.** A resistance of 50Ω and an inductance of 0.1H are connected in series with battery of 10V. Find current in circuit at any time 't'.

**b.** In a network circuit of R-L series R=50Ω and L=10H, a constant voltage 150V is applied at t=0 by closing the switch. Find the current in the circuit at t=0.10sec.

❖❖❖❖❖❖❖

# 6

# THE LAPLACE TRANSFORM

**Unit Structure**





## 6.0 Objectives

After going through this unit, you will be able to:

- Understand the concept of Laplace Transformation, Theorems on Important Properties of Laplace Transformation
- Solve the problem based on Elementary Laplace Transforms with its type.
- Understand the concept of First shifting and Second shifting theorem
- Understand Convolution Theorem Laplace Transform of an Integral and Derivatives

## 6.1 Introduction

In mathematics, the Laplace transform, named after its inventor Pierre-Simon Laplace, is an integral transform that converts a function of a real variable t (often time) to a function of a complex variable s. It is an essential part of mathematical background required of engineers and scientists. This method has advantage of directly giving the solution of differential equations with given boundary values without the necessity of finding the general solution and then evaluating from it the arbitrary constants. It also provide ready tables of Laplace transforms which reduce the problem of solving differential equations to plain algebraic manipulations.

Whenever a mathematical operator works on a function, the function is changed or transformed into another function. For example when the differential operator $D \ \left(\frac{d}{dx}\right)$ works on $f(x) = tan,x$ ,

it produces a new function $\phi(x) \equiv D\ f(x) = \ sec^2 x$.

## 6. 2 Definition

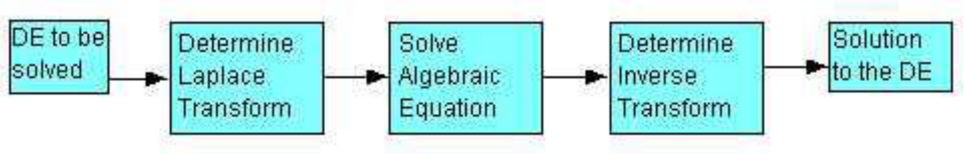
Laplace transform is yet another operational tool for solving constant coefficients linear differential equations. The process of solution consists of three main steps:

i) The given “hard" problem is transformed into a “simple" equation.

ii) This simple equation is solved by purely algebraic manipulations.

iii) The solution of the simple equation is transformed back to obtain the solution of the given problem.

In this way the Laplace transformation reduces the problem of solving a differential equation to an algebraic problem. The third step is made easier by tables, whose role is similar to that of integral tables in integration.




If $f(t)$ is a function of $t$ , then the definite integral $\int_0^\infty e^{-st} f(t)dt$ ,

if it exists, will be a function of the parameter $s$ , and is denoted by $f(s)$.

There is a one to one correspondence between $f(t)$ and $\overline{f}\ (s)$,
and the relation transforms $f(t)$,

a function of $t$ into a new function $f(s)$, which is a function of another variable $s$.

$f(t)$ is called the object function , which is defined for $t \geq 0$, $\bar{f}(s)$ is the resultant or image function , $s$ is the parameter of the the transform, which should be sufficiently large to make the integral convergent.

The relation between $f(t)$ and $f(s)$, $$f(s) = \int_0^{\infty} e^{-st}\,[f(t)]dt \quad ------(1)$$

symbolically it is written as $\mathcal{L}\{f(t)\} = f(s)$, and $\bar{f}(s)$ is called the Laplace transform of *f(t)*.

$\mathcal{L}\{AF_1(t) + BF_2(t)\} = A\mathcal{L}\{F_1(t)\} + B\mathcal{L}\{F_2(t)\}$ Laplace Linear Transformation

## 6. 3 Table of Elementary Laplace Transform

| *f(t)* | *f(s)* |
|---|---|
| *1* | $\frac{1}{s}$ |
| $e^{at}$ | $\frac{1}{s-a}, s > a$ |
| *Sin at* | $\frac{a}{s^2+a^2}$ |
| *Cos at* | $\frac{s}{s^2-a^2}$ |
| *sinh at* | $\frac{a}{s^2-a^2}$ |
| *cosh at* | $\frac{s}{s^2-a^2}$ |
| $\frac{t}{2a} \sin at$ | $\frac{s}{(s^2+a^2)^2}$ |
| $\frac{1}{2a^3}(\sin at - at \cos at)$ | $\frac{1}{(s^2+a^2)^2}$ |
| $t^n$ | $\frac{(n+1)!}{s^{n+1}}$ |

## 6.4 Theorems on Important Properties of Laplace Transformation

### I. Linearity Property :

If a, b, c be any constants and f, g, h any functions of t , then

$$L\{af(t) + bg(t) - ch(t)\} = aL\{f(t)\} + b\{g(t)\} - cL\{h(t)\},$$

L is called linear operator

**II. First Shifting Theorem :**

If $\mathcal{L}\{f(t)\} = \bar{f}(s)$, then $\mathcal{L}\{e^{-at}f(t)\} = \bar{f}(s+a)$

Proof : $\mathcal{L}\{e^{-at}f(t)\} = \int_0^\infty e^{-st}\{e^{-at}f(t)\}\,dt$

$$= \int_0^\infty e^{-(s+a)t} f(t)\,dt$$

$$= \int_0^\infty e^{-pt} f(t)dt \;\; (\,where\ p = s + a) \quad = f(p) = \bar{f}(s+a)$$

***Example 1**: Find the laplace transform of i) $e^{-bt}\ cosat$*

**Solution :** We know $\mathcal{L}(\cos at) = \dfrac{s}{s^2 + a^2}$

$$\mathcal{L}\{e^{-bt}\ cosat\} = \frac{s+b}{(s+b)^2 + a^2}$$

***Example 2**: Find the laplace transform of i) $t^2\ e^{3t}$*

**Solution :** We know $\mathcal{L}(t^2) = \dfrac{2!}{s^3}$

$$\mathcal{L}\{t^2\ e^{3t}\} = \frac{2!}{(s-3)^2}$$

***Example 3**: Find the laplace transform of i) $\sin 2t \sin 3t$ ii) $cos^2 2t$ iii) $sin^3$*

**Solution :**

i) since $sin\ 2t\ sin\ 3t = \frac{1}{2}\,[\cos t - \cos 5t]$

$\therefore\ \mathcal{L}\{\sin 2t \sin 3t\} = \dfrac{1}{2}\,[\mathcal{L}(\cos t) - \mathcal{L}(\cos 5t)]$

$$= \frac{1}{2}\left[\frac{s}{s^2+1^2} - \frac{s}{s^2+5^2}\right]$$

$$= \frac{12s}{(s^2+1)(s^2+25)}$$

ii) since $cos^2 2t = \dfrac{1}{2}(1 + cos4t)$

$$\therefore\ \mathcal{L}\{cos^2 2t\} = \frac{1}{2}[\mathcal{L}(1) + \mathcal{L}(\cos 4t)] = \frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2 + 16}\right)$$

iii) since $\sin 6t = 3\sin 2t - 4\, sin^3 2t\ \ or\ \ sin^3 2t = \frac{3}{4}\sin 2t - \frac{1}{4}\sin 6t$

$$\therefore\ \mathcal{L}\{sin^3 2t\} = \frac{3}{4}[\mathcal{L}(\sin 2t) - \frac{1}{4}\mathcal{L}(\sin 6t)]$$

$$= \frac{3}{4}.\frac{2}{s^2 + 2^2} - \frac{1}{4}.\frac{2}{s^2 + 6^2}$$

$$= \frac{48}{(s^2 + 4)(s^2 + 36)}$$

***Example 4**: Find the laplace transform of*

$$i)\ e^{-3t}(2\cos 5t - 3\sin 5t) \quad ii)\ e^{2t}cos^2 t$$

**Solution** : i) $\mathcal{L}\{e^{-3t}(2\cos 5t - 3\sin 5t)\}$

$$= 2\mathcal{L}(e^{-3t}\cos 5t) - 3\mathcal{L}(e^{-3t}\sin 5t)$$

$$= 2.\frac{s+3}{(s+3)^2 + 5^2} - 3.\frac{5}{(s+3)^2 + 5^2}$$

$$= \frac{2s - 9}{s^2 + 6s + 34}$$

ii) Since $\mathcal{L}\{cos^2 t\} = \frac{1}{2}\mathcal{L}(1 + \cos 2t) = \frac{1}{2}\left\{\frac{1}{s} + \frac{s}{s^2 + 4}\right\}$

$\therefore$ By shifting property , we get $\mathcal{L}\{e^{2t}\ cos^2 t\} = \frac{1}{2}\left\{\frac{1}{s-2} + \frac{s-2}{(s-2)^2 + 4}\right\}$

***Example 5**: If $\mathcal{L}\{f(t)\} = \bar{f}(s)$ show that*

$$i)\ \mathcal{L}[(\sinh at)f(t)] = \frac{1}{2}[\bar{f}(s-a) - \bar{f}(s+a)]$$

$$ii)\ \mathcal{L}[(\cosh at)f(t)] = \frac{1}{2}[\bar{f}(s-a) + \bar{f}(s+a)]$$

*Hence evluate* $(i)\ \sinh 2t \sin 3t\ (ii)\ \cosh 3t \cos 2t$

**Solution**: We have $\mathcal{L}\{(\sinh at)\, f(t)\} = \mathcal{L}\left\{\frac{1}{2}(e^{at} - e^{-at})f(t)\right\}$

$$= \frac{1}{2}[\mathcal{L}\{e^{at}f(t)\} - \mathcal{L}\{e^{-at}f(t)\}]$$

$$= \frac{1}{2}[\bar{f}(s-a) - \bar{f}(s+a)], \textit{by shifting property}$$

$$Similarly\ \mathcal{L}\{(\cosh at)\, f(t)\} = \frac{1}{2}\ [\mathcal{L}\{e^{at} f(t)\} + \mathcal{L}\{e^{-at} f(t)\}]$$

$$= \frac{1}{2}\left[\bar{f}(s-a) + \bar{f}(s+a)\right], by\ shifting\ property$$

$$(i) we\ have\ \mathcal{L}\,(\sin 3t) = \frac{3}{s^2 + 3^2}$$

$$\mathcal{L}\,(\sinh 2t \sin 3t\,) = \frac{1}{2}\left\{\frac{3}{(s-2)^2 + 3^2} - \frac{3}{(s+2)^2 + 3^2}\right\}$$

$$= \frac{12s}{s^4 + 10\, s^2 + 169}$$

$$(ii)\ \ we\ have\ \mathcal{L}\,(\cos 2t) = \frac{s}{s^2 + 2^2}$$

$$\mathcal{L}\,(\cosh\ 3t \cos 2t\,) = \frac{1}{2}\left\{\frac{s-3}{(s-3)^2 + 2^2} + \frac{s+3}{(s+3)^2 + 2^2}\right\}$$

$$= \frac{2s(s^2 - 5)}{s^4 - 10\, s^2 + 169}$$

## 6. 4. 2 Second Shifting Theorem

**III. Second Shifting Theorem :**

If $\mathcal{L}\,\{f(t)\} = \bar{f}(s)$ and $F(t) = \begin{cases} f(t-a) & t>a \\ 0 & t<a \end{cases}$ then $\mathcal{L}\,\{F(t)\} = e^{-as}\, \bar{f}(s)$

Proof : $\mathcal{L}\,\{F(t)\} = \int_0^\infty e^{-st} F(t) dt$

$$= \int_0^a e^{-st}\ F(t)\ dt + \int_a^\infty e^{-st}\ F(t)\ dt$$

$$= \int_0^a e^{-st}(0) dt\ + \int_a^\infty e^{-st}\ f(t-a)\ dt$$

$$= \int_a^\infty e^{-st}\ f(t-a)\ dt$$

$$= \int_0^\infty e^{-s(u+a)}\ f(u)\ du, \qquad [\,u = t-a\,]$$

$$= e^{-as} \int_0^\infty e^{-su}\ f(u)\ du\ =\ e^{-as}\ \bar{f}(s)$$

***hence,*** $\boldsymbol{\mathcal{L}\,\{F(t)\} = e^{-as}\, \bar{f}(s)}$. ***where*** $\mathbf{F(t)} = \begin{cases} \boldsymbol{f(t-a)} & \boldsymbol{t>a} \\ \mathbf{0} & \boldsymbol{t<a} \end{cases}$

***Example* 6**: *Find* $\mathcal{L}\,\{F(t)\}$ *for* $F(t) = \begin{cases} (t-1)^3 & t>1 \\ 0 & 0< t<1 \end{cases}$

**Solution** : Here $f(t) = t^3$ , hence $\bar{f}(s) = \frac{3!}{s^4}$

*By above theorem as* $a = 1,\qquad \mathcal{L}\{F(t)\} = \dfrac{3!\,e^{-s}}{s^4}$

**IV . If** $\mathcal{L}\{f(t)\} = \bar{f}(s)$, **then** $\mathcal{L}\{t^n f(t)\} = (-1)^n \dfrac{d^n}{ds^n}\bar{f}(s)$ , ***where*** $n = 1, 2, 3 \ldots$

***This is called multiplication by*** $t^n$

**Corr. The resultIf** $\mathcal{L}\{t\, f(t)\} = -\dfrac{d}{ds}\,\bar{f}(s) = -f'(s)$

***the differentiation of the transform of a function corresponds to the multiplication of the function by*** $-t$

***Example 7***: *Find* $\mathcal{L}\{F(t)\}$ *for* $(i)\dfrac{t}{2a}\ sinh\ at\quad (ii)t^2 \cos at$

**Solution** : $(i) f(t) = \dfrac{\sinh at}{2a}\ , f(s) = \dfrac{1}{2}\ \dfrac{1}{s^2 - a^2}$

$$\therefore\ \mathcal{L}\left\{t\ \frac{1}{2a}\sinh at\right\} = (-1)\frac{d}{ds}\left\{\frac{1}{2}\ .\frac{1}{s^2 - a^2}\right\}$$

$$= (-1)\frac{1}{2}\ \frac{-2s}{(s^2 - a^2)^2}$$

$$= \frac{s}{(s^2 - a^2)^2}$$

$(ii) f(t) = \cos at\ , \bar{f}(\mathbf{s}) = \dfrac{s}{s^2 + a^2}$

$$\therefore\ \mathcal{L}\{t^2 \cos at\} = (-1)^2 \frac{d^2}{ds^2}\left\{\frac{s}{s^2 + a^2}\right\}$$

$$= \frac{2s\,(s^2 - 3a^2)}{(s^2 + a^2)^3}$$

***Example 8***: *Find the Laplace transforms of*

$(i)t\ \cos at\quad (ii)t^2 \sin at\quad (iii)t^3 e^{-3t}\quad (iv)te^{-t}\sin 3t$

***Solution*** : $(i)$ *Since* , $\mathcal{L}(\cos at) = \dfrac{s}{s^2 + a^2}$

$$\mathcal{L}(\mathrm{t}\cos at) = -\frac{d}{ds}\left(\frac{s}{s^2 + a^2}\right) = -\frac{(s^2 + a^2 - s.2s)}{(s^2 + a^2)}$$

$$= -\frac{(a^2 - s^2)}{(s^2 + a^2)} = \frac{(s^2 - a^2)}{(s^2 + a^2)}$$

$(ii)\ Since\ , \mathcal{L}(\sin at) = \dfrac{a}{s^2 + a^2}$

$$\mathcal{L}(\mathrm{t}^2 \sin at) = (-1)^2 \frac{d^2}{ds^2}\left(\frac{a}{s^2 + a^2}\right)$$

$$= \frac{d}{ds}\left\{\frac{-2as}{(s^2 + a^2)^2}\right\} = \frac{2a\,(3s^2 - a^2)}{(s^2 + a^2)^3}$$

$(iii)\ Since\ , \mathcal{L}(e^{-3t}) = \dfrac{1}{s+3}$

$$\mathcal{L}(\mathrm{t}^3\, e^{-3t}) = (-1)^3 \frac{d^3}{ds^3}\left(\frac{1}{s+3}\right) = \frac{(-1)^3 . 3!}{(s+3)^{3+1}} = \frac{6}{(s+3)^4}$$

$(iv)\ Since\ , \mathcal{L}(\sin 3t) = \dfrac{3}{s^2 + 3^2}$

$$\mathcal{L}(t\ \sin 3t) = -\frac{d}{ds}\left(\frac{s}{s^2 + 3^2}\right) = \frac{6s}{(s^2+9)^2}$$

*Using shifting property, we get*

$$\mathcal{L}(e^{-t} t\ \sin 3t) = \frac{6(s+1)}{[(s+1)^2 + 9]^2} = \frac{6(s+1)}{(s^2 + 2s + 10)^2}$$

**V . If** $\mathcal{L}\{f(t)\} = \bar{f}(s)$**, then** $\mathcal{L}\left\{\dfrac{f(t)}{t}\right\}$

$$= \int_s^\infty \bar{f}(s)ds\ , \textit{provided}\ \lim_{t \to +0} \frac{f(t)}{t}\ \textit{exists}$$

***This is called divison by t***

***Example 9***: *Find the Laplace transforms of*

$(i)\ \dfrac{(1 - e^t)}{t}$ $\qquad$ $(ii)\ \dfrac{\cos at - \cos bt}{t} + t \sin at$

***Solution*** : $(i)\ Since\ , \mathcal{L}(1 - e^t) = \mathcal{L}(1) - \mathcal{L}(e^t) = \dfrac{1}{s} - \dfrac{1}{s-1}$

$$\mathcal{L}\left(\frac{1 - e^t}{t}\right) = \int_s^\infty \left(\frac{1}{s} - \frac{1}{s-1}\right) ds\ = |\log s - \log(s-1)|_0^\infty$$

$$= \left|\log\left(\frac{s}{s-1}\right)\right|_0^\infty = -\log\left[\frac{1}{1-(1/s)}\right] = \log\left(\frac{s-1}{s}\right)$$

$$(ii)\ Since\ , \mathcal{L}(\cos at - \cos bt\ ) = \frac{s}{s^2+\ a^2} - \frac{s}{s^2+\ b^2}$$

$$and\ \ \mathcal{L}(\sin at\ ) = \frac{s}{s^2+\ a^2}$$

$$\therefore\ \mathcal{L}\left(\frac{\cos at - \cos bt}{t}\right) + \mathcal{L}(\text{t} \sin at\ )$$

$$= \int_s^\infty \left(\frac{s}{s^2+\ a^2} - \frac{s}{s^2+\ b^2}\right) ds - \frac{\text{d}}{\text{ds}}\left(\frac{s}{s^2+\ a^2}\right)$$

$$= \left|\frac{1}{2}\log(s^2+\ a^2) - \frac{1}{2}\log(s^2+\ b^2)\right|_s^\infty - \text{a}\left(\frac{-2s}{(s^2+\ a^2)^2}\right)$$

$$= \frac{1}{2}\lim_{s\to\infty}\log\frac{s^2+\ a^2}{s^2+\ b^2} - \frac{1}{2}\log\frac{s^2+\ a^2}{s^2+\ b^2} + \frac{2as}{(s^2+\ a^2)^2}$$

$$= \frac{1}{2}\log\left(\frac{1+0}{1+0}\right) - \frac{1}{2}\log\left(\frac{s^2+\ a^2}{s^2+\ b^2}\right) + \frac{2as}{(s^2+\ a^2)^2}$$

$$= \log\left(\frac{s^2+\ a^2}{s^2+\ b^2}\right)^{1/2} + \frac{2as}{(s^2+\ a^2)^2} \qquad ------$$

$$-\ (\log 1 = 0\ )$$

**VI . If $\mathcal{L}\{f(t)\} = \bar{f}(s)$, then $\mathcal{L}\{f(at)\}$**

$$= \frac{1}{a}\bar{f}\left(\frac{s}{a}\right) \quad (\ \textbf{\textit{Chnage of scale Property}}\ )$$

***Example* 10**: $If\ \mathcal{L}\{f(t)\} = \dfrac{8+12s-2s^2}{(s^2+4)^2}, find\ \mathcal{L}\{f(2t)\}$

**Solution**: From above result ,

$$\mathcal{L}\{f(2t)\} = \frac{1}{2}\left\{\frac{8+12\left(\frac{s}{2}\right)-2\left(\frac{s}{2}\right)^2}{\left(\left(\frac{s}{2}\right)^2+4\right)^2}\right\}$$

$$= \frac{4(16+12s-s^2)}{(s^2+16)^2}$$

**VII . Transform of Error Function**

$$We\ know\ erf\left(\sqrt{x}\right) = \frac{2}{\sqrt{\pi}}\int_o^{\sqrt{x}} e^{-t^2}\ dt$$

$$\mathcal{L}\{erf(\sqrt{t})\} = \frac{1}{s\sqrt{s-1}}$$

## 6.4.3 The convolution Theorem

**VII. The convolution Theorem :**

This theorem is useful to find a function F(t) whose

transform $\overline{\mathrm{F}}(\mathrm{s})$is not the transform of a known function , by expressing $\overline{\mathrm{F}}(\mathrm{s})$ as t

products of two functions of each of which is the transform of aknown function

$$\overline{\mathrm{F}}(\mathrm{s}) = f_1(s)\, f_2(s)$$

*where* $\overline{f_1}(s)$ *and* $\overline{f_2}(s)$ *are transforms of known functions* $f_1(t)$ *and* $f_2(t)$

*The theorem states that*

$$\mathcal{L}\left\{\int_0^t f_1\,(t-u) f_2(u) du\right\} = f_1(s) f_2(s) = \overline{f_1}(s)\overline{f_2}(s)$$

$$= \mathcal{L}\left[\int_0^t f_1\,(u) f_2(t-u) du\right]$$

*This theorem is useful to find inverse transformation.*

***Example 11***: *Verify the convolution theoreom for he pair of functions*

$$f_1(t) = t, f_2(t) = e^{at}$$

***Solution***: $\overline{f_1}(s) = \frac{1}{s^2}, \overline{f_2}(s) = \frac{1}{s-a}$

$$\therefore\ \overline{f_1}(s)\overline{f_2}(s) = \frac{1}{s^2(s-a)}$$

$$Now, \int_0^t f_1\,(u) f_2(t-u) du = \int_0^t u. e^{a(t-u)} du$$

$$= \left[-\frac{u}{a}\, e^{a(t-u)} - \frac{1}{a^2} e^{a(t-u)}\right]_0^t$$

$$= \frac{1}{a^2}[e^{at} - at - 1]$$

$$\therefore \mathcal{L}\left\{\int_0^t f_1\,(u) f_2(t-u) du\right\} = \mathcal{L}\left\{\frac{1}{a^2}[e^{at} - at - 1]\right\}$$

$$= \frac{1}{a^2}\left[\frac{1}{s-a} - \frac{a}{s^2} - \frac{1}{s}\right]$$

$$= \frac{1}{s^2(s-a)} = \bar{f}_1(s)\bar{f}_2(s)$$

## **6.4.4 Laplace Transform of an Integral**

*By definition ,*

$$\mathcal{L}\left\{\int_0^t f\,(u)du\right\} = \int_0^\infty e^{-st}\left[\int_0^t f\,(u)du\right]dt \qquad ----------(i)$$

$$since\ \frac{d}{dt}\left[\int_0^t f\,(u)du\right] = f(t),$$

*we get by integrating by parts , the result (i) as*

$$\mathcal{L}\left\{\int_0^t f\,(u)du\right\} = \left[-\frac{1}{s}\,e^{-st}\int_0^t f\,(u)du\right]_{\mathbf{0}}^{\infty} + \frac{1}{s}\int_0^\infty e^{-st} f\,(t)dt$$

$$= \frac{1}{s}\bar{f}(s)$$

$$Thus\ , \boldsymbol{\mathcal{L}}\left\{\int_{\mathbf{0}}^{\boldsymbol{t}} \boldsymbol{f}\,(\boldsymbol{u})\boldsymbol{du}\right\} = \frac{\mathbf{1}}{\boldsymbol{s}}\bar{\boldsymbol{f}}(\boldsymbol{s})$$

*i.e. function is integrated onver ( 0, t)the transforming of the integral is obtained by dividing the transform of the function by s .*

**OR**

$$If\ \mathcal{L}(t) = \bar{f}(s), then\ \mathcal{L}\left\{\int_0^t f\,(u)du\right\} = \frac{1}{s}\bar{f}(s),$$

$$Let\ \emptyset(t) = \int_0^t f\,(u)du\ , then\ \emptyset'(t) = f(t) and\ \emptyset(0) = 0$$

$$\therefore\ \mathcal{L}\{\emptyset'(t)\} = s\,\bar{\emptyset}(s) - \emptyset(0)$$

$$Or\ \bar{\emptyset}(s) = \frac{1}{s}\mathcal{L}\{\emptyset'(t)\}\quad i.e.\mathcal{L}\left\{\int_0^t f\,(u)du\right\} = \frac{1}{s}\bar{f}(s)$$

***Example* 12**: *Verify* $\mathcal{L}\left\{\int_0^t u^2\, e^{-u} du\right\} = \frac{1}{s}\mathcal{L}\{t^2\, e^{-t}\}$

***Solution***: $\int_0^t u^2\, e^{-u} du = [-(u^2+2u+2)e^{-u}]_0^t$

$$= 2 - (t^2+2t+2)e^{-t}$$

$$\therefore \mathcal{L}\left\{\int_0^t u^2\, e^{-u} du\right\}$$

$$= \mathcal{L}\{2 - (t^2+2t+2)e^{-t}\} \qquad ----------(i)$$

$$\mathcal{L}(t^2 e^{-t}) = (-1)^2 \frac{d^2}{ds^2}\left(\frac{1}{s+1}\right) = \frac{2}{(s+1)^3}$$

$$\mathcal{L}(2te^{-t}) = 2.(-1)\frac{d}{ds}\left(\frac{1}{s+1}\right) = \frac{2}{(s+1)^2}$$

$\therefore$ From (i), we get

$$\mathcal{L}\left\{\int_0^t u^2\, e^{-u} du\right\} = \frac{2}{s} - \left[\frac{2}{(s+1)^3} + \frac{2}{(s+1)^2} + \frac{2}{(s+1)^1}\right]$$

$$= \frac{2}{s(s+1)^3} = \frac{1}{s}\mathcal{L}\{t^2 e^{-t}\}$$

***Example* 13**: *Evaluate the following*:

(i) $\int_0^\infty te^{-3t}\sin t\, dt$ (ii) $\int_0^\infty \frac{\sin mt}{t}\, dt$

(iii) $\int_0^\infty e^t\left(\frac{\cos at - \cos bt}{t}\right) dt$ (iv) $\mathcal{L}\left\{\int_0^t \frac{e^{-t}\sin t}{t}\, dt\right\}$

***Solution*** : (i) $\int_0^\infty te^{-3t}\sin t\, dt = \int_0^\infty te^{-st}(t\sin t)\, dt \quad where\ s = 3$

$$= \mathcal{L}(t\ sint\ ), by\ definition$$

$$= (-1)\frac{d}{ds}\left(\frac{1}{s^2+1}\right)$$

$$= \frac{2s}{(s^2+1)^2} = \frac{2\times 3}{(3^2+1)^2} = \frac{3}{50}$$

$(ii) \int_0^\infty \frac{\sin mt}{t}\, dt$

$$\mathcal{L}(\sin mt) = \frac{m}{(s^2+m^2)} = f(s), \ (say)$$

$$\mathcal{L}\left(\frac{\sin mt}{t}\right) = \int_s^\infty f(s)ds = \int_s^\infty \frac{m\,ds}{s^2+m^2} = \left|tan^{-1}\frac{s}{m}\right|_s^\infty$$

**OR** $by\ definition \int_0^\infty e^{-st}\frac{\sin mt}{t}\, dt = \frac{\boldsymbol{\pi}}{\mathbf{2}} - tan^{-1}\frac{s}{m}$

$Now, \lim_{s\to 0} tan^{-1}(s/m) = 0\ if\ m > 0\ or\ \pi\ if\ m < 0$

$Thus\ taking\ limit\ as\ s \to 0, we\ get$

$$\int_0^\infty \frac{\sin mt}{t}\, dt = \frac{\pi}{2}\ if\ m > 0\ or -\pi/2\ if\ m < 0$$

$(iii) \int_0^\infty e^t \left(\frac{\cos at - \cos bt}{t}\right) dt$

$We\ know\ that\ \mathcal{L}(\cos at) = \frac{s}{(s^2+a^2)}, \mathcal{L}(\cos bt) = \frac{s}{(s^2+b^2)}$

$$\mathcal{L}\left(\frac{\cos at - \cos bt}{t}\right) = \int_s^\infty \left(\frac{s}{(s^2+a^2)} - \frac{s}{(s^2+b^2)}\right) ds$$

$$= \frac{1}{2}\left\{log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right\}_s^\infty = \frac{1}{2}log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

$$This\ implies \int_0^\infty e^{-st}\left(\frac{\cos at - \cos bt}{t}\right) dt = \frac{1}{2}log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

$$Taking\ s = 1, we\ get \int_0^\infty e^{-t}\left(\frac{\cos at - \cos bt}{t}\right) dt$$

$$= \frac{1}{2}log\left(\frac{1+b^2}{1+a^2}\right)$$

$(iv)\mathcal{L}\left\{\int_0^t \frac{e^{-t}\sin t}{t}\, dt\right\}$

$$Since\ \mathcal{L}\left(\frac{\sin t}{t}\right) = \int_0^\infty \frac{ds}{s^2+1} = tan^{-1}s = \frac{\pi}{2} - tan^{-1}s = cot^{-1}s$$

$$\mathcal{L}\left\{e^t\left(\frac{\sin t}{t}\right)\right\}$$

$$= cot^{-1}(s-1) \qquad -----\ by\ shifting\ property$$

$$\mathcal{L}\left[\int_0^t \left\{e^t\left(\frac{\sin t}{t}\right)\right\} dt\right] = \frac{1}{s} cot^{-1}(s-1)$$

### <u>6.4.5 *Laplace Transform of Derivative*</u>

We can express the transform of any derivative of the function f(t)interms of the function itself and in term of the values of the lower order derivative of the function at $t = 0$

( i.e. values approached by the derivatives as $t \to 0$ from positive values).

If $\mathcal{L}[f(t)] = \bar{f}(s)$and f(t)is continuous and is of exponential order $s_0$

$\left[i.e. \lim_{m\to\infty} e^{-ms} f(m) = 0\,, for\ s > s_0\right]$, then $\boldsymbol{\mathcal{L}\{f'(t)\} = s\bar{f}(s) - f(0)}$

*Where* $f(0)$*is the value of* $f(t)$ *at* $t = 0$.

$\boldsymbol{\mathcal{L}\{f'(t)\} = s\bar{f}(s) - f(0)}$ $------------ (\boldsymbol{I})$

***Corollary*** : −

$If\ \mathcal{L}\{f(t)\} = \bar{f}(s)\ then\ \mathcal{L}\{f''(t)\} = s^2\bar{f}(s) - sf(0) - f'(0)$

$Let\ F(t) = f'(t)\ then$

$$\mathcal{L}\{f''(t)\} = \mathcal{L}\{F'(t)\}$$

$$= s\,\mathcal{L}\{F(t)\} - F(0) \qquad ------ by\ (\boldsymbol{I})$$

$$= s\,\mathcal{L}\{f'(t)\} - f'(0)$$

$$= s\left[\bar{f}(s) - f(0)\right] - f'(0) \qquad ------ by\ (\boldsymbol{I})$$

$$= s^2\,\bar{f}(s) - sf(0) - f'(0) \qquad ------ by\ (\boldsymbol{I})$$

*By using mathematical induction, we can show that*

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\,\bar{f}(s) - s^{n-1}\,\bar{f}(0) - s^{n-2}\,f'(0) \ldots\ldots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

## 6.5 Additional Problems

***Example 14***: *Find the laplace transform of each of the following functio*

$(i) \cos t \cos 2t \quad (ii)\ t^2 - 3t + 5 \quad (iii)\ t^2 \sin at \quad (iv) e^{4t} cosh\ 5t$

***Solution*** : $(i)\ \mathcal{L}\{\cos t \cos 2t\} = \mathcal{L}\left\{\frac{1}{2}(\cos 3t + \cos t)\right\}$

$$= \frac{1}{2}\{\mathcal{L}(\cos 3t) + \mathcal{L}(\cos t)\}$$

$$= \frac{1}{2}\left\{\frac{s}{s^2+(3)^2} + \frac{s}{s^2+(1)^2}\right\}$$

$$= \left\{\frac{s\,(s^2+5)}{(s^2+1)\,(s^2+9)}\right\}$$

$(ii)$ $\mathcal{L}\{\,t^2 - 3t + 5\,\}$

$$We\ know, \quad \mathcal{L}\{t^{n-1}\} = \frac{(n-1)!}{s^n}$$

$$\mathcal{L}\{t^2\} = \frac{2!}{s^3} = \frac{2}{s^3} \qquad , n = 3$$

$$\mathcal{L}\{t\} = \frac{1!}{s^2} = \frac{1}{s^2} \qquad , n = 2$$

$$\mathcal{L}\{t^2\} = \frac{1}{s} \qquad , n = 1$$

$$\therefore \mathcal{L}\{\,t^2 - 3t + 5\,\} = \mathcal{L}\,(t^2) - 3\mathcal{L}(t) + 5\mathcal{L}(1)$$

$$= \frac{2}{s^3} - \frac{3}{s^2} + \frac{5}{s} = \frac{5s^2 - 3s + 2}{s^3}$$

$(iii)$ $t^2 \sin at$

$$\mathcal{L}\{t^2 \sin at\} = (-1^2)\,\frac{d^2}{ds^2}\cdot\left\{\frac{s}{s^2+a^2}\right\} = \frac{2a(3s^2-a^2)}{(s^2+a^2)^3}$$

$(iv)$ $e^{4t} cosh\, 5t$

$$\mathcal{L}\{e^{4t} cosh\, 5t\} = \frac{s-4}{(s-4)^2 - 5^2} = \frac{s-4}{s^2 - 8s - 9}$$

***Example 15***: *Find* $\mathcal{L}\{f(t)\}, if\ f(t) = \{^{\cos(t-\alpha)\ ,\ \ t>\alpha}_{0\ \ \ \ \ \ \ \ \ \ \ ,<\alpha}$

***Solution*** : *By definition ,*

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$$

$$= \int_0^\alpha e^{-st}\,(0)dt + \int_\alpha^\infty e^{-st} \cos\,(t-\alpha)\,dt$$

$$= \int_0^\infty e^{-s(u+\alpha)} \cos\ u\,du \qquad [where\ (u = t-\alpha)]$$

$$= e^{-\alpha s}\int_0^\infty e^{-su} \cos\ u\,du$$

$$= e^{-\alpha e}\,\mathcal{L}\{\cos u\} = e^{-s\alpha}\frac{s}{s^2+1}$$

***Example 16: Find the laplace transform of each of the following functions***

$$(i)\ t^{5/2}\ (ii)\ e^{-3t}t^{-1/2}\ \ (iii)\ \text{erf}\sqrt{t}$$

***Solution :*** $(i)\ t^{5/2}$

$$we\ have\ \mathcal{L}(t^n) = \frac{(n+1)!}{s^{n+1}}$$

$$\mathcal{L}\left(t^{5/2}\right) = \frac{\left({}^{7}\!/_{2}\right)!}{s^{7/2}} = \frac{15}{8}\frac{\left({}^{1}\!/_{2}\right)!}{s^{7/2}} = \frac{15}{8}\sqrt{\frac{\pi}{s^7}}$$

$(ii)\ e^{-3t}t^{-1/2}$

$$\mathcal{L}\left(t^{-1/2}\right) = \frac{\left({}^{1}\!/_{2}\right)!}{s^{1/2}} = \sqrt{\frac{\pi}{s}}$$

$$\therefore \mathcal{L}\left(e^{-3t}t^{-1/2}\right) = \sqrt{\frac{\pi}{s+3}}$$

$(iii)\ \text{erf}\sqrt{t}$

*By definition of error function*

$$erf\left(\sqrt{t}\right) = \frac{2}{\sqrt{\pi}}\int_{o}^{\sqrt{t}} e^{-x^2}\,dx$$

$$= \frac{1}{\sqrt{\pi}}\int_{0}^{t} u^{-1/2}\,e^{-u}\,du \qquad [\,x^2 = u\,]$$

$$\mathcal{L}\{erf\left(\sqrt{t}\right)\} = \frac{1}{\sqrt{\pi}}\,\mathcal{L}\left\{\int_{0}^{t} u^{-1/2}\,e^{-u}\,du\right\}$$

$$= \frac{1}{\sqrt{\pi}}\frac{1}{s}\,\mathcal{L}\left\{u^{-1/2}\,e^{-u}\,du\right\} \qquad - Lap.Tra.on\ Integrals$$

$$= \frac{1}{\sqrt{\pi}}\frac{1}{s}\frac{\left({}^{1}\!/_{2}\right)!}{(s-1)^{1/2}} = \frac{1}{s\sqrt{s-1}}$$

***Example*** 17: *Given* $\mathcal{L}\left\{2\sqrt{\frac{t}{\pi}}\right\} = \frac{1}{s^{3/2}}$ *show that* $\mathcal{L}\left\{\frac{1}{\sqrt{\pi t}}\right\} = \frac{1}{\sqrt{s}}$

***Solution*** : *Let* $f(t) = 2\sqrt{\frac{t}{\pi}}$

$$\therefore F'(t) = \frac{2}{\sqrt{\pi}}\frac{1}{2}\, t^{-1/2} = \frac{1}{\sqrt{\pi t}}$$

$$\therefore \mathcal{L}(f'(t)) = \mathcal{L}\frac{1}{\sqrt{\pi}}$$

$$= s\,\bar{f}(s) - f(0) \quad ----Lap.Tra.on\ Derivatives$$

$$= s\mathcal{L}\left\{2\sqrt{\frac{t}{\pi}}\right\} = s.\frac{1}{s^{3/2}} = \frac{1}{\sqrt{s}}$$

***Example*** 18: *Evaluate* $\int_0^\infty t\, e^{-3t} \sin t\, dt$

***Solution*** : $\int_0^\infty t\, e^{-3t} \sin t\, dt = \int_0^\infty e^{-st}\,(\,t \sin t\,)\; dt \qquad (\ where\ s = 3\ )$

$$= \mathcal{L}\{\,t \sin t\,\} \qquad -by\ definition$$

$$= (-1)\,\frac{d}{ds}\left\{\frac{1}{s^2+1}\right\}$$

$$= \frac{2s}{(s^2+1)^2}$$

$$= \frac{2*3}{((3)^2+1)^2} = \frac{3}{50} \qquad [\,replacing\ s = 3\,]$$

## 6.6 Exercise

1. Obtain the Laplace Transform of each of the following functions:

(i) $(t^2+1)^2$ $\left(\textbf{Ans :}\ \frac{s^4 + 4s^2 + 24}{s^5}\right)$

(ii) $(\,t+1)^2 e^t$ $\left(\textbf{Ans :}\ \frac{s^2+1}{(s-1)^3}\right)$

(iii) $t^2 \cos kt$ $\left(\textbf{Ans :}\ \frac{2s\,(s^2 - 3k^2)}{(s^2+k^2)^3}\right)$

(iv) $sin^3 t$ $\left(\textbf{Ans :}\ \frac{6}{(s^2+1)(s^2+9)}\right)$

$(v)\ \cos at \sinh at$ $\left(\textbf{Ans}: \frac{a(s^2-2a^2)}{s^4+4a^4}\right)$

$(vi)(\sin 2t-\cos 2t)^2$ $\left(\textbf{Ans}: \frac{1}{s}-\frac{4}{(s^2+16)}\right)$

2. BY using fundamental definition, find the Laplace transform of $f(t)$, where

$(i)\ f(t)=\begin{cases} t, & 0<t<4 \\ 5, & t>4 \end{cases}$ $\left(\textbf{Ans}: \frac{a(1-e^{-bs})}{s}\right)$

$(ii)\ f(t)=\begin{cases} a, & 0<t<b \\ 0, & t>b \end{cases}$ $\left(\textbf{Ans}: \frac{1}{s^2}+\left(\frac{1}{s}-\frac{1}{s^2}\right)e^{-4s}\right)$

$(iii)\ f(t)=\begin{cases} (t-1)^2, & t>1 \\ 0, & 0<t<1 \end{cases}$ $\left(\textbf{Ans}: \frac{2e^{-s}}{s^3}\right)$

3. Find the Laplace transform of $f(t)$:

$(i)\ \frac{1}{t}(1-\cos at)$ $\left(\textbf{Ans}: \frac{1}{2}\log\left[\frac{s^2+a^2}{s^2}\right]\right)$

$(ii)\ \frac{1}{t}(\cos at-\cos bt)$ $\left(\textbf{Ans}: \frac{1}{2}\log\left[\frac{s^2+b^2}{s^2+a^2}\right]\right)$

$(iii)\ \frac{\sinh t}{t}$ $\left(\textbf{Ans}: \frac{1}{2}\log\left[\frac{s+1}{s-1}\right]\right)$

$(iv)\ \int_0^t e^t \frac{\sin t}{t}\,dt$ $\left(\textbf{Ans}: \frac{1}{s}\cot^{-1}(s-1)\right)$

4. If $\mathcal{L}f(t)=\frac{s^2-s+1}{(2s+1)^2(s-1)}$, find $\mathcal{L}(2t)$ $\left(\textbf{Ans}: \frac{s^2-2s+4}{4(s+1)^2(s-2)}\right)$

5. Evaluate:

(i) $\int_0^\infty t^3 e^{-t}\sin t\ dt$ $(\textbf{Ans}: 0)$

(i) $\int_0^\infty e^{-2t}\sin^3 t\ dt$ $\left(\textbf{Ans}: \frac{6}{65}\right)$

(i) $\int_0^\infty te^{-3t}\sin t\ dt$ $\left(\textbf{Ans}: \frac{3}{50}\right)$

## 6.7 Summary

In this unit we learn Laplace Transform definition, Elementary Laplace Transforms, Theorems on Important Properties of Laplace Transformation

$$\boldsymbol{f}(\boldsymbol{s}) = \int_0^\infty \mathbf{e}^{-\mathbf{st}} \, [\boldsymbol{f}(\boldsymbol{t})] \boldsymbol{dt} \;, \mathcal{L}\{f(t)\}$$

$$= f(s), and \;\; \bar{f}(s) \text{ is called the Laplace transform of } f(t)$$

$$\mathcal{L}\{AF_1(\mathrm{t}) + BF_2(\mathrm{t})\}$$

$$= A\mathcal{L}\{F_1(t)\} + B\mathcal{L}\{F_2(t)\} \qquad \text{Laplace Linear Transformation}$$

**<u>Table of Elementary Laplace Transform</u>**

| *f(t)* | *f(s)* |
|---|---|
| $1$ | $\frac{1}{s}$ |
| $e^{at}$ | $\frac{1}{s-a}, s > a$ |
| $Sin\ at$ | $\frac{a}{s^2+a^2}$ |
| $Cos\ at$ | $\frac{s}{s^2-a^2}$ |
| $sinh\ at$ | $\frac{a}{s^2-a^2}$ |
| $cosh\ at$ | $\frac{s}{s^2-a^2}$ |
| $\frac{t}{2a}\ sin\ at$ | $\frac{s}{(s^2+a^2)^2}$ |
| $\frac{1}{2a^3}(\,sin\ at - at\ cos\ at\,)$ | $\frac{1}{(s^2+a^2)^2}$ |
| $t^n$ | $\frac{(n+1)!}{s^{n+1}}$ |

**First Shifting Theorem** : If $\mathcal{L}\{f(t)\} = \bar{f}(s)$, then $\mathcal{L}\{e^{-at} f(t)\}$

$$= \bar{f}(s+a)$$

**Second Shifting Theorem** : If $\mathcal{L}\{f(t)\} = \bar{f}(s)$ and F(t)

$$= \begin{cases} f(t-a) & t>a \\ 0 & t<a \end{cases} \text{then } \mathcal{L}\{F(t)\} = e^{-as}\,\bar{f}(s)$$

If $\mathcal{L}\{f(t)\} = \bar{f}(\mathrm{s})$, then $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\bar{f}(s) \quad , where\ n = 1,2,3 \ldots$

***This is called multiplication by $t^n$***

If $\mathcal{L}\{f(t)\} = \bar{f}(s)$, then $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \bar{f}(s)ds$ , *provided* $\lim_{t\to+0}\frac{f(t)}{t}$ *exists*

***This is called divison by t***

If $\mathcal{L}\{f(t)\} = \bar{f}(s)$, then $\mathcal{L}\{f(at)\}$

$$= \frac{1}{a}\,\bar{f}\left(\frac{s}{a}\right) \quad (\textbf{\textit{Chnage of scale Property}})$$

**Transform of Error Function** : $\mathcal{L}\{erf(\sqrt{t})\} = \frac{1}{s\sqrt{s-1}}$

**The convolution Theorem:**

$$\mathcal{L}\left\{\int_0^t f_1\,(t-u)f_2(u)du\right\} = f_1(s)f_2(s) = \overline{f_1}(s)\overline{f_2}(s)$$

$$= \mathcal{L}\left[\int_0^t f_1\,(u)f_2(t-u)du\right]$$

***Laplace Transform of an Integral***: $\mathcal{L}\left\{\int_0^t f\,(u)du\right\} = \frac{1}{s}\bar{f}(s)$

***Laplace Transform of Derivative*** : $\mathcal{L}\{f'(t)\} = s\bar{f}(s) - f(0)$

## 6.8 References

1. A Text Book of Applied Mathematics Vol I - P. N. Wartikar and J. N. Wartikar

2. Applied Mathematics II - P. N. Wartikar and J. N. Wartikar

3. Higher Engineering Mathematics - Dr. B. S. Grewal

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Unit 3

# 7

# INVERSE LAPLACE TRANSFORM

**Unit Structure**



## 7.0 Objectives

After going through this unit, you will be able to:

- Understand the concept of Inverse Laplace Transformation, shifting theorem and use of Convolution Theorem
- Solve the problem based on Ordinary Linear Differential Equations with Constant Coefficients
- Understand the concept Solution of Simultaneous Ordinary Differential Equations,
- Understand Laplace Transformation of Special Function, Periodic Functions, Heaviside Unit Step Function, Dirac-delta Function

## 7.1 Introduction: Inverse Laplace Transform

Having find the Laplace Transforms of few functions, let us now determine the inverse transforms of given functions. We are now in a position to find the Laplace transform $\bar{f}\ (s)$for the given object function $f(t)$.

We shall now consider the inverse problem, i.e. given
$\bar{f}\ (s)$, to find the object function $f(t)$ of which $f(s)$ is the Laplace Transform.

**Definition:** If $\mathcal{L}\{f(t)\} = f\ (s)$, then $f(t)$ is called the inverse Laplace Transform of $\bar{f}\ (s)$ and this inverse relation is deonted by.

$$\boldsymbol{\mathcal{L}^{-1}\{\bar{f}\ (s)\} = f(t)}$$

| | |
|---|---|
| $\mathcal{L}^{-1}\left[\frac{1}{s}\right] = 1$ | $\mathcal{L}^{-1}\left[\frac{1}{s-a}\right] = e^{at}$ |
| $\mathcal{L}^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{(n-1)!}, n = 1,2,3..$ | $\mathcal{L}^{-1}\left[\frac{1}{(s-a)^n}\right] = \frac{e^{at}\ t^{n-1}}{(n-1)!}$ |
| $\mathcal{L}^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\ Sin\ at$ | $\mathcal{L}^{-1}\left[\frac{s}{s^2+a^2}\right] = cos\ at$ |
| $\mathcal{L}^{-1}\left[\frac{1}{s^2-a^2}\right] = Sinh\ at$ | $\mathcal{L}^{-1}\left[\frac{s}{s^2-a^2}\right] = cosh\ at$ |
| $\mathcal{L}^{-1}\left[\frac{1}{(s-a)^2+b^2}\right] = \frac{1}{b}\ e^{at} Sin\ bt$ | $\mathcal{L}^{-1}\left[\frac{s-a}{(s-a)^2+b^2}\right] = e^{at} Cos\ bt$ |
| $\mathcal{L}^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{1}{2a} t\ Sin\ at$ | |
| $\mathcal{L}^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = \frac{1}{2a^3}(\ Sin\ at - at\ cos\ at)$ | |

$$\mathcal{L}^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = \frac{1}{2a} t\ Sin\ at$$

$$\mathcal{L}(t\ Sin\ at) = \frac{2as}{(s^2+a^2)^2} \quad and\ \mathcal{L}(t\ cos\ at) = \frac{s^2-a^2}{(s^2+a^2)^2}$$

$$\therefore\ t\ Sin\ at = 2a\mathcal{L}^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] \quad Hence\ \ \mathcal{L}^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = \frac{1}{2a} t\ Sin\ at$$

$$t\ cos\ at = \mathcal{L}^{-1}\left[\frac{s^2-a^2}{(s^2+a^2)^2}\right] = \mathcal{L}^{-1}\left[\frac{(s^2+a^2)-2a^2}{(s^2+a^2)^2}\right]$$

$$t\ cos\ at = \mathcal{L}^{-1}\left[\frac{1}{(s^2+a^2)}\right] - 2a^2\mathcal{L}^{-1}\left[\frac{1}{(s^2+a^2)^2}\right]$$

$$t\ cos\ at = \frac{1}{a}\ Sin\ at - 2a^2\mathcal{L}^{-1}\left[\frac{1}{(s^2+a^2)^2}\right]$$

$$\therefore\ \mathcal{L}^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = \frac{1}{2a^3}(\ Sin\ at - at\ cos\ at)$$

***Example* 1**: Find the inverse transforms of $(i)\,\dfrac{1}{s+4}$ $(ii)\,\dfrac{2s+6}{s^2+4}$

**Solution :** $(i)\,\dfrac{1}{s+4}$, $\mathcal{L}^{-1}\left[\dfrac{1}{s+4}\right] = e^{-4t}$, ***OR*** $f(t) = e^{-4t}$

$$(ii)\,\frac{2s+6}{s^2+4} = \frac{2s}{s^2+4} + \frac{6}{s^2+4} = 2\frac{s}{s^2+4} + 3\frac{2}{s^2+4}$$

We know $\mathcal{L}(\cos 2t) = \dfrac{s}{s^2+4}, \mathcal{L}(\sin 2t) = \dfrac{2}{s^2+4}$

$$\therefore\ \mathcal{L}^{-1}\left[\frac{2s+6}{s^2+4}\right] = 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + 3\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$$

$$= 2\cos 2t + 3\sin 2t$$

***Example* 2**: Find the inverse transforms of $(i)\,\dfrac{s^2-3s+4}{s^3}$ $(ii)\,\dfrac{s+2}{s^2-4s+13}$

**Solution :** $(i)\,\dfrac{s^2-3s+4}{s^3}$,

$$\mathcal{L}^{-1}\left[\frac{s^2-3s+4}{s^3}\right] = \mathcal{L}^{-1}\left[\frac{1}{s}\right] - 3\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] + 4\mathcal{L}^{-1}\left[\frac{1}{s^3}\right]$$

$$= 1 - 3t + 4.\frac{t^2}{2!} = 1 - 3t + 2t^2$$

$(ii)\,\dfrac{s+2}{s^2-4s+13}$,

$$\mathcal{L}^{-1}\left[\frac{s+2}{s^2-4s+13}\right] = \mathcal{L}^{-1}\left[\frac{s+2}{(s-2)^2+9}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{s-2+4}{(s-2)^2+3^2}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{s-2}{(s-2)^2+3^2}\right] + 4\,\mathcal{L}^{-1}\left[\frac{1}{(s-2)^2+3^2}\right]$$

$$= e^{2t}\cos 3t + \frac{4}{3}\,e^{2t}\sin 3t$$

There are different methods to find inverse Laplace transform by using the known Laplace transforms of elementary functions.

## 7.1.1 Shifting Theorem

$(\boldsymbol{I})\ If\ \mathcal{L}^{-1}\left\{\bar{f}\,(s)\right\} = f(t), then\ \mathcal{L}^{-1}\left\{\bar{f}(s-a)\right\} = e^{at} f(t) = e^{at}\,\mathcal{L}^{-1}\left\{\bar{f}\,(s)\right\}$

$(\boldsymbol{II}) If\ \mathcal{L}^{-1}\left\{\bar{f}\,(s)\right\} = f(t) and\ f(0) = 0, then\ \mathcal{L}^{-1}\left\{s\bar{f}(s)\right\} = \frac{d}{dt}\{f(t)\} = f'(t)$

*i.e.* if known standard transform $\bar{f}(s)$is multiplied by $s$ ,

the inverse transform is the differentiation of $f(t)$

In general , $\mathcal{L}^{-1}\left\{s^n\bar{f}(s)\right\} == \frac{d^n}{dt^n}\{f(t)\}$ , provided $f(0) = f'(0) = \cdots$
$= f^{n-1}(0) = 0$

Sometimes along with the above result we require to use following

$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\bar{f}(s) = (-1)^n\, \bar{f}^{(n)}(s)$ which can be expressed as

$$\mathcal{L}^{-1}\left\{f^{(n)}(s)\right\} = (-1)^n t^n f(t)$$

$(\boldsymbol{III}) If\ \mathcal{L}^{-1}\left\{\bar{f}\,(s)\right\} = f(t)\ then\ \ \mathcal{L}^{-1}\left\{\frac{\bar{f}(s)}{s}\right\} = \int_0^t f(t)dt$

$$Also\quad \mathcal{L}^{-1}\left\{\frac{\bar{f}(s)}{s^2}\right\} = \int_0^t \left\{\int_0^t f(t)dt\right\} dt$$

$$\mathcal{L}^{-1}\left\{\frac{\bar{f}(s)}{s^2}\right\}$$
$$= \int_0^t \left\{\int_0^t \left(\int_0^t f(t)\ dt\right) dt\right\} dt \quad \text{and so on.}$$

$(\boldsymbol{IV}) If\ \mathcal{L}^{-1}\left\{\bar{f}\,(s)\right\} = f(t)$ then $tf(t) = \mathcal{L}^{-1}\left\{-\frac{d}{ds}\left[\bar{f}\,(s)\right]\right\},$

it follows from $\mathcal{L}(tf(t)\,) =-\frac{d}{ds}\left[\bar{f}\,(s)\right]$

$(\boldsymbol{V})\mathcal{L}\left(\frac{f(t)}{t}\right) = \int_s^\infty \bar{f}\,(s)\,ds,$ This is useful in finding $f(t)$when $f(s)$ is given ,

provided inverse transform of $\int_s^\infty \bar{f}\,(s)$ can be conveniently calculated

***Example* 3**: *Find* $f(t), if\ \bar{f}(s) = \dfrac{s+7}{s^2+2s+5}$

**Solution**

: We complete a square with the first two term in the denominator, thus

$$s^2 + 2s + 5 = (s+1)^2 + (2)^2$$

$$\text{Hence}\ , \bar{f}(s) = \frac{s+7}{s^2+2s+5} = \frac{(s+1)}{(s+1)^2+(2)^2} + 3\,\frac{2}{(s+1)^2+(2)^2}$$

$$\text{We know}\ \mathcal{L}(\cos 2t) = \frac{s}{s^2+(2)^2}, \mathcal{L}(\sin 2t) = \frac{2}{s^2+(2)^2}$$

Hence by shifting theorem we have ,

$$\mathcal{L}^{-1}\left\{\frac{(s+1)}{(s+1)^2+(2)^2}\right\} = e^{-t}\cos\ 2t$$

$$\mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2+(2)^2}\right\} = e^{-t}\sin 2t$$

$$\therefore f(t) = \mathcal{L}^{-1}\left\{\frac{s+7}{s^2+2s+5}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{(s+1)}{(s+1)^2+(2)^2}\right\} + 3\,\mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2+(2)^2}\right\}$$

$$= e^{-t}\cos\ 2t + 3e^{-t}\sin 2t$$

***Example* 4**: Find the inverse laplace transforms of the following:

$$\text{(i)}\ \frac{s^2}{(s-2)^3} \qquad\qquad (ii)\ \frac{(s+2)^2}{(s^2+4s+8)^2}$$

**Solution :** $(i)\ s^2 = (s-2)^2 + 4(s-2) + 4$

$$\therefore \qquad \frac{s^2}{(s-2)^3} = \frac{1}{(s-2)} + \frac{4}{(s-2)^2} + \frac{4}{(s-2)^3}$$

$$\therefore \qquad \mathcal{L}^{-1}\left\{\frac{s^2}{(s-2)^3}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{(s-2)}\right\} + 4\,\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2}\right\} + 4\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^3}\right\}$$

$$= e^{2t} + 4e^{2t}t + 2e^{2t}t^2 \qquad \text{( Using Shifting property )}$$

$$(ii)\ \mathcal{L}^{-1}\left\{\frac{(s+2)^2}{(s^2+4s+8)^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{(s+2)^2}{(s^2+4s+4+4)^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{(s+2)^2}{((s+2)^2+4)^2}\right\}$$

$$= e^{-2t}\ \mathcal{L}^{-1}\left\{\frac{s^2}{(s^2+4)^2}\right\} = e^{-2t}\ \mathcal{L}^{-1}\left\{\frac{s^2+4-4}{(s^2+4)^2}\right\}$$

$$= e^{-2t}\ \mathcal{L}^{-1}\left\{\frac{1}{(s^2+4)} - \frac{4}{(s^2+4)^2}\right\}$$

$$= \frac{e^{-2t}\sin 2t}{t} - 4e^{-2t}\left\{\frac{1}{4}\left(\frac{\sin 2t}{4} - \frac{t\cos 2t}{2}\right)\right\}$$

$$= e^{-2t}\left\{\frac{\sin 2t}{2} - \frac{\sin 2t}{4} + \frac{t\cos 2t}{2}\right\}$$

$$= e^{-2t}\left\{\frac{\sin 2t}{4} + \frac{t\cos 2t}{2}\right\}$$

## 7.1.2 Partial fraction Methods

Generally in many problems $\bar{f}(s)$is a rational fraction $\frac{\bar{F}(s)}{\bar{G}(s)}$ with degree of $\bar{F}(s)$less than that of $\bar{G}(s)$and this fraction can be expressed as sum on partial fractions of the type

$$\frac{A}{(as+b)^r}, \frac{A}{(as^2+bs+b)^r}\ (r = 1,2,\ldots)$$

and finding the Laplace transform of each of the partial fractions , we find $\mathcal{L}^{-1}\{\bar{f}(s)\}$

***Example* 4**: Find the inverse Laplace transform of each the following functions:

(i) $\frac{2s^2-6s+5}{s^3-6s^2+11s-6}$ (ii) $\frac{4s+5}{(s-1)^2(s+2)}$

(iii) $\frac{6s^3-21s^2+20s-7}{(s+1)(s-2)^3}$ (iv) $\frac{s^2+2s-4}{(s^2+2s+5)(s^2+2s+2)}$

**Solution :** $(i)\ \dfrac{2s^2-6s+5}{s^3-6s^2+11s-6}$, here the deniominator is $s^3-6s^2+11s-6$

here the deniominator is $s^3-6s^2+11s-6$

$$= (s-1)(s-2)(s-3)$$

$$\frac{2s^2-6s+5}{s^3-6s^2+11s-6} = \frac{2s^2-6s+5}{(s-1)(s-2)(s-3)}$$

$$= \frac{A}{(s-1)} - \frac{B}{(s-2)} + \frac{C}{(s-3)}$$

$$A = \frac{[2*1^2-6*1+5]}{(1-2)(1-3)} = \frac{1}{2}$$

$$B = \frac{[2*2^2-6*2+5]}{(2-1)(2-3)} = -1$$

$$C = \frac{[2*3^2-6*3+5]}{(3-1)(3-2)} = \frac{5}{2}$$

$$\therefore\ \frac{2s^2-6s+5}{s^3-6s^2+11s-6} = \frac{1/2}{(s-1)} - \frac{1}{(s-2)} + \frac{5/2}{(s-3)}$$

We have $\mathcal{L}^{-1}\left\{\dfrac{1}{(s-1)}\right\} = e^t,\ \mathcal{L}^{-1}\left\{\dfrac{1}{(s-2)}\right\}$

$$= e^{2t},\ \mathcal{L}^{-1}\left\{\frac{1}{(s-3)}\right\} = e^{3t}$$

$$\therefore f(t) = \mathcal{L}^{-1}\left\{\frac{2s^2-6s+5}{(s-1)(s-2)(s-3)}\right\}$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{(s-1)}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s-2)}\right\} + \frac{5}{2}\mathcal{L}^{-1}\left\{\frac{1}{(s-3)}\right\}$$

$$= \frac{1}{2}e^t - e^{2t} + \frac{5}{2}e^{3t}$$

$(ii)\ \dfrac{4s+5}{(s-1)^2(s+2)}$, here the deniominator is $(s-1)^2(s+2)$

$$Let\ \frac{4s+5}{(s-1)^2(s+2)} = \frac{A}{(s-1)} + \frac{B}{(s-1)^2} + \frac{4(-2)+5}{(-2-1)^2(s+2)}$$

Multiplying both sides by $(s-1)^2(s+2)$, we get

$$(s-1)^2(s+2)\,(4s+5) = A(s-1)(s+2) + B(s+2) - \frac{1}{3}(s-1)^2$$

Put $s = 1$ in above equation , we get, $9 = 3B, \therefore \mathrm{B} = 3$

Equating the coefficients of $s^2$ from both the sides, $\quad 0 = A - \frac{1}{3}$,

$\therefore \mathrm{A} = \frac{1}{3}$

$$\therefore \ \mathcal{L}^{-1}\left\{\frac{4s+5}{(s-1)^2(s+2)}\right\}$$

$$= \frac{1}{3}\,\mathcal{L}^{-1}\left(\frac{1}{(s-1)}\right) + 3\,\mathcal{L}^{-1}\left[\frac{1}{(s-1)^2}\right] - \frac{1}{3}\,\mathcal{L}^{-1}\left[\frac{1}{(s+2)}\right]$$

$$= \frac{1}{3}e^{t} + 3te^{t} - \frac{1}{3}e^{-2t}$$

$$(iii)\ \frac{6s^3 - 21s^2 + 20s - 7}{(s+1)(s-2)^3} = \frac{2}{(s+1)} + \frac{4}{(s-2)} + \frac{3}{(s-2)^2} - \frac{1}{(s-2)^3}$$

We have $\mathcal{L}^{-1}\left\{\frac{1}{(s-a)^n}\right\}$

$= \frac{t^{n-1}}{(n-1)!}e^{3t}$ , Using tbale and shifting theorm

Hence, $f(t) = \mathcal{L}^{-1}\left\{\frac{6s^3 - 21s^2 + 20s - 7}{(s+1)(s-2)^3}\right\}$

$$= 2\mathcal{L}^{-1}\left\{\frac{1}{(s+1)}\right\} + 4\mathcal{L}^{-1}\left\{\frac{1}{(s-2)}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s-2)^3}\right\}$$

$$= 2e^{-t} + 4e^{2t} + 3te^{2t} - \frac{t^2}{2!}e^{2t}$$

$$= 2e^{-t} + (4 + 3t - \frac{1}{2}t^2)e^{2t}$$

$$(iv)\ \frac{s^2 + 2s - 4}{(s^2 + 2s + 5)(s^2 + 2s + 2)},$$

*quadratic factors cannot be resolved into real factors with*

*real numbers, hence*

$$\frac{s^2+2s-4}{(s^2+2s+5)(s^2+2s+2)} = \frac{3}{(s^2+2s+5)} - \frac{4}{(s^2+2s+2)}$$

$$= \frac{\left(\frac{3}{2}\right).2}{(s+1)^2+(2)^2} - \frac{2}{(s+1)^2+(1)^2}$$

Using shifting theorem, $f(t) = \mathcal{L}^{-1}\left\{\frac{s^2+2s-4}{(s^2+2s+5)(s^2+2s+2)}\right\}$

$$= \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2+(2)^2}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+(1)^2}\right\}$$

$$= \frac{3}{2}e^{-t}\sin 2t - 2e^{-t}\sin t$$

$(\boldsymbol{II}) If\ \mathcal{L}^{-1}\left\{\bar{f}\,(s)\right\} = f(t) and\ f(0) = 0, then\ \boldsymbol{\mathcal{L}^{-1}}\left\{\boldsymbol{s\bar{f}(s)}\right\} = \frac{\boldsymbol{d}}{\boldsymbol{dt}}\{\boldsymbol{f(t)}\} = \boldsymbol{f'(t)}$

$i.e.$ if known standard transform $\bar{f}(s)$is multiplied by $s$ ,

the inverse transform is the differentiation of $f(t)$

$In\ general\ , \mathcal{L}^{-1}\left\{s^n\bar{f}(s)\right\} == \frac{d^n}{dt^n}\{f(t)\},$

$provided\ f(0) = f'(0) = \cdots = f^{n-1}(0) = 0$

$Sometimes\ along\ with\ the\ above\ result\ we\ require\ \ to\ use\ following$

$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\bar{f}(s) = (-1)^n\, \bar{f}^{(n)}(s)\ which\ can\ be\ expressed\ as$

$$\boldsymbol{\mathcal{L}^{-1}\left\{f^{(n)}(s)\right\} = (-1)^n t^n f(t)}$$

***Example* 5**: Find the inverse Laplace transform of each the following functions:

(i) $\frac{s^2}{(s^2+a^2)^2}$ (ii) $\frac{s^2}{(s+a)^3}$

(iii) $log\left(1+\frac{a^2}{s^2}\right)$ (iv) $tan^{-1}\left(\frac{2}{s^2}\right)$

**Solution :** $(i)\ \bar{f}(s) = \frac{1}{s^2+a^2}\ \ \therefore\ \mathcal{L}^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin at = f(t)$

$Now, \bar{f}'(s) = \frac{-2s}{(s^2+a^2)^2}, Using\ \boldsymbol{\mathcal{L}^{-1}}\{\boldsymbol{f^{(n)}(s)}\}$
$= \boldsymbol{(-1)^n t^n f(t)}\ we\ get$

$$\mathcal{L}^{-1}\left\{\frac{-2s}{(s^2+a^2)^2}\right\} = (-1)^{(1)}t^1\frac{1}{a}\sin at$$

$$\therefore\ \mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)}\right\} = \frac{1}{2a}\text{t}\sin at$$

$$\boldsymbol{Now, Using\ \mathcal{L}^{-1}\left\{s^n\bar{f}(s)\right\} = \frac{d^n}{dt^n}\{f(t)\}, we\ have}$$

$$\mathcal{L}^{-1}\left\{\frac{s^2}{(s^2+a^2)^2}\right\} = \mathcal{L}^{-1}\left\{s.\frac{s^2}{(s^2+a^2)^2}\right\}$$

$$= \frac{d}{dt}\left\{\frac{1}{2a}\text{t}\sin at\right\} = \frac{1}{2a}(a\,\text{t}\cos\text{at} + \sin at)$$

$$(ii)\ \frac{s^2}{(s+a)^3}$$

$$Now, \bar{f}(\text{s}) = \frac{1}{(s+a)} \quad , \ f(t) = e^{-at}$$

$$Now, \bar{f}'(\text{s}) = \frac{1}{(s+a)^2}\ , \quad \bar{f}''(\text{s}) = \frac{2}{(s+a)^3}$$

$$Using\ \boldsymbol{\mathcal{L}^{-1}\{f^{(n)}(s)\} = (-1)^n t^n f(t)}\ we\ get$$

$$\mathcal{L}^{-1}\left\{\frac{2}{(s+a)^2}\right\} = (-1)^2t^2e^{-at}$$

$$\therefore\ \mathcal{L}^{-1}\left\{\frac{1}{(s+a)^3}\right\} = \frac{1}{2}t^2e^{-at}$$

$$\boldsymbol{Now, Using\ \mathcal{L}^{-1}\left\{s^n\bar{f}(s)\right\} = \frac{d^n}{dt^n}\{f(t)\}, we\ have}$$

$$\therefore\ \mathcal{L}^{-1}\left\{\frac{s^2}{(s+a)^3}\right\} = \frac{d^2}{dt^2}\left\{\frac{1}{2}t^2e^{-at}\right\} = \frac{1}{2}[a^2t^2 - 4at + 2]e^{-at}$$

$$\text{(iii)}\ f(s) = log\left(1 + \frac{a^2}{s^2}\right) = \log(s^2+a^2) - 2\log s$$

$$\bar{f}'(s) = \frac{2s}{s^2+a^2} - \frac{2}{s} = F(s)$$

$$Using\ \boldsymbol{\mathcal{L}^{-1}\{f^{(n)}(s)\} = (-1)^n t^n f(t)}\ we\ get$$

$$\therefore\ \mathcal{L}^{-1}\bar{f}'(s) = 29\cos at - 1) = -t\,f(t)$$

$$\therefore\ f(t) = \frac{2}{t}(1 - \cos at)$$

$$(iv)\ tan^{-1}\left(\frac{2}{s^2}\right)$$

$$f(s) = tan^{-1}\left(\frac{2}{s^2}\right),$$

$$\bar{f}\,'(s) = \frac{1}{1+\frac{4}{s^4}}\left(-\frac{4}{s^3}\right) = -\frac{4s}{s^4+4}$$

$$= -\frac{4s}{(s^2-2s+2)\,(s^2+2s+2)}$$

$$= -\left[\frac{1}{(s^2-2s+2)} - \frac{1}{(s^2+2s+2)}\right]$$

$$= -\left[\frac{1}{(s-1)^2+1} - \frac{1}{(s+1)^2+1}\right]$$

$$\therefore\ \mathcal{L}^{-1}\bar{f}\,'(s) = -[e^t \sin t - e^{-t}\sin t]$$

$$= -\left[\frac{e^t - e^{-t}}{2}\right] 2\sin t = -2\sin t \sinh t$$

$$\therefore\ \mathcal{L}^{-1}\bar{f}'(s) = -t\,f(t) = -2\sin t \sinh t$$

$$\therefore\ \mathcal{L}^{-1}tan^{-1}\left(\frac{2}{s^2}\right) = f(t) = \frac{2}{t}\sin t \sinh t$$

## 7.1.3 Use of Convolution Theorem

If the function $\bar{f}\,(s)$, whose inverse transform is required, can be expressed as a product of $\bar{F}\,(s) * \bar{G}(s)$, where inverse transforms $\bar{F}(s)$ and $\bar{G}\,(s)$ are known, we use convolution theorem

**If $\mathcal{L}^{-1}\bar{F}\,(s) = F(t), \mathcal{L}^{-1}\bar{G}\,(s) = G(t)$ and $\bar{f}\,(s) = \bar{F}\,(s) * \bar{G}(s)$ then**

$$\{\mathcal{L}^{-1}\bar{f}\,(s)\} = \mathcal{L}^{-1}\{\bar{F}\,(s) * \bar{G}(s)\} = \int_0^t F(t-u)\,G(u)\,du$$

Corollary: Since $\mathcal{L}^{-1}\left(\frac{1}{s}\right) = 1$ **and** $\mathcal{L}^{-1}\bar{f}(s) = f(t)$

*Let* $\bar{F}\,(s) = \frac{1}{s}$ and $\bar{G}(s) = \bar{f}\,(s)$, *hence from above result we get,*

$$\mathcal{L}^{-1}\left\{\frac{f(s)}{s}\right\} = \int_0^t 1.f(u)du$$

***Note***: *$F(t)$ and $G(t)$ are interchangeable*

***Example 6***: *Obtain the inverse Laplace transform of the following*:

$$(\boldsymbol{i})\ \frac{\mathbf{1}}{\boldsymbol{s^2(s+1)^2}} \qquad (\boldsymbol{ii})\ \frac{\boldsymbol{s}}{\boldsymbol{(s^2+a^2)^2}}$$

***Solution***: $(i)\ \frac{1}{s^2(s+1)^2} = \frac{1}{s^2}\ \frac{1}{(s+1)^2}$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t\ = F(t)\ \ and\ \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} = te^{-t} = G(t)$$

$$Using, \{\boldsymbol{\mathcal{L}^{-1}\bar{f}\ (s)}\} = \boldsymbol{\mathcal{L}^{-1}\{\bar{F}\ (s) * \bar{G}(s)\}} = \int_{\mathbf{0}}^{\boldsymbol{t}} \boldsymbol{F(t-u)\ G(u)\ du}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\} = \int_0^t (t-u)\ u\ e^{-u}\ du$$

$$= [-(ut-u^2)\ e^{-u} - \ (t-2u)e^{-u} - (-2)\ e^{-u}]_0^t$$

$$= t\ e^{-t} + \ 2e^{-t} + t - 2$$

$$\boldsymbol{\mathcal{L}^{-1}\left\{\frac{f(s)}{s}\right\}} = \int_{\mathbf{0}}^{\boldsymbol{t}} \mathbf{1}.\boldsymbol{f(u)du}, using\ repeatedly,\ \ \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\}$$

$$= t\ e^{-t} = f(t)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\ \frac{1}{(s+1)^2}\right\}$$

$$= \int_0^t u\ e^{-u}\ du = [-u\ e^{-u} - e^{-u}]_0^t = 1-(t+1)e^{-t}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\ \frac{1}{(s+1)^2}\right\}$$

$$= \int_{\mathbf{0}}^{\boldsymbol{t}} [1-(u+1)\ e^{-u}]du$$

$$= \ [u + (u+1)\ e^{-u} + e^{-u}\ ]_0^t = te^{-t} + 2e^{-t} + t - 2$$

$$(\boldsymbol{ii})\ \frac{s}{(s^2+a^2)^2} = \frac{1}{s^2+a^2} \cdot \frac{s}{s^2+a^2}$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} = cos\ at = F\ (t)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{\sin at}{a} = G\,(t)$$

**If** $\boldsymbol{\mathcal{L}^{-1}\bar{F}\,(s) = F(t), \mathcal{L}^{-1}\bar{G}\,(s) = G(t)\ and\ \bar{f}\,(s)}$
$\boldsymbol{= \bar{F}\,(s) * \bar{G}(s) \qquad then}$

$$\boldsymbol{\left\{\mathcal{L}^{-1}\bar{f}\,(s)\right\} = \mathcal{L}^{-1}\{\bar{F}\,(s) * \bar{G}(s)\}}$$

$$\boldsymbol{= \int_0^t F(t-u)\,G(u)\,du}$$

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s^2+a^2)}\cdot\frac{s}{(s^2+a^2)}\right\}$$

$$= \int_0^t \cos a\,(\,t-u)\,\frac{\sin au}{a}\,du$$

$$= \frac{1}{2a}\int_0^t [\sin at + \sin(\,2au - at\,)]\,du$$

$$= \frac{1}{2a}\left[u\sin at - \frac{1}{2a}\cos(\,2au - at\,)\right]_0^t$$

$$= \frac{1}{2a}\,t\sin at$$

***Example* 7**: Find the inverse transform of the following:

$$(i)\ \frac{1}{(s-2)^4\,(s+3)}$$

***Solution***: $(i)\ \mathcal{L}^{-1}\left\{\frac{1}{(s-2)^4\,(s+3)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-2)^4\,(s-2+5)}\right\}$

$$= e^{2t}\,\mathcal{L}^{-1}\left\{\frac{1}{s^4\,(s+5)}\right\}$$

By convolution theorem : $\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\} = \frac{t^3}{3!}$ $,\mathcal{L}^{-1}\left\{\frac{1}{s+5}\right\} = e^{-5t}$

$$\therefore\ \mathcal{L}^{-1}\left\{\frac{1}{s^4\,(s+5)}\right\} = \int_0^t \frac{u^3}{6}\,e^{-5(t-u)}du = \frac{e^{-5t}}{6}\int_0^t u^3\,e^{5u}du$$

$$= \frac{e^{-5t}}{6}\left[\left(\frac{1}{5}u^3 - \frac{3}{25}u^2 + \frac{6}{125}u - \frac{6}{125}\right)e^{5u}du\right]_0^t$$

$$= \frac{1}{6}\left(\frac{1}{5}t^3 - \frac{3}{25}t^2 + \frac{6}{125}t - \frac{6}{125}\right) - \frac{e^{-5t}}{625}$$

## 7.2 Solution of Ordinary Linear Differential Equations with Constant Coefficients

The Laplace transform method of solving differential equations yields particular solution without the necessity of first finding the general solution and then evaluating the arbitrary constant.

This is specially useful for solving linear differential equations with constant coefficients.

Procedure to solve a Linear Differential Equations with Constant Coefficients by transform method.

1. Take the Laplace transform of both sides of the differential equation using Laplace Transform of derivative (From Previous chapter) and the given initial conditions
2. Transpose the terms with minus signs to the right.
3. Divide by the coefficient of $\bar{y}$, getting $\bar{y}$ as a known function of s
4. Resolve this function of s into partial fractions and take the inverse transforms of both sides.

This gives y as a function of t which is the desired solution satisfying the given conditions.

**Example 8**: Solve by the method of transforms, the equation

$$y''' + 2y'' - y' - 2y = 0 \; given \; y(0) = y'(0) = 0 \; and \; y''(0) = 6$$

**Solution**:

Let take the Laplace transform of both sides for $y''' + 2y'' - y' - 2y = 0$

$$[s^3\bar{y} - s^2y(0) - sy'(0) - y''(0)] + 2[s^2\bar{y} - sy(0) - y'(0)] - [s\bar{y} - y(0)] - 2\bar{y} = 0$$

Using the given conditions, it reduces to

$$(s^3 + 2s^2 - s - 2)\bar{y} = 6$$

$$\therefore \; \bar{y} = \frac{6}{(s^3 + 2s^2 - s - 2)}$$

$$= \frac{6}{(s-1)(s+1)(s+2)} = \frac{6}{(s-1)} + \frac{6}{(s+1)} + \frac{6}{(s+2)}$$

$$= \frac{6}{(s-1)(6)} + \frac{6}{(-2)(s+1)} + \frac{6}{3(s+2)}$$

On inversion , we get $y$

$$= \mathcal{L}^{-1}\left(\frac{1}{(s-1)}\right) - 3\mathcal{L}^{-1}\left(\frac{1}{(s+1)}\right) + 2\mathcal{L}^{-1}\left(\frac{1}{(s+2)}\right)$$

$Or\ \ y = e^{t} - 3e^{-t} + 2e^{-2t}$ which is the desired result.

**Example 9**: Use transform method to solve

$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = e^t \ with\ x = 2, \frac{dx}{dt} = -1\ at\ t = 0$$

**Solution**:

Let take the Laplace transform of both sides for $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = e^t$

$$[s^2\bar{x} - sx(0) - x'(0)] - 2(s\bar{x} - x(0)) + \bar{x} = \frac{1}{s-1}$$

using the given conditions, it reduces to

$$(s^2 - 2s + 1)\bar{x} = \frac{1}{s-1} + 2s - 5 = \frac{2s^2 - 7s + 6}{s-1}$$

$$\therefore\ \bar{x} = \frac{2s^2 - 7s + 6}{(s-1)(s^2 - 2s + 1)} = \frac{2s^2 - 7s + 6}{(s-1)(s-1)^2} = \frac{2s^2 - 7s + 6}{(s-1)^3}$$

$$\therefore\ \bar{x} = \frac{2}{s-1} - \frac{3}{(s-1)^2} + \frac{1}{(s-1)^3}$$ on breaking into partial fractions

On inversion , we get $x$

$$= 2\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) - 3\mathcal{L}^{-1}\left(\frac{1}{(s-1)^2}\right) + \mathcal{L}^{-1}\left(\frac{1}{(s-1)^3}\right)$$

$$= 2e^t - \frac{3e^t.t}{1!} + \frac{e^t.t^2}{2!} = 2e^t - 3te^t + \frac{1}{2}t^2e^t$$

**Example 10 :** Solve $(D^2 + n^2)x = a\, sin\,(nt + \alpha), x = Dx = 0$ at $t = 0$

**Solution:** Let take the Laplace transform of both sides for

$$(D^2 + n^2)x = a\, sin\,(nt + \alpha)$$

$$[s^2\bar{x} - sx(0) - x'(0)] + n^2\bar{x} = a\,\mathcal{L}\{\sin nt\,.\cos\alpha + \cos nt\,.\sin\alpha\}$$

on using the given conditions

$$(s^2 + n^2)\bar{x} = a\,cos\,\alpha.\frac{n}{s^2 + n^2} + a\sin\alpha.\frac{s}{s^2 + n^2}$$

$$\therefore \bar{x} = an\,cos\,\alpha.\frac{1}{(s^2 + n^2)^2} + a\sin\alpha.\frac{s}{(s^2 + n^2)^2}$$

On inversion, we obtain

$$x = an\,cos\,\alpha.\frac{1}{2n^3}(\sin nt - nt\cos nt) + a\sin\alpha.\frac{t}{2n}\sin nt$$

$$= a\frac{\{\sin nt\cos\alpha - nt\cos(nt + \alpha)\}}{2n^2}$$

**Example 11 :** Solve $(D^3 - 3D^2 + 3D - 1)y = t^2e^t$ given that

$$y(0) = 1, y'(0) = 0, y''(0) = -2$$

**Solution:** Let take the Laplace transform of both sides , we get

$$[s^3\bar{y} - s^2y(0) - sy'(0) - y''(0)] - 3\,[s^2\bar{y} - sy(0) - y'(0)] + 3[s\bar{y} - y(0)] - \bar{y}$$

$$= \frac{2}{(s-1)^3}$$

On using given conditions, it reduces to

$$\bar{y} = \frac{s^2 - 3s + 1}{(s-1)^3} + \frac{2}{(s-1)^6} = \frac{(s-1)^2 - (s-1) - 1}{(s-1)^3} + \frac{2}{(s-1)^6}$$

$$\bar{y} = \frac{1}{(s-1)} + \frac{1}{(s-1)^2} - \frac{1}{(s-1)^3} + \frac{2}{(s-1)^6}$$

On inversion, we obtain

$$\mathcal{L}^{-1}\{\bar{y}\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-1)}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^3}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^6}\right\}$$

$$y = e^t\left(1 - t - \frac{1}{2}t^2 + \frac{1}{60}t^5\right)$$

**Example 12 :** Solve $\frac{d^2x}{dt^2} + 9x = \cos 2t\,, if\ x(0) = 1\,, x\left(\frac{\pi}{2}\right) = -1$

**Solution**: Since $x'(0)$is not given, we assume $x'(0) = a$

Taking the Laplace transform of both sides of the equation $\frac{d^2x}{dt^2} + 9x$ $= \cos 2\mathrm{t}$, we have

$$\mathcal{L}(x'') + 9\mathcal{L}(x) = \mathcal{L}\,(\cos 2t) \quad , i.e.\,[s^2\bar{x} - sx(0) - x'(0)] + 9\bar{x} = \frac{s}{s^2+4}$$

$$(s^2+9)\bar{x} = s + a + \frac{s}{s^2+4} \qquad \boldsymbol{OR}\ \ \bar{x} = \frac{s+a}{(s^2+9)} + \frac{s}{(s^2+4)\,(s^2+9)}$$

$$\boldsymbol{OR}\ \ \bar{x} = \frac{s+a}{(s^2+9)} + \frac{s}{(s^2+4)\,(s^2+9)}$$

$$\boldsymbol{OR}\ \ \bar{x} = \frac{a}{(s^2+9)} + \frac{1}{5}.\frac{s}{(s^2+4)} + \frac{4}{5}.\frac{s}{(s^2+9)}$$

On inversion, we obtain

$$\mathcal{L}^{-1}\{\bar{x}\} = \mathcal{L}^{-1}\left\{\frac{a}{(s^2+9)}\right\} + \frac{1}{5}.\mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)}\right\} + \frac{4}{5}.\mathcal{L}^{-1}\left\{\frac{s}{(s^2+9)}\right\}$$

$$x = \frac{a}{3}\sin 3t + \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t$$

$$When\ t = \left(\frac{\pi}{2}\right), -1 = -\frac{a}{3} - \frac{1}{5} \quad OR\ \frac{a}{3} = \frac{4}{5} \qquad since\ x\left(\frac{\pi}{2}\right) = -1$$

Here the solution is $x = \frac{1}{5}(\cos 2t + 4\sin 3t + 4\cos\ 3t\ )$

## 7.3 Solution of Simultaneous Ordinary Differential Equations

**The Laplace transform method is applicable to solve two or more simultaneous ordinary differential equations.**

**Example 13 :** Solve the simultaneous equations

$$\frac{dx}{dt} + 5x - 2y = t, \frac{dy}{dt} + 2x + y = 0\ being\ given\ x = y = 0\ when\ t = 0.$$

**Solution**: Taking the Laplace transforms of the given equations,

$$\frac{dx}{dt} + 5x - 2y = t, \frac{dy}{dt} + 2x + y = 0 \, , we \;\; get$$

$$[\, s\bar{x}\, x(0)] + \; 5\bar{x} - 2\bar{y} = \frac{1}{s^2} \qquad\qquad [\because \quad x(0) = 0\, ]$$

$$i.e. (s+5)\bar{x} - 2\bar{y} = \frac{1}{s^2} \qquad ------(i)$$

$$[\, s\bar{y} - \; y(0)] + 2\bar{x} + \bar{y} = 0 \qquad\qquad [\because \quad y(0) = 0\, ]$$

$$i.e. 2\bar{x} + \; (s+1)\bar{y} = 0 \qquad ------(ii)$$

Solving $(i)$ $and$ $(ii) for \quad , we\ get$

$$\bar{x} = \begin{vmatrix} 1/s^2 & -2 \\ 0 & s+1 \end{vmatrix} + \begin{vmatrix} s+5 & -2 \\ 2 & s+1 \end{vmatrix}$$

$$\bar{x} = \frac{s+1}{s^2\,(s+3)^2}$$

$$\bar{x} = \frac{1}{27\, s} + \frac{1}{9\, s^2} - \frac{1}{27(s+3)} - \frac{2}{9\,(s+3)^2}$$

Substituting the value of $\bar{x}$ in $(ii)$, we get

$$\bar{y} = -\frac{2}{s^2\,(s+3)^2} = \frac{4}{27\, s} - \frac{2}{9s^2} - \frac{4}{27(s+3)} - \frac{2}{9\,(s+3)^2}$$

On inversion , we get

$$x = \frac{1}{27} + \frac{t}{9} - \frac{1}{27}e^{-3t} - \frac{2}{9}te^{-3t} \quad , \qquad and \;\; y = \frac{4}{27} - \frac{2t}{9} - \frac{4}{27}e^{-3t} - \frac{2}{9}te^{-3t}$$

**Example 14** : The coordinators $(x, y)$of a particle moving along a place curve at any time t are given by $\frac{dy}{dt} + 2x = sin\, 2t, \frac{dx}{dt} - 2y = cos\, 2t\, , (\, t > 0)$.

If at $t = 0, x = 1$ and $y = 0$ , show by transforms, that the particle moves along the curve $4x^2 + \; 4xy + 5y^2 = 4$

**Solution**: $Taking\ the\ Laplace\ transforms\ of\ the\ given\ equations,$

$and\ noting\ that\ y(0) = 0\, , x(0) = 1\, , we\ get$

$$[\,s\bar{y} - y(0)] + 2\bar{x} = \frac{2}{s^2 + 2^2} \quad or \;\; 2\bar{x} + s\bar{y} = \frac{2}{s^2 + 4} \; and \;\; -(i)$$

$$[\,s\bar{x} - x(0)] - 2\bar{y} = \frac{2}{s^2+2^2} \quad or \; s\bar{x} - 2\bar{y} = \frac{2}{s^2+4} + 1 \quad ---(ii)$$

Multiplying $(i) by\ s$ and $(ii) by\ 2$ and substracting , we get

$$(s^2 + 4)\bar{y} = -2 \;\; or \; \bar{y} = -\frac{2}{(s^2 + 4)}$$

On inversion, $\quad y = -2\mathcal{L}^{-1}\left[\frac{1}{(s^2 + 4)}\right] = -\sin 2t$

From the given first equation,

$$2x = sin\ 2t - \frac{dy}{dt} = sin\ 2t - \frac{d}{dt}(-sin\ 2t)$$

$$or \;\; 2x = sin\ 2t + 2\ cos\ 2t \;\; or \;\; 4x^2$$
$$= (sin\ 2t + 2\ cos\ 2t\,)^2 \quad ------------(iii)$$

$$Also \qquad 4xy = (sin\ 2t + 2\ cos\ 2t)(\,-2\ sin\ 2t)$$
$$= -2\,(\,sin^2 2t + 2\ sin\ 2t\ cos\ 2t\,) -------(iv)$$

$$and \qquad 5y^2 = 5\ sin^2\ 2t \qquad\qquad -------(iv)$$

$adding\ (iii), (iv), and\ (v), we\ obtain$

$$4x^2 + 4xy + 5y^2$$
$$= sin^2\ 2t + 4\ sin\ 2t\ cos\ 2t + 4\ cos^2 2t - 2\ sin^2 2t$$
$$- 4\ sin\ 2t\ cos\ 2t + 5\ sin^2 2t$$

$$= 4\ sin^2\ 2t + 4\ cos^2 2t = 4$$

**Example 15 :** The small oscillations of a certain system with two degrees of freedom are given by equations $D^2x + 3x - 2y = 0,\; D^2x + D^2y + 3x + 5y = 0$ where *D=d/dt*.

If $x = 0, y = 0\,, x = 3\,, y = 2$ when $t = 0$ , find $x$ and $y$ when $t = \frac{1}{2}$.

**Solution**: Taking the Laplace transforms of the given equations,

$$[s^2\bar{x} - sx(0) - x'(0)] + 3\bar{x} - 2\bar{y} = 0$$

$$i.e., (\,s^2 + 3)\bar{x} - 2\bar{y} = 3 \qquad -----------(i)$$

$$and \;\; [s^2\bar{x} - sx(0) - x'(0)] + [s^2\bar{y} - sy(0) - y'(0)] - 3\bar{x} + 5\bar{y} = 0$$

$$i.e., (s^2-3)\overline{x} - (s^2+5)\overline{y} = 5 \quad --------(ii)$$

*Solving* $(i)$ *and* $(ii)$ *for* $\bar{x}$ *and* $\bar{y}$*, we get*

$$\bar{x} = \begin{vmatrix} 3 & -2 \\ 5 & s^2+5 \end{vmatrix} + \begin{vmatrix} s^2+3 & -2 \\ s^2-3 & s^2+5 \end{vmatrix} = \frac{3s^2+25}{(s^2+1)\,(s^2+9)}$$

$$\bar{x} = \frac{11}{4}.\frac{1}{s^2+1} + \frac{1}{4}.\frac{1}{(s^2+9)}$$

$$\bar{y} = \begin{vmatrix} s^2+3 & 3 \\ s^2-3 & 5 \end{vmatrix} + \begin{vmatrix} s^2+3 & -2 \\ s^2-3 & s^2+5 \end{vmatrix} = \frac{2s^2+24}{(s^2+1)\,(s^2+9)}$$

$$\bar{y} = \frac{11}{4}.\frac{1}{s^2+1} + \frac{3}{4}.\frac{1}{(s^2+9)}$$

*On inversion, we get*

$$x = \frac{11}{4}\sin t + \frac{1}{12}\sin 3t \quad , \quad y = \frac{11}{4}\sin t - \frac{1}{4}\sin 3t$$

## 7.4 Laplace Transformation of Special Function

In some physical and engineering problems, it is required to find the solution of a differential equation of the system which it is acted on by:

(i) a periodic force or periodic voltage
(ii) a impulsive force or voltage acting instantaneously at a certain time, or a concentrated load acting at a point,
(iii) a force acting on a part of the system or voltage acting for finite interval of time

- **Periodic Functions**
- **Heaviside Unit Step Function**
- **Dirac – delta Function (Unit Impulse Function)**

### 7.4.1 Periodic Functions

The perodic function $f(t)$ of period $T$ is defined as

$$f(t+T) = f(t), T > 0 \quad --------(I)$$

For e.g. (i) $f(t) = \sin t$ is a periodic function of period $T = 2\pi$, as

$$f(t+T) = \sin(t+2\pi) = \sin t = f(t)$$

$$f(t+T) = sin(t+2\pi) = sin\, t = f(t)$$

For e. g. (ii) Square Wave Function

$$f(t) = 1, 0 \leq 1 < a \qquad \textit{with period } 2a$$
$$= -1, a < t < 2a$$

The Laplace transform of a periodic function $f(t)$ defined by $(I)$ is given by

$$\bar{f}(s) = \mathcal{L}\{f(t)\}$$

$$= \int_0^\infty e^{-st} f(t)dt$$

$$= \frac{1}{1-e^{-sT}} \int_0^T e^{-su} f(u)du \ ------(II) \ \textit{for period } T$$

**Example 16 :** The " Square Wave Function " of period $2a$ is defined by

$$f(t) = 1, \quad 0 \leq t < a$$
$$= \quad -1, a < t < 2a$$

Find the Laplace transform of *f(t)*

**Solution**: $\bar{f}(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt,$

$$= \frac{1}{1-e^{-sT}} \int_0^T e^{-su} f(u)du \ ------ \ \textit{for period } T$$

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-2a}} \int_0^{2a} e^{-su} f(u)du \ ------ \ \textit{for period } T = 2a$$

$$= \frac{1}{1-e^{-2a}} \left[ \int_0^a e^{-su} 1. du + \int_0^{2a} e^{-su}(-1). du \right]$$

$$= \frac{1}{s} \frac{(1-e^{-as})^2}{(1-e^{-2as})} = \frac{(1-e^{-as})}{(1+e^{-as})} = \frac{1}{s} tanh \frac{as}{2}$$

## 7.4.2 Heaviside Unit Step Function

There are some fractions of which the inverse transform can not be determined from the formulae so far derived.To over come the such cases,the Unit Step Function ( Heaviside′s Unit Function ) is introduce.

### 1. Unit Step Function ( Heaviside′s Unit Function )

Definiton :

The unit step function $u(t-a)$ is defined as follows:

$$u(t-a) = \begin{cases} 0 \text{ for } t < a \\ 1 \text{ for } t \geq a \end{cases}$$

where , $a$ is always positive . It is also denoted as $H(t-a)$.

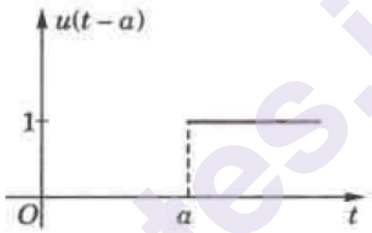



Fig (Unit Step Function )



### 2) Transform of unit function.

$$\mathcal{L}\{u\,(t-a)\} = \int_0^\infty e^{-st}\,u(t-a)dt =$$

$$= \int_0^a e^{-st}\,.0\,dt + \int_a^\infty e^{-st}\,.1dt = 0 + \left|\frac{e^{-st}}{-s}\right|_a^\infty$$

Thus $\mathcal{L}\{u\,(t-a)\} = \frac{e^{-as}}{-s}$

The product $f(t)\,u(t-a) = \begin{cases} 0 & for\ t < a \\ f(t) & for\ t \geq a \end{cases}$

The function $f(t-a).u(t-a)$ represents the graph of $f(t)$shifted through a distance $\boldsymbol{a}$ to the right and is of special importance

**Second shifting property**

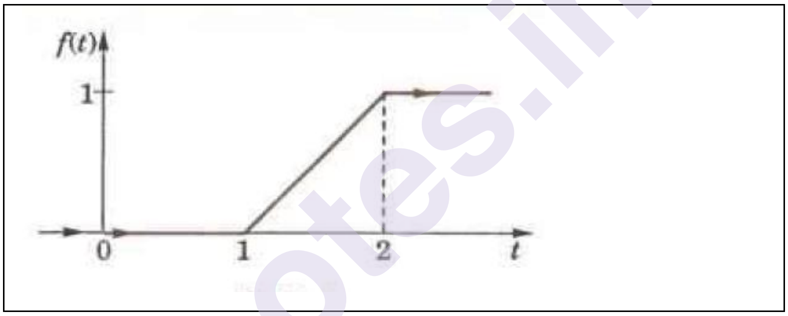
$$\text{If } \mathcal{L}\{f(t)\} = \bar{f}(s), then\ \mathcal{L}\{f(t-a).u(t-a)\} = e^{-as}\bar{f}(s)$$

$$\mathcal{L}\{f(t-a).u(t-a)\} = \int_0^\infty e^{-st} f(t-a)\,u(t-a)dt$$

$$= \int_0^a e^{-st} f(t-a)\,(0)dt + \int_a^\infty e^{-st} f(t-a)\,dt \qquad [\text{Put } t-a=u]$$

$$= \int_0^\infty e^{-s(u+a)} f(u)\,du = e^{-sa}\int_0^\infty e^{-su} f(u)\,du = e^{-as}\bar{f}(s)$$

**Example** Express the following function ( From the below figure) in terms of unit step function and find its Laplace transform






**Solution**: We have

$$f(t) = \begin{cases} 0, & 0 < t < 1 \\ t-1, & 1 < t < 2 \\ 0, & t > 2 \end{cases}$$

**OR** $f(t) = (t-1)[u(t-1) - u(t-2)] + u(t-2)$

$$= (t-1)u(t-1) - (t-2)u(t-2)$$

By second shifting property , $\mathcal{L}\{f(t-a).u(t-a)\} = e^{-as}\mathcal{L}\{f(t)\}$

**Also** $\mathcal{L}\{f(t)\} = \mathcal{L}(t) = \dfrac{1}{s^2}$

$$\therefore\ \mathcal{L}\{(t-1)u(t-1)\} = e^{-s}\frac{1}{s^2}\ \ and\ \ \mathcal{L}\{(t-2)u(t-2)\} = e^{-2s}\frac{1}{s^2}$$

**Hence** $\mathcal{L}\{f(t)\} = \mathcal{L}\{(t-1)u(t-1) - (t-2)u(t-2)\} = \dfrac{e^{-s} - e^{-2s}}{s^2}$

**Example 18 :** Using unit step function , find the Laplace transform of

$$f(t) = \begin{cases} \sin t\ , & 0 \le t < \pi \\ \sin 2t\ , & \pi \le t < 2\pi \\ \sin 3t & t \ge 2\pi\ , \end{cases}$$

**Solution :**

$$f(t) = \sin t\,[u(t-0) - u(t-\pi)] + \sin 2t\,[u(t-\pi) - u(t-2\pi)] + \sin 3t\,.\,u(t-2\pi)$$

$$= \sin t + (\sin 2t - \sin t)\,u(t-\pi) + (\sin 3t - \sin 2t)\,u(t-2\pi)$$

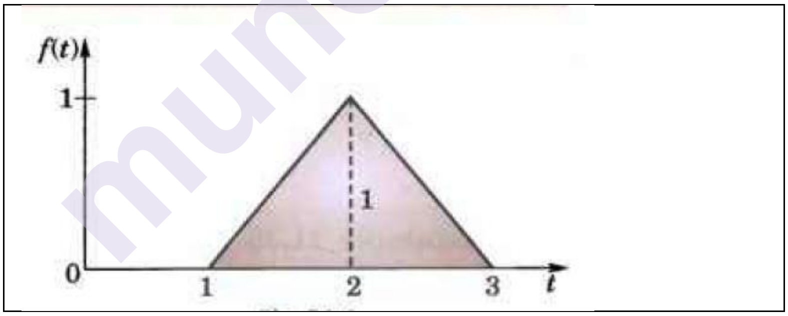
Since $\mathcal{L}[f(t-a)u(t-a)] = e^{-as}\bar{f}(s)$ and $\mathcal{L}[\sin at] = \dfrac{a}{s^2+a^2}$

$$\mathcal{L}[f(t)] = \mathcal{L}[\sin t] + \mathcal{L}[(\sin 2t - \sin t).\,u(t-\pi)] + \mathcal{L}[(\sin 3t - \sin 2t).\,u(t-2\pi)]$$

$$= \frac{1}{s^2+1} + e^{-\pi s}\left(\frac{2}{s^2+4} - \frac{1}{s^2+1}\right) + e^{-2\pi s}\left(\frac{3}{s^2+9} - \frac{2}{s^2+4}\right)$$

**Example 19** (**i**) Express the following function ( From the below figure) in terms of unit step function and find its Laplace transform.

(**ii**)Obtain the Laplace transform of $e^{-t}[1 - u(t-2)]$.




**Solution :** (**i**) We have

$$f(t) = \begin{cases} t-1, & 1 < t < 2 \\ 3-t\ , & 2 < t < 3 \end{cases}$$

**OR** $f(t) = (t-1)\{u(t-1) - u(t-2)\} + (3-t)\{u(t-2) - u(t-3)\}$

$$= (t-1)\,u(t-1) - 2(t-2)u(t-2) + (t-3)u(t-3)$$

Since $\mathcal{L}\{f(t-a).\,u(t-a)\} = e^{-as}\bar{f}(s)$

$$\therefore \mathcal{L}\{f(t)\} = e^{-s}.\frac{1}{s^2} - 2e^{-2s}.\frac{1}{s^2} + e^{-3s}.\frac{1}{s^2} \qquad [\because f(t) = t\,]$$

$$= \frac{e^{-s}(1-e^{-s})^2}{s^2}$$

**(ii)** $\mathcal{L}\{e^{-t}[1-u(t-2)]\} = \mathcal{L}\{e^{-t}\} - \mathcal{L}\{e^{-t}\,u(t-2)\}$

$$= \frac{1}{s+1} - e^{-2}\mathcal{L}\{e^{-(t-2)}u(t-2)\}$$

Taking $f(t) = e^{-t}\,, \bar{f}\,(s) = \dfrac{1}{s+1}$

and using $\mathcal{L}\{f(t-a).u(t-a)\} = e^{-as}\bar{f}(s)$

$$\mathcal{L}\{e^{-(t-2)}u(t-2)\} = e^{-2s}.\frac{1}{s+1}$$

Hence, $\mathcal{L}e^{-t}\{1-u(t-2)\} = \dfrac{\{1-e^{-2(s+1)}\}}{s+1}$

**Example 20** : Using Laplace Transform , evaluate the following

$$\int_0^\infty e^t(1+2t-t^2+t^3)H(t-1)dt$$

**Solution** : We have $\mathcal{L}\{(1+2t-t^2+t^3)H(t-1)\}$

$$= e^{-s}\ \mathcal{L}\{1+2(t+1)-(t+1)^2+(t+1)^3\}$$

$$= e^{-s}\ \mathcal{L}\{3+3t+2t^2+t^3\}$$

$$= e^{-s}\left(3.\frac{1}{s} + 3.\frac{1}{s^2} + 2.\frac{2!}{s^3} + \frac{3!}{s^4}\right)$$

$$= e^{-s}\left(\frac{3}{s} + \frac{3}{s^2} + \frac{4}{s^3} + \frac{6}{s^4}\right)$$

By Definition, this implies that

$$\int_0^\infty e^{-st}(1+2t-t^2+t^3)H(t-1)dt = e^{-s}\left(\frac{3}{s} + \frac{3}{s^2} + \frac{4}{s^3} + \frac{6}{s^4}\right)$$

Taking s = 1 , we obtain

$$\int_0^\infty e^t(1+2t-t^2+t^3)H(t-1)dt = e^{-1}(3+3+4+6) = \frac{16}{e}$$

**Example 21** : Evaluate (i) $\mathcal{L}^{-1}\left\{\frac{e^{-s}-3e^{3-s}}{s^2}\right\}$ (ii) $\mathcal{L}^{-1}\left\{\frac{se^{-as}}{s^2-\omega^2}\right\}$, $a > 0$

**Solution** : $\mathcal{L}^{-1}\left\{e^{-s}.\frac{1}{s^2}\right\} = \begin{Bmatrix} t-1, & t>1 \\ 0, & t<1 \end{Bmatrix} = (t-1)u(t-1)$

$$\mathcal{L}^{-1}\left\{e^{-3s}.\frac{1}{s^2}\right\} = \begin{Bmatrix} t-3, & t>3 \\ 0, & t<3 \end{Bmatrix} = (t-3)u(t-3)$$

$$\therefore \text{(i)}\mathcal{L}^{-1}\left\{\frac{e^{-s}-3e^{3-s}}{s^2}\right\} = \mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2}\right\} - 3\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s^2}\right\}$$

$$= (t-1)u(t-1) - 3(t-3)u(t-3)$$

(ii) $\mathcal{L}^{-1}\left\{\frac{se^{-as}}{s^2-\omega^2}\right\}$, we know $\mathcal{L}^{-1}\left\{\frac{s}{s^2-\omega^2}\right\} = cosh\,\omega t$

$$\therefore \mathcal{L}^{-1}\left\{\frac{se^{-as}}{s^2-\omega^2}\right\} = \begin{cases} cosh\,\omega(t-a), & t>a \\ 0, & t<a \end{cases}$$

$= cosh\,\omega(t-a)\ u\,(t-a)$, by second shifting property

**Example 22** : Find the inverse Laplace transform of :

(i) $\frac{se^{-s/2}+\pi e^{-s}}{s^2+\pi^2}$ (ii) $\frac{e^{-cs}}{s^2(s+a)}$ , $(c > 0)$

**Solution** : **(i)** $\mathcal{L}^{-1}\left\{\frac{s}{s^2+\pi^2}\right\} = cos\,\pi t$ , $\mathcal{L}^{-1}\left\{\frac{\pi}{s^2+\pi^2}\right\} = sin\,\pi t$

and $\mathcal{L}^{-1}\{e^{-as}\bar{f}(s)\} = f(t-a).u(t-a)$

$$\therefore \mathcal{L}^{-1}\left\{\frac{se^{-s/2}+\pi e^{-s}}{s^2+\pi^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{e^{-s/2}.\frac{s}{s^2+\pi^2}\right\} + \mathcal{L}^{-1}\left\{e^{-s}.\frac{\pi}{s^2+\pi^2}\right\}$$

$$= \cos\pi\left(t-\frac{1}{2}\right).u\left(t-\frac{1}{2}\right) + \sin\pi(t-1).u(t-1)$$

$$= \sin\pi t\,.u\left(t-\frac{1}{2}\right) - \sin\pi t.u(t-1)$$

$$= \left\{u\left(t-\frac{1}{2}\right) - u(t-1)\right\}\sin\pi t$$

$$\textbf{(ii)}\ \mathcal{L}^{-1}\left\{\frac{e^{-cs}}{s^2(s+a)}\right\} = \mathcal{L}^{-1}\left\{e^{-cs}\left(-\frac{1}{a^2}\cdot\frac{1}{s}+\frac{1}{a}\cdot\frac{1}{s^2}+\frac{1}{a^2}\cdot\frac{1}{s+a}\right)\right\}$$

$$\text{Using } \mathcal{L}^{-1}\{e^{-as}\bar{f}(s)\} = f(t-a).u(t-a)$$

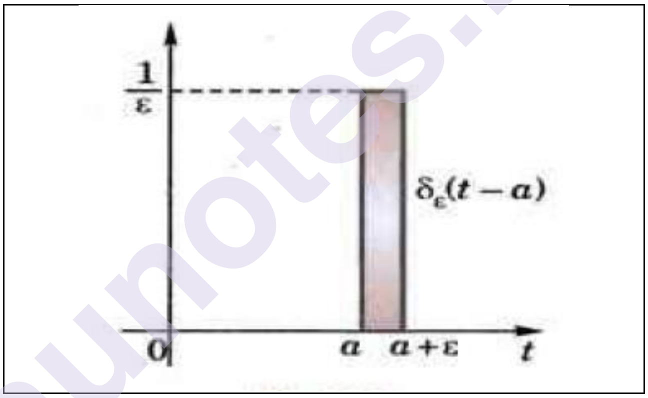
$$\mathcal{L}^{-1}\left\{\frac{e^{-cs}}{s^2(s+a)}\right\}$$

$$= -\frac{1}{a^2}\{1.u(t-c)\}+\frac{1}{a}\{(t-c).u(t-c)\}+\frac{1}{a^2}\{e^{-a(t-c)}.u(t-c)\}$$

$$= \frac{1}{a^2}\{a(t-c)-1+e^{-a(t-c)}\}\,u(t-c)$$

## $\underline{\textbf{7.4.3 Dirac} - \textbf{delta Function(Unit Impulse Function)}}$

The idea of a very large force acting for a very short time is of frequent occurrence in mechanics. This Unit impulse ( Dirac Delta) function is useful in this case.




**1. Unit impulse** ( **Dirac Delta**) function is considered as the limiting form of the function

$$\delta_\varepsilon(t-a) \begin{array}{ll} = 1/\varepsilon\ , & \boldsymbol{a} \le \boldsymbol{t} \le \boldsymbol{a}+\varepsilon \\ = \ 0, & otherwise \end{array}$$

as $\varepsilon \to 0$. It is clear from above figure that as $\varepsilon \to 0$ , the height of the strip increases indefinitely and the width decreases in such a way that its area is always unity.

Thus the unit impulse function $\delta(t-a)$ is defined as follows:

$$\delta(t-a) = \ \infty \ \text{ for } \ t=a\,; = 0 \text{ for } \ t \ \neq a\,,$$

Such that $\int_0^{\infty} \delta(t-a)\,dt = 1 \;\; (a \geq 0)$

As an illustration, a load $w_0$ acting at the point $x = a$ of the beam may be considered as the limiting case of uniform loading $w_0/\varepsilon$ per unit length over the portion of the beam between $x = a$ and $x = a + \varepsilon$. Thus

$$w(x) \begin{array}{ll} = w_0/\varepsilon\,, & \boldsymbol{a < x < a + \varepsilon} \\ = \;\; 0, & otherwise \end{array}$$

$i.e. \quad w(x) = w_0\, \delta(x+a)$

**2. Transform of unit impulse ( Dirac Delta) function.**

If $f(t)$ be a function of $t$ at $t = a$, then

$$\int_0^{\infty} f(t)\, \delta_\varepsilon(t-a).dt = \int_0^{a+\varepsilon} f(t)\, \frac{\mathbf{1}}{\varepsilon}\, .dt$$

$= (a+\varepsilon-a) f(\eta) \frac{1}{\varepsilon} = f(\eta), \quad where\ a < \eta < a+\varepsilon$, by Mean Value theorem for integrals

As $\varepsilon \to \mathbf{0}$, ***we get*** $\int_0^{\infty} f(t)\;\delta(t-a)dt = f(a)$.

In particular, when $f(t) = e^{-st}$, we have $\mathcal{L}\{\delta(t-a)\} = e^{-as}$

| |
|---|
| Mean value Theorem , $\dfrac{f(b)-f(a)}{b-a}$ <br> $= f'(c)$ ( for some $c$, $a < c < b$ ) <br> Provided that $f$ is differentiable on $a < x < b$, <br> and continuous $a \leq x \leq b$ |

**Example 23**: Evaluate (i) $\int_0^\infty sin\,2t\ \delta(t-\pi/4)\ dt$ (ii) $\mathcal{L}\left\{\frac{1}{t}\delta(t-a)\right\}$

**Solution**: (i) We know that $\int_0^\infty f(t)\,\delta(t-a)dt = f(a)$

$$\therefore \int_0^\infty sin\,2t\ \delta(t-\pi/4)\ dt = \sin(2.\pi/4) = \sin(\pi/2) = 1$$

(ii) We know that $\mathcal{L}\{\delta(t-a)\} = e^{-as}$

$$\therefore\ \mathcal{L}\left\{\frac{1}{t}\delta(t-a)\right\} = \int_s^\infty \mathcal{L}\{\delta(t-a)\}\ ds = \int_s^\infty e^{-as}\ ds$$

$$= \left|\frac{e^{-as}}{-a}\right|_s^\infty = \frac{1}{a}e^{-as}$$

**Example 24**: An impulsive voltage $E\delta(t)$ is applied to a circuit consisting of $L, R, C$ in series with zero initial conditions. If $i$ be the current at any subsequent time $t$ find the limit of i as t → 0

**Solution**: The equation of the circuit governing the current $i$ is

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int_0^t i\ dt = E\delta(t) \text{ where } i = 0, \text{when } t = 0$$

Taking Laplace transform of both sides, we get

$$L\,[s\,\bar{\imath} - i(0)] + R\bar{\imath} + \frac{1}{C}\frac{1}{s}\bar{\imath} = E \quad \text{(Using transform of derivative and integrals)}$$

**Or** $\left(s^2 + \frac{R}{L}s + \frac{1}{CL}\right)\bar{\imath} = \frac{E}{L}s$ **Or** $(s^2 + 2as + a^2 + b^2)\bar{\imath} = \frac{E}{L}s$

where $\frac{R}{L} = 2a$ and $\frac{1}{CL} = a^2 + b^2$

**Or** $\bar{\imath} = \frac{E}{L}\frac{(s+a)-a}{(s+a)^2+b^2} = \frac{E}{L}\left\{\frac{(s+a)}{(s+a)^2+b^2} - a\frac{1}{(s+a)^2+b^2}\right\}$

On inversion, we get

$$\bar{\imath} = \frac{E}{L}\left\{e^{-at}\cos bt - \frac{a}{b}e^{-at}\sin bt\right\}$$

Taking limit as $t \to 0$, $\quad i \to E/L$

Although the current $i = 0$ initially,

yet a large current will develop instantaneously due to impulsive voltage applied at $t = 0$. In fact the limit of this current which is $E/L$

## 7.5 Exercise

1. Find the Inverse Laplace Transform of each of the following functions:

(i) $\dfrac{1}{(s-1)^5}$ $\left( \textbf{Ans}: \dfrac{1}{24} t^4 e^{-t} \right)$

(ii) $\dfrac{4s+15}{16s^2-25}$ $\left( \textbf{Ans}: \dfrac{1}{4} cosh \dfrac{5}{4} t + \dfrac{3}{4} sinh \dfrac{5}{4} t \right)$

(iii) $\dfrac{3(s^2-1)^2}{2s^5}$ $\left( \textbf{Ans}: \dfrac{3}{2} - \dfrac{3}{2} t^2 + \dfrac{1}{16} t^4 \right)$

(iv) $\dfrac{1}{s^{3/2}}$ $\left( \textbf{Ans}: \ 2\sqrt{\dfrac{t}{\pi}} \right)$

(v) $\dfrac{1}{\sqrt{2s+3}}$ $\left( \textbf{Ans}: \dfrac{1}{\sqrt{2\pi t}} \ e^{-3t/2} \right)$

2. Find the Inverse Laplace Transform of each of the following functions:

(i) $\dfrac{4s+12}{s^2+8s+16}$ $\left( \textbf{Ans}: \ 4e^{-4t} (1-t) \right)$

(ii) $\dfrac{3s+7}{s^2-2s-3}$ $\left( \textbf{Ans}: 4e^{3t} - e^{-t} \right)$

(iii) $\dfrac{s^2+1}{s^3+3s^2+2s}$ $\left( \textbf{Ans}: \dfrac{1}{2} - 2e^{-t} + \dfrac{5}{2} e^{-2t} \right)$

(iv) $\dfrac{s+29}{(s+4)(s^2+9)}$ $\left( \textbf{Ans}: e^{-4t} - cos\ 3t + \dfrac{5}{3} sin\ 3t \right)$

(v) $\dfrac{s+2}{s^3(s-1)^2}$ $\left( \textbf{Ans}: (3t-8)\ e^t + t^2 + 5t + 8 \right)$

(vi) $\dfrac{1}{s^3(s^2+1)}$ $\left( \textbf{Ans}: \dfrac{1}{2}\ t^2 + cos\ t - 1 \right)$

(vii) $\dfrac{s^2-a^2}{(s^2+a^2)^2}$ $\left( \textbf{Ans}: t\ cos\ at \right)$

(viii) $\dfrac{s}{(s^2+1)(s^2+4)}$ $\left( \textbf{Ans}: \dfrac{1}{3} (cos\ t - cos\ 2t) \right)$

(ix) $\dfrac{1}{(s^2+a^2)^2}$ $\left( \textbf{Ans}: \dfrac{1}{2a^3}\ (sin\ at - at\ cos\ at) \right)$

(x) $\dfrac{s}{s^4+4a^4}$ $\left( \textbf{Ans}: \dfrac{1}{2a^2}\ (sin\ at\ sinh\ at) \right)$

3. Use convolution theorem to obtain inverse Laplace *transform of each of the following*

(i) $\frac{a}{s(s-a)}$ $\quad$ ( **Ans** : $e^{at}-1$)

(ii) $\frac{1}{s(s^2+a^2)}$ $\quad$ $\left(\textbf{Ans}: \frac{1}{a^2}(1-\cos at)\right)$

(iii) $\frac{1}{s\sqrt{s+4}}$ $\quad$ $\left(\textbf{Ans}: \frac{1}{2}\operatorname{erf}(2\sqrt{t})\right)$

(iv) $\frac{1}{s}\log\frac{(s+3)}{(s+2)}$ $\quad$ $\left(\textbf{Ans}: \int_0^t \frac{e^{-2x}-e^{-3x}}{x}\,dx\right)$

4. Solve the following differential equations ( $t>0$) *with given initial values*

(i) $(D+1)^2 y = \sin t$ , *with* $y = \frac{dy}{dt} = 1$ *at* $t=0$

$\left(\textbf{Ans}: y = \frac{5}{2}t\,e^{-t} + \frac{3}{2}e^{-t} - \frac{1}{2}\cos t\right)$

(ii) $(D+1)^2 y = \sin t$ , *with* $y = \frac{dy}{dt} = 1$ *at* $t=0$

$\left(\textbf{Ans}: y = \frac{5}{2}t\,e^{-t} + \frac{3}{2}e^{-t} - \frac{1}{2}\cos t\right)$

(ii) $(D^2+4D+8)y = 1$ , *with* $y = 0, D_y = 1$ *at* $t = 0$

$\left(\textbf{Ans}: y = \frac{1}{8}(1 - e^{-2t}\cos 2t - 3e^{-2t}\sin 2t)\right)$

(iii) $(D+1)y = t^2 e^{-t}$, *given* $y = 3$ *when* $t = 0$ ,

$\left(\textbf{Ans}: y(t) = e^{-t}\left(\frac{t^3}{3}+3\right)\right)$

(iv) $\frac{d^2y}{dt^2} - \frac{dy}{dt} - 6y = 2$ *with* $y(0) = 1\ y'(0) = 0$ ,

$\left(\textbf{Ans}: y = -\frac{1}{3} + \frac{8}{15}e^{3t} + \frac{4}{5}e^{-2t}\right)$

(v) $\frac{d^2y}{dt^2} + y = \sin t$ *with* $y(0) = 1\ y'(0) = -\frac{1}{2}$ ,

$\left(\textbf{Ans}: y = \left(1-\frac{t}{2}\right)\cos t\right)$

5. *Solve the simultaneous equations*

(i) $\left.\begin{aligned}\frac{dx}{dt} &= 2x - 3y\\ \frac{dy}{dt} &= y - 2x\end{aligned}\right\}$ *subject to conditions* $x(0) = 8, y(0) = 3$ ,

( **Ans** : $x = 5e^{-t} + 3e^{4t}$ , $y = 5e^{-t} - 2e^{4t}$)

(ii) $\left.\begin{aligned} \frac{dx}{dt} - \frac{dy}{dt} + 2y &= \cos 2t \\ \frac{dx}{dt} + \frac{dy}{dt} - 2x &= \sin 2t \end{aligned}\right\}$ *subject to conditions* $x(0) = 0\,, y(0) = -1\,,$

$\left(\textbf{Ans}: x = \frac{1}{2}e^t(\cos t + \sin t) - \frac{1}{2}\cos 2t\ , y = -e^t(\cos t - \sin t) - \sin 2t\right)$

(ii) $\left.\begin{aligned} \frac{d^2x}{dt^2} + x + y &= 0 \\ 4\,\frac{d^2x}{dt^2} - x &= 0 \end{aligned}\right\}$ $x(0) = -2a\,, y(0) = a\,,\ x'(0) = 2b\,, y'(0) = -b$

$\left(\textbf{Ans}: x = 2a\cos\frac{t}{\sqrt{2}} + 2\sqrt{2}\,b\sin\frac{t}{\sqrt{2}}\right)$

## 7.6 Summary

In this unit we learn Inverse Laplace Transform definition, Shifting Theorem, Partial fraction Methods, Use of Convolution Theorem.

***Inverse*** : $\mathcal{L}^{-1}\{\bar{f}\,(s)\} = f(t)$

$\mathcal{L}^{-1}\left[\frac{1}{s}\right] = 1$ $\qquad$ $\mathcal{L}^{-1}\left[\frac{1}{s-a}\right] = e^{at}$

$\mathcal{L}^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{(n-1)!}, n = 1,2,3..$ $\qquad$ $\mathcal{L}^{-1}\left[\frac{1}{(s-a)^n}\right] = \frac{e^{at}\,t^{n-1}}{(n-1)!}$

$\mathcal{L}^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\ Sin\ at$ $\qquad$ $\mathcal{L}^{-1}\left[\frac{s}{s^2+a^2}\right] = cos\ at$

$\mathcal{L}^{-1}\left[\frac{1}{s^2-a^2}\right] = Sinh\ at$ $\qquad$ $\mathcal{L}^{-1}\left[\frac{s}{s^2-a^2}\right] = cosh\ at$

$\mathcal{L}^{-1}\left[\frac{1}{(s-a)^2+b^2}\right] = \frac{1}{b}\ e^{at} Sin\ bt$ $\qquad$ $\mathcal{L}^{-1}\left[\frac{s-a}{(s-a)^2+b^2}\right] = e^{at} Cos\ bt$

$\mathcal{L}^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{1}{2a} t\ Sin\ at$

$\mathcal{L}^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = \frac{1}{2a^3}(\,Sin\ at - at\ cos\ at)$

**Shifting Theorem**:

$(\boldsymbol{I})\ If\ \mathcal{L}^{-1}\{\bar{f}\,(s)\} = f(t), then\ \mathcal{L}^{-1}\{\bar{f}(s-a)\} = e^{at} f(t) = e^{at}\mathcal{L}^{-1}\{\bar{f}\,(s)\}$

$(\boldsymbol{II}) If\ \mathcal{L}^{-1}\{\bar{f}\,(s)\} = f(t) and\ f(0) = 0, then\ \mathcal{L}^{-1}\{s\bar{f}(s)\} = \frac{d}{dt}\{f(t)\}$
$= f'(t)$

$(\boldsymbol{III}) If\ \mathcal{L}^{-1}\{\bar{f}\,(s)\} = f(t)\ then\ \ \mathcal{L}^{-1}\left\{\frac{\bar{f}(s)}{s}\right\} = \int_0^t f(t)dt$

$(\boldsymbol{IV}) If\ \mathcal{L}^{-1}\{\bar{f}\,(s)\} = f(t)\ then\ \ tf(t) = \mathcal{L}^{-1}\left\{-\frac{d}{ds}[\bar{f}\,(s)]\right\}$

$(\boldsymbol{V})\mathcal{L}\left(\frac{f(t)}{t}\right) = \int_s^\infty \bar{f}\,(s)\,ds$

**Partial fraction Methods**

Generally in many problems $\bar{f}(s)$is a rational fraction $\frac{\bar{F}\,(s)}{\bar{G}(s)}$ with

degree of $\bar{F}\,(s)$less than that of $\bar{G}(s)$and this fraction can be expressed as sum on partial fractions of the type

$\frac{A}{(as+b)^r}, \frac{A}{(as^2+bs+b)^r}\ \ (r\ = 1,2, \dots)$

and finding the Laplace transform of each of the partial fractions , we find $\mathcal{L}^{-1}\{\bar{f}\,(s)\}$

$\mathcal{L}^{-1}\{s^n\bar{f}(s)\} == \frac{d^n}{dt^n}\{f(t)\}, provided\ f(0) = f'(0) = \cdots = f^{n-1}(0) = 0$

$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\bar{f}(s) = (-1)^n\,\bar{f}^{(n)}(s)\ which\ can\ be\ expressed\ as$

$$\boldsymbol{\mathcal{L}^{-1}\{f^{(n)}(s)\} = (-1)^n t^n f(t)}$$

**Use of Convolution Theorem**

**If** $\boldsymbol{\mathcal{L}^{-1}\bar{F}\,(s) = F(t), \mathcal{L}^{-1}\bar{G}\,(s) = G(t)\ and\ \bar{f}\,(s) = \bar{F}\,(s) * \bar{G}(s)\qquad then}$

$$\boldsymbol{\{\mathcal{L}^{-1}\bar{f}\,(s)\} = \ \mathcal{L}^{-1}\{\bar{F}\,(s) * \bar{G}(s)\} = \int_0^t F(t-u)\,G(u)\,du}$$

Corollary: Since $\boldsymbol{\mathcal{L}^{-1}\left(\frac{1}{s}\right) = 1\ and\ \mathcal{L}^{-1}\bar{f}(s) = f(t)}$

$$\boldsymbol{\mathcal{L}^{-1}\left\{\frac{f(s)}{s}\right\} = \int_0^t 1.f(u)du}$$

**Laplace Transformation of Special Function**

- **Periodic Functions**
- **Heaviside Unit Step Function**
- **Dirac − delta Function(Unit Impulse Function)**

**Periodic Functions**

The perodic function $f(t)$ of period $T$ is defined as

$$f(t+T)=f(t), T>0$$

$$\bar{f}(s) = \mathcal{L}\{f(t)\}$$

$$= \int_0^\infty e^{-st} f(t)dt$$

$$= \frac{1}{1-e^{-sT}} \int_0^T e^{-su} f(u)du \quad ------ \quad for\ period\ T$$

**Heaviside Unit Step Function**

The unit step function $u(t-a)$ is defined as follows:

$$u(t-a) = \begin{cases} 0 \text{ for } t < a \\ 1 \text{ for } t \geq a \end{cases}$$

where , $a$ is always positive . It is also denoted as $H(t-a)$.

**Dirac − delta Function(Unit Impulse Function)**
**Unit impulse ( Dirac Delta)**

function is considered as the limiting form of the function

$$\delta_\varepsilon(t-a) \begin{array}{ll} = 1/\varepsilon\ , & \boldsymbol{a} \leq \boldsymbol{t} \leq \boldsymbol{a}+\varepsilon \\ = 0, & otherwise \end{array}$$

as $\varepsilon \to 0$. It is clear from above figure that as $\varepsilon \to 0$ , the height of the strip increases indefinitely and the width decreases in such a way that its area is always unity.

## 7.7 References


1. A Text Book of Applied Mathematics Vol I - P. N. Wartikar and J. N. Wartikar
2. Applied Mathematics II - P. N. Wartikar and J. N. Wartikar
3. Higher Engineering Mathematics - Dr. B. S. Grewal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Unit 4

# 8

# MULTIPLE INTEGRALS

**Unit Structure**



## 8.0 Objectives

*After reading this chapter, you should be able to:*

1. *Understand double integrals & notations.*
2. *Solve problems based on double integrals.*
3. *Understand double integral in polar co-ordinates,*
4. *Know the concept of triple integrals,*

## 8.1 Double Integral: Introduction and Notation

It is presumed that the students are familiar with " the limit of a sum as an integer."

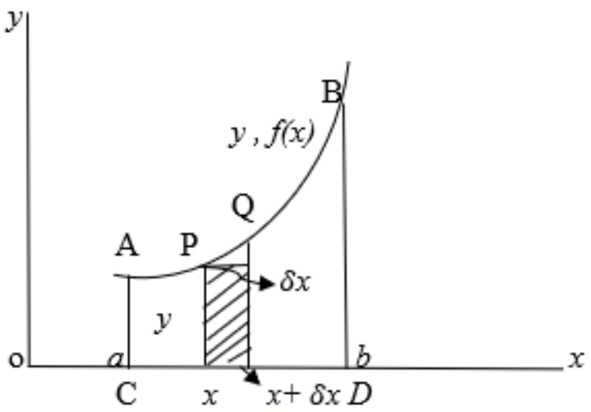


**Fig 1**

$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} y\delta x \qquad \text{and this is expressed as} \qquad \int_a^b y\, dx$$

Thus

$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} y\delta x = \int_a^b y\, dx$$

Let us now consider the integration of a function of two variables over a given area.

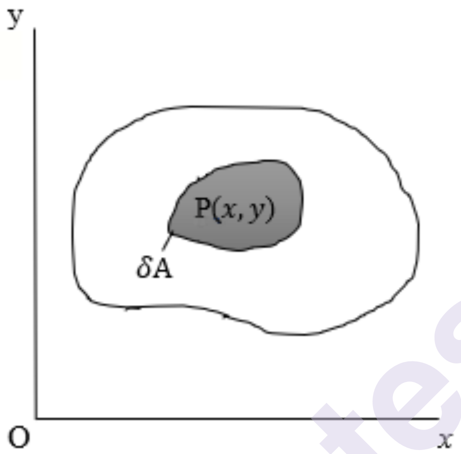


**Fig 2**

To make the idea clear, we shall consider a plane lamina in the xOy plane, the surface density $\sigma$ of which is a function of the position of the point $P(x, y)$. Thus surface density $\sigma = f(x, y)$.

To find the mass of the lamina, we shall take a small area $\delta A$ about the point $P(x, y)$.

The mass of this elementary area is $f(x, y)\ \delta A$. To find the total mass of the lamina, we shall find out expressions such as $(x, y)\ \delta A$, all over the lamina, form the sum $\sum f(x, y)\ \delta A$, and to be more accurate, $\delta A$ must be taken a small as possible.

That is

The mass of the lamina = $\lim_{\delta A \to 0} \sum f(x, y)\ \delta A$ … … (8.1)

where summation extends all over the lamina.

Let us take $\delta A$ in a more convenient way so that the summation in (8.1) can be carried out.

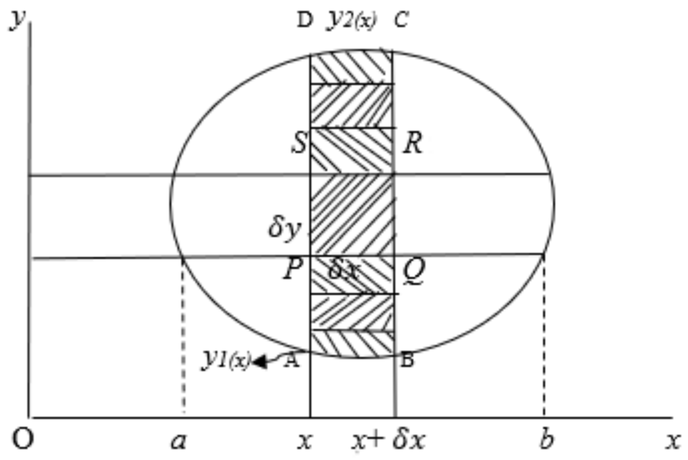


**Fig. 3**

Divide the lamina by a system of straight lines parallel to the *x* and *y* axis into a mesh of elementary rectangles. Take the rectangle with one corner at P(*x*,*y*).

Then the area of rectangle PQRS $\delta$A= $\delta x\,.\,\delta y$

And the mass of the elementary rectangle = $f(x,y)\ \delta x \delta y$.

By (8.1) the mass of the lamina M is

$$\mathrm{M} = \lim_{\substack{\delta x \to 0 \\ \delta y \to 0}} \sum\sum f(x,y)\ \delta x \delta y \qquad \ldots \qquad \ldots \qquad (8.2)$$

We shall evaluate the expression on the R.H.S. of the (8.2) in a systematic way.

Taking the sum of $f(x,y)\ \delta x \delta y$ over the strip ABCD, we have for the mass of the elementary strip ABCD

$$= \lim_{\delta y \to 0} \sum_{\mathrm{A}}^{\mathrm{D}} f(x,y)\ \delta x \delta y \qquad \ldots \qquad \ldots \qquad (8.3)$$

Where in this summation we note that *x* and $\delta x$ are constants. We can therefore write (8.3) as

$$= \delta x \lim_{\delta y \to 0} \sum_{y_{\mathrm{A}}}^{y_{\mathrm{D}}} f(x,y)\ \delta y \qquad \ldots \qquad \ldots \qquad (8.4)$$

And by introductory remarks on the limit of the sum as an integral we write (8.4) as

$$= \delta x \int_{y_1(x)}^{y_2(x)} f(x,y)\ dy \qquad \ldots \qquad \ldots \qquad (8.5)$$

Where $y_1(x)$ and $y_2(x)$ are the values of $y$ at A and D and both depend on the position of the ordinate, that is on $x$.

It is to be remembered in the integral of (8.5) that $x$ is to be regarded as a constant in the integration w.r.t. $y$ and since the limits of the integral are the functions of $x$,

So $\int_{y_1(x)}^{y_2(x)} f(x,y)\, dy$ will be some function of $x$, say $\emptyset(x)$. We thus say that let

$$\int_{y_1(x)}^{y_2(x)} f(x,y)\, dy = \emptyset(x) \quad \ldots \quad \ldots \quad (8.6)$$

So that from (8.5) , we can write the mass of the elementary strip ABCD as $[\emptyset(x).\, \delta x\,]$

Next taking the mass of each strip such as ABCD parallel to the $y$ - axis ,over the area of the lamina, we have

$$\text{Mass of the lamina} = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} \emptyset(x)\, \delta x.$$

$$= \int_{x=a}^{x=b} \emptyset(x) dx. \quad \ldots \quad (8.7)$$

Substituting for $\emptyset(x)$ from (8.6) in (8.7), we get

$$\text{Mass of the lamina} = \int_{x=a}^{x=b} \left\{ \int_{y_1(x)}^{y_2(x)} f(x,y)\, dy \right\} dx \quad \ldots \quad (8.8)$$

The expression on the R.H.S. of the equation (8.8) is called a double integral for obvious reason and is written in various ways as follows

$$\int_a^b \int_{y_1(x)}^{y_2(x)} f(x,y)\, dy\, dx. \quad \ldots \quad (8.9a)$$

or

$$\int_a^b \int_{y_1(x)}^{y_2(x)} f(x,y)\, dx\, dy. \quad \ldots \quad (8.9b)$$

Where the integral signs are written in order of integration taken from the right,

$$\text{or} \qquad \int_a^b dx \int_{y_1(x)}^{y_2(x)} f(x,y)\ dy. \qquad \ldots \qquad (8.9c)$$

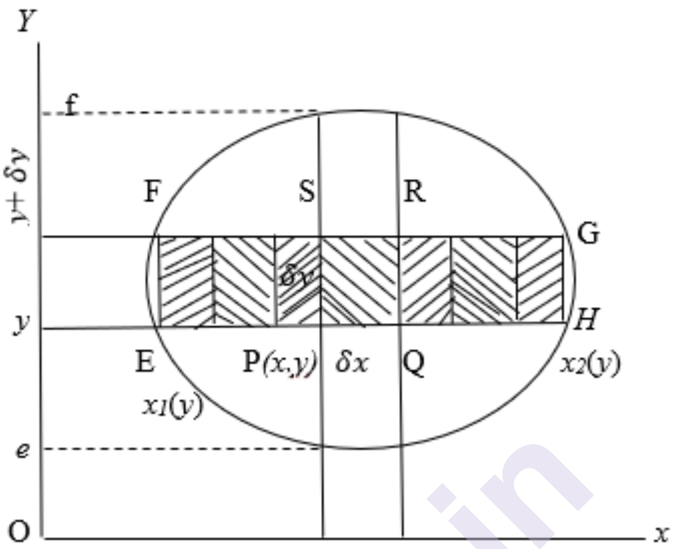


**Fig. 4**

This last way of writing the integral is more convenient, as it expresses clearly the order in which the integration is performed i.e. we first integrate w.r.t. $y$ considering $x$ as a constant and then we integrate w.r.t. $x$. It may also be noted that when we take the elementary strips parallel to the $y$-axis, we first integrate w.r.t. $y$.

If instead of taking the elementary strip parallel to the $y$-axis we take it parallel to the $x$-axis such as EFGH shown in the adjacent figure, we have by a similar reasoning to the above

$$\text{Mass of the lamina} = \int_e^f dy \int_{x_1(y)}^{x_2(y)} f(x,y)\, dx \qquad \ldots \qquad (8.10)$$

In which we have to first integrate w.r.t. $x$ and then w.r.t. $y$, thus changing the order of the integration. Both the integrals (8.9) and (8.10) represent the mass of the lamina and so are equal. The total area of the lamina is known as the region of integration.

The function $f(x,y)$ was considered as the surface density of the lamina, just for the sake of understanding clearly the idea of double integral. However $f(x,y)$ may be any function of the position of a point in the loop-area, and the double integral of this function over the area of the loop is given by (8.9) or (8.10) that is

$$\int_a^b dx \int_{y_1(x)}^{y_2(x)} f(x,y)\, dy \;\; \textbf{or} \int_e^f dy \int_{x_1(y)}^{x_2(y)} f(x,y)\, dx \qquad \ldots (8.11)$$

## 8.2 Change of the order of integration; Evaluation of Double integrals

The method of evaluating the double integrals (8.11) is actually clear from the theory developed in the previous section. We note that in the evaluation of the double integrals, we integrate first w.r.t. one variable ($y$ or $x$ depending upon the limits, and the elementary strip) and considering the other variable as constant and then integrate with respect to the remaining variable.

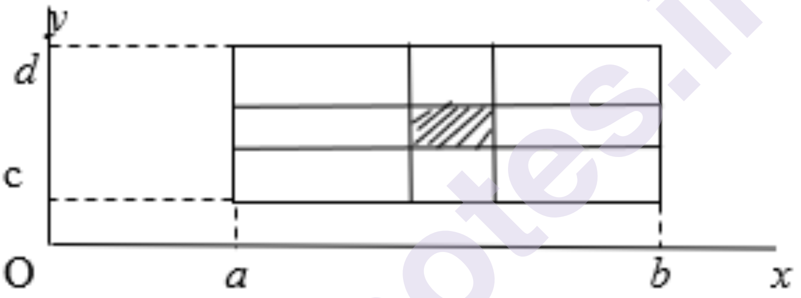


**Fig. 5**

If the limits of integration are the constants such as in the region of integration being a rectangle, then the change in the order of integration does not require change of the limits of integration.

Thus from the adjacent figure, we see that

$$\int_a^b dx \int_c^d f(x,y)\, dy = \int_c^d dy \int_a^b f(x,y)\, dx \qquad \ldots \qquad (8.12)$$

But if the limits be the variable as in the general case taken in section 8.1 then in changing the order of integration a corresponding change is to be made in the limits of integration as seen from (8.11). Sometimes in changing the order of integration we are required to split up the region of integration and the new integral is expressed as a sum of a number of double integrals. The examples solved below make this ideas clear. The change of the order of integration is sometimes convenient in the evaluation of the double integrals. This is also illustrated in problems solved below. In changing the order of integration, it is convenient to draw rough sketch of the region of integration, which will help to fix up the new limits of integration.

**Example 1.** Evaluate $\int (x^2 - y^2)\, dA$ over the area of the triangle whose vertices are the points (0,1),(1,1) and (1,2).

The equations of the sides of the triangle whose vertices are at A(0,1), B(1,1), C(1,2) are $x = 1$, $y = 1$ and $x = y - 1$ … (i) as shown in the figure 6.

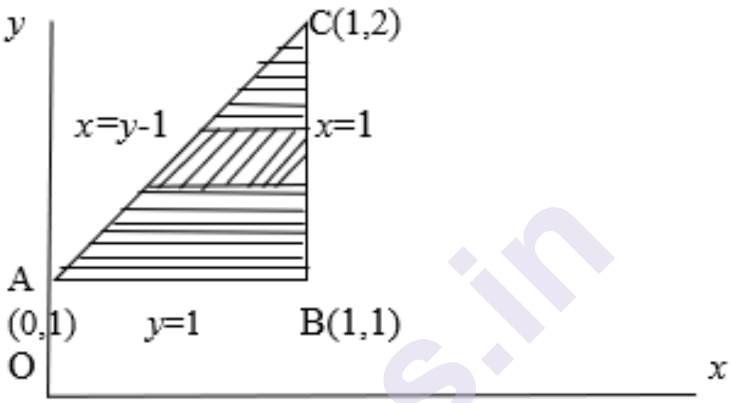


**Fig. 6**



If we take an elementary strip parallel to the $x$-axis, we will be integrating the given function with respect to $x$. The ends of this strip are bounded by the lines $x = y - 1$ and $x = 1$, so that these are the limits of integration with respect to $x$. Next we integrate w.r.t. $y$ from y = 1 to y = 2, which then covers the whole area of the triangle ABC.

Thus if $I = \int (x^2 - y^2)\, dA$ taken over the area of the triangle ABC

Then,

$$I = \int_1^2 dy \int_{y-1}^{1} (x^2 - y^2)\, dx \qquad .. \qquad .. \qquad \text{(ii)}$$

To evaluate the first integral, we regard $y$ as a constant,

$$I = \int_1^2 dy \left[\frac{x^3}{3} - y^2 x\right]_{y-1}^{1}$$

$$= \int_1^2 \left\{\frac{1}{3} - y^2 - \frac{(y-1)^3}{3} + y^2(y-1)\right\} dy$$

$$= \int_1^2 \left\{ \frac{1}{3} - 2y^2 - \frac{(y-1)^3}{3} + y^3 \right\} dy$$

$$= \left[ \frac{y}{3} - \frac{2y^3}{3} - \frac{(y-1)^4}{12} + \frac{y^4}{4} \right]_1^2$$

$$= \left[ \frac{2}{3} - \frac{16}{3} - \frac{1}{12} + 4 - \frac{1}{3} + \frac{2}{3} - \frac{1}{4} \right]$$

$$= -\frac{2}{3}$$

It will be interesting to try the above example by taking strips parallel to the y-axis, which is left to the students as an exercise leading to the same result as above.

**Example 2. Evaluate**

$$\int_0^a dy \int_0^{a-\sqrt{a^2-y^2}} \frac{xy \log(x+a)}{(x-a)^2} \, dx$$

In the integral as it stands, the integration is first w.r.t. $x$ and this integration, as is clear is complicated. As integration w.r.t. $y$ is simple, we therefore change the order of integration, for which sake we find out the region of integration for the given problem.

In the given Interval where the integration is first w.r.t. $x$, the elementary strips are parallel to the $x$-axis and these strips extend from $x = 0$ (i.e. the $y$ – axis) to $x = a - \sqrt{a^2 - y^2}$ i.e. to the boundary of the circle $(x-a)^2 + y^2 = a^2$. Moreover as $x = a$ minus $\sqrt{a^2 - y^2}$, it extends upto the side (i) of the circle and not upto (ii) for which $x = a$ plus $\sqrt{a^2 - y^2}$.

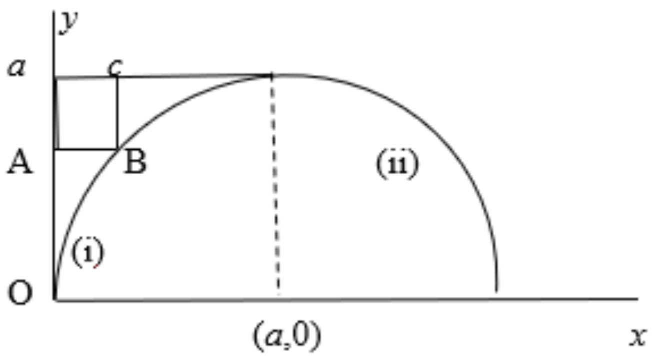


**Fig. 7**

An elementary strip such as this is shown in the figure 7 by AB. Next we integrate w.r.t. $y$ from $y = 0$ to $y = a$ and so the strips such as AB, bounded on one side by the $y$-axis and on the other by the circumference of the circle are taken from $y = 0$ to $y = a$. Thus the region of integration is the shaded part in the figure.

If we change the order of integration, integrating first w.r.t. y then the elementary strip is parallel to the y-axis, such as BC in the figure which extends from circumference of the circle $(x - a)^2 + y^2 = a^2$ i.e. $y = \sqrt{2ax - x^2}$ to the line $y = a$. These are therefore the limits of integration w.r.t. y. To have same region of integration as in the given integral. We must take such strips from $x = 0$ to $x = a$, which are the limits of integration w.r.t. $x$. Thus changing the order of integration, the given integral say I, can be written as

$$\mathrm{I} = \int_0^a dx \int_{\sqrt{2ax-x^2}}^{a} \frac{xy\log(x+a)}{(x-a)^2}\,dy$$

Integrating w.r.t. $y$ considering $x$ as constant, we have

$$\mathrm{I} = \int_0^a dx\,\frac{x\log(x+a)}{(x-a)^2}\left[\frac{y^2}{2}\right]_{\sqrt{2ax-x^2}}^{a}$$

$$= \frac{1}{2}\int_0^a dx\,\frac{x\log(x+a)}{(x-a)^2}\,[a^2 - 2ax + x^2]$$

$$= \frac{1}{2}\int_0^a x\log(x+a)\,dx$$

This can be integrated by parts, with log $(x + a)$ as a part to be differentiated which gives

$$\mathrm{I} = \frac{a^2}{\theta}[2\log a + 1]$$

**Example 3.** Change the order of Integration in

$$\int_0^{2a}\int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} f(x,y)\,dxdy$$

The order of integration in the given integral is first w.r.t. $y$ and then w.r.t. $x$

The elementary strips here are parallel to the $y$-axis (such as A B) and extend from $y = \sqrt{2ax - x^2}$, [ i.e. the circle $x^2 + y^2 - 2ax = 0$. with centre at $(a,0)$ and radius $a$] to $y = \sqrt{2ax}$ [i.e. the parabola $y^2 = 2ax$] and such strips are taken from $x = 0$ to

$x = 2a$. The shaded area between the parabola and the circle is therefore the region of integration.

In changing the order of integration, we integrate first w.r.t. $x$, with elementary strips parallel to the $x$-axis, such as CD. In covering the same region as above, the ends of these strips extend to different curves. We therefore divide the region by the line $y = a$ into three parts (I),(II),(III) as shown in the figure.

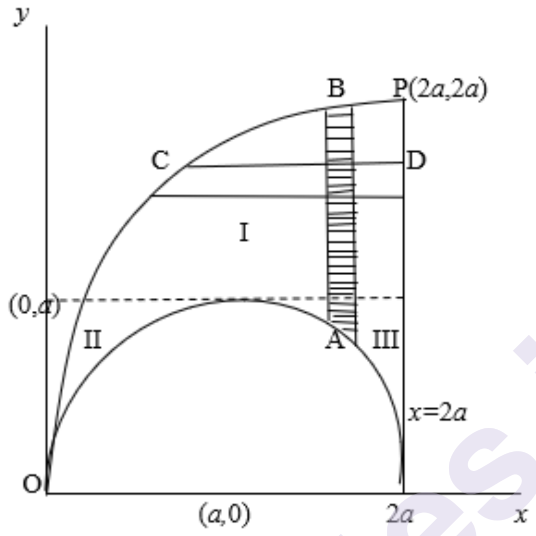


**Fig. 8**

For the region (I), the strip extend from the parabola $y^2 = 2ax$ i.e. $x = \frac{y^2}{2a}$ to the straight line

$x = 2a$, so these are the limits of integration w.r.t. $x$. Such strips are to be taken from $y = a$ to $y = 2a$, to cover the region (I) completely. So the part of the integral in this region $I_1$ is

$$I_1 = \int_a^{2a} dy \int_{\frac{y^2}{2a}}^{2a} f(x,y)\,dx \qquad .. \qquad .. \qquad \text{(i)}$$

From the region (II), the strips extend from the parabola $y^2 = 2ax$ i.e. $x = \frac{y^2}{2a}$ to the circle

$x^2 + y^2 - 2ax = 0$ i.e. $x = a \pm\sqrt{a^2 - y^2}$ in which we take the negative sign with the radical as is obvious from the figure, so the limits of integration w.r.t. x are $x = \frac{y^2}{2a}$ to $x = a - \sqrt{a^2 - y^2}$ and such strips are taken from $y = 0$ to $y = a$, to cover this region completely. The contribution to the integral from this region $I_2$ is therefore

$$\mathrm{I}_2 = \int_0^a dy \int_{\frac{y^2}{2a}}^{a-\sqrt{a^2-y^2}} f(x,y)\,dx \qquad .. \qquad .. \qquad \text{(ii)}$$

For the region (III), the strips extend from the circle $x^2 + y^2 - 2ax = 0$ [i.e. $x = a \pm \sqrt{a^2 - y^2}$; in this we have to take the positive sign with the radical as is clear from the figure] to the line $x=2a$, so that the limits of integration w.r.t $x$ are $x = a + \sqrt{a^2 - y^2}$ to $x=2a$; and such strips are to be taken from $y = 0$ to $y = a$, which covers in the integration the region (III) Denoting this part of integral by $\mathrm{I}_3$, we have

$$\mathrm{I}_3 = \int_0^a dy \int_{a+\sqrt{a^2-y^2}}^{2a} f(x,y)\,dx \qquad .. \qquad .. \qquad \text{(iii)}$$

Thus if we change the order of integration, we have to divide the region of integration, and the given integral is equal to $\mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3$ or from (i), (ii), (iii)

$$\int_0^{2a} dx \int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} f(x,y)\,dy = \int_a^{2a} dy \int_{\frac{y^2}{2a}}^{2a} f(x,y)\,dx + \int_0^a dy \int_{\frac{y^2}{2a}}^{a-\sqrt{a^2-y^2}} f(x,y)\,dx$$

$$+ \int_0^a dy \int_{a+\sqrt{a^2-y^2}}^{2a} f(x,y)\,dx$$

This example illustrates that in changing the order of integration sometimes not only limits are to be changed, but it is necessary to split up the region of integration.

**Example 4.** Change the order of integration for the integral

$$\int_0^a \int_{\frac{x^2}{a}}^{2a-x} xy\,dx\,dy$$

and evaluate the same with reversed order of integration.

The given integral is

$$\int_0^a dx \int_{\frac{x^2}{a}}^{2a-x} y\,dy \qquad .. \qquad .. \qquad (i)$$

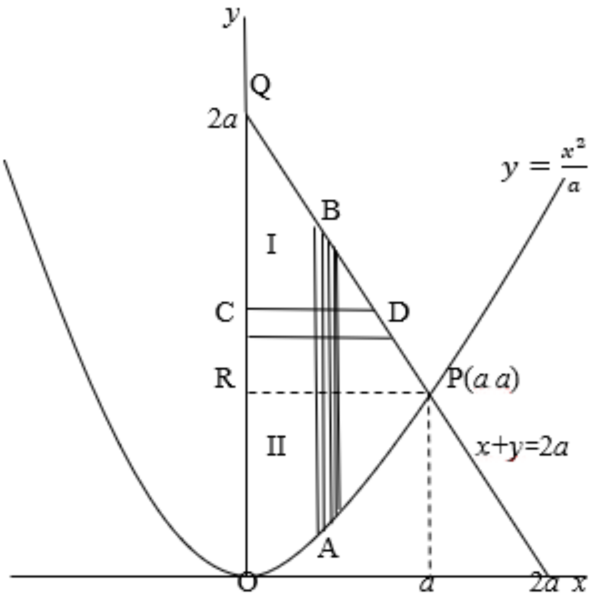


**Fig. 9**

In this the integration is first w.r.t. $y$ with strips such as AB, parallel to the y-axis with extremities lying on the parabola $y = \frac{x^2}{a}$ and the straight line $y = 2a - x$. These strips are taken from $x = 0$ to $x = a$, that gives the region of integration, the curvilinear triangle OPQ, shaded in the figure 9.

In changing the order of integration, the integration is to be taken first w.r.t. $x$ with elementary strip parallel to $x$ axis, such as CD, and that needs dividing the region of integration by the line $y = a$, i.e. the line PR, into two parts the triangle PQR and the curvilinear triangle OPR denoted in the figure by (I) and (II) respectively.

For the region (I), the limits of integration w.r.t. $x$ are $x = 0$ to $x = 2a - y$ and the limits of the next integration w.r.t. $y$ are $y = a$ to $y = 2a$, so the contribution to the given integral from region (I) is

$$I_1 = \int_a^{2a} dy \int_0^{2a-y} xy\, dx \qquad .. \qquad .. \qquad \text{(ii)}$$

For the region (II), the limits of integration w.r.t. $x$ are $x = 0$ to $x = \sqrt{ay}$ and those w.r.t. $y$ are $y = 0$ to $y = a$, so the contribution to the given integral from the region (II) is

$$I_2 = \int_0^{a} dy \int_0^{\sqrt{ay}} xy\, dx \qquad .. \qquad .. \qquad \text{(iii)}$$

Hence, reversing the order of integration, from (i), (ii) and (iii),

$$\int_0^a dx \int_{\frac{x^2}{a}}^{2a-x} xy\, dy = \int_a^{2a} dy \int_0^{2a-y} xy\, dx$$

$$+ \int_0^a dy \int_0^{\sqrt{ay}} xy\, dx \qquad .. \qquad .. \qquad \text{(iv)}$$

Now, with usual method of evaluating the double integral

$$\int_a^{2a} dy \int_0^{2a-y} xy\, dx = \int_a^{2a} dy.y \left[\frac{x^2}{2}\right]_0^{2a-y} = \frac{1}{2}\int_0^{2a} y(2a-y)^2 dy$$

$$= \frac{5}{24}a^4 \qquad .. \qquad .. \qquad \text{(v)}$$

and

$$\int_0^a dy \int_0^{\sqrt{ay}} xy\, dx = \int_0^a dy.y \left[\frac{x^2}{2}\right]_0^{\sqrt{ay}} = \frac{1}{2}\int_0^a ay^2 dy$$

$$= \frac{1}{6}a^4 \qquad .. \qquad .. \qquad \text{(vi)}$$

From (iv), (v) and (vi),

$$\int_0^a dx \int_{\frac{x^2}{a}}^{2a-x} xy\, dx = \frac{5}{24}a^4 + \frac{1}{6}a^4 = \frac{3}{8}a^4$$

## 8.3 Double integral in polar co-ordinates

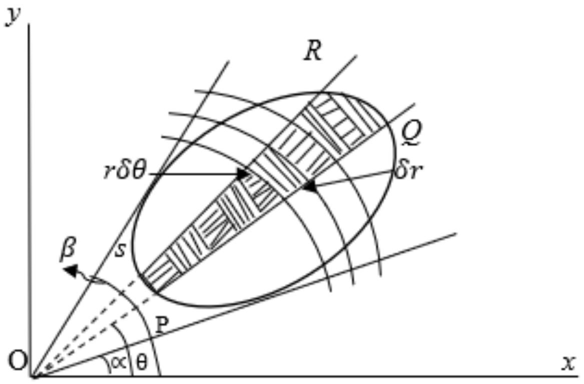


**Fig 10**

In case we use polar co-ordinates, divide the region of integration by curves $r$ = const. (which are circles) and η = const. (which are straight- lines)

This gives a mesh of the form shown, where the elementary area is $\delta r . r\ \delta\theta$

Thus if $f(r, \theta)$ be a function of position, we have over the wedge PQ, the sum as

$$\lim_{\delta r \to 0} \delta\theta \sum_{P}^{Q} f\,(r, \theta). r\ \delta r$$

$$= \delta\theta \int_{r_1(\theta)}^{r_2(\theta)} f\,(r, \theta)\, r dr \qquad \ldots \qquad \ldots \qquad (8.13)$$

Where $r_1(\theta)$ and $r_2(\theta)$ are equations of the two parts of curves where θ is kept constant, while integrating w.r.t. $r$. Finally summing for all the wedges between $\theta = \propto$ and $\theta = \beta$, we get

$$\operatorname{Lim}_{\delta\theta \to 0} \sum_{\alpha}^{\beta} \delta\theta \sum_{r_1(\theta)}^{r_2(\theta)} f\,(r, \theta)\, r dr$$

$$= \int_{\alpha}^{\beta} d\theta \int_{r_1(\theta)}^{r_2(\theta)} f\,(r, \theta)\, r dr \qquad \ldots \qquad \ldots \qquad (8.14)$$

The order of integration may be changed with appropriate changes in the limits.

**Example 1.** Evaluate

$$\iint \frac{r dr d\theta}{\sqrt{a^2 + r^2}}$$

over one loop of the lemniscate $r^2 = a^2 \cos 2\theta$

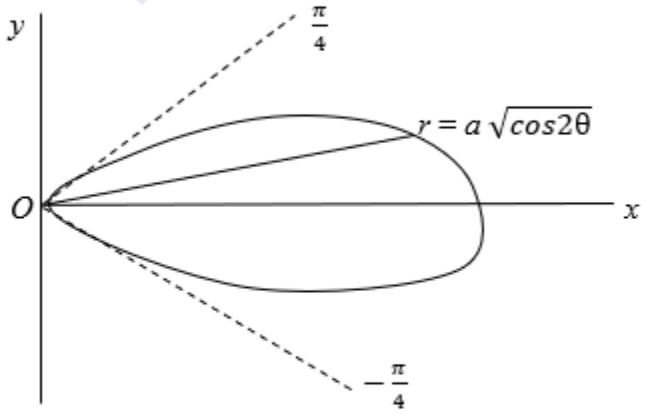


**Fig. 11**

$$I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\theta \int_{0}^{\sqrt{cos2\theta}} \frac{r\,dr}{\sqrt{a^2 + r^2}}$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\theta \left[\sqrt{a^2 + r^2}\right]_0^{a\sqrt{cos2\theta}}$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\theta \left\{a\sqrt{1 + cos2\theta}| - a\right\}$$

$$= a \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} [\sqrt{2}\,|\cos\theta - 1] d\theta$$

$$= a\left[\sqrt{2}\,|\,\mathrm{Sin}\,\theta - \theta\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= a\left[2 - \frac{\pi}{2}\right] = 2a\left[1 - \frac{\pi}{4}\right]$$

**Example 2** . Evaluate

$$\int_0^a dx \int_{\sqrt{ax - x^2}}^{\sqrt{a^2 - x^2}} \frac{dy}{\sqrt{a^2 - x^2 - y^2}},$$

by changing to polar coordinates.

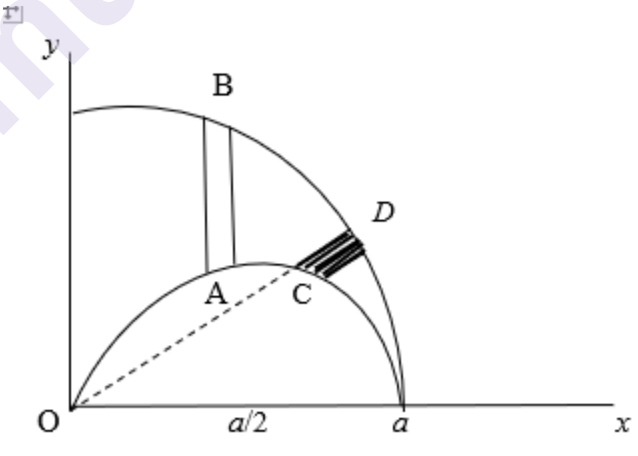


**Fig. 12**

Here the elementary strips, such as AB are parallel to the y axis and extend from

$$y = \sqrt{ax - x^2}$$

[which is the circle $x^2 + y^2 - ax = 0$,

with centre at $\left(\frac{a}{2},0\right)$ and radius $\frac{a}{2}$ ] to

y = $\sqrt{a^2 - x^2}$ [ i.e. the circle $x^2 + y^2 = a^2$ , with center at the origin and radius $a$.] such strips are taken from $x = 0$ to $x = a$, and so the area between the two circles, is the region of integration.

To change the given integral to polar coordinates, we substitute $x = r$ cosθ, $y = r$ sinθ, and $dxdy$ by its equivalent elementary area in polar coordinates $rdrd$θ. The equations of the circle in polar coordinates are $r = a$ cosθ and $r = a$ and the ends of the elementary wedge, such as CD along the radius vector lies on these circles and so give the limits of integration w.r.t. $r$ and to cover the same region of integration as in given integral. 0 varies from 0 to $\frac{\pi}{2}$.

Thus the transformed integral I is

$$I = \int_0^{\frac{\pi}{2}} d\theta \int_{a\cos\theta}^{a} \frac{r}{\sqrt{a^2 - r^2}}\, dr$$

$$= \int_0^{\frac{\pi}{2}} d\theta \left[-\sqrt{a^2 - r^2}\right]_{a\cos\theta}^{a}$$

$$= \int_0^{\frac{\pi}{2}} a\, sin\theta\, d\theta = a$$

## 8.4 Triple integrals :-

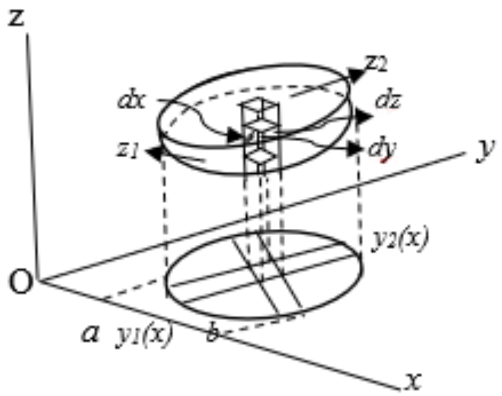


**Fig. 13**

Let $f\ (x, y, z)$ be any function of the position of a point $(x, y, z)$ in space [say the density of the body]. Divide the body by a system of planes into small rectangular blocks. The element of volume at P $(x, y, z)$ is then d$xdydz$.

The mass of the elementary cuboid at P $= f\ (x, y, z).\ dxdydz$

Then

$$\lim_{dz \to 0} \sum_{z_1}^{z_2} f\ (x, y, z)\ dxdydz$$

$$= dxdy \int_{z_1(x,y)}^{z_2(x.y)} f\ (x, y, z)\ dz \quad \ldots \quad \ldots \quad (8.15)$$

where $z_1(x, y)$ and $z_2(x. y)$ are the equations of the lower and upper surfaces of the bounding volume. The result (8.15) gives the mass of the elementary column on $dx dy$ in the $x$O$y$ plane as the base. In the integral (8.15), $x$, $y$ are constants.

We now have to sum for all the columns standing on the area in the $x$O$y$ plane vertically below the surface. Taking first all the columns in a slice parallel to the $y$-$z$ plane which means integration w.r.t. $y$ while keeping $x$ constant, we get

$$\left[ \int_{y_1(x)}^{y_2(x)} \left\{ \int_{z_1(x,y)}^{z_2(x.y)} f\ (x, y, z)\ dz \right\} dy \right] dx \quad \ldots \quad \ldots \quad (8.16)$$

and finally summing for all the slices from $x = a$ to $x = b$, we have

$$\boxed{\int_v f\ (x, y, z)\ dv = \int_a^b dx \int_{y_1(x)}^{y_2(x)} dy \int_{z_1(x,y)}^{z_2(x.y)} f\ (x, y, z)\ dz}$$

$$\ldots \quad (8.17)$$

The evaluation of a space or volume integral involves three successive integration and so is called a triple integral. The order of integration may be changed with appropriate changes in the limits.

In polar co-ordinates the volume of an elementary cuboid

$$dv = r^2 \sin\theta\ dr\ d\theta\ d\emptyset$$

and the integral (8.17) takes the form

$$\iiint f(r, \theta, \emptyset) r^{2} \sin\theta\ dr\ d\theta\ d\emptyset$$

And in cylindrical co-ordinates , the elementary volume is

$$dv = \rho \, d\rho \, d\emptyset \, dz$$

and the integral (8.17) takes the from

$$\iiint f(\rho, \emptyset, z)\rho \, d\rho \, d\emptyset \, dz$$

with appropriate limits.

**Example 1**. Show that the volume bounded by the cylinder $y^2 = z$ , $y = x^2$ And the planes $z = 0$ , $x + y + z = 2$ is equal to

$$\int_0^1 \int_{x^2}^{\sqrt{x}} \int_0^{2-x-y} dx \, dy \, dz$$

and evaluate it.

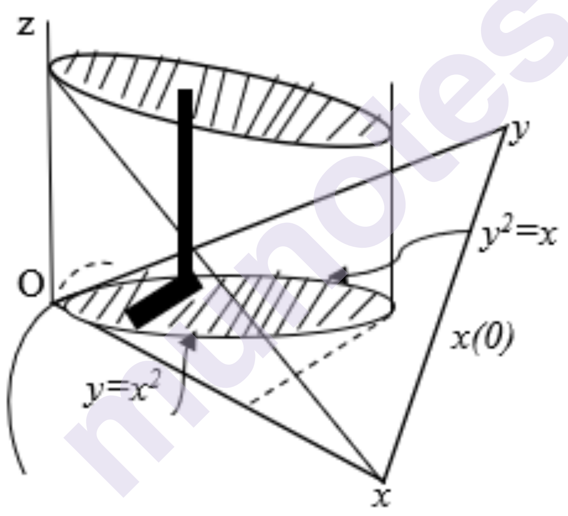


**Fig. 14**

The cylinder stands on the area common to the parabolas with generators parallel to the z-axis, and the volume required is the portion of this cylinder cut-off by the planes $z = 0$ and $x+y+z=2$ i.e. $z = 2 - x - y$

Integrating first w.r.t. $z$ we obtain the volume of the elementary column, on $dx \, dy$ as the base, where limits for $z$ are $z = 0$ to $z = 2- x- y$.

Thus the volume of elementary column on the *dxdy* as the base is

$$dxdy \int_{0}^{2-x-y} dz \qquad .. \qquad .. \qquad \text{(i)}$$

Taking a slice parallel to the yOx plane , of all such columns, leads on the integration w.r.t. y from $y = x^2$ to $y = \sqrt{x}$ ( ref. fig.14), we thus have the volume of an elementary slice parallel to the yOz plane as

$$\text{dx} \int_{x^2}^{\sqrt{x}} dy \int_{0}^{2-x-y} dz \qquad ... \qquad ... \qquad \text{(ii)}$$

Summing the volumes of such slices, bounded by the curves $y = x^2$, $y = \sqrt{x}$ , from $x = 0$ to $x = 1$, gives the total volume of the cylinder in question and is

$$\int_{0}^{1} dx \int_{x^2}^{\sqrt{x}} dy \int_{0}^{2-x-y} dz \qquad ... \qquad ... \qquad \text{(iii)}$$

which is the same as the given integral. To evaluate it we use the same principles as used in the evaluation of a double integral. Thus

$$\int_{0}^{1} dx \int_{x^2}^{\sqrt{x}} dy \int_{0}^{2-x-y} dz = \int_{0}^{1} dx \int_{x^2}^{\sqrt{x}} dy \ [z]_0^{2-x-y}$$

$$= \int_{0}^{1} dx \int_{x^2}^{\sqrt{x}} (2 - x - y) dy$$

$$= \int_{0}^{1} dx \left[ (2 - x)y - \frac{y^2}{2} \right]_{x^2}^{\sqrt{x}}$$

$$= \int_{0}^{1} \left\{ (2 - x)\sqrt{x} - \frac{x}{2} - (2 - x)x^2 + \frac{x^4}{2} \right\} dx$$

$$= \left[ \frac{4x^{\frac{3}{2}}}{3} - \frac{2x^{\frac{5}{2}}}{5} - \frac{x^2}{4} - \frac{2x^3}{3} + \frac{x^4}{4} + \frac{x^5}{10} \right]_0^1 = \frac{11}{30}$$

## 8.5 Summary

The eight chapter of this book introduces the students with concepts of double integral, evaluation of double integrals: change of the order of the integration and double integral in polar co-ordinates with notations, which is important in understanding, implementation in application areas of integrals. Triple integrals is also explained with solved problems and illustrations.

## 8.6 Exercises

**Evaluate the following Integrals**

1. $\iint y\,dxdy$ over

   i) the area bounded by $y = x^2$ and $x + y = 2$

   ii) the area bounded by $x = 0$, $y = x^2$ and $x + y = 2$ in the first quadrant.

2. $\iint xy(x+y)\,dxdy$ over the area bounded by the parabola $x^2 = y \text{ and } y^2 = -x$.

3. i) $\iint (x^2 + y^2)dxdy$

   ii) $\iint x^2\,y\,dxdy$ over the area in the positive quadrant of ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

4. $\iint \frac{dxdy}{x^4 + y^2}$ where $x > 1$ and $y > x^2$

**Change the order of integrals and evaluate**

5. $\int_0^2 \int_0^{\frac{x^2}{4}} xydxdy$

6. $\int_0^1 \int_y^{\sqrt{y}} xydxdy$

7. $\int_0^a \int_{\frac{x}{a}}^{\sqrt{\frac{x}{a}}} (x^2 + y^2)dxdy$

8. $\int_0^1 dx \int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2 + y^2}} - dy$

**Answers**

1. i) $\frac{36}{5}$ ii) $\frac{16}{5}$ 2. $\frac{114}{420}$ 3. i) $\frac{\pi ab}{16}(a^2b^2)$ ii) $\frac{a^4b^2}{24}$ 4. $\frac{\pi}{4}$

5. $\frac{1}{3}$ 6. $\frac{1}{24}$ 7. $\frac{a}{4}\left[\frac{a^2}{7} + \frac{1}{5}\right]$ 8. $\left[1 - \frac{1}{\sqrt{2}}\right]$

**Show the region of integration and change the order of integration**

9. $\int_{-a}^{a} \int_0^{\frac{y^2}{a}} f(x,y)dydx$

10. $\int_{-2}^{1} \int_{x^2}^{2-x} f(x,y)dxdy$

11. $\int_0^a dy \int_{\sqrt{a^2-y^2}}^{y+a} f(x,y)dx$

12. $\int_a^b dx \int_0^{\frac{c^2}{x}} f(x,y)dxdy$

**Evaluate**

13. $\iint rdrd\theta\ \ dxdy$ over the cardioide $r = 1 + \cos\theta$

14. $\iint r^3drd\theta\ \ dxdy$ over the area included between the circles $r = 2\sin\theta$ and $r = 4\sin\theta$

15. $\iint r^4cos^3\theta drd\theta$ over the interior of circle $r = 2a\cos\theta$

**Express the following integrals in polar coordinates, showing the region of integration and evaluate.**

16. $\int_0^a \int_0^{\sqrt{a^2-y^2}} y^2\sqrt{x^2+y^2}\,dxdy$

17. $\int_0^{4a} \int_{\frac{y^2}{4a}}^{y} \frac{x^2-y^2}{x^2+y^2}\,dxdy$

18. $\int_0^2 \int_{1-\sqrt{2x-x^2}}^{1+\sqrt{2x-x^2}} \frac{dxdy}{(x^2+y^2)^2}$

19. $\int_0^a \int_y^a \frac{x\,dxdy}{x^2+y^2}$

**Change to polar coordinates and evaluate**

20. $\iint \frac{x^2-y^2}{(x^2+y^2)^{\frac{3}{2}}}\,dxdy$ over the region of the circle $x^2+y^2=2ax$ in the first quadrant.

21. $\iint y^2\,dxdy$ over the area which lies outside the circle $x^2+y^2-ax=0$ but inside circle $x^2+y^2-2ax=0$.

22. Evaluate $\iint \frac{dxdy}{(1+x^2+y^2)^3}$ over one loop of the lemniscate $(x^2+y^2)^2=x^2-y^2$

**Answers**

| | | | | |
|---|---|---|---|---|
| 13. $\frac{3\pi}{2}$ | 14. $\frac{45\pi}{2}$ | 15. $\frac{7\pi}{4}a^5$ | 16. $\frac{\pi a^5}{20}$ | 17. $8a^2\left(\frac{\pi}{2}-\frac{5}{3}\right)$ |
| 18. $\pi$ | 19. $\frac{\pi a}{4}$ | 20. $\frac{2a}{3}$ | 21. $\frac{15\pi}{64}a^4$ | 22. $\frac{\pi-2}{4}$ |

23. Show that $\iiint \frac{dxdydz}{(x+y+z+1)^2}=\frac{1}{2}\left(\log 2-\frac{5}{8}\right)$, integration *being taken* throughout the volume of the tetrahedran bounded by the coordinate planes a plane $x+y+z+1=1$

## 8.7 References

1. A Text Book of Applied Mathematics Vol I - P. N. Wartikar and J. N. Wartikar
2. Applied Mathematics II - P. N. Wartikar and J. N. Wartikar
3. Higher Engineering Mathematics - Dr. B. S. Grewal

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Unit 4

# 9

# APPLICATIONS OF INTEGRATION

**Unit Structure**



## 9.0 Objectives

*After reading this chapter, you should be able to:*

1. *Know the concept Areas & volume of solids.*
2. *Formulae of these in terms of integrals.*
3. *Single & multiple integrals & their use in examples*
4. *Solve problems based on area & volume integrals*

## 9.1 Introduction

In this chapter we shall study the applications of integral calculus to the problems involving areas, volumes and surface of solids, centre of gravity, hydrostatic centre of pressure, moment of inertia, mean and root mean square values etc. Formulae for these in terms of integrals, single and multiple are developed and their use in the example on these topics is illustrated.

## 9.2 Areas

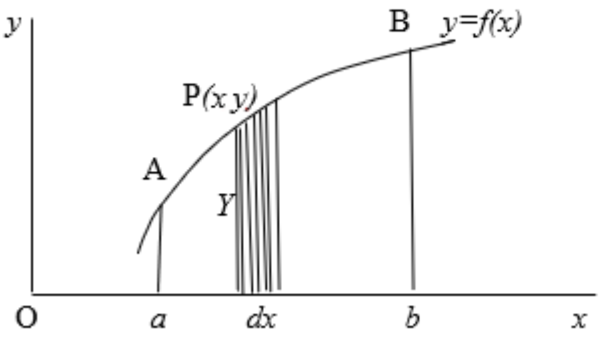


**Fig 1**

The area A, included by the curve $y = f(x)$ the $x$-axis and the ordinates $x = a$ and $x = b$ is given by

$$A = \int_a^b y\,dx \qquad \ldots\ldots \quad (9.1)$$

Similarly the area A', included by the curve $y = f(x)$, the $y$-axis, $y = c$ and $y = d$ is

$$A' = \int_c^d x\,dy \qquad \ldots \qquad \ldots \qquad (9.2)$$

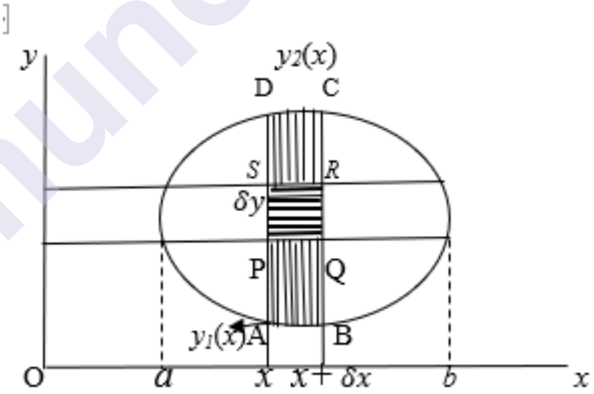


**Fig. 2**

In case of a loop as shown in figure 2, the area of an elementary rectangle at P $(x, y)$ is $dxdy$ and so the area of the loop is given by

$$\text{Area of the loop} = \int_a^b \int_{y_1(x)}^{y_2(x)} dxdy \qquad \ldots\ldots \quad (9.3)$$

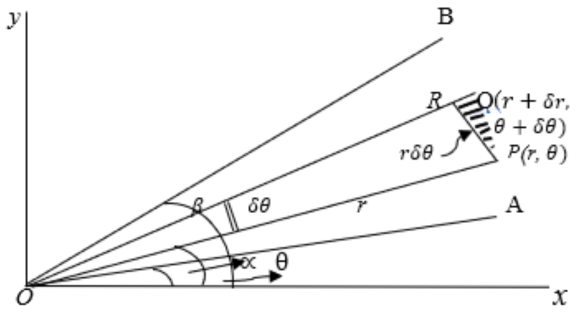


**Fig. 3**

If the equation of curve is given in polar coordinates by $r = f(\theta)$, then as $\delta\theta \to 0$, the area of the elementary triangle OPQ is $\frac{1}{2}r^2\,\delta\theta$ [ for dropping PR perpendicular to OQ, PQ = $r\delta\theta$ can be taken as the base of the $\Delta$ OPR of which the height is $r$, and its area is $\frac{1}{2}r^2\,\delta\theta$ ; as $\delta\theta \to 0$,

$\Delta$ OPR $\to$ $\Delta$ OPQ ] and so

$$\text{Area of the sector OAB} = \frac{1}{2}\int_{\alpha}^{\beta} r^2\, d\theta \qquad \ldots \quad (9.4)$$

**Example 1.** Trace the curve $y^2a^4 = x^5(2a - x)$ and show that its area is equal to $\frac{5\pi}{4}a^2$.

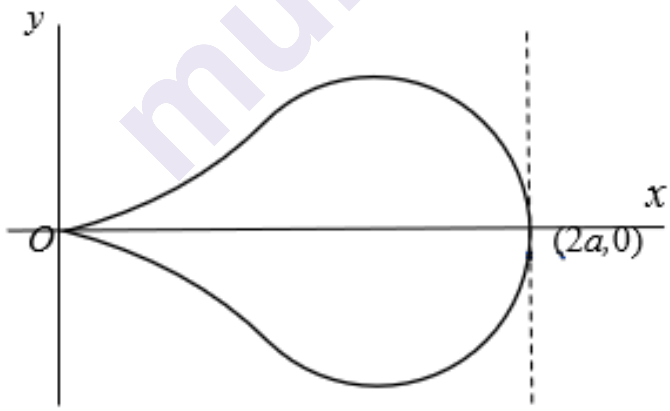


**Fig. 5**

The tracing done by the methods of curve-tracing gives the curve as a symmetrical loop on the $x$ – axis between $x = 0$ and $x = 2a$.

$\int_0^{2a} ydx$ gives the area of the upper half of the loop and so the area A of the loop is

$$A = 2\int_0^{2a} ydx \qquad \ldots \qquad (i)$$

From the equation of the curve $y = \frac{x^{\frac{5}{2}}(2a-x)^{\frac{1}{2}}}{a^2}$, substituting this in (i),

$$A = 2\int_0^{2} \frac{x^{\frac{5}{2}}(2a-x)^{\frac{1}{2}}}{a^2} dx \qquad \ldots \qquad \ldots \qquad (ii)$$

For integration, we put $x = 2a\sin^2\theta$, so that $dx = 4a\, sin\,\theta\cos\theta\ d\theta$ and when $x = 0$, $\theta = 0$ and when $x = 2a$, $\theta = \frac{\pi}{2}$.

$$\therefore\ A = 64a^2.\int_0^{\frac{\pi}{2}} \sin^6\theta\cos^2\theta\, d\theta \qquad \ldots \qquad \ldots \qquad (iii)$$

By the reduction formulae, we can write the value of this integral, so

$$\therefore\ A = 64a^2.\frac{(5.3.1).(1)}{8.6.4.2}\frac{\pi}{2} = \frac{5\pi}{4}a^2$$

**Example 2.** Trace the curve $a^2x^2 = y^3(2a-y)$ and show that its area is equal to $\pi a^2$.

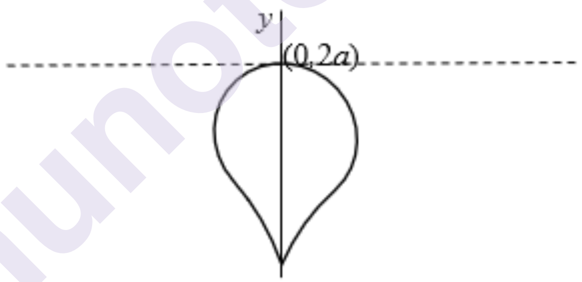


**Fig. 6**

Here the loop is on the $y$-axis, and so we use the formula (9.2) for the area. Thus the area of the loop is

$$A = 2\int_0^{2a} xdy$$

$$A = 2\int_0^{2a} \frac{y^{\frac{3}{2}}(2a-y)^{\frac{1}{2}}}{a}\, dy$$

Substituting $y = 2a\sin^2\theta$,

$$A = 32a^2 \int_0^{\frac{\pi}{2}} \sin^4\theta \cos^2\theta \, d\theta$$

$$= 32\, a^2 . \frac{(3.1).(1)}{6.4.2} \frac{\pi}{2} = \pi a^2.$$

**Example 3**. Prove that the area of the loop of the curve $x^5 + y^5 = 5ax^2y^2$ is $\frac{5}{2}a^2$.

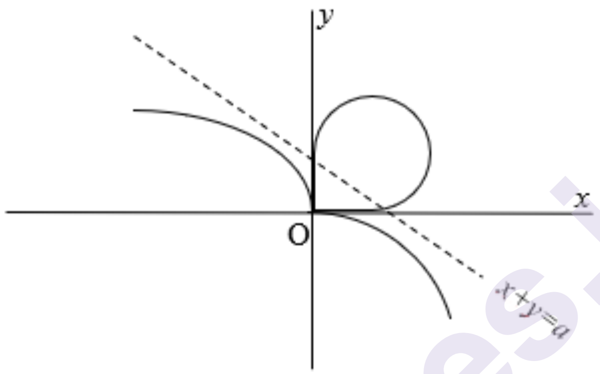


**Fig. 7**

From the equation of the curve, it is clear that the loop does not lie on the *x* or *y* axis and so is inclined to them. In case of inclined loop, we change the equation to polar co-ordinates with $x = r\cos\theta$ , $y = r\sin\theta$.

The equation of the curve in polar coordinates is

$$r = \frac{5a\sin^2\theta\cos^2\theta}{\sin^5\theta + \cos^5\theta} \qquad \dots \qquad \text{(i)}$$

*r* is zero when $\theta = 0$ and $\frac{\pi}{2}$ , so the loop of the curve lies between these two limits. Using formula (9.4), the area A of the loop is

$$A = \frac{1}{2}\int_0^{\frac{\pi}{2}} r^2 d\theta \qquad \dots \qquad \dots \qquad (ii)$$

Substituting for *r* from (i) in (ii),

$$A = \frac{25a^2}{2}\int_0^{\frac{\pi}{2}} \frac{\sin^4\theta\cos^4\theta}{(\sin^5\theta + \cos^5\theta)^2} \, d\theta$$

Dividing the numerator and denominator by $\cos^{10}\theta$ ,

$$A = \frac{25a^2}{2}\int_0^{\frac{\pi}{2}} \frac{\sec^2\theta\,.\tan^4\theta}{(1+\tan^5\theta)^2}\,.d\theta$$

Put $z = 1+\tan^5\theta$, $dz = 5\ \sec^2\theta\tan^4\theta\ d\theta$. When $\theta = 0$, $z = 1$ and

when $\theta = \frac{\pi}{2}$, $z = \infty$ ,

$$\therefore\ A = \frac{5a^2}{2}\int_1^{\infty}\frac{dz}{z^2} = \frac{5a^2}{2}\left[-\frac{1}{z}\right]_1^{\infty} = \frac{5a^2}{2}.$$

**Example 4.** In the cycloid $x = a\ (\theta + \sin\theta), y = a(1-\cos\theta)$ find the area between its base and portion of the curve from cusp to cusp.

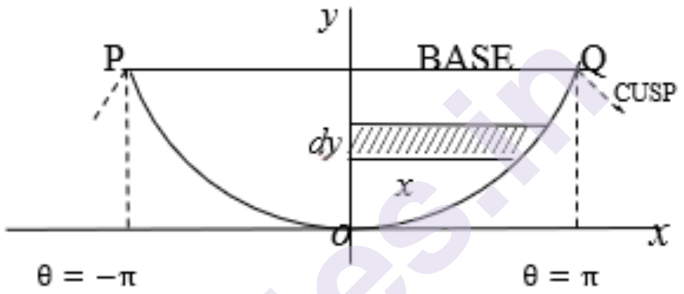


**Fig. 8**

The sketch of the curve is shown in the figure with cusps at P and Q and the base PQ.

The area required is that of the curvilinear figure POQ.

$$\text{Required Area } A = 2\int xdy$$

$$= 2\int_0^{\pi} x\,\frac{dy}{d\theta}\,d\theta \qquad \ldots \qquad \ldots \qquad (i)$$

From the equation of the cycloid $x = a\ (\theta + \sin\theta)$, $\frac{dy}{d\theta} = a\sin\theta$ substituting in (i)

$$A = 2\int_0^{\pi} a^2\ (\theta + \sin\theta)\sin\theta\ d\theta$$

$$= 2a^2\int_0^{\pi}[\theta\sin\theta\ + \sin^2\theta]\ d\theta \qquad \ldots \qquad \ldots \qquad (ii)$$

$$\text{Now} \int_0^{\pi} \theta \sin\theta \; d\theta \;=\; [-\theta\cos\theta + \sin\theta\;]_0^{\pi}$$

$$= \quad \pi \qquad \ldots \qquad \ldots \qquad (iii)$$

$$\text{and} \int_0^{\pi} \sin^2\theta \, d\theta \;=\; 2\int_0^{\frac{\pi}{2}} \sin^2\theta d\theta \;=\; 2.\frac{\pi}{4}$$

$$= \quad \frac{\pi}{2} \qquad \ldots \qquad \ldots \qquad (iv)$$

Substituting these values of the integrals in (ii)

$$\text{A} \;=\; 2a^2\left[\pi + \frac{\pi}{2}\right] = \; 3\pi a^2.$$

**Example 5**. Find the area between $y^2 = \frac{x^3}{a-x}$ and its asymptote. The nature of the curve is shown in the figure with asymptote $x = a$ [Asymptote is the line to which the curve approaches]

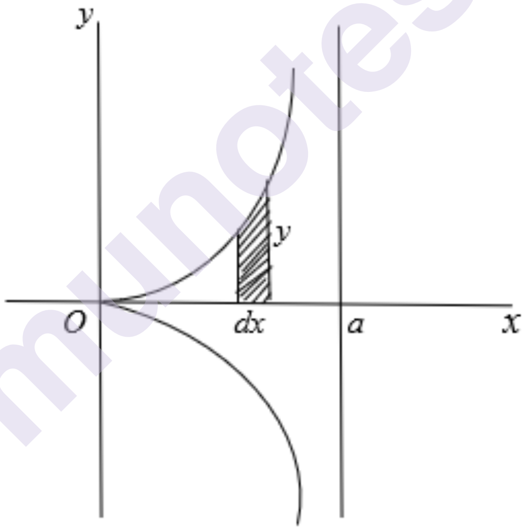


**Fig. 9**

The required area A is :

$$\text{A} \;=\; 2\int_0^{a} y dx$$

$$= 2\int_0^a \frac{x^{\frac{3}{2}}}{(a-x)^{\frac{1}{2}}}\,dx$$

with $x = a\sin^2\theta$,

$$A = 4a^2\int_0^{\frac{\pi}{2}} \sin^4\theta\, d\,\theta$$

$$= 4a^2 \cdot \frac{3}{4}\cdot\frac{1}{2}\,\frac{\pi}{2} = \frac{3}{4}\pi a^2.$$

**Example 6**. Find the area of the loop of the curve

$$r = a\cos 3\theta + b\sin 3\theta.$$

Let $\propto = \tan^{-1}\frac{a}{b}$, so that $a = \sqrt{a^2+b^2}\sin\alpha$,

$b = \sqrt{a^2+b^2}\cos\alpha$ so that the equation of the curve can be written as
$r = \sqrt{a^2+b^2}(\sin\alpha\cos 3\theta + \cos\alpha\sin 3\theta)$.

or $\quad r = \sqrt{a^2+b^2}\sin(3\theta+\alpha)$ ... ... (i)

To find the position of the loop, we have when $r = 0$, $3\theta + \alpha = n\pi$ (where $n$ is an integer).

Taking consecutive values of $n$ as 0 and 1,

one of the loop lie between $\theta = -\frac{\alpha}{3}$ and $\theta = \frac{\pi-\alpha}{3}$.

$\therefore$ The area of the loop $= A = \frac{1}{2}\int_{-\frac{\alpha}{3}}^{\frac{\pi-\alpha}{3}} r^2 d\theta$ ... ... (ii)

Substituting for $r$ from (i)

$$A = \frac{(a^2+b^2)}{2}\int_{-\frac{\alpha}{3}}^{\frac{\pi-\alpha}{3}} \sin^2(3\theta+\alpha)\,d\theta$$

In this put $\emptyset = 3\theta + \alpha$ ; so that

$$A = \frac{(a^2+b^2)}{6}\int_0^{\pi}\sin^2\emptyset\, d\emptyset$$

$$= \frac{(a^2+b^2)}{3}\int_0^{\frac{\pi}{2}} \sin^2\emptyset \, d\emptyset$$

$$= \frac{(a^2+b^2)}{3}.\frac{1}{2}.\frac{\pi}{2} \;=\; \frac{\pi}{12}\,(a^2+b^2).$$

**Example 7**. Find by double integration the area included between the curves $y = 3x^2 - x - 3$

and $y = -2x^2 + 4x + 7$.

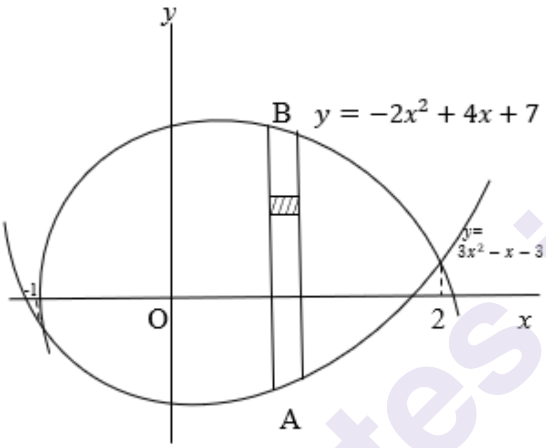


**Fig. 10**

The abscissa of the points of intersection of the two parabolas, a rough sketch of which is given in the adjacent diagram are given by

$$3x^2 - x - 3 = \; -2x^2 + 4x + 7$$

$$\text{i.e. } x^2 - x - 2 = 0$$

$$\therefore \quad x = -1, 2.$$

Taking the elementary strip parallel to the *y*-axis, such as AB, bounded by the two parabolas we integrate first w.r.t. *y,* and then integrating w.r.t. *x* from *x* = -1 *to x* = 2, gives for the area A required.

$$\mathrm{A} = \int_{-1}^{2}\int_{3x^2-x-3}^{-2x^2+4x+7} dxdy$$

$$= \int_{-1}^{2} dx\,[y]_{3x^2-x-3}^{-2x^2+4x+7} \qquad = 5\int_{-1}^{2}(-x^2+x+2)\,dx$$

$$= 5\left\{-\frac{8}{3}+\frac{4}{2}+4-\left(\frac{1}{3}+\frac{1}{2}-2\right)\right\}$$
$$= \frac{5}{2}.$$

**Example 8**. Find by double integration the area included between the curve $r = a(\sec\theta + cos\theta)$ and its asympotote $r = a \sec\theta$.

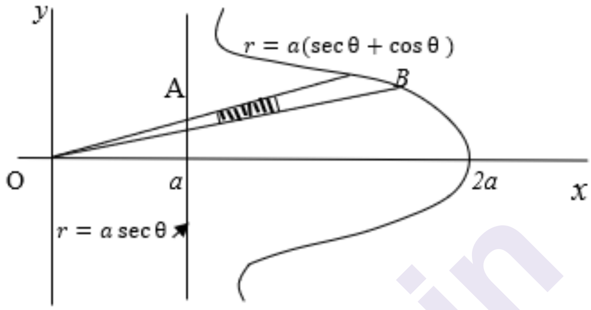


**Fig. 11**

By transforming the equations to cartesian, coordinates, the curves are easily traced, as shown in figure.

Taking a wedge such as AB, its extremities lie on the curve r = a sec$\theta$ and $r = a(\sec\theta + \cos\theta)$ and to get the area between the asymptote and the curve, θ varies from -$\frac{\pi}{2}$ $to$ $\frac{\pi}{2}$; or by symmetry the area A required is :

$$A = 2\int_0^{\frac{\pi}{2}} \int_{a\, sec\theta}^{a(sec\theta+cos\theta)} r\, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} [r^2]_{a\,\sec\theta}^{a(sec\theta+cos\theta)}\, d\theta$$

$$= a^2\int_0^{\frac{\pi}{2}} \{(\sec\theta+\cos\theta)^2 - \sec^2\theta\}\, d\theta = a^2\int_0^{\frac{\pi}{2}} [2+\cos^2\theta]\, d\theta$$

$$= a^2\left[\pi+\frac{\pi}{4}\right] = \frac{5\pi}{4}a^2$$

## 9.3 VOLUMES OF SOLIDS

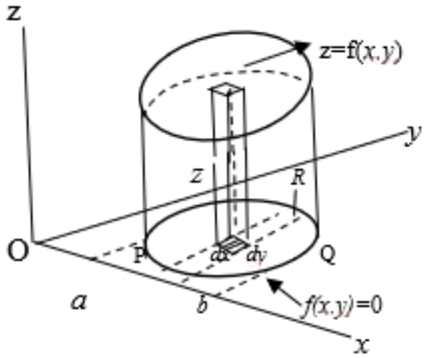


**Fig. 12**

Let $z = f(x,y)$ be the equation of the surface, of which the orthogonal projection in the $xOy$ plane is the contour PQR, whose equation is $f(x,y) = 0$. The volume of an elementary parallellopiped on $dxdy$ bounded by the surface, $z= f(x,y)$ and sides parallel to the $z$ axis is

$zdxdy = f(x,y)\, dx\, dy$.

The summation of all such terms over the area of closed curve PQR gives the volume of the solid cylinder bounded by the given surface and the plane $xOy$ with generators parallel to the $z$- axis as

$$\text{Volume } = \iint f(x,y)dx\,dy \qquad \ldots \qquad \ldots \qquad (9.6)$$

to be taken on the area of the contour PQR.

To express the volume of a solid as a triple integral, we note that the volume of an elementary cuboid is $dx\,dy\,dz$ ; and so the volume of the solid is given by

$$\text{Volume} = \iiint dx\,dy\,dz \qquad \ldots \qquad \ldots \qquad (9.7)$$

Where the limits of integration w.r.t. $z$ (if we integrate first w.r.t. $z$) are $z_1$ and $z_2$ obtained from its equations to the top and bottom of the given surface and then the double integration is w.r.t. $x$ and $y$ is performed over the area of projection of the given solid on the $xOy$ plane.

If $\rho = f(x,y,z)$ is the density of the solid at the point P( $x,y,z$), then the mass of the solid is

$$\iiint f(x,y,z)\ dx\,dy\,dz \qquad \ldots \qquad \ldots \qquad (9.8)$$

with appropriate limits of integrations.

**Example 1.** Find by double integration the volume of the sphere $x^2 + y^2 + z^2 = a^2$ cut off by the plane $z = 0$ and the cylinder $x^2 + y^2 = ax$.

Taking the polar co-ordinate in the *x*O*y* plane, elementary area at P $(r, \theta)$ is $rdrd\theta$. If the line at P drawn parallel to the $z$ – axis has length $z$, the volume of the elementary parallelopiped at P $zrdrd\theta$, and the volume of the cylinder on the circle $x^2 + y^2 = ax$, $z = 0$ bounded at the top by the surface of the sphere $x^2 + y^2 + z^2 = a^2$ is with proper limits of integration.

$$\iint z\, r\, dr\, d\theta \qquad \ldots \qquad \ldots \qquad \text{(i)}$$

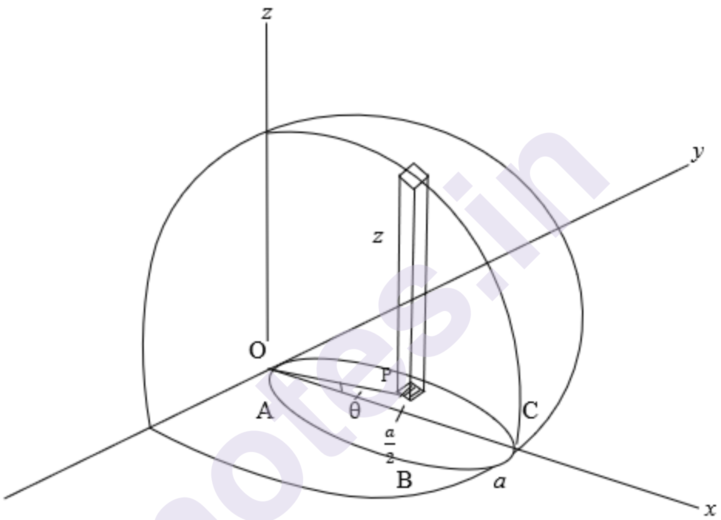


**Fig. 13**

As $x^2 + y^2 = r^2$ so the equation of the sphere is $z^2 + r^2 = a^2$ or $z = \sqrt{a^2 - r^2}$. The region of integration is the circle $x^2 + y^2 - ax = 0$ which has its center at $\left(\frac{a}{2}, 0{,}0\right)$ and radius is $\frac{a}{2}$. Its polar equation is $r = a\cos\theta$. So the limits of integration w.r.t. $r$ are 0 and $a\cos\theta$ and w.r.t. $\theta$ are $-\frac{\pi}{2}$ *and* $\frac{\pi}{2}$. With these considerations and using (i), the volume V required is (by symmetry)

$$\text{V} = 2\int_0^{\frac{\pi}{2}} d\theta \int_0^{a\cos\theta} \sqrt{a^2 - r^2}\,|\, rdr \qquad \ldots \qquad \ldots \qquad \text{(ii)}$$

To evaluate the first integral put $t^2 = a^2 - r^2$, so we have

$$\int_0^{a\cos\theta} \sqrt{a^2 - r^2}\,|\, rdr = -\int_a^{a\sin\theta} t^2 dt = -\left[\frac{t^3}{3}\right]_a^{a\sin\theta}$$

$$= \frac{1}{3}a^3[1 - \sin^2\theta] \qquad \ldots \qquad \ldots \qquad \text{(iii)}$$

Using this in (ii), the volume required is

$$V = \frac{2a^3}{3}\int_0^{\frac{\pi}{2}} [1 - \sin^2\theta]d\theta \; = \frac{a^3}{1}(3\pi - 4)$$

**Example 2.** Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ and $z = 0$.

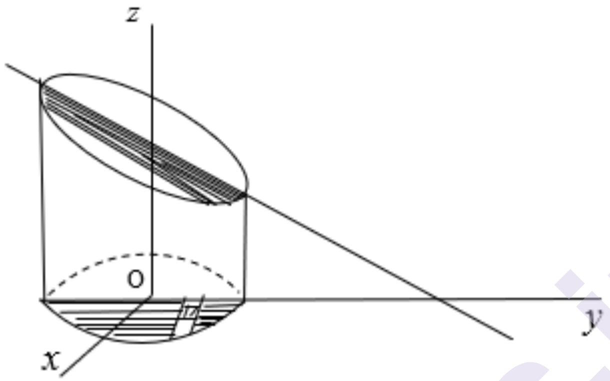


**Fig.14**

From Fig. 14 it is self-evident that $z = 4 - y$ is to be integrated over the circle $x^2 + y^2 = 4$ in the *xy* - plane.

To cover the shaded half of this circle, $x$ varies from 0 to $\sqrt{(4 - y^2)}$ and $y$ varies from -2 to 2.

∴ Required Volume

$$= 2\int_{-2}^{2}\int_0^{\sqrt{(4-y^2)}} z\,dx\,dy \quad = 2\int_{-2}^{2}\int_0^{\sqrt{(4-y^2)}} (4 - y)\,dx\,dy$$

$$= 2\int_{-2}^{2}(4 - y)\,[x]_0^{\sqrt{(4-y^2)}}dy \quad = 2\int_{-2}^{2}(4 - y)\sqrt{(4 - y^2)}\,dy$$

$$= 2\int_{-2}^{2}4\sqrt{(4 - y^2)}\,dy \; - \; 2\int_{-2}^{2}y\sqrt{(4 - y^2)}\,dy$$

$$= 8\int_{-2}^{2}\sqrt{(4 - y^2)}\,dy$$

[The second term vanishes as the integrand is an odd function]

$$= 8\left[\frac{y\sqrt{(4-y^2)}}{2} + \frac{4}{2}\mathrm{Sin}^{-1}\frac{y}{2}\right]_{-2}^{2} \quad = \quad 16\,\pi.$$

**Example 3.** Find the volume of ellipsoid $\frac{x^2}{a^2}+\frac{x^2}{b^2}+\frac{x^2}{c^2}=1.$

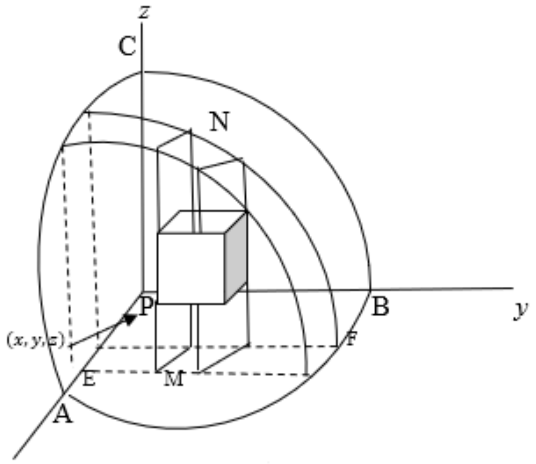


**Fig. 15**

Let OABC be the positive octant of the given ellipsoid which is bounded by the planes OAB ($z$ = 0), OBC ($x$ = 0), OCA ($y$ = 0), and the surface ABC, i.e.

$$\frac{x^2}{a^2}+\frac{y^2}{b}+\frac{z^2}{c^2}=1.$$

Divide this region R into rectangular parallelopipeds of volume $\delta x \delta y \delta z$. Consider such an element at P$(x, y, z)$ (Fig. 15)

$$\therefore \text{ the required volume } = 8\iiint_R dx\,dy\,dz$$

In this region R,

(i) $z$ varies from 0 to MN, where

$$\text{MN} = c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$$

(ii) $y$ varies from 0 to EF, where

$$\text{EF} = b\sqrt{1-\frac{x^2}{a^2}}$$

from the equation of the ellipse OAB, i. e. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$

(iii) $x$ varies from 0 to OA = $a$.

Hence the volume of the whole ellipsoid

$$= 8\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dx\, dy\, dz \quad = 8\int_0^a dx \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} dy\, [z]_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}$$

$$= 8c\int_0^a dx \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \sqrt{\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)}\, dy$$

$$= \frac{8c}{b}\int_0^a dx \int_0^{\rho} \sqrt{(\rho^2-y^2)}\, dy \;\text{ when } \rho = b\sqrt{1-\frac{x^2}{a^2}}$$

$$= \frac{8c}{b}\int_0^a dx \left[\frac{y\sqrt{(\rho^2-y^2)}}{2} + \frac{\rho^2}{2}\sin^{-1}\frac{y}{\rho}\right]_0^{\rho} \quad = \frac{8c}{b}\int_0^a \frac{b^2}{2}(1-\frac{x^2}{a^2})\frac{\pi}{2}dx$$

$$= 2\pi bc\int_0^a \left(1-\frac{x^2}{a^2}\right)dx \;=\; 2\pi bc\left[x-\frac{x^3}{3a^2}\right]_0^a$$

$$= \frac{4\pi abc}{3}.$$

## 9.4 Summary

The ninth chapter of this book discusses the applications of integral calculus to the problems involving areas, volumes and surface of solids. Formulae of these concepts in single and multiple integrals are developed and their use in the examples are illustrated with diagrams. At the end, unsolved problems as exercise are left to students for practice.

## 9.5 Exercises

1. Find the area enclosed by the curves bounded by

   $x^2 = 4ay$ and $x^2 + 4a^2 = \frac{8a^3}{y}$

2. Find the whole area between the curve $x^2y^2$
   $= a^2(y^2 - x^2)$ and its asymptote.

3. Find by double integration the area between the curve
   $y^2 = x^2 - 6x + 3$ and $y = 2x - 9$

4. Find by double integration the area between the curve $y^2 = \frac{4a^2(2a-x)}{x}$ and its asmptote

5. Find the area between the curve $y^2 - 4x$ *and* $2x - 3y + \frac{1}{2} = 0$.

6. Find the area included between the curves $9xy = 4$ *and* $2x + y = 2$ by double integration.

7. Find the area common to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *and* $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$.

8. Find the double integration area included between the curves $y^2 = 4a(x+a)$ and $y^2 = 4b(b-x)$.

9. Show that the area of a loop of the curve $r = a \cos n\theta$ is $\frac{\pi a^2}{4n}$ and the state total area in case n is odd , n is even. Also find the area contained between the circle $r = a$ and $r = a \cos 5\theta$.

10. Find the area bounded by the curve $r = 2a \cos 3\theta$ and lying outside the circle $r = a$.

11. Find the whole area of the curve represented by the equation $r = a + b \cos \theta$, assuming $a > b$.

12. Find the area of the curve $r^2 = a^2 \cos 2\theta$.

13. Show that the area of the loop of the curve $y^2(a+x) = x^2(3a-x)$ is equal to the area between the curve and its asymptot

14. Show by double integration that the area between the parabola $y^2 = 4ax$ and $x^2 = 4ay$ *is* $\frac{16}{3}a^2$

15. Show that the area enclosed by the curves $x y^2 = a^2(a - x)$ and $(a - x) y^2 = a^2 x$ *is* $(\pi - 2) a^2$.

16. Prove that the area of the part of the ellipse

$\frac{x^2}{a^2} + \frac{x^2}{b^2} = 1$, $(a > b)$ which is within the parabola

$b^2x^2 = (a^2 - b^2)ay$ is given by $\frac{1}{3}\, b^2e + \sin^{-1}e$

where e is the eccentricity of the ellipse.

17. Show that the area of loop of the curve

(i) $r\cos\theta = a\cos 2\theta$ is $\dfrac{a^2(4 - \pi)}{2}$.

(ii) $r = a\,\theta\cos\theta$ is $\dfrac{\pi a^2}{96}(\pi^2 - 6)$.

18. Show that the area of the loop of the curve

$r^2(2c^2\cos\theta - 2ac\sin\theta\cos\theta + a^2\sin^2\theta) = a^2c^2$ is $\pi\ ac$

**Answers**

1. $\frac{2a^2}{3}(3\pi - 2)$ 2. $4a^2$ 3. $10\frac{2}{3}$ 4. $4\pi a^2$

5. $\frac{1}{2}$ 6. $\frac{1}{3} - \frac{4}{9}\log 2$ 7. $4ab\tan^{-1}\frac{b}{a}$ 8. $\frac{8}{3}(a + b)\sqrt{ab}$

9. n – odd. $\frac{\pi a^2}{4}$; n – even $\frac{\pi a^2}{2}, \frac{3}{4}\ \pi a^2$ 10. $a^2(\frac{\pi}{3} + \frac{\sqrt{3}}{2})$

11. $\pi\left(a^2 + \frac{1}{2}b^2\right)$. 12. $a^2$

19. If the density at a point varies as the square of the distance of the point from the $xy$- plane, find the mass of the volume common to the sphere $x^2 + y^2 + z^2 = a^2$ and cylinder $x^2 + y^2 = ax$.

20. Find the volume bounded by the surface $z = c\,(1 - \frac{x}{a})\,(1 - \frac{y}{b})$

and the positive quadrant of the elliptic cylinder $\frac{x^2}{a^2} + \frac{x^2}{b^2} = 1$, $z = 0$

21. Find the volume of the solid bounded by the surfaces z

$$= 4 - x^2 - \frac{1}{2}y^2 \quad \text{and } z = 3x^2 + \frac{1}{2}y^2.$$

22. Find the volume common to the right circular cylinders $x^2 + y^2$

$$= a^2, \quad x^2 + z^2 = a^2.$$

23. A right circular cylindar of radius $\frac{a}{2}$ and height $a$ is formed by the plane $z = 0, = a.$ and the surface $x^2 + y^2 = ax$. Find the volume of the portion of the cylinder inside the cone $x^2 + y^2 = z^2$.

**Answers**

19. $\frac{2a^5}{15}(\pi - \frac{16}{15})$ 20. $\frac{abc}{4}(\pi - \frac{13}{6})$ 21. $4\sqrt{2}\,\pi$ 22. $\frac{16}{3}a^3$

23. $\frac{a^3}{36}(9\pi - 16)$

## 9.6 References

1. A Text Book of Applied Mathematics Vol I - P. N. Wartikar and J. N. Wartikar
2. Applied Mathematics II - P. N. Wartikar and J. N. Wartikar
3. Higher Engineering Mathematics - Dr. B. S. Grewal

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Unit 5

# 10

# BETA AND GAMMA FUNCTIONS

Unit Structure



## 10.0 Objectives

After going through this unit, you will be able to:

- Understand the concept of Gamma function , properties of Gamma function
- Solve the problem based on Gamma function with its type.
- Understand the concept of Beta function , properties of Beta function
- Understand the relation between Gamma and Beta Function
- Know the concept of Duplication formula

## 10.1 Introduction

At this stage students are well versed with elementary methods of integration and evaluation of real definite integrals. In this chapter we introduce some advanced techniques. Beta and Gamma integrals or typically called Beta and Gamma functions are the special kind of integrals which find their applications in theory of probability, integral transforms, fluid mechanics and so on. Certain kind of real definite integrals can be evaluated by using Beta and Gamma Functions. Their use is prominent in evaluation of multiple integrals. In this chapter we shall discuss some properties of Beta and Gamma Functions and Duplication formula.

Leonhard Euler

Historically, the idea of extending the factorial to non-integers was considered by Daniel Bernoulli and Christian Goldbach in the 1720s.It was solved by Leonhard Euler at the end of the same decade.

Euler discovered many interesting properties, such as its reflection formula:.

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi - x)}$$

James Stirling, contemporary of Euler, also tried to extend thefactorial and came up with the Stirling formula, which gives a good approximation of n! but it is not exact. Later on, Carl Gauss, the prince of mathematics, introduced the Gamma function for complex numbers using the Pochhammer factorial. In the early 1810s, it was Adrien Legendre who first used the **Γ** symbol and named the Gamma function.



## 10. 2 Gamma Functions

Consider the definite integral $\int_0^{\infty} e^{-x}x^{n-1}dx$ it is denoted by the symbols Γ(n) [ we read is as Gamma 'n' ] and is called as Gamma Function of n. Thus

$$\mathbf{\Gamma(n) = \int_0^{\infty} e^{-x}x^{n-1}dx \quad (n > 0)} \qquad \text{------------------------------(1)}$$

Gamma Function is also called as Euler's Integral of the second kind. It defines a function of n for positive values of n.

## 10.3 Applications of Gamma Functions:

In a Gamma distribution, the gamma function is used to determine time based occurrences such as

1. The time between occurrences of earthquakes .
2. Life length of electronic component.
3. Waiting time between any two consecutive events.
4. Gamma function arises in various probability distribution function.

## 10.4 Properties of Gamma Functions:

**1.** $\boldsymbol{\Gamma(n) = 2\int_0^\infty e^{-x^2} x^{2n-1} dx}$

Proof: $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$ Put $x = t^2$ , $dx = 2t\, dt$

$= \int_0^\infty e^{-t^2} t^{2n-2}\, 2t\, dt$

| $x$ | 0 | $\infty$ |
|---|---|---|
| $t$ | 0 | $\infty$ |

$= 2\int_0^\infty e^{-t^2} t^{2n-1}\, dt$

$$\boldsymbol{\Gamma(n) = 2\int_0^\infty e^{-t^2} t^{2n-1}\, dt}$$ ------------------------------------**(2)**

[ It may be borne in mind that variable of integration is immaterial in a definite integral ]

Relations **(1)** and **(2)** are both considered as definitions of Gamma functions.

**2.** $\boldsymbol{\Gamma(1) = 1}$

Proof: By definition , $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$ put n =1

$\Gamma(1) = \int_0^\infty e^{-x} x^0 dx = \int_0^\infty e^{-x}\, dx = [-e^x]_0^\infty = (-e^\infty + e^0) = 0 + 1 = 1$

$$\boldsymbol{\Gamma(1) = 1}$$

**3. Reduction Formulae for Gamma Function :**

**Γ(n+1) = n Γ(n)**

Proof: By definition , $\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}dx$ Replace n by n+1

$$\Gamma(n+1) = \int_0^\infty e^{-x}x^{n}dx$$

Now, integrating by parts

$$\Gamma(n+1) = [x^n(-e^x)]_0^\infty - \int_0^\infty nx^{n-1}(e^{-x})dx$$

Now,

$\lim_{x\to\infty}\frac{x^n}{e^x} = 0$, also if $n > 0$, $\frac{x^n}{e^x} = 0$ for $x = 0$ $\therefore \left[\frac{x^n}{e^x}\right]_0^\infty = 0$

$$\Gamma(n+1) = 0 + n\int_0^\infty e^{-x}x^{n-1}\,dx = n\,\Gamma(n)$$

| **Γ(n+1) = n Γ(n)** |
|---|

If n is a positive integer ,

Γ(n+1 ) = n (n-1) Γ(n-1 ) ∵ Γ(n) = (n-1) Γ(n-1 )

= n (n-1) (n-2) Γ(n-2 )

= n (n-1) (n-2) (n-3) (n -4 ) ……………..3.2.1.Γ(1 )

= n (n-1) (n-2) (n-3) (n -4 ) ……………..3.2.1 ∵ Γ(1) = 1

= n!

Γ(n+1 ) = n! if n is a positive integer

| **Γ(n+1) = n Γ(n) in general , n is rational number**<br>**= n! if n is a positive integer** |
|---|

4.

| **Γ(0) = ∞** |
|---|

$\because \Gamma(n) = \frac{\Gamma(n+1)}{n}$ , $\Gamma(0) = \frac{\Gamma(1)}{0} = \frac{1}{0} = \infty$

5. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

6. $\because \Gamma(n+1) = n!$

$$\therefore \Gamma(5) = \Gamma(4+1) \text{ ------- } (\Gamma(n+1))$$
$$= 4! \text{ ----- n is positive integer}$$
$$= 24$$

$$\Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2} + 1\right) \text{ ------- } (\Gamma(n+1))$$
$$= \frac{1}{2}\Gamma\left(\frac{1}{2}\right) \text{ -----( n is rational number )}$$
$$= \frac{1}{2}\sqrt{\pi} \text{ ------ } \because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma\left(\frac{5}{2}\right) = \Gamma\left(\frac{3}{2} + 1\right)$$
$$= \left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)$$
$$= \left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2} + 1\right)$$
$$= \left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) = \left(\frac{3}{4}\right)\sqrt{\pi}$$

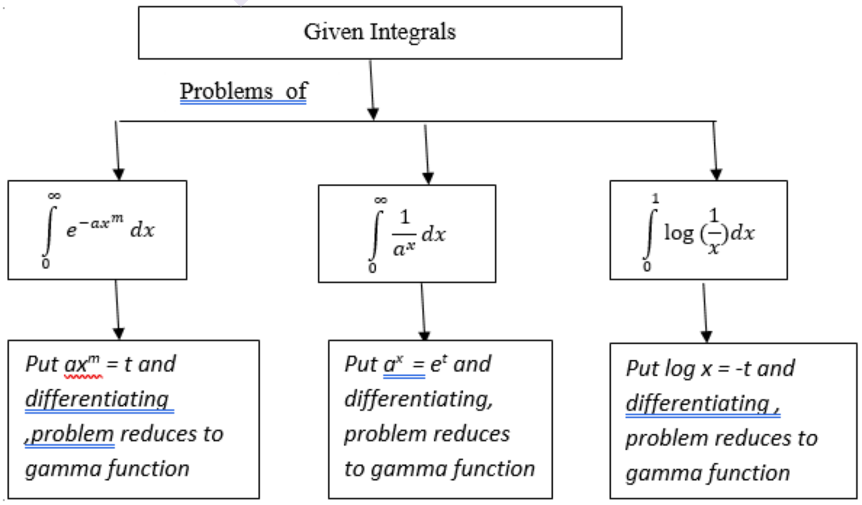
$$\Gamma\left(\frac{11}{2}\right) = \left(\frac{9}{2}\right)\left(\frac{7}{2}\right)\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\sqrt{\pi}$$

For negative fraction n , we use

$$\Gamma(n) = \frac{\Gamma(n+1)}{n}$$

$$\Gamma\left(-\frac{5}{3}\right) = \frac{\Gamma\left(-\frac{5}{3}+1\right)}{-\frac{5}{3}} = \left(-\frac{3}{5}\right)\Gamma\left(-\frac{2}{3}\right) = \left(-\frac{3}{5}\right)\frac{\Gamma\left(-\frac{2}{3}+1\right)}{-\frac{2}{3}} =$$
$$\left(-\frac{3}{5}\right)\left(-\frac{3}{2}\right)\Gamma\left(\frac{1}{3}\right)$$
$$= \left(\frac{9}{10}\right)\Gamma\left(\frac{1}{3}\right)$$

## 10.5 Flow Chart of Gamma Function



**Type I – $\int_0^\infty e^{-ax^m}\,dx$**

Method of Solving: , Put $ax^m=t$ , then differentiate, check limit points, reduces the given integral as gamma function, then we can solve by using definition of gamma function.

**Example 1:** Evaluate $\int_0^\infty x^7 e^{-2x^2}\,dx$

**Solution :** Let I= $\int_0^\infty x^7 e^{-2x^2}\,dx$ ---------------------------(A)

Put $2x^2 = t$ or $x^2 = t/2$

$$x^2 = \frac{t}{2} \text{ – – – – – – – – – – – – – } (i)$$

$$\therefore x = \sqrt{\frac{t}{2}} = \frac{t^{\frac{1}{2}}}{\sqrt{2}} \text{ – – – – – – – – } (ii)$$

Differentiating w.r.t. ‘ t’ we get

$$\frac{dx}{dt} = \frac{1}{\sqrt{2}}\frac{1}{2}t^{\frac{-1}{2}} = \frac{1}{2\sqrt{2}}\,t^{\frac{-1}{2}}$$

$$dx = \frac{1}{2\sqrt{2}}\,t^{\frac{-1}{2}}dt$$

Now limit point from (i) or (ii)

Let $x=0$ => $0 = \frac{t}{2}$ => $t= 0$ i.e. $x=0$, => $t= 0$

*And* $x = \infty$ => $\infty = \frac{t}{2}$ => $t = \infty$ ∴

| | | |
|---|---|---|
| $x$ | $o$ | $\infty$ |
| $t$ | $0$ | $\infty$ |

∴ ( A) becomes

$$I = \int_0^\infty \left(\frac{t^{\frac{1}{2}}}{\sqrt{2}}\right)^7 \frac{1}{2\sqrt{2}}\,t^{\frac{-1}{2}}e^{-t}\,dt$$

$$= \int_0^\infty \frac{t^{\frac{7}{2}}}{(2^{1/2})^7}\,\frac{1}{2\sqrt{2}}\,t^{\frac{-1}{2}}e^{-t}\,dt$$

$$= \frac{1}{2^{7/2}}\frac{1}{(2)2^{1/2}}\int_0^\infty t^{7/2}\,t^{\frac{-1}{2}}e^{-t}\,dt$$

$$= \frac{1}{2^{7/2}} \frac{1}{2^{3/2}} \int_0^\infty t^3\, e^{-t}\, dt$$

$$= \frac{1}{2^{(7+3)/2}} \int_0^\infty t^3\, e^{-t}\, dt$$

$$= \frac{1}{2^5} \int_0^\infty t^{4-1}\, e^{-t}\, dt$$

$$= \frac{1}{32} \int_0^\infty e^{-t}\, t^{4-1} dt$$

Now using definition of gamma function

$$\boldsymbol{\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx \quad (n > 0)}$$

($\because$ n= 4 , variable is ‘t’)

$$\therefore = \frac{1}{32} \int_0^\infty e^{-t}\, t^{4-1} dt = \frac{1}{32}\, \Gamma(4) = \frac{1}{32}.3! = \frac{6}{32} = \frac{3}{16}$$

$$\therefore \mathrm{I} = \int_0^\infty x^7 e^{-2x^2}\, dx = \frac{3}{16}$$

**Example2:** Evaluate $\mathrm{I} = \int_0^\infty x^9 e^{-2x^2}\, dx$

**Solution:** Let I= $\int_0^\infty x^9 e^{-2x^2}\, dx$--------------------------(A)

Put $2x^2 = t$ or $x^2 = t/2$

$$x^2 = \frac{t}{2} - - - - - - - - - - - - - (i)$$

$$\therefore x = \sqrt{\frac{t}{2}} = \frac{t^{\frac{1}{2}}}{\sqrt{2}}$$

Differentiating w.r.t. ‘ t’ we get

$$\frac{dx}{dt} = \frac{1}{\sqrt{2}} \frac{1}{2} t^{\frac{-1}{2}} = \frac{1}{2\sqrt{2}}\, t^{\frac{-1}{2}}$$

$$dx = \frac{1}{2\sqrt{2}}\, t^{\frac{-1}{2}} dt$$

Now limit point from (i) Let $x=0 \;=>\; 0=\frac{t}{2} \;=> t=0$ i.e. $x=0, => t=0$

*And* $x=\infty => \infty=\frac{t}{2} => t=\infty \;\therefore$

| | | |
|---|---|---|
| $x$ | $o$ | $\infty$ |
| $t$ | $0$ | $\infty$ |

∴ Integral Solution ( A) becomes

$$I = \int_0^\infty \left(\frac{t^{\frac{1}{2}}}{\sqrt{2}}\right)^9 \frac{1}{2\sqrt{2}}\, t^{\frac{-1}{2}} e^{-t}\, dt$$

$$= \int_0^\infty \frac{1}{(\sqrt{2})^9}\frac{1}{2\sqrt{2}} t^{9/2}\, t^{\frac{-1}{2}} e^{-t}\, dt$$

$$= \frac{1}{(\sqrt{2})^9}\frac{1}{2\sqrt{2}} \int_0^\infty t^{9/2}\, t^{\frac{-1}{2}} e^{-t}\, dt$$

$$= \frac{1}{2^{9/2}}\frac{1}{(2)2^{1/2}} \int_0^\infty t^{9/2}\, t^{\frac{-1}{2}} e^{-t}\, dt$$

$$= \frac{1}{2^{9/2}}\frac{1}{(2)2^{1/2}} \int_0^\infty t^4\, e^{-t}\, dt$$

$$= \frac{1}{(2)2^5} \int_0^\infty t^4\, e^{-t}\, dt$$

$$= \frac{1}{64} \int_0^\infty t^{5-1}\, e^{-t}\, dt$$

$$= \frac{1}{64}\,\Gamma(5) \because \text{ by definition )}$$

$$= \frac{1}{64}\, 4! = \frac{3}{8}$$

$$\therefore \mathrm{I} = \int_0^\infty x^9 e^{-2x^2}\, dx = \frac{3}{8}$$

**Example3:** Evaluate

$$I = \int_0^\infty x^2 e^{-k^2 x^2} \, dx$$

**Solution:**

$$Let\ I = \int_0^\infty x^2 e^{-k^2x^2} \, dx -------------(A)$$

Put $h^2 x^2 = t$ or $x^2 = t / k^2$

$$x^2 = \frac{t}{k^2} ------------- (i)$$

$$\therefore x = \sqrt{\frac{t}{k^2}} = \frac{\sqrt{t}}{k}$$

Differentiating w.r.t. ' t' we get

$$\frac{dx}{dt} = \frac{1}{k}\,\frac{1}{2} t^{-1/2} = \frac{1}{2k}\, t^{\frac{-1}{2}}$$

$$dx = \frac{1}{2k}\, t^{\frac{-1}{2}} dt$$

Now limit point from (i) Let $x=0 \Rightarrow 0 = \frac{t}{k^2} \Rightarrow t= 0$ i.e. $x=0, \Rightarrow t= 0$

*And* $x = \infty \Rightarrow \infty = \frac{t}{k^2} \Rightarrow t = \infty$ ∴

| | | |
|---|---|---|
| $x$ | $o$ | $\infty$ |
| $t$ | $0$ | $\infty$ |

∴ Integral Solution ( A) becomes

$$I = \int_0^\infty (\frac{\sqrt{t}}{k})^2 \frac{1}{2k}\, t^{\frac{-1}{2}} e^{-t}\, dt$$

$$= \int_0^\infty \frac{t}{k^2}\,\frac{1}{2k}\, t^{\frac{-1}{2}} e^{-t}\, dt$$

$$= \frac{1}{2k^3}\int_0^\infty t^{\frac{1}{2}} e^{-t}\, dt$$

$$= \frac{1}{2k^3}\int_0^\infty e^{-t}\, t^{\frac{3}{2}-1}\, dt$$

$$= \frac{1}{2k^3}\,\Gamma\left(\frac{3}{2}\right) \qquad (\text{ by definition })$$

$$= \frac{1}{2k^3}\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{1}{4k^3}\sqrt{\pi} \qquad\qquad \because \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\ ,$$

$\Gamma(n+1) = n\,\Gamma(n)$ , if n is rational number

**Example 4:** Evaluate

**Solution:**

$$Let\ I = \int_0^\infty \sqrt[4]{x}e^{-\sqrt{x}}\, dx = \int_0^\infty x^{1/4} e^{-x^{1/2}}\, dx - - - - -(A)$$

Put $x^{1/2} = t$ or $x = t^2$ ( Squaring on both sides )

$x = t^2 \quad - - - - - - - - - - - - - (i)$

Differentiating w.r.t. ‘ t’ we get

$$\frac{dx}{dt} = 2t \quad \therefore\ dx = 2t\, dt$$

Now limit point from (i) Let $x=0$ => $t = 0$ i.e. $x=0$, => $t= 0$

And $x = \infty$ => $t = \infty$ ∴

| | | |
|---|---|---|
| $x$ | $o$ | $\infty$ |
| $t$ | $0$ | $\infty$ |

∴ Integral Solution ( A) becomes

$$I = \int_0^\infty (t^2)^{1/4} e^{-t}\ 2t\, dt$$

$$= \int_0^\infty t^{1/2}\, 2t\ e^{-t}\ dt$$

$$= 2\int_0^\infty t^{3/2}\ e^{-t}\ dt$$

$$= 2\int_0^\infty e^{-t}\, t^{3/2}\, dt$$

$$= 2\int_0^\infty e^{-t}\, t^{\frac{5}{2}-1}\, dt$$

$$= 2\,\Gamma\left(\frac{5}{2}\right) \qquad \text{( by definition of gamma function )}$$

$$= 2\,\Gamma\left(\frac{5}{2}+1\right) = 2\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right) = 2\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)$$

$$= 3\left(\frac{1}{2}\right)\sqrt{\pi} = \left(\frac{3}{2}\right)\sqrt{\pi}$$

**Type II** $= \displaystyle\int_0^\infty \frac{dx}{a^x}$

Method of solving put $a^x = e^t$

Take log on both sides $\log a^x = \log e^t \Rightarrow x \log a = t \log e$

$x = \dfrac{t}{\log a} \quad \because \log e = 1$

Differentiating w.r.t. 't' we get

$\dfrac{dx}{dt} = \dfrac{1}{\log a} \quad \therefore dx = \dfrac{dt}{\log a}$

- Then checking limit points
- Substitution given integral ( becomes ) reduces to gamma function.

**Example1**: Evaluate $I = \displaystyle\int_0^\infty \frac{x^3}{3^x}\, dx$

**Solution**: Let $I = \displaystyle\int_0^\infty \frac{x^3}{3^x}\, dx$

put $3^x = e^t$, Taking log on both sides

$log\ 3^x = log\ e^t, \Rightarrow x\log 3 = t\log e$

$x = \dfrac{t}{\log 3}$ $---------(i) \quad \because \log e = 1$

Differentiating w.r.t. 't' we get

$$\frac{dx}{dt} = \frac{1}{\log 3} \quad \Rightarrow dx = \frac{dt}{\log 3}$$

Now limits points from (i)

When $x = 0 \Rightarrow \mathrm{t} = 0$ and $x = \infty \Rightarrow \mathrm{t} = \infty$

| | | |
|---|---|---|
| $x$ | $o$ | $\infty$ |
| $t$ | $0$ | $\infty$ |

$$\mathrm{I} = \int_0^\infty \frac{x^3}{3^x} dx = \int_0^\infty \left[\frac{t}{\log 3}\right]^3 \frac{1}{e^t} \frac{dt}{\log 3}$$

$$= \int_0^\infty \frac{t^3}{(\log 3)^4} e^{-t}\, dt \quad = \int_0^\infty \frac{t^{4-1}}{(\log 3)^4} e^{-t}\, dt = \frac{1}{(\log 3)^4} \int_0^\infty t^{4-1} e^{-t}\, dt$$

$$= \frac{1}{(\log 3)^4} \Gamma(4) \quad = \frac{3!}{(\log 3)^4} = \frac{6}{(\log 3)^4}$$

**Example 2**: Evaluate $\mathrm{I} = \int_0^\infty \frac{x^4}{4^x} dx$

**Solution**: Let $\mathrm{I} = \int_0^\infty \frac{x^4}{4^x} dx$

put $4^x = e^t$ , Taking log on both sides

$log\ 4^x = log\ e^t , \Rightarrow x \log 4 = t \log e$

$x = \frac{t}{\log 4}$ $- - - - - - - - -(i)$ $\quad \because \log \mathrm{e} = 1$

Differentiating w.r.t. 't' we get

$$\frac{dx}{dt} = \frac{1}{\log 4} \quad \Rightarrow dx = \frac{dt}{\log 4}$$

Now limits points from (i)

When $x = 0 \Rightarrow \mathrm{t} = 0$ and $x = \infty \Rightarrow \mathrm{t} = \infty$

| | | |
|---|---|---|
| $x$ | $o$ | $\infty$ |
| $t$ | $0$ | $\infty$ |

$$I = \int_0^\infty \frac{x^4}{4^x} dx = \int_0^\infty \left[\frac{t}{\log 4}\right]^4 \frac{1}{e^t} \frac{dt}{\log 4}$$

$$= \int_0^\infty \frac{t^4}{(\log 4)^5} e^{-t}\, dt \quad = \int_0^\infty \frac{t^{5-1}}{(\log 4)^5} e^{-t}\, dt = \frac{1}{(\log 4)^5} \int_0^\infty t^{5-1} e^{-t}\, dt$$

$$= \frac{1}{(\log 4)^5} \Gamma(5) \quad = \frac{4!}{(\log 4)^5} = \frac{24}{(\log 4)^5}$$

**Type III** $= \int_0^1 \log\left(\frac{1}{x}\right) dx \quad \boldsymbol{OR} \int_0^1 (-\log x)\, dx$

Method of solving : put $\log\frac{1}{x} = t \quad \boldsymbol{OR}\ \log x = -t \quad \boldsymbol{OR}\ x = e^{-t} - - - - (i)$

Differentiating w.r.t. 't' we get

$\frac{dx}{dt} = -e^{-t} \quad \therefore\ dx = -e^{-t}\, dt$

- Then checking limit points

Now limits points from (i)

When $x = 0 \Rightarrow e^{-t} = 0$ , t = ∞

When $x = 1 \Rightarrow e^{-t} = 1$ , t = 0

| | | |
|---|---|---|
| *x* | *o* | *1* |
| *t* | ∞ | *0* |

- Substitution given integral ( becomes ) reduces to gamma function.

**Example 1**: Evaluate $I = \int_0^1 \frac{x\, dx}{\sqrt{\log\frac{1}{x}}}$

**Solution**: Let $I = \int_0^1 \frac{x\, dx}{\sqrt{\log\frac{1}{x}}} - - - - - - - - - - - - - (A)$

Let $\log\frac{1}{x} = t$ ***OR*** $\frac{1}{x} = e^t$ ***OR*** $x = e^{-t}$

$x = e^{-t}$ ---------- (i)

$$\frac{dx}{dt} = -e^{-t} \quad \therefore \ dx = -e^{-t}\,dt$$

Now limits points from (i)

When $x = 0 \Rightarrow e^{-t} = 0, \ t = \infty$

When $x = 1 \Rightarrow e^{-t} = 1, \ t = 0$

| x | o | 1 |
|---|---|---|
| t | ∞ | 0 |

$\therefore$ Integral (A) becomes $= \int_{\infty}^{0} \frac{e^{-t}\,(-e^{-t}\,dt)}{\sqrt{t}} = \int_{\infty}^{0} -e^{-2t}\, t^{-1/2}\, dt$

*By using property of integration,* $= \int_{0}^{\infty} e^{-2t}\, t^{-1/2}\, dt = \int_{0}^{\infty} e^{-2t}\, t^{\frac{1}{2}-1}\, dt$

*Using* $\int_{0}^{\infty} e^{-ky}\, y^{n-1}\, dy = \frac{\Gamma(n)}{k^n}$

$$\therefore \int_{0}^{\infty} e^{-2t}\, t^{\frac{1}{2}-1}\, dt = \frac{\Gamma\left(\frac{1}{2}\right)}{2^{\frac{1}{2}}} = \frac{\sqrt{\pi}}{\sqrt{2}} = \sqrt{\frac{\pi}{2}}$$

**Example 2**: Evaluate $I = \int_{0}^{1} \frac{dx}{\sqrt{x\,logx}}$

**Solution**: Let $I = \int_{0}^{1} \frac{dx}{\sqrt{x\,logx}}$ ------------ (A)

Let $\log\frac{1}{x} = t$ ***OR*** $\frac{1}{x} = e^t$ ***OR*** $x = e^{-t}$

$x = e^{-t}$ ---------- (i)

$$\frac{dx}{dt} = -e^{-t} \quad \therefore \ dx = -e^{-t}\,dt$$

Now we check limits points from (i)

When $x = 0 \Rightarrow e^{-t} = 0, \ t = \infty$

When $x = 1 \Rightarrow e^{-t} = 1, \ t = 0$

| | | |
|---|---|---|
| $x$ | $0$ | $1$ |
| $t$ | $\infty$ | $0$ |

$$\therefore \text{Integral (A) becomes} = \int_{\infty}^{0} \frac{-dt\, e^{-t}}{\sqrt{e^{-t}\, t}} = -\int_{\infty}^{0} e^{t/2}\, t^{-1/2}\, dt\ e^{-t}$$

$$By\ using\ property\ of\ integration, = \int_{0}^{\infty} e^{-t/2}\, t^{-1/2}\, dt$$

$$Using \int_{0}^{\infty} e^{-ky}\, y^{n-1}\, dy = \frac{\Gamma(n)}{k^n}$$

$$\therefore \int_{0}^{\infty} e^{-t/2}\, t^{\frac{1}{2}-1}\, dt \ \left(Here, n = 1/2\ , k = 1/2\right) \Rightarrow = \frac{\Gamma\left(\frac{1}{2}\right)}{(1/2)^{\frac{1}{2}}} = \frac{\sqrt{\pi}}{\sqrt{1/2}}$$

$$= \sqrt{2\pi}$$

## 10.6 Beta Function

Beta Function: The definite integral $\int_0^1 x^{m-1}(1-x)^{n-1}\, dx, m > 0, n > 0$ is a function of $m$ and $n$ called Beta Function, denoted by B(m,n) ( we read it as Beta (m,n))

$$\mathbf{B(m,n)} = \int_{0}^{1} \boldsymbol{x^{m-1}(1-x)^{n-1}\, dx, m >} \ 0, n > 0$$

The Beta function is also called as Euler's integral of the first kind. Beta function of negative numbers is not defined.

E.g. 1 ) $B\left(3, \frac{3}{2}\right) = \int_{0}^{1} x^2 (1-x)^{1/2}\, dx,$

2 ) $B\left(5, \frac{5}{2}\right) = \int_{0}^{1} t^4 (1-t)^{3/2}\, dt$

## 10.7 Properties of Beta Function :

**1.** $B(m,n) = B(n,m)$

Proof: $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx, m > 0, n > 0$

$$= \int_0^1 (1-x)^{m-1}(1-(1-x)^{n-1}\,dx,$$

$$\because \int_0^a f(x)dx = \int_0^a f(a-x)dx \quad , Here\ a = 1$$

$$\therefore B(m,n) = \int_0^1 (1-x)^{m-1}.x^{n-1}\,dx = 1\int_0^1 x^{n-1}(1-x)^{m-1}\,dx = B(n,m)$$

| B( m, n ) = B( n, m ) |
|---|

**2.** $\int_0^1 x^m (1-x)^n\,dx = B(m+1, n+1)$

**3.** $B(m,n) = 2\int_0^{\pi/2} sin^{2m-1}\theta \ \ cos^{2n-1}\theta\,d\theta$

Proof: $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx, put\ x = sin^2\theta, dx$

$= 2\sin\theta\, cos\theta\, d\theta$

| x | o | 1 |
|---|---|---|
| $\theta$ | 0 | $\pi/2$ |

$$= \int_0^{\pi/2} sin^{2m-2}\theta \ \ (1- sin^2\theta)^{n-1}\ 2sin\theta cos\theta\,d\theta$$

$$B(m,n) = 2\int_0^{\pi/2} sin^{2m-1}\theta \ \ cos^{2n-1}\theta\,d\theta$$

We consider this as a definition of Beta Function.

Further, let $2m-1 = p, 2n-1 = q \ \therefore m = \frac{p+1}{2}, n = \frac{q+1}{2}$ then

$$B\left(\frac{p+1}{2},\frac{q+1}{2}\right) = 2\int_0^{\pi/2} \sin^p\theta\ \cos^q\theta\ d\theta$$

Standard Formula : $\int_0^{\pi/2} \sin^p\theta\ \cos^q\theta\ d\theta = \frac{1}{2}B\left(\frac{p+1}{2},\frac{q+1}{2}\right)$

**4**. $B(m,n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$

Proof: $B(m,n) = \int_0^{1} x^{m-1}(1-x)^{n-1}\ dx$ , put $x = \frac{t}{1+t}$ ( i.e. $x(1+t)$

$= t$ , $\therefore x + xt = t$)

$\therefore x = t - xt$ OR $t = \frac{x}{1-x}$ ( *Please Note substitiuon*)

$\therefore$ when $x = 0$, $t = \frac{0}{1-0}$ *and* when $x = 1, t = \frac{0}{1-1} = \frac{1}{0} = \infty$

| $x$ | 0 | 1 |
|---|---|---|
| $t$ | 0 | $\infty$ |

Also $dx = \frac{(1+t)(1) - t(1)t}{(1+t)^2}\, dt = \frac{1}{(1+t)^2}\, dt$

$$B(m,n) = \int_0^{\infty} \frac{t^{m-1}}{(1+t)^{m-1}}.\left(1-\frac{t}{1+t}\right)^{n-1}.\frac{dt}{(1+t)^2}$$

$$= \int_0^{\infty} \frac{t^{m-1}\,dt}{(t+1)^{m-1}(1+t)^{n-1}(1+t)^2} = \frac{t^{m-1}dt}{(1+t)^{m+n}}$$

$$B(m,n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$$

( Note: We consider this result also as another definition of Beta Function)

$$B(m+1,n+1) = \int_0^{\infty} x^m(1-x)^n\ dx$$

**5.** Relationbetween Beta and Gamma Function, $B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

**6.** $\int_0^{\pi/2} sin^p\theta \; cos^q\theta \; d\theta = \frac{1}{2}\mathrm{B}\left(\frac{p+1}{2},\frac{q+1}{2}\right) = \frac{\Gamma\left(\frac{p+q}{2}\right)\Gamma(\frac{q+1}{2})}{\Gamma(\frac{p+q+2}{2})}$

Put $p = 0, q = 0$ $\int_0^{\frac{\pi}{2}} d\theta = \frac{1}{2}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)} \Rightarrow \frac{\pi}{2} = \frac{1}{2}\left(\Gamma\left(\frac{1}{2}\right)\right)^2 \quad \therefore \Gamma\left(\frac{1}{2}\right)$
$= \sqrt{\pi}$

## 10.8 Problem based on Beta Function

**Type I : Examples based on definition of Beta Function**

$$\mathrm{B(m,n)} = \int_0^1 x^{m-1}\,(1-x)^{n-1}\;dx$$

**Example 1**: Evaluate $\int_0^1 x^3\left(1-\sqrt{x}\right)^5\;dx$

**Solution**: Let $I = \int_0^1 x^3\left(1-\sqrt{x}\right)^5\;dx$

$put\;\sqrt{x} = t\,, \therefore x = t^2\; Differentiating\; w.r.t.'t'\; weget\;\; \therefore \frac{dx}{dt} = 2t \Rightarrow dx$
$= 2t\;dt$

Now checking limit point by using $x = t^2$

$\therefore$ when $x = 0,\; t^2 = 0 \;\; \Rightarrow t = 0\,, and$ when $x = 1, t^2 = 1 \;\; \Rightarrow t = 1$

| | | |
|---|---|---|
| *x* | *0* | *1* |
| *t* | *0* | *1* |

$\therefore$ the given integral I becomes $I = \int_0^1 (t^2)^3\,(1-t)^5\;2t\;dt$

$$\therefore \quad I = \int_0^1 (t)^6\,(1-t)^5\;2t\;dt = 2\int_0^1 t^7\,(1-t)^5\;\;dt$$

$$\therefore \quad I = \int_0^1 t^{8-1}(1-t)^{6-1}\ dt$$

$$= 2\,B\,(8,6) - - - - - - - - (By\ Definition)$$

Now using Relationbetween Beta and Gamma Function, $B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

$$\therefore \quad I = \frac{\Gamma(8)\Gamma(6)}{\Gamma(8+6)} = 2.\frac{(7!)(5!)}{(13!)} = \frac{1}{5148}$$

**Example 2**: Evaluate $\int_0^1 \frac{dx}{\sqrt[3]{1-x^3}}$

**Solution**: Let $I = \int_0^1 \frac{dx}{\sqrt[3]{1-x^3}}$

$put\ x^3 = t, \therefore x = t^{1/3}\ Differentiating\ w.r.t.'t'\ weget\ \therefore \frac{dx}{dt} = \frac{1}{3}t^{\frac{1}{3}-1}$

$$= \frac{1}{3}t^{-2/3}$$

$\therefore$ dx $= \frac{1}{3}t^{-2/3}dt$ ,  Now checking limit point by using $x = t^{1/3}$

$\therefore$ when $x = 0$,

$t^{1/3} = 0 \Rightarrow t = 0$, *and* when $x = 1, t^{1/3} = 1 \Rightarrow t = 1$

| | | |
|---|---|---|
| *x* | *0* | *1* |
| *t* | *0* | *1* |

$\therefore$ the given integral I becomes $I = \int_0^1 \frac{\frac{1}{3}t^{-2/3}\,dt}{(1-t)^{1/3}}$

$$\therefore I = \frac{1}{3}\int_0^1 t^{-2/3}(1-t)^{-1/3}\,dt = \int_0^1 t^{\frac{1}{3}-1}(1-t)^{\frac{2}{3}-1}\,dt$$

$$\therefore I = \frac{1}{3}\,B\left(\frac{1}{3},\frac{2}{3}\right) - - - - - - By\ Definition$$

Now using Relationbetween Beta and Gamma Function, $B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

$$\therefore I = \frac{1}{3}\frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}+\frac{2}{3}\right)} = \frac{1}{3}\frac{\frac{1}{3}}{\Gamma(1)} = \frac{1}{3}\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right) \quad \because \Gamma(1) = 1$$

$$\therefore I = \frac{1}{3}\Gamma\left(\frac{1}{3}\right)\Gamma\left(1-\frac{1}{3}\right) \qquad o < p < 1, Using\ \Gamma(p)\Gamma(1-p) = \frac{\pi}{sinp\pi}$$

$$\therefore I = \frac{1}{3}\frac{\pi}{\sin\frac{\pi}{3}} = \frac{1}{3}\frac{\pi}{\frac{\sqrt{3}}{2}} = \frac{2\pi}{3\sqrt{3}}$$

**Type II – Examples Based on**

$$\int_0^{\pi/2} sin^p\theta\ cos^q\theta\ d\theta = \frac{1}{2}B\left(\frac{p+1}{2},\frac{q+1}{2}\right) = \frac{\Gamma\left(\frac{p+q}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}$$

**Example 1**: Find $\int_0^{\pi/2} \sqrt{tan\theta}\ d\theta$

**Solution**: Let $I = \int_0^{\pi/2} \sqrt{tan\theta}\ d\theta = \int_0^{\pi/2} \sqrt{\frac{sin\theta}{cos\theta}}\ d\theta$

$$\therefore I = \int_0^{\pi/2} sin^{1/2}\theta\ cos^{-1/2}\theta\ d\theta$$

put $p = {}^{1}/_{2}, q = {}^{-1}/_{2}$ Using result $\int_0^{\frac{\pi}{2}} sin^p\theta\ cos^q\theta\ d\theta$

$$= \frac{1}{2}B\left(\frac{p+1}{2},\frac{q+1}{2}\right)$$

$$\therefore I = \frac{1}{2}B\left(\frac{\frac{1}{2}+1}{2},\frac{\frac{-1}{2}+1}{2}\right) = \frac{1}{2}B\left(\frac{3}{4},\frac{1}{4}\right)$$

$$\therefore I = \frac{1}{2}\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}+\frac{1}{4}\right)} = \frac{1}{2}\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma(1)} = \frac{1}{2}\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right) \quad \because \Gamma(1) = 1$$

$$\text{Using } \Gamma(p)\Gamma(1-p) = \frac{\pi}{sinp\pi}$$

$$\therefore \mathrm{I} = \frac{1}{2}\Gamma\left(1-\frac{1}{4}\right)\Gamma\left(\frac{1}{4}\right) = \frac{1}{2}\,\frac{\pi}{\sin\frac{\pi}{4}} = \frac{1}{2}\,\frac{\pi}{\frac{1}{\sqrt{2}}} = \frac{\sqrt{2}\pi}{2} = \frac{\pi}{\sqrt{2}}$$

**Note**: Similarly we can show $\int_0^{\pi/2} \sqrt{cot\theta}\ d\theta = \frac{\pi}{\sqrt{2}}$

**Example 2**: Find $\int_0^{\pi/2} \sqrt{cot\theta}\ d\theta$

**Solution**: Let $\mathrm{I} = \int_0^{\pi/2} \sqrt{tan\theta}\, d\theta = \int_0^{\pi/2} \sqrt{\frac{cos\theta}{sin\theta}}\ d\theta$

$$\therefore\ \mathrm{I} = \int_0^{\pi/2} sin^{-1/2}\theta\ cos^{1/2}\theta\, d\theta$$

put $\mathrm{p} = {-1}/{2}$, $\mathrm{q} = {1}/{2}$ Using result $\int_0^{\frac{\pi}{2}} sin^p\theta\ cos^q\theta\, d\theta$

$$= \frac{1}{2}\mathrm{B}\left(\frac{\mathrm{p}+1}{2}, \frac{\mathrm{q}+1}{2}\right)$$

$$\therefore\ \mathrm{I} = \frac{1}{2}\mathrm{B}\left(\frac{\frac{-1}{2}+1}{2}, \frac{\frac{1}{2}+1}{2}\right) = \frac{1}{2}\mathrm{B}\left(\frac{1}{4}, \frac{3}{4}\right)$$

$$\therefore\ \mathrm{I} = \frac{1}{2}\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}+\frac{1}{4}\right)} = \frac{1}{2}\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma(1)} = \frac{1}{2}\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right) \quad \because\ \Gamma(1) = 1$$

Using $\Gamma(\mathrm{p})\Gamma(1-\mathrm{p}) = \frac{\pi}{sinp\pi}$

$$\therefore \mathrm{I} = \frac{1}{2}\Gamma\left(1-\frac{1}{4}\right)\Gamma\left(\frac{1}{4}\right) = \frac{1}{2}\,\frac{\pi}{\sin\frac{\pi}{4}} = \frac{1}{2}\,\frac{\pi}{\frac{1}{\sqrt{2}}} = \frac{\sqrt{2}\pi}{2} = \frac{\pi}{\sqrt{2}}$$

**Type III** – Examples based on $\mathrm{B}(\mathrm{m},\mathrm{n}) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$

**Example 1**: **Find** $\int_0^{\infty} \frac{x^8 - x^{14}}{(1+x)^{24}}\, dx$

**Solution** : Let I = $\int_0^\infty \frac{x^8 - x^{14}}{(1+x)^{24}}\, dx$

$$= \int_0^\infty \frac{x^8}{(1+x)^{24}}\, dx - \int_0^\infty \frac{x^{14}}{(1+x)^{24}}\, dx$$

$$= \int_0^\infty \frac{x^{9-1}}{(1+x)^{9+15}}\, dx - \int_0^\infty \frac{x^{15-1}}{(1+x)^{15+9}}\, dx$$

$$= \mathrm{B}(9, 15) - B(15, 9) \quad \because \mathrm{B(m, n) = B(m, n)}$$

$\therefore\ I = 0$

**Example : Find** $\int_0^\infty \frac{x^9(1-x^5)}{(1+x)^{25}}\, dx$

**Solution** : Let I = $\int_0^\infty \frac{x^9(1-x^5)}{(1+x)^{25}}\, dx = \int_0^\infty \frac{(x^9 - x^{14})}{(1+x)^{25}}\, dx$

$$= \int_0^\infty \frac{x^9}{(1+x)^{25}}\, dx - \int_0^\infty \frac{x^{14}}{(1+x)^{25}}\, dx$$

$$= \int_0^\infty \frac{x^{10-1}}{(1+x)^{10+15}}\, dx - \int_0^\infty \frac{x^{15-1}}{(1+x)^{15+10}}\, dx$$

$$= \mathrm{B}(10, 15) - B(15, 10) \quad \because \mathrm{B(m, n) = B(m, n)}$$

$$\therefore\ I = 0$$

**Example 3 : Prove that** $\int_0^\infty \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}$

**Solution** : Let I = $\int_0^\infty \frac{dx}{1+x^4} = \int_0^\infty \frac{dx}{1+(x^2)^2}$

$put\ x^2 = \tan\theta\ , \therefore x = \sqrt{tan\theta}\ Differentiating\ w.r.t.'x'\ weget$

$\therefore 2x\frac{dx}{d\theta} = sec^2\theta \quad \therefore 2x\, dx = sec^2\theta\, d\theta \quad \therefore dx = \frac{sec^2\theta\, d\theta}{2x} = \frac{sec^2\theta\, d\theta}{2\sqrt{tan\theta}}$

$\therefore\therefore dx = \frac{sec^2\theta\, d\theta}{2\sqrt{tan\theta}}$ , Now checking limit point by using $x^2 = \tan\theta$

$\therefore$ when $x = 0$,

$$0 = \tan\theta \;\Rightarrow\; \theta = 0\,, and \text{ when } x = \infty\,, \Rightarrow\; \theta = \frac{\pi}{2} \because\; tan\frac{\pi}{2} = \infty$$

| $x$ | $0$ | $1$ |
|---|---|---|
| $\theta$ | $0$ | $\pi/2$ |

$\therefore$ the given integral I becomes , $= \int_0^{\pi/2} \dfrac{\dfrac{sec^2\theta\, d\theta}{2\sqrt{tan\theta}}}{1 + tan^2\theta}$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{sec^2\theta}{\sqrt{tan\theta}}\,\frac{d\theta}{(sec^2\theta)} = \frac{1}{2}\int_0^{\pi/2} \frac{1}{\sqrt{tan\theta}}\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \sqrt{cot\theta}\;\, d\theta = \int_0^{\frac{\pi}{2}} sin^{-1/2}\theta\; cos^{1/2}\theta\; d\theta$$

put p = $-1/2$, q = $1/2$ Using result $\int_0^{\frac{\pi}{2}} sin^p\theta\; cos^q\theta\; d\theta$

$$= \frac{1}{2}\mathrm{B}\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$\therefore\; \mathrm{I} = \frac{1}{2}\,\frac{1}{2}\mathrm{B}\left(\frac{\frac{-1}{2}+1}{2}, \frac{\frac{1}{2}+1}{2}\right) = \frac{1}{4}\mathrm{B}\left(\frac{1}{4}, \frac{3}{4}\right)$$

$$\therefore\; \mathrm{I} = \frac{1}{4}\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}+\frac{1}{4}\right)} = \frac{1}{4}\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma(1)} = \frac{1}{4}\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right) \;\because\; \Gamma(1) = 1$$

$$\text{Using } \Gamma(p)\Gamma(1-p) \quad = \frac{\pi}{sinp\pi}$$

$$\therefore \mathrm{I} = \frac{1}{4}\Gamma\left(1-\frac{1}{4}\right)\Gamma\left(\frac{1}{4}\right) = \frac{1}{4}\,\frac{\pi}{\sin\frac{\pi}{4}} = \frac{1}{4}\,\frac{\pi}{\frac{1}{\sqrt{2}}} = \frac{1}{2\sqrt{2}\sqrt{2}}\,\frac{\pi}{\frac{1}{\sqrt{2}}} = \frac{\pi}{2\sqrt{2}}$$

**(New Type )Example 4 : Evaluate** $\int_3^5 (x-3)^{1/2}(5-x)^{1/2}dx$

**Solution** : Let I = $\int_{3}^{5} (x-3)^{1/2}(5-x)^{1/2}dx$

$put\ (x-3) = (\ Upper\ Limit - Lower\ Limit\ )\ t$

$\therefore (x-3) = (\ 5-3)\ t \quad \therefore (x-3) = 2t \quad \therefore x = (2t+3)$

$\therefore \frac{dx}{dt} = 2\ , \qquad \therefore dx = 2\ dt$ Now checking limit point by using $x = (2t+3)$

$\therefore$ when $x = 3, \qquad 2t = 0 \Rightarrow t = 0\ , and$ when $x = 5\ , \Rightarrow 5 = 2t + 3 \Rightarrow t$
$= 1$

| | | |
|---|---|---|
| *x* | *3* | *5* |
| *t* | *0* | *1* |

$\therefore\ I = \int_{0}^{1} (2t)^{1/2}(5-(2t+3))^{1/2}\ 2dt$

$= \int_{0}^{1} 2\sqrt{2}\ t^{1/2}(2-2t)^{1/2}\ dt$

$= 2\sqrt{2}\int_{0}^{1} t^{1/2}\ (2)^{1/2}(1-t)^{1/2}\ dt$

$= 2\sqrt{2}\ \sqrt{2}\int_{0}^{1} t^{1/2}\ (1-t)^{1/2}\ dt$

$= 4\int_{0}^{1} t^{\frac{3}{2}-1}\ (1-t)^{\frac{3}{2}-1}\ dt$

$= 4B(\frac{3}{2}, \frac{3}{2})$

$= 4\ \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}+\frac{3}{2}\right)} = 4\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(3)} = 4\frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\Gamma(3)}$

$= \frac{4}{2!}\frac{1}{2}\Gamma(\pi)\ \frac{1}{2}\Gamma(\pi) = \frac{\pi}{2}$

## 10.9 Duplication Formula of Gamma Functions

$$\mathbf{\Gamma(m)\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m)}$$

**Proof**: Consider , $\frac{1}{2}\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)} = \int_0^{\frac{\pi}{2}} sin^p\theta\ cos^q\theta\, d\theta$

$= \frac{1}{2}B\left(\frac{p+1}{2},\frac{q+1}{2}\right)$

Put $= = 2m-1, q = q = 2m-1 \quad \left(\text{i.e. } m = \frac{p+1}{2}, m = \frac{q+1}{2}\right)$

$$\frac{1}{2}\frac{\Gamma(m)\Gamma(m)}{\Gamma(2m)} = \int_0^{\frac{\pi}{2}} sin^{2m-1}\theta\ cos^{2m-1}\theta\, d\theta$$

$$\frac{\Gamma(m)\Gamma(m)}{\Gamma(2m)} = \frac{2}{2^{2m-1}}\int_0^{\frac{\pi}{2}} (2sin\,\theta cos\,\theta)^{2m-1}\, d\theta$$

$$= \frac{2}{2^{2m-1}}\int_0^{\frac{\pi}{2}} (sin2\,\theta)^{2m-1}\, d\theta \quad , \quad Put\ 2\theta = t\ , \therefore\ d\theta$$

$= \frac{1}{2}dt$

| $\theta$ | $0$ | $\pi/2$ |
|---|---|---|
| $t$ | $0$ | $\pi$ |

$$= \frac{1}{2^{2m-1}}\int_0^{\frac{\pi}{2}} (sin\,t)^{2m-1}\, dt = \frac{1}{2^{2m-1}}\, 2\int_0^{\frac{\pi}{2}} sin^{2m-1}\, t.dt \quad [\because f(\pi - t)$$

$= f(t)]$

$$= \frac{2}{2^{2m-1}}\int_0^{\frac{\pi}{2}} sin^{2m-1}\, t.cos^0\, t.dt$$

$$= \frac{\Gamma(m)\Gamma(m)}{\Gamma(2m)} = \frac{2}{2^{2m-1}}\frac{1}{2}\frac{\Gamma\left(\frac{2m-1+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{\Gamma\left(\frac{2m-1+0+2}{2}\right)}$$

$$= \frac{1}{2^{2m-1}}\frac{\Gamma(m)\sqrt{\pi}}{\Gamma\left(m+\frac{1}{2}\right)}$$

$$\therefore \boldsymbol{\Gamma(m)\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m)}$$

## 10.10 Exercise

1. Prove that $\int_0^\infty \sqrt{x}\, e^{-\sqrt[3]{x}}\ dx = \frac{315}{16}\sqrt{\pi}$ (**Hint** : $x = t^3$ )

2. Prove that $\int_0^\infty x^7\, e^{-2x^2}\ dx = \frac{3}{16}$ (**Hint** : $2x^2 = t$ )

3. Prove that $\int_0^\infty x^2\, e^{-h^2x^2}\ dx = \frac{\sqrt{\pi}}{4h^3}$ (**Hint** : $h^2x^2 = t$ )

4. Prove that $\int_0^\infty \sqrt{y}\, e^{-y^3}\ dy = \frac{\sqrt{\pi}}{3}$ (**Hint** : $y^3 = t$ )

5. Prove that $\int_0^1 \frac{dx}{\sqrt{x \log \frac{1}{x}}} = \sqrt{2\pi}$ $\left(\textbf{Hint} : \log\frac{1}{x} = t\right)$

6. Prove that $\int_0^1 \frac{dx}{\sqrt{-\log x}} = \sqrt{\pi}$ (**Hint** : $-\log x = t$ )

7. Evaluate $\int_0^1 x^3\ (1-\sqrt{x})^5\, dx$ (**Hint** : $\sqrt{x} = t$ ) ***Ans***: $\frac{1}{5148}$

8. Evaluate $\int_0^n n\ (n-x)^p\, dx$ (**Hint** : $x = nt$ ) ***Ans***: $n^{n+p+1}B(n+1, p+1)$

9. Prove that B(m, n)

$$= \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$\left(\textbf{Hint}: \text{Put } x = \frac{1}{1+t} \text{ in the defnition of } B(m,n)\right)$$

10. Show that $\int_0^2 x\,(8-x^3)^{1/3}\,dx = \frac{2\pi}{3\sqrt{3}}$

$$\left(\textbf{Hint}: \text{Put } x^3 = t, Use\ \Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin p\pi}, 0 < p < 1\right)$$

## 10.12 Summary

In this unit we learn Gamma and Beta Function and its Duplication Formula

**Gamma Function**

$\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}dx \quad (n > 0)$

$\Gamma(n) = 2\int_0^\infty e^{-t^2}t^{2n-1}\,dt$

$\Gamma(1) = 1$

$\Gamma(n+1) = n\,\Gamma(n)$

$\Gamma(n+1) = n\,\Gamma(n)$ in general , n is rational number

$= n!$ if n is a positive integer

$\Gamma(0) = \infty$ , $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ , $\Gamma(n+1) = n!$

Type I – $\int_0^\infty e^{-ax^m}\,dx$ , Type II $= \int_0^\infty \frac{dx}{a^x}$ , Type III $=$ $\int_0^1 \log\left(\frac{1}{x}\right)dx$ $OR$ $\int_0^1(-logx)\,dx$

**Beta Function**

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx, m > 0, n > 0$$

$B(m,n) = B(n, m)$

$$\int_0^1 x^m(1-x)^n\,dx = B(m+1, n+1)$$

$$B(m,n) = 2\int_0^{\pi/2} sin^{2m-1}\theta \quad cos^{2n-1}\theta\, d\theta$$

$$B(m,n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$$

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

$$\int_0^{\frac{\pi}{2}} sin^p\theta \; cos^q\theta \; d\theta = \frac{1}{2}B\left(\frac{p+1}{2},\frac{q+1}{2}\right) = \frac{\Gamma\left(\frac{p+q}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}$$

**Duplication Formula**

$$\Gamma(m)\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m)$$

## 10.11 References


1. A Text Book of Applied Mathematics Vol I - P. N. Wartikar and J. N. Wartikar
2. Applied Mathematics II - P. N. Wartikar and J. N. Wartikar
3. Higher Engineering Mathematics - Dr. B. S. Grewal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Unit 5

# 11

# DIFFERENTIATION UNDER THE INTEGRAL SIGN ( DUIS ) & ERROR FUNCTIONS

**Unit Structure**





## 11.0 Objectives

After going through this unit, you will be able to:

- Understand the concept of Differential Under the integral sign ( DUIS) and Error Functions
- Solve the problem based on Leibnitz's Rule.
- Know the concept of Differentiation and Integration of Error Function.

## 11.1 Introduction

Not all integrals can be evaluated using analytical techniques, such as integration by substitution, by parts or by partial fractions. People come up with different ways of solving the integrals and DUIS is one of them. It is an effective technique used in evaluation of real definite integrals. When a definite integral $I = \int_a^b f(x,\alpha)dx$, which is to be integrated w.r.t. variable x and contains parameter , by using DUIS.

There are different rules when limits of integral are constants or functions of parameter $\alpha$ When DUIS technique is used , the definite integral evaluation results into an ordinary differential equation, the solution of this equation results in the evaluation of definite integral. The technique is very useful in Laplace Transform. Error function integral is close to Probability Integral and is used in probability distribution. Complementary error functions are involved in finding inverse Laplace transforms of complicated functions.

## 11.2 Rule I : Integral With Limits ( a,b) as Constants

If $\mathrm{I}(\alpha) = \int_a^b f(x,\alpha)dx$, Where a and b are constants , then

$$\frac{\mathrm{dI}}{\mathrm{d}\alpha} = \int_a^b \frac{\partial}{\partial\alpha} f(x,\alpha)dx$$

**Proof**: $\frac{\mathrm{dI}}{\mathrm{d}\alpha} = \lim_{\delta\alpha\to 0} \frac{\mathrm{I}((\alpha+\delta\alpha) - I(\alpha)}{\delta\alpha}$

$$= \lim_{\delta\alpha\to 0} \frac{1}{\delta\alpha}\left[\int_a^b f(x,\alpha+\delta\alpha)dx - \int_a^b f(x,\alpha)dx\right]$$

$$= \lim_{\delta\alpha\to 0} \frac{1}{\delta\alpha} \int_a^b [f(x,\alpha+\delta\alpha) - f(x,\alpha)]dx$$

$$= \lim_{\delta\alpha\to 0} \int_a^b \left[\frac{f(x,\alpha+\delta\alpha) - f(x,\alpha)}{\delta\alpha}\right] dx$$

$$= \int_a^b \lim_{\delta\alpha\to 0} \left[\frac{f(x,\alpha+\delta\alpha) - f(x,\alpha)}{\delta\alpha}\right] dx$$

$$\therefore \quad \frac{\mathrm{dI}}{\mathrm{d}\alpha} = \int_a^b \frac{\partial}{\partial\alpha} f(x,\alpha)dx \qquad (\,by\ Definition\ of partial\ derivative\,)$$

$$\textbf{Rule} - \textbf{I : If I}(\boldsymbol{\alpha}) = \int_a^b \boldsymbol{f(x,\alpha)dx\ then} \quad \frac{\mathbf{dI}}{\mathbf{d}\boldsymbol{\alpha}} = \int_a^b \frac{\boldsymbol{\partial}}{\boldsymbol{\partial\alpha}} \boldsymbol{f(x,\alpha)dx}$$

It may be noted that if integral involves two parameters *'x'* and '$\alpha'$' integration is to be carried out w.r.t variable '*x'* treating $\alpha'$ as constant. Rule (I ) gives method to differentiate integral w.r.t. parameter $\alpha$

**Example1**: Evaluate $\int_0^\infty \frac{e^{-x}}{x}(1-e^{-ax})dx \ \ (a > -1)$

**Solution**: $\phi(a) = \int_0^\infty \frac{e^{-x}}{x}(1-e^{-ax})dx$

Differentiating w.r.t. a, $\frac{d\phi}{da} = \int_0^\infty \frac{\partial}{\partial a}\left[\frac{e^{-x}}{x}(1-e^{-ax})\right]dx$

$$\frac{d\phi}{da} = \int_0^\infty \frac{e^{-x}}{x}(-e^{-ax})(-x)\,dx$$

$$= \left[\frac{e^{-(a+1)x}}{-(a+1)}\right]_0^\infty = \frac{e^{-\infty}}{-(a+1)} - \frac{e^0}{-(a+1)} \quad \because a+1 > 0 \ i.e.\ a > -1$$

$$= \mathbf{0} + \frac{1}{(a+1)}$$

$$\therefore d\phi = \frac{da}{(a+1)}$$

$\phi(a) = \log(a+1) + C$

To determine C, put $a = 0 \ \therefore \phi(0) = 0 + C$

But , $\phi(0) = \int_0^\infty \frac{e^{-x}}{x}(1-1)(-x)\,dx = 0 \quad \therefore C = 0$

*Hence*, $\phi(a) = \log(a+1)$

**Example2** : Prove that Evaluate $\int_0^1 \frac{x^a - 1}{\log x}dx = \log(1+a)\,;\ a \geq 0$

**Solution**: Let $\phi(a) = \int_0^1 \frac{x^a - 1}{\log x}dx$

Differentiating w.r.t. a, $\frac{d\phi}{da} = \int_0^1 \frac{\partial}{\partial a}\left[\frac{x^a - 1}{\log x}\right]dx$

$$\frac{d\phi}{da} = \int_0^1 \frac{1}{\log x}\,.x^a \log x\,.dx$$

$$\frac{d\phi}{da} = \int_0^1 x^a\,dx = \left[\frac{x^{a+1}}{a+1}\right]_0^1 \quad \because \quad a \geq 0$$

.

$$\therefore \quad d\phi = \frac{1}{(a+1)}\,da$$

$\therefore \quad \phi(a) = \log(a+1) + C$

To determine C, put $a = 0 \;\therefore \phi(0) = \; 0 + C$

But , $\phi(0) = \int_0^1 \frac{x^0 - 1}{\log x}dx = 0 \quad , \quad \therefore C = 0$

$$Hence, \phi(a) = \log(a+1)$$

**Example 3** : Prove that $\int_0^\infty \frac{\cos\lambda x}{x}\,(e^{-ax} - e^{-bx})dx$

$$= \frac{1}{2}\log\left(\frac{b^2+\lambda^2}{a^2+\lambda^2}\right) \;; a > 0 \,, b > 0$$

**Solution**: Let $\quad \phi(a) = \int_0^\infty \frac{\cos\lambda x}{x}\,(e^{-ax} - e^{-bx})dx$

Differentiating w.r.t. a, $\quad \frac{d\phi}{da} = \int_0^\infty \frac{\partial}{\partial a}\left[\frac{\cos\lambda x}{x}(e^{-ax} - e^{-bx})\right]dx$

$$\frac{d\phi}{da} = \int_0^\infty \frac{\cos\lambda x}{x}\,[e^{-ax}(-x) - 0]dx = -\int_0^\infty e^{-ax}\,\cos\lambda x\;dx$$

$$= -\left[\frac{e^{-ax}}{a^2+\lambda^2}\,(-a\,cos\lambda x + \;\lambda\,sin\lambda x)\right]_0^\infty$$

By Using , $\quad \int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\,(a\,cosbx + \; b\,sinbx)$

$$\frac{d\phi}{da} = \; -\left[0 - \frac{e^0}{a^2+\lambda^2}(-a+0)\right] = \frac{-a}{a^2+\lambda^2}$$

$$d\phi \; = \frac{-a}{a^2+\lambda^2}\,da$$

$\therefore \quad \phi(a) = \frac{-1}{2}\int \frac{2a}{a^2+\lambda^2}\,da$

$$\therefore \ \phi(a) = \frac{-1}{2}\,[\log\,(a^2+\lambda^2)] + C$$

To determine C, put $a = b \ \therefore \phi(b) = \frac{-1}{2}\,[\log\,(b^2+\lambda^2)] + C$

$$\phi(b) = \int_0^{\infty} \frac{cos\lambda x}{x}\,(e^{-bx} - e^{-bx})\,dx + C = 0$$

$$\therefore \ C = \frac{1}{2}\ \log\,(b^2+\lambda^2)$$

$$\therefore \ \phi(a) = -\frac{1}{2}\ \log\,(a^2+\lambda^2) + \frac{1}{2}\ \log\,(b^2+\lambda^2) = \frac{1}{2}\log\left(\frac{b^2+\lambda^2}{a^2+\lambda^2}\right)$$

**Example 4** : Show that $\int\limits_0^{\infty} \frac{tan^{-1}(ax)}{x(1+x^2)}.dx = \frac{\pi}{2}\log(1+a\,)$

**Solution**: Let $\phi(a) = \int\limits_0^{\infty} \frac{tan^{-1}(ax)}{x(1+x^2)}.dx$

Differentiating w.r.t. a, $\frac{d\phi}{da} = \int\limits_0^{\infty} \frac{\partial}{\partial a}\left[\frac{tan^{-1}(ax)}{x(1+x^2)}\right] dx$

$$\frac{d\phi}{da} = \int_0^{\infty} \frac{1.(x)}{(1+a^2x^2)}.\frac{1}{x(1+x^2)}\,dx$$

$$\frac{d\phi}{da} = \int_0^{\infty} \frac{dx}{(1+a^2x^2)(1+x^2)}$$

$$\frac{d\phi}{da} = \int_0^{\infty} \left(\frac{\frac{1}{(1-1/a^2}}{1+a^2x^2} + \frac{\frac{1}{1-a^2}}{1+x^2}\right) dx$$

$$\frac{d\phi}{da} = \frac{1}{1-a^2}\left[\int_0^{\infty} \frac{dx}{1+x^2} - \int_0^{\infty} \frac{a^2}{1+a^2x^2}\right] dx$$

$$\frac{d\phi}{da} = \frac{1}{1-a^2}\,[tan^{-1}x - a\,tan^{-1}(ax)]_0^{\infty}\,dx$$

$$= \frac{1}{1-a^2}\left[\frac{\pi}{2} - a.\frac{\pi}{2}\right] = \frac{\pi}{2}\,\frac{(1-a)}{(1-a)(1+a)} = \frac{\pi}{2}.\frac{1}{(1+a)}$$

$$d\phi = \frac{\pi}{2}.\frac{da}{(1+a)} \ \therefore \ \phi(a) = \frac{\pi}{2}\log(1+a) + C$$

To determine C, we put a = 0 $\therefore \ \phi(0) = \ C$

$$\therefore \ \phi(0) = \int_0^\infty \frac{tan^{-1}(0)}{x(1+x^2)}\, dx \ = 0 \ \ \therefore C = 0$$

$$\phi(a) = \frac{\pi}{2}\log(1+a)$$

**Example 5** : Evaluate $\int_0^\pi \frac{dx}{a+b\,cosx}$ , $a > 0, \ |b| < a \ \ and \ \ deduce\ following$

$$\int_0^\pi \frac{dx}{(a+b\,cosx)^2} = \frac{\pi a}{(a^2-b^2)^{3/2}} \ \ and \ \ \int_0^\pi \frac{cosx\,dx}{(a+b\,cosx)^2} = \frac{-\pi b}{(a^2-b^2)^{3/2}}$$

**Solution**: Let $\quad I = \int_0^\pi \frac{dx}{a+b\,cosx} \ Put\ t = tan\frac{x}{2}\ , dx = \frac{2dt}{1+t^2}\ , cosx = \frac{1-t^2}{1+t^2}$

| x | 0 | π |
|---|---|---|
| t | 0 | ∞ |

$$I = \int_0^\infty \frac{\frac{2dt}{1+t^2}}{a+b\frac{1-t^2}{1+t^2}} = 2\int_0^\infty \frac{dt}{a+at^2+b+bt^2}$$

$$= \ 2\int_0^\infty \frac{dt}{(a+b)+(a-b)t^2} = \frac{2}{(a-b)}\int_0^\infty \frac{dt}{\frac{(a+b)}{(a-b)}+t^2}$$

$$= \frac{2}{(a-b)}\left[\frac{1}{\sqrt{\frac{a+b}{a-b}}}\, tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\ \ t\right)\right]_0^\infty$$

$$= \frac{2}{\sqrt{a^2-b^2}}\,(\,tan^{-1}\infty - \ \ tan^{-1}0) = \frac{2}{\sqrt{a^2-b^2}}\,\frac{\pi}{2}$$

$$\therefore \int_0^\pi \frac{dx}{a+b\,cosx} = \frac{\pi}{\sqrt{a^2-b^2}} \text{-------------- (1)}$$

*Differentiating (1) both sides w.r.t. a*

$$\int_0^\pi \frac{\partial}{\partial a}\left[\frac{1}{a+b\,cosx}\right] dx \ = \ \frac{\pi}{2}\,(a^2-b^2)^{3/2}\,.2a$$

$$\int_0^\pi -\frac{1}{(a+bcosx)^2}\,dx \ = \ -\pi a\ \ (a^2-b^2)^{3/2}$$

$$\therefore \int_0^{\pi} \frac{1}{(a+bcosx)^2}dx = \frac{\pi a}{(a^2-b^2)^{3/2}} --------------- (2)$$

*Differentiating (1) both sides w.r.t. b*

$$\int_0^{\pi} \frac{\partial}{\partial a}\left[\frac{1}{a+b\ cosx}\right]dx = \frac{\pi}{2}(a^2-b^2)^{3/2}.(-2b)$$

$$\int_0^{\pi} \frac{-1.cosx}{(a+bcosx)^2}dx = \frac{\pi b}{(a^2-b^2)^{3/2}}$$

$$i.e.\ \int_0^{\pi} \frac{cosx}{(a+bcosx)^2}dx = \frac{-\pi b}{(a^2-b^2)^{3/2}} ----------- (3)$$

*Hence (1), (2) , (3) are the required results.*

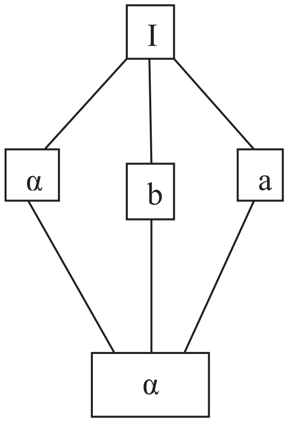
## 11.3 Rule – II Integral With Limits as Functions of the Parameter : Leibnintz's Rule

If $I(\alpha)$

$= \int_{a(\alpha)}^{b(\alpha)} f(x,\alpha)dx,$ where a and bare functions of the parameter $\alpha$ , *then* ,

$$\frac{dI}{d\alpha} = \int_{a(\alpha)}^{b(\alpha)} \frac{\partial}{\partial\alpha}\{f(x,\alpha)\}dx + f(b,\alpha)\frac{db}{d\alpha} - f(a,\alpha)\frac{da}{d\alpha}$$

**Proof**: Since the parameter α enters into the function I( α)due to the interal f(x, α)and due to the limits a, b which are functions of α, we express this by denoting I( α)as I( α) = ϕ( α, b, a), from the below tree diagram, we get




$$\frac{dI}{d\alpha} = \frac{\partial I}{\partial\alpha}(I) + \frac{\partial I}{\partial b}\frac{db}{d\alpha} + \frac{\partial I}{\partial a}\frac{da}{d\alpha} --------------- (I)$$

$$\text{Now, } I(\alpha) = \int_{a(\alpha)}^{b(\alpha)} f(x,\alpha)dx \;\; BY\ using\ Rule\ I, \frac{\partial I}{\partial \alpha}\text{(I)}$$

is obtained by treating a, b as constants,

$$\text{We have, } \frac{\partial I}{\partial \alpha} = \int_{a(\alpha)}^{b(\alpha)} \frac{\partial}{\partial \alpha} f(x,\alpha)dx \quad ------------------(II)$$

$$\text{Let the definite integral be representd as } \int f(x,\alpha)dx = \psi\,(x,\alpha),$$

$$\text{i. e.} \frac{\partial}{\partial x}\,[\,\psi\,(x,\alpha)] = f(x,\alpha) -------------------\,(III)$$

$$\text{Hence } \phi(\,\alpha, b, a) = I(\alpha)$$

$$= \int_{a(\alpha)}^{b(\alpha)} f(x,\alpha)dx = [\,\psi\,(x,\alpha)]_{a(\alpha)}^{b(\alpha)} = \psi\,(b,\alpha) - \psi\,(a,\alpha) -- \\ ----------------------(IV)$$

Hence from IV , we get ,

$$\frac{\partial I}{\partial b} = \frac{\partial \phi}{\partial b} = \frac{\partial}{\partial b}\psi\,(b,\alpha) = f(b,\alpha) \quad (\text{from III}) ----------\,(V)$$

$$\frac{\partial I}{\partial a} = \frac{\partial \phi}{\partial a} = -\frac{\partial}{\partial b}\psi\,(a,\alpha) = -f(a,\alpha) \quad (\text{from III}) --------\,(VI)$$

Hence substituting from equations (II), (V), (VI)in (I)we get

**Rule II**: $$\frac{dI}{d\alpha} = \frac{d}{d\alpha}\int_{a(\alpha)}^{b(\alpha)} f(x,\alpha)dx = \int_{a(\alpha)}^{b(\alpha)} \frac{\partial}{\partial a} f(x,\alpha)dx + f(b,\alpha)\frac{db}{d\alpha} - f(a,\alpha)\frac{da}{d\alpha}$$

**Example 1**

: Verify the rule of differentiation under integral sign for the integral

$$\int_a^{a^2} \log(\,ax)dx$$

**Solution :** $$\phi(a) = \int_a^{a^2} \log(\,ax)dx$$

$$\frac{d\phi}{da} = \int_a^{a^2} \frac{\partial}{\partial a}\log(ax)dx + \left\{\frac{d}{da}(a^2)\right\}\log(a.a^2) - \left\{\frac{d}{da}(a)\right\}.\log a^2$$

$$= \int_a^{a^2} \frac{1}{ax}.x.dx + 2a.\log(a^3) - 2\log a$$

$$= \left[\frac{1}{a}x\right]_a^{a^2} + 6a.\log(a) - 2\log a$$

$$= \frac{1}{a}(a^2 - a) + 6a.\log(a) - 2\log a$$

$$= (a-1) + 6a.\log(a) - 2\log a \text{ ------------------ } (1)$$

$$\phi(a) = \int_a^{a^2} \log(ax).1.dx = [\log(ax).x]_a^{a^2} - \int_a^{a^2} \frac{1}{ax}.a.x.dx$$

$$= a^2\log a^3 - a\log a^2 - [x]_a^{a^2} = 3a^2\log a - 2a\log a - (a^2 - a)$$

$$\frac{d\phi}{da} = 6a\log a + 3a^2.\frac{1}{a} - 2\log a - 2a.\frac{1}{a} - (2a-1)$$

$$\frac{d\phi}{da} = 6a\log a - 2\log a + a - 1 \text{ ------------------ } (2)$$

From (1)and (2)the rule is verified

**Example 2 : Show that** $\phi(a) = \int_{\pi/6a}^{\pi/2a} \frac{\sin ax}{x} dx$ is independent of a

**Solution :** To show that $\phi(a)$

$$= \int_{\pi/6a}^{\pi/2a} \frac{\sin ax}{x} dx$$ is independent of a, we find $\phi'(a)$Using DUIS RuleII

$$\frac{d\phi}{da} = \int_{\pi/6a}^{\pi/2a} \frac{\partial}{\partial a}\left(\frac{\sin ax}{x}\right)dx + \left\{\frac{d}{da}\left(\frac{\pi}{2a}\right)\right\}.\frac{\sin\left(a\left(\frac{\pi}{2a}\right)\right)}{\left(\frac{\pi}{2a}\right)}$$

$$-\left\{\frac{d}{da}\left(\frac{\pi}{6a}\right)\right\}.\frac{\sin\left(a\left(\frac{\pi}{6a}\right)\right)}{\left(\frac{\pi}{6a}\right)}$$

$$\frac{d\phi}{da} = \int_{\pi/6a}^{\pi/2a} \frac{\cos ax.x.dx}{x} + \left(-\frac{\pi}{2a^2}\right).\frac{1}{(\pi/2a)} - \left(\frac{\pi}{6a^2}\right).\frac{1/2}{(\pi/6a)}$$

$$= \left[\frac{\sin ax}{a}\right]_{\pi/6a}^{\pi/2a} - \frac{1}{a} + \frac{1}{2a} = \frac{1}{a}\left[\sin\frac{\pi}{2} - \sin\frac{\pi}{6}\right] - \frac{1}{a} + \frac{1}{2a}$$

$$= \frac{1}{a} - \frac{1}{2a} + \frac{1}{2a} - \frac{1}{a} = 0$$

Thus , $\frac{d\phi}{da} = 0$ implies that $\phi(a)$ independent of a

**Example 3**:

Verify the rule of differentiation under integral sign for the integral

$$\int_0^{a^2} \tan^{-1}\frac{x}{a}\,dx$$

**Solution :** $\phi(a) = \int_0^{a^2} \tan^{-1}\frac{x}{a}\,dx$

$$\phi'(a) = \int_0^{a^2} \frac{\partial}{\partial a}\left(\tan^{-1}\frac{x}{a}\right)dx + \left\{\frac{d}{da}(a^2)\right\}\tan^{-1}\left(\frac{a^2}{a}\right) - \left\{\frac{d}{dx}(0)\right\}\tan^{-1}(0)$$

$$= \int_0^{a^2} \frac{1}{1+\frac{x^2}{a^2}}\left(\frac{x}{a^2}\right)dx + 2a\tan^{-1}(a) = -\int_0^{a^2}\frac{x}{a^2+x^2}\,dx + 2a\tan^{-1}(a)$$

$$= -\frac{1}{2}\int_0^{a^2}\frac{2x\,.\,dx}{a^2+x^2} + 2a\tan^{-1}a = -\frac{1}{2}\left[\log(a^2+x^2)\right]_0^{a^2} 2a\tan^{-1}a$$

$$= -\frac{1}{2}\left[\log(a^2+a^4) - \log a^2\right] + 2a\tan^{-1}a$$

$$= -\frac{1}{2}\log\frac{a^2(1+a^2)}{a^2} + 2a\tan^{-1}a$$

$$\therefore\ \phi'(a) = -\frac{1}{2}\log(1+a^2) + 2a\tan^{-1}a \qquad ------------(1)$$

Next by integration by parts

$$\phi(a) = \int_0^{a^2} \tan^{-1}\left(\frac{x}{a}\right).1.dx$$

$$= \left[\tan^{-1}\left(\frac{x}{a}\right)\ (x)\right]_0^{a^2} - \int_0^{a^2} \frac{1}{1+\frac{x^2}{a^2}}.\frac{1}{a}.x.dx$$

$$= a^2\tan^{-1}a - 0 - a\int_0^{a^2} \frac{x\,dx}{a^2 + x^2} = a^2\tan^{-1}a - \frac{a}{2}[\log(a^2 + x^2)]_0^{a^2}$$

$$= a^2\tan^{-1}a - \frac{a}{2}\log\frac{a^2(1+a^2)}{a^2} = a^2\tan^{-1}a - \frac{a}{2}\log(1+a^2)$$

$$\phi(a) = a^2\tan^{-1}a - \frac{a}{2}\log(1+a^2)$$

$$\therefore\ \phi'(a) = 2a\tan^{-1}a + a^2.\frac{1}{1+a^2} - \frac{1}{2}\log(1+a^2) - \frac{a}{2}\left(\frac{2a}{1+a^2}\right)$$

$$\therefore\ \phi'(a) = 2a\tan^{-1}a - \frac{1}{2}\log(1+a^2) = -------------(2)$$

From (1)and (2)the rule of differentiation under integral sign or the interal is verified.

**Example 4**: If $y = \int_0^x f(t)\sin a\,(x-t)dt$, show that $\frac{d^2y}{dx^2} + a^2y = a\,f(x)\,dx$

**Solution** : $y = \int_0^x f(t)\sin a\,(x-t)dt$,

Differentiating w.r.t. x,

$$\frac{dy}{dx} = \int_0^x \frac{\partial}{\partial x}[f(t)\sin a\,(x-t)]\,dt + \left\{\frac{d}{dx}\ (x)\right\} f(x)\sin 0 - \left\{\frac{d}{dx}\ (0)\right\}\ f(0)\sin 0$$

$$= \int_0^x a\,f(t)\cos a\,(x-t)]\,dt + 0 - 0$$

Again differentiating w.r.t. x,

$$\frac{d^2y}{dx^2} = \int_0^x \frac{\partial}{\partial x}[a\,f(t)\cos a\,(x-t)]\,dt + \left[\frac{d}{dx}(x)\right] a\,f(x)\ \cos 0$$

$$- \frac{d}{dx}(0).\,a\,f(0).\cos 0$$

$$\frac{d^2y}{dx^2} = \int_0^x a\,f(t)(-\sin a\,(x-t))\,.\,a\,.\,dt + a\,.\,f(x) - 0$$

$$\frac{d^2y}{dx^2} = -a^2 \int_0^x f(t)\sin a\,(x-t)dt + a\,.\,f(x) = \; -a^2y + a\,f(x)$$

$$\frac{d^2y}{dx^2} + a^2y = \; a\,f(x)$$

## 11.4 Error Function:-Definition

**Definition**: Error function x is defined as

$\frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}du$ and is denoted by erf(x) .

We write **erf(x)**

$$= \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}du \quad ----------- (1)$$

This function or integral is also called Error Function integral or Probability integral and is encounterd in many branches of Mathematics, Physics or Engineering.

**Complementary Error Function**:

Complementary error function x is defined

as $\frac{2}{\sqrt{\pi}}\int_x^\infty e^{-u^2}du$ and is denoted by erfc(x).

We write **erf c(x)**

$$= \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-u^2}du \quad ----------- (2)$$

**Alternate Definition of Error Function**: In integral of (1),

if we put $u^2 = t$, $\quad 2udu = dt$ or $du = \frac{dt}{2\sqrt{t}}$ ;

| | | |
|---|---|---|
| u | 0 | x |
| t | 0 | $x^2$ |

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^{x^2} e^{-t}\frac{dt}{2\sqrt{t}} = \frac{1}{\sqrt{\pi}}\int_0^{x^2} e^{-t}\,t^{-1/2}\,dt$$

$$\mathbf{erf(x)} = \frac{\mathbf{1}}{\sqrt{\boldsymbol{\pi}}}\int_{\mathbf{0}}^{\mathbf{x^2}} \mathbf{e^{-t}\, t^{-1/2}\, dt} \quad ----------------- (\mathbf{3})$$

This is also considered as definition of Error function x and

either (1) or (3) used for erf(x) according to the need of the problem

**11.5 Properties of Error Functions**

**1.** $\mathrm{erf}(\infty) = \frac{2}{\sqrt{\pi}}\int_0^{\infty} e^{-u^2} du \quad ------------- \quad (\text{Put } u^2 = y)$

$$= \frac{2}{\sqrt{\pi}}\int_0^{\infty} e^{-y}\frac{1}{2}y^{-1/2}dy = \frac{1}{\sqrt{\pi}}\int_0^{\infty} e^{-y}y^{-1/2}dy$$

$$= \frac{1}{\sqrt{\pi}}\sqrt{\pi} = 1$$

$$\mathbf{erf(\infty) = 1} \quad -----------(\mathbf{4})$$

$$\mathbf{erf(\infty) = 1}$$

**2.** $\mathrm{erf}(0) = \frac{2}{\sqrt{\pi}}\int_0^{0} e^{-u^2} du = 0$

$$\mathbf{erf(0) = 0} \quad -----------(\mathbf{5})$$

$$\mathbf{erf(0) = 0}$$

**3.** $\mathrm{erf}(x) + \mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}}\left[\int_0^{x} e^{-u^2}du + \int_x^{\infty} e^{-u^2}du\right] = \frac{2}{\sqrt{\pi}}\left[\int_0^{\infty} e^{-u^2}du\right]$

$$= \mathrm{erf}(\infty) = 1$$

$$\mathbf{erf(x) + erfc(x) = 1} \quad ---------- (\mathbf{6})$$

$$\mathbf{erf(x) + erfc(x) = 1}$$

**4 . Error Function is an odd function :** $\mathbf{erf(-x) = -\,erf(x)}$

**Proof**: $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\left[\int_0^{x} e^{-u^2}du\right] \quad --------- \text{Replace x by} - \text{x}$

$$\therefore \quad \mathrm{erf}(-x) = \frac{2}{\sqrt{\pi}}\left[\int_0^{-x} e^{-u^2}du\right] \quad \text{put } u = -y\,;\, du = -dy$$

| | | |
|---|---|---|
| u | 0 | -x |
| y | 0 | 0 |

$$\therefore \quad \text{erf}(-x) = \frac{2}{\sqrt{\pi}}\left[\int_0^x e^{-y^2}(-dy)\right] = \frac{2}{\sqrt{\pi}}\int_0^x e^{-y^2}(-dy)$$

$$\mathbf{erf(-x) = -erf(x)} \qquad ----------(7)$$

$$\mathbf{erf(-x) = -erf(x)}$$

**5 . Expression for erf(x) in series :**

**Proof**: $\text{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}du$

SInce $e^{-t} = \frac{t^0}{0!} - \frac{t^1}{1!} + \frac{t^2}{2!} - \frac{t^3}{3!} \ldots\ldots\ldots\ldots\ldots.. = 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!}\ldots\ldots\ldots\ldots\ldots..$

$\therefore \quad \text{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \left[1 - u^2 + \frac{u^4}{2!} - \frac{u^6}{3!} + \ldots\ldots\right] du$ ( By putting t $= -u^2$ in $e^{-t}$ )

$$= \frac{2}{\sqrt{\pi}}\left[u - \frac{u^3}{3} + \frac{u^5}{10} - \frac{u^7}{42} + \ldots\ldots\right]_0^x$$

$$\mathbf{erf(x)} = \frac{2}{\sqrt{\pi}}\left[x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \ldots\ldots\right] \qquad ----------(\mathbf{8})$$

$$\mathbf{erf(x) = \frac{2}{\sqrt{\pi}}\left[x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \ldots\ldots\right]}$$

This series is uniformly convergent and hence erf(x) is a continuous function of x. Values of erf(x) can be tabulated using above series.

**6 . Altermnate definition of Complementary error function :**

By Result $\text{erf}(\infty) = 1$ , $\text{erf}(\infty) = \frac{1}{\sqrt{\pi}}\int_0^\infty e^{-t}\,t^{-1/2}\,dt = 1$

This can be rewritten as , $\frac{1}{\sqrt{\pi}}\left\{\int_0^{x^2} e^{-t}\,t^{-1/2}\,dt + \int_{x^2}^{\infty} e^{-t}\,t^{-1/2}\,dt\right\} = 1$

$$\frac{1}{\sqrt{\pi}}\int_0^{x^2} e^{-t}\,t^{-1/2}\,dt + \frac{1}{\sqrt{\pi}}\int_{x^2}^{\infty} e^{-t}\,t^{-1/2}\,dt$$

$$= 1 \quad ------------ (\mathbf{9})$$

Here first integral on L. H. S. of (9)is erf(x) and second integral $\frac{1}{\sqrt{\pi}}\int_{x^2}^{\infty} e^{-t}\,t^{-1/2}$

complementary error function x or written as erfc(x).

$$\therefore\ \text{erfc}(x)\frac{1}{\sqrt{\pi}}\int_{x^2}^{\infty} e^{-t}\,t^{-1/2}\,dt \quad -------------------- (\mathbf{10})$$

$$\mathbf{erfc(x)\ \frac{1}{\sqrt{\pi}}\int_{x^2}^{\infty} e^{-t}\,t^{-1/2}\,dt}$$

Thus from (9), we note that , erf(x) + erfc(x) = 1

## 11.6 Differentiation and Integration of Error function

**Differentiation of Error function:**

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}du$$

$$\text{erf}(ax) = \frac{2}{\sqrt{\pi}}\int_0^{ax} e^{-u^2}du$$

Using second rule of differentitaion under the integral sign, and noting that integration is w. r. t. u and differentiation is carreid out w. r. t. x.

$$\frac{d}{dx}\text{erf}(ax) = \frac{2}{\sqrt{\pi}}\left[\int_0^{ax}\frac{\partial}{\partial x}\,e^{-u^2}du + \left\{\frac{d}{dx}(ax)\right\}e^{-a^2x^2} - \left\{\frac{d}{dx}(0)\right\}e^{-0}\right]$$

$$\frac{d}{dx}\text{erf}(ax) = \frac{2}{\sqrt{\pi}}\left[0 + a.\,e^{-a^2x^2} - 0\right] = \frac{2a.\,e^{-a^2x^2}}{\sqrt{\pi}}$$

$$\mathbf{\frac{d}{dx}erf(ax) = \ = \frac{2a.\,e^{-a^2x^2}}{\sqrt{\pi}}} \quad --------------------(\mathbf{11})$$

$$\mathbf{\frac{d}{dx}erf(ax) = \ = \frac{2a.e^{-a^2x^2}}{\sqrt{\pi}}}$$

**Integration of Error function:**

$$\int_0^t erf(ax)\ dx = \int_0^t 1.erf(ax)dx$$

Integrating by parts treating unity as second function

and erf(ax) as first function

$$= [erf(ax).x]_0^t - \int_0^t \frac{d}{dx}erf(ax).x.dx$$

$$= t.erf(at) - 0 - \int_0^t \frac{2a.e^{-a^2x^2}}{\sqrt{\pi}}.x.dx \quad \left(\because \frac{d}{dx}erf(ax) = \ = \frac{2a.e^{-a^2x^2}}{\sqrt{\pi}}\right)$$

$$= t.erf(at) + \frac{1}{\sqrt{\pi}}.\frac{1}{a}\int_0^t e^{-a^2x^2}(-2a^2\,x\,dx\,)$$

$$= t.erf(at) + \frac{1}{a\sqrt{\pi}}\left[e^{-a^2x^2}\right]_0^t$$

$$= t.erf(at) + \frac{1}{a\sqrt{\pi}}\left(e^{-a^2t^2} - 1\right) = t.erf(at) + \frac{1}{a\sqrt{\pi}}e^{-a^2t^2} - \frac{1}{a\sqrt{\pi}}$$

$$\therefore \int_0^t erf(ax)\ dx$$

$$= t.erf(at) + \frac{1}{a\sqrt{\pi}}e^{-a^2t^2} - \frac{1}{a\sqrt{\pi}} \qquad ---------- (\mathbf{12})$$

$$\mathbf{\int_0^t erf(ax)\ dx = t.erf(at) + \frac{1}{a\sqrt{\pi}}e^{-a^2t^2} - \frac{1}{a\sqrt{\pi}}}$$

**Example 1**: Show that $\int_0^t erf(ax)\,dx + \int_0^t erf\,c(ax)\ dx = t$

**Solution**: $\int_0^t erf(ax)\,dx + \int_0^t erf\,c(ax)\ dx$

$$= \int_0^t [\mathrm{erf}(ax) + \mathrm{erf\,c}(ax)]\,dx$$

$$= \int_0^t (1).\,dx = [x]_0^t = t \qquad \{\because \mathrm{erf}(ax) + \mathrm{erfc}(ax) = 1\}$$

**Example 2**: Prove that $\mathrm{erfc}(-x) + \mathrm{erfc}(x) = 2$

**Solution** : We have $\mathrm{erf}(x) + \mathrm{erfc}(x) = 1$, Let replace x by $-x$

$$\mathrm{erf}(-x) + \mathrm{erfc}(-x) = 1$$

$$\mathrm{erf}(-x) + \mathrm{erfc}(-x) = 1$$

$$-\mathrm{erf}(x) + \mathrm{erfc}(-x) = 1 \qquad \{\because \mathrm{erf}(-x) = -\mathrm{erf}(x)\}$$

$$\mathrm{erfc}(-x) = 1 + \mathrm{erf}(x)$$

$$\mathrm{erfc}(-x) + \mathrm{erfc}(x) = 1 + \mathrm{erf}(x) + \mathrm{erfc}(x)$$

$$\mathrm{erfc}(-x) + \mathrm{erfc}(x) = 1 + 1 = 2$$

**Example 3** : Prove that $\frac{1}{x}\frac{d}{da}\mathrm{erfc}(ax) = -\frac{1}{a}\frac{d}{dx}\mathrm{erf}(ax)$

**Solution** : $\mathrm{erf\,c}(ax) = \frac{2}{\sqrt{\pi}} \int_{ax}^{\infty} e^{-u^2}\,du$

$$\frac{d}{da}\mathrm{erf\,c}(ax) = \frac{2}{\sqrt{\pi}}\left\{\frac{\partial}{\partial a}\int_{ax}^{\infty}\left(e^{-u^2}\right).du + \frac{d}{da}(\infty).e^{-\infty} - \frac{d}{da}(ax).e^{-a^2x^2}\right\}$$

$$= \frac{2}{\sqrt{\pi}}\left\{0 + 0 - xe^{-a^2x^2}\right\} = -\frac{2xe^{-a^2x^2}}{\sqrt{\pi}}$$

$$\frac{1}{x}\frac{d}{da}\mathrm{erf\,c}(ax) = -\frac{2}{\sqrt{\pi}}e^{-a^2x^2} \qquad \text{--------------- (1)}$$

$$\frac{d}{dx}\mathrm{erf}(ax) = \frac{2}{\sqrt{\pi}}\left\{\frac{\partial}{\partial x}\int_0^{ax}\left(e^{-u^2}\right).du + \frac{d}{da}(ax).e^{-a^2x^2} - \frac{d}{dx}(0).e^0\right\}$$

$$= \frac{2}{\sqrt{\pi}}\left\{0 + a.e^{-a^2x^2} - 0\right\} = \frac{2ae^{-a^2x^2}}{\sqrt{\pi}}$$

$$-\frac{1}{a}\frac{d}{dx}\mathrm{erf}(ax) = -\frac{2}{\sqrt{\pi}}e^{-a^2x^2} \qquad \text{---------------(2)}$$

From (1)and (2), it is prove that $\frac{1}{x}\frac{d}{da}\text{erfc}(ax) = -\frac{1}{a}\frac{d}{dx}\text{erf}(ax)$

**Example 4**: Show that $\int_a^b e^{-x^2}dx = \frac{\sqrt{\pi}}{2}[\text{erf}(b) - \text{erf}(a)]$

**Solution** : By definition $\text{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}du$

if $x = \infty$ , then, $\text{erf}(\infty) = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-u^2}du$

$$\therefore\ 1 = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-x^2}dx \qquad \{\because\ \text{erf}(\infty) = 1\}$$

Assuming that $b > a$, we can write ,

$$1 = \frac{2}{\sqrt{\pi}}\left\{\int_0^a e^{-x^2}dx + \int_a^b e^{-x^2}dx + \int_b^\infty e^{-x^2}dx\right\}$$

$$1 = \frac{2}{\sqrt{\pi}}\int_0^a e^{-x^2}dx + \frac{2}{\sqrt{\pi}}\int_a^b e^{-x^2}dx + \frac{2}{\sqrt{\pi}}\int_b^\infty e^{-x^2}dx$$

$$1 = \text{erf}(a) + \frac{2}{\sqrt{\pi}}\int_a^b e^{-x^2}dx + \text{erfc}(b)$$

$$1 - \text{erfc}(b) = \text{erf}(a) + \frac{2}{\sqrt{\pi}}\int_a^b e^{-x^2}dx$$

$$\text{erf}(b) - \text{erf}(a) = \frac{2}{\sqrt{\pi}}\int_a^b e^{-x^2}dx \qquad \{\because\ \text{erf}(b) + \text{erfc}(b) = 1\}$$

$$\int_a^b e^{-x^2}dx = \frac{\sqrt{\pi}}{2}[\text{erf}(b) - \text{erf}(a)]$$

**Example 5**: Show that $\int_0^\infty e^{-x^2-2bx}dx = \frac{\sqrt{\pi}}{2}.e^{b^2}[1 - \text{erf}(b)]$

**Solution** : $I = \int_0^\infty e^{-x^2-2bx}\,dx = \int_0^\infty e^{-x^2-2bx-b^2+b^2}\,dx = e^{b^2}\int_0^\infty e^{-(x+b)^2}\,dx$

put $x + b = u$ , $dx = du$

| | | |
|---|---|---|
| x | 0 | $\infty$ |
| u | b | $\infty$ |

$$I = e^{b^2}\int_b^\infty e^{-u^2}du = e^{b^2}\frac{\sqrt{\pi}}{2}.\frac{2}{\sqrt{\pi}}\int_b^\infty e^{-u^2}du$$

$$= \frac{\sqrt{\pi}}{2}e^{b^2}.\,\text{erfc}(b) = \frac{\sqrt{\pi}}{2}e^{b^2}[1-\text{erf}(b)]$$

**Example 6**: If $\alpha(x) = \sqrt{\frac{2}{\pi}}\int_0^x e^{\frac{t^2}{-2}}dt$ show that $\text{erf}(x) = \alpha[x\sqrt{2}]$

**Solution** : $\alpha(x) = \sqrt{\frac{2}{\pi}}\int_0^x e^{\frac{t^2}{-2}}dt$

$$\therefore\ \alpha(x\sqrt{2}) = \sqrt{\frac{2}{\pi}}\int_0^{x\sqrt{2}} e^{\frac{t^2}{-2}}dt$$

put $t^2 = 2u^2$ , $2t\,dt = 4u\,du$ , $\quad dt = \frac{2u\,du}{t} = \frac{2u\,du}{\sqrt{2}\,.\,u} = \sqrt{2}\,du$

| | | |
|---|---|---|
| t | 0 | $x\sqrt{2}$ |
| u | 0 | x |

$$= \sqrt{\frac{2}{\pi}}\int_0^x e^{-u^2}\sqrt{2}.\,du$$

$$= \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}\sqrt{2}.\,du = \text{erf}\,(x)$$

## 11.7 Exercise

1. Prove that $\int_0^1 \frac{x^a - x^b}{\log x}\, dx = \log\left(\frac{a+1}{b+1}\right)\ ; a > 0, b > 0$

2. Assuming that $\int_0^\infty \frac{sinx}{x}\, dx = \frac{\pi}{2}$, evaluate $\int_0^\infty \frac{1-\cos ax}{x^2}\, dx$

3. Prove that $\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\, dx = \log\left(\frac{b}{a}\right)\ ; a > 0\,, b > 0$

**Hint**: $\phi'(a) = -\frac{1}{a}\ \therefore \phi(a) = -\log\ a + C$, Put $a = b, C = \log\ b$

4. Prove that $\int_0^\infty \frac{e^{-ax}\sin x}{x^2}\, dx = cot^{-1}a.$ *Deduce that* $\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}$

5. Prove that $\int_0^\infty \frac{1-\cos ax}{x^2}\, dx = \frac{\pi a}{2}$

6. If $f(x) = \int_a^x (x-t)^2\ G(t)dt$ then show that $\frac{d^3f}{dx^3} - 2\,G(x) = 0$

**Hint**: Here x is a parameter , $f'(x) = \int_a^x (2)(x-t)G(t)dt\ , f''^{(x)} = \int_a^x G(t)dt$

$$f'''(x) = 2\left[\int_a^x \frac{\partial}{\partial x} G(t)dt + \left\{\frac{dx}{dx}\right\} G(x) - \left\{\frac{da}{dx}\right\} G(a)\ \right]$$

7. If $F(t) = \int_t^{t^2} e^{tx^2} dx$, then show that $\frac{dF}{dt} = \frac{1}{2t}\left[5t^2e^{t^5} - 3te^{t^3} - F(t)\right]$

8. Show that $\frac{d}{da}.\int_{\sqrt{a}}^{1/a} \cos ax^2\, dx$

$$= -\int_{\sqrt{a}}^{1/a} x^2 . \sin ax^2\, dx - \frac{1}{a^2}\cos\frac{1}{a} - \frac{1}{2\sqrt{a}}\cos a^2$$

9. If $\phi(a) = \int_{a}^{a^2} \frac{\sin ax}{x} dx$ , find $\frac{d\phi}{da}$

10. Verify the rule of differntiation under interal sign

for the integral $\int_{a}^{a^2} \frac{1}{x+a} dx$

11. Find $\text{erf}(0)$ , $\text{erf}(\infty)$ , $\text{erfc}(0)$

12. $\frac{d}{dx}\, \text{erfc}(ax^n)$

13. $\frac{d}{dx}\, \text{erfc}(\sqrt{x})$

14. Show that $\int_0^\infty e^{-(x+a)^2} dx = \frac{\sqrt{\pi}}{2}[1 - \text{erf}(a)]$

15. Define $\text{erf}(x)$ , $\text{erfc}(x)$, $\text{erf}(\sqrt{t})$, $\text{erfc}(\sqrt{t})$.

## 11.8 Summary

**Rule – I : If** $\boldsymbol{I(\alpha) = \int_{a}^{b} f(x,\alpha)\,dx}$ ***then*** $\boldsymbol{\frac{dI}{d\alpha} = \int_{a}^{b} \frac{\partial}{\partial \alpha} f(x,\alpha)\,dx}$

**Rule – II: (LEIBNITZ's RULE)** $\frac{dI}{d\alpha} = \frac{d}{d\alpha}\int_{a(\alpha)}^{b(\alpha)} f(x,\alpha)dx$

$$= \int_{a(\alpha)}^{b(\alpha)} \frac{\partial}{\partial a} f(x,\alpha)dx + f(b,\alpha)\frac{db}{d\alpha} - f(a,\alpha)\frac{da}{d\alpha}$$

**Error Function :** $\text{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2} du$

**Complementary Error Function :** $\text{erf}\,c(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-u^2} du$

**Alternate Definition of Error Function :** $\text{erf}(x) = \frac{1}{\sqrt{\pi}}\int_0^{x^2} e^{-t}\, t^{-1/2}\, dt$

**Properties of Error Fnction :**

$\mathbf{erf}(\infty) = \mathbf{1}$

$\mathbf{erf}(0) = \mathbf{0}$

$\mathbf{erf}(x) + \mathbf{erfc}(x) = \mathbf{1}$

$\mathbf{erf}(-x) = -\mathbf{erf}(x)$

$$\mathbf{erf}(x) = \frac{2}{\sqrt{\pi}}\left[x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \ldots\ldots\right]$$

$$\mathbf{erfc}(x) = \frac{1}{\sqrt{\pi}}\int_{x^2}^{\infty} e^{-t}\, t^{-1/2}\, dt$$

**Differentiation of Error function:**

$$\frac{d}{dx}\mathbf{erf}(ax) = = \frac{2a.\,e^{-a^2x^2}}{\sqrt{\pi}}$$

**Integration of Error function:**

$$\int_0^t \mathbf{erf}(ax)\; dx = t\,.\,\mathbf{erf}(at) + \frac{1}{a\sqrt{\pi}}\, e^{-a^2t^2} - \frac{1}{a\sqrt{\pi}}$$

## 11.9 References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