1

SIMPLE AND COMPOUND INTEREST

Unit Structure

1.0 OBJECTIVES

After going through this chapter you will able to know:

- Calculation of simple interest.
- Calculation of compound interest.
- Difference between nominal and effective rate of interest.

1.1 INTRODUCTION

We have to work with money everyday life. When a borrower borrows money from a lender or any financial institution or bank, there is some extra amount that is charged on the total amount that is borrowed. This extra amount is termed as an interest rate. Interest charged can be of two type's simple interest and compound interest.

To summarize, the concept of simple interest is the amount paid for the money borrowed for a fixed period of time. While in the case of compound interest, whenever the interest isup for payment, it is added back to the principal amount. In this chapter we will highlight the differences between simple interest and compound interest.

1.2 SIMPLE INTEREST

Principal: The sum borrowed by a person is called its *principal*. It is denoted by P.

Period: The time span for which money is lent is called *period*. It is denoted by n.

Interest: The amount paid by a borrower to the lender for the use of money borrowed for a certain period of time is called *Interest*. It is denoted by *I*.

Rate of Interest: This is the interest to be paid on the amount of Rs. 100 per annum (i.e. per year). This is denoted by *r*.

Total Amount: The sum of the principal and interest is called as the total amount and is denoted by *A*. Thus, $A = P + I$.
i.e. Interest paid $I = A - P$.

Simple Interest:
The interest which is payable on the principal only is called as *simple interest* (S.I.). For example the interest on Rs. 100 at 11% after one year is Rs.11 and the amount is 100 + 11 = Rs. 111.

It is calculated by the formula: $I = \frac{Pnr}{100}$

Amount at the end of n^{th} year = $A = P + I = P + \frac{Pnr}{100} = P\left(1+\frac{nr}{100}\right)$

Example1: If Mr. Amol borrows Rs. 1000 for 4 years at 5% rate of interest, find (i) simple interest and (ii) total amount.
Answer: Given P = Rs. 1000, $n = 4$ and $r = 5$

(i) $I = \frac{Pnr}{100} = \frac{1000 \text{ x } 4 \text{ x } 5}{100}$ = Rs. 200

(ii) $A = P + I = 1000 + 200$ = Rs. 1200

Example2: If Mr. Aasim borrows Rs. 500 for 5 years and pays an interest of Rs. 175, find rate of interest.
Answer: Given $P = 500$, $n = 5$ and I = Rs. 175

Now, $I = \frac{Pnr}{100}$ $\quad\quad \Rightarrow r = \frac{I \text{ x } 100}{P \text{ x } n}$

$\therefore r = \frac{175 \text{ x } 100}{500 \text{ x } 5} = 7$

Example3: Find the period for Rs. 2500 to yield Rs. 900 in simple interest at 12%
Answer: Given P = Rs. 2500, $I = 900$, $r = 12$

Now, $I = \frac{Pnr}{100}$ $\quad\quad \Rightarrow n = \frac{I \text{ x } 100}{P \text{ x } r}$

$\therefore n = \frac{900 \text{ x } 100}{2500 \text{ x } 12} = 3$

The period is 3 years.

Example 4: Mr. Akshay lent Rs. 5000 to Mr. Amit and Rs. 4000 to Mr. Sunil for 5 years and received total simple interest of Rs. 4950. Find (i) the rate of interest and (ii) simple interest of each.
Answer: Let the rate of interest be *r*.

S.I. for Anil = $\frac{5000 \text{ x } 5 \text{ x } r}{100} = 250r$ … (1)

and S.I. for Sunil = $\frac{4000 \text{ x } 5 \text{ x } r}{100} = 200r$ … (2)

from (1) and (2), we have,
total interest from both = $250r + 200r$
$= 450r$

But total interest received be Mr. Akshay = Rs. 4950

$\therefore 450r = 4950 \quad \Rightarrow r = \frac{4950}{450} = 11$

$\therefore$ the rate of interest =

Example 5: The S.I. on a sum of money is one-fourth the principal. If the period is same as that of the rate of interest then find the rate of interest.

Answer: Given $I = \frac{P}{4}$ and $n = r$

Now, w.k.t. $I = \frac{Pnr}{100}$

$\therefore \frac{P}{4} = \frac{P \text{ x } r \text{ x } r}{100} \quad \Rightarrow \frac{100}{4} = r^2$

$\therefore r^2 = 25 \quad \Rightarrow r = 5.$

$\therefore$ The rate of interest = 5%

Example 6: If Rs. 8400 amount to Rs. 11088 in 4 years, what will Rs. 10500 amount to in 5 years at the same rate of interest?

Answer:

(i) Given $n = 4$, $P =$ Rs. 8400, $A =$ Rs. 11088

$\therefore I = A - P = 11088 - 8400 =$ Rs. 2688

Let r be the rate of interest.

$\therefore$ $\boldsymbol{r = 8\%}$

Now, $I = \frac{Pnr}{100} \quad \Rightarrow 2688 = \frac{8400 \text{ x } 4 \text{ x } r}{100}$

(ii) To find A when $n = 5$, $P =$ Rs. 10500, $r = 8$

$$A = P\left(1 + \frac{nr}{100}\right) = 10500 \text{ x } \left(1 + \frac{5 \text{ x } 8}{100}\right) = 10500 \text{ x } \frac{140}{100} = 14700$$

$\therefore$ The required amount = Rs. 14,700

Example 7: Mr. Arvind Kamble earns Rs. 1400 after 5 years by lending at a certain rate of interest Rs. 2300 to Mr. Vinod and Rs. 2750 to Mr. Kuldeep at a rate of interest 1% more than that of Mr. Vinod. Find the rates of interest.

Answer: Let the rate of interest for Mr. Vinod be r %.

Hence the rate of interest for Mr. Kuldeep is $(r + 1)$ %.

S.I. from Mr. Vinod = $\frac{2300 \text{ x } 5 \text{ x } r}{100} = 115.r$ … (1)

S.I. from Mr. Kuldeep = $\frac{2750 \text{ x } 5 \text{ x } (r+1)}{100} = 137.5(r + 1)$ … (2)

The total interest received by Mr. Arvind from both = $115r + 137.5(r + 1)$

$= 115r + 137.5r + 137.5$

$= 252.5r + 137.5 = 1400$ … given

$\therefore 252.5r = 1262.5$

$\therefore$ The rate of interest for Mr. Vinod = **5%** and the rate of interest for Mr. Kuldeep = **6%**

1.3 COMPOUND INTEREST

The interest which is calculated on the amount in the previous year is called compound interest.

For example, the compound interest on Rs. 100 at 8% after one year is Rs. 8 and after two years is 108 + 8% of 108 = Rs. 116.64

If P is the principal, r is the rate of interest p.a. then the amount at the end of n^{th} year called as *compound amount* is given by the formula:

$$A = P\left(1+\frac{r}{100}\right)^n$$

The *compound interest* is given by the formula:

$$\mathbf{CI = A - P}$$

Note:

1. The interest may be compounded annually (yearly), semi-annually (half yearly), quarterly or monthly. Thus, the general formula to calculate the amount at the end of n years is as follows:

$$A = P\left(1+\frac{r}{p \text{ x } 100}\right)^{np}$$

Here p: number of times the interest is compounded in a year.
$p = 1$ if interest is compounded annually

$p = 2$ if interest is compounded semi-annually
$p = 4$ if interest is compounded quarterly
$p = 12$ if interest is compounded monthly

2. It is easy to calculate amount first and then the compound interest as compared with finding interest first and then amount in case of simple interest.

Example 8: Find the compound amount and compound interest of Rs. 800 invested for 10 years at 8% if the interest is compounded annually.

Answer: Given $P = 800$, $r = 8$, $n = 10$.
Since the interest is compounded annually, we have

$$A = P\left(1+\frac{r}{100}\right)^{n} = 800 \text{ x } \left(1+\frac{8}{100}\right)^{10} = 800 \text{ x } 2.1589 \approx \text{Rs. } 1727$$

Example 9: Find the compound amount and compound interest of Rs. 1000 invested for 5 years at 6% if the interest is compounded (*i*) annually, (*ii*) semi annually, (*iii*) quarterly and (*iv*) monthly.

Answer: Given $P = \text{Rs. } 1000$, $r = 6$, $n = 5$
(*i*) If the interest is compounded annually:

$$A = P\left(1+\frac{r}{100}\right)^{n} = 1000 \text{ x } \left(1+\frac{6}{100}\right)^{5} = 1000 \text{ x } 1.3382 \approx \text{Rs. } 1338$$

$CI = A - P = 1338 - 1000 = \text{Rs. } 338$

(*ii*) If the interest is compounded semi-annually:

$$A = P\left(1+\frac{r}{2 \text{ x } 100}\right)^{2n} = 1000 \text{ x } \left(1+\frac{6}{200}\right)^{10} = 1000 \text{ x } 1.3440 = \text{Rs. } 1340$$

$CI = A - P = 1340 - 1000 = \text{Rs. } 340$

(*iii*) If the interest is compounded quarterly:

$$A = P\left(1+\frac{r}{4 \text{ x } 100}\right)^{4n} = 1000 \text{ x } \left(1+\frac{6}{400}\right)^{20} = 1000 \text{ x } 1.347 = \text{Rs. } 1347$$

$CI = A - P = 1347 - 1000 = \text{Rs. } 347$

(*iv*) If the interest is compounded monthly:

$$A = P\left(1+\frac{r}{12 \text{ x } 100}\right)^{12n} = 1000 \text{ x } \left(1+\frac{6}{1200}\right)^{60} = 1000 \text{ x } 1.349 = \text{Rs. } 1349$$

$CI = A - P = 1349 - 1000 = \text{Rs. } 349$

Example 10: Find the principal which will amount to Rs. 11,236 in 2 years at 6% compound interest compounded annually.

Answer: Given $A = \text{Rs. } 11236$, $n = 2$, $r = 6$ and $P = ?$

Now, $A = P\left(1+\frac{r}{100}\right)^{n}$

$\therefore 11236 = P\left(1+\frac{6}{100}\right)^2 = P \text{ x } 1.1236$

$\therefore P = \frac{11236}{1.1236} = 10{,}000$

$\therefore$ the required principal is Rs. 10,000.

Example 11: Dr. Ashwinikumar wants to invest some amount for 4 years in a bank. Bank *X* offers 8% interest if compounded half yearly while bank *Y* offers 6% interest if compounded monthly. Which bank should Dr. Ashwinikumar select for better benefits?

Answer: Given $n = 4$,
Let the principal Mr. Ashwinikumar wants to invest be P = Rs. 100
From Bank *X*:$r = 8$ and interest is compounded half-yearly, so $p = 2$.

$$\therefore A = P\left(1+\frac{r}{2 \text{ x } 100}\right)^{2n} = 100 \text{ x } \left(1+\frac{8}{200}\right)^4 = 116.9858 \qquad \dots (1)$$

From Bank *Y*: $r = 6, p = 12$

$$\therefore A = P\left(1+\frac{r}{12 \text{ x } 100}\right)^{12n} = 100 \text{ x } \left(1+\frac{6}{1200}\right)^{48} = 127.0489 \qquad \dots (2)$$

Comparing (1) and (2), Dr. Ashwinikumar should invest his amount in bank *Y* as it gives more interest at the end of the period.

Example 12: In how many years would Rs. 75,000 amount to Rs. 1,05,794.907 at 7% compound interest compounded semi-annually?

Answer: Given A = Rs. 105794.907, P = Rs. 75000, $r = 7$, $p = 2$

$$A = P\left(1+\frac{r}{2 \text{ x } 100}\right)^{2n}$$

$$\therefore 105794.907 = 75000 \text{ x } \left(1+\frac{7}{200}\right)^{2n}$$

$$\therefore \frac{105794.907}{75000} = (1.035)^{2n}$$

$\therefore 1.41059876 = (1.035)^{2n}$

$\therefore (1.035)^{10} = (1.035)^{2n} \qquad \Rightarrow 2n = 10$

$\therefore$ $\boldsymbol{n = 5}$

Example 13: A certain principal amounts to Rs. 4410 after 2 years and to Rs.4630.50 after 3 years at a certain rate of interest compounded annually. Find the principal and the rate of interest.

Answer: Let the principal be P and rate of interest be r.

Now, we know that $A = P\left(1+\frac{r}{100}\right)^n$

From the given data we have,

$$4410 = P\left(1+\frac{r}{100}\right)^2 \quad \text{and} \quad 4630.5 = P\left(1+\frac{r}{100}\right)^3$$

$\therefore 4410 = P(1 + 0.01r)^2 \ldots (1)$

$4630.5 = P(1 + 0.01r)^3 \ldots (2)$

Dividing (2) by (1), we have

$$\frac{4630.5}{4410} = \frac{P(1+0.01r)^3}{P(1+0.01r)^2} \Rightarrow 1.05 = 1 + 0.01r$$

$\therefore\ 0.05 = 0.01r$

$\Rightarrow \boldsymbol{r = 5}$

Example 14: Find the rate of interest at which a sum of Rs. 2000 amounts to Rs. 2690 in 3 years given that the interest is compounded half yearly.

($\sqrt[6]{1.345} = 1.05$)

Answer: Given P = Rs. 2000, A = Rs. 2680, $n = 3$, $p = 2$

Now, $A = P\left(1+\frac{r}{2 \text{ x } 100}\right)^{2n}$

$$\therefore 2690 = 2000 \text{ x } \left(1+\frac{r}{200}\right)^6$$

$$\therefore \frac{2690}{2000} = \left(1+\frac{r}{200}\right)^6 \qquad \Rightarrow 1.345 = \left(1+\frac{r}{200}\right)^6$$

$$\therefore \sqrt[6]{1.345} = 1 + \frac{r}{200} \qquad \Rightarrow 1.05 = 1 + \frac{r}{200}$$

$\therefore r = 0.05 \text{ x } 200 = 10\%$

Thus, the rate of compound interest is 10 %.

Example 15: If the interest compounded half yearly on a certain principal at the end of one year at 8% is Rs. 3264, find the principal.

Answer: Given CI = Rs. 3264, $n = 1$, $p = 2$ and $r = 8$

Now, $CI = A - P = P\left(1+\frac{8}{200}\right)^2 - P$

i.e. $3264 = P[\,(1.04)^2 - 1] = 0.0816P$

$$\therefore P = \frac{3264}{0.0816} = 40000$$

Thus, the principal is Rs. 40,000.

1.4 EFFECTIVE RATES OF INTEREST

Let a person invest Rs. 100 at 6% rate of interest p.a. compounded half yearly. At the end of the year the compound amount is $100(1.03)^2 = 106.09$. This means that the annual rate of return is 6.09% which is actually more than normal (also called nominal) rate of interest 6%. Such a rate of interest is called as *effective rate of interest*. It is denoted by R_e.

The formula to compute the effective rate of interest is as follows:

$$R_e = \left(1 + \frac{r}{p \text{ x } 100}\right)^p - 1$$

<u>Note</u>:

1. If the period of conversion is annual then the effective rate is same as the normal rate of interest.

2. Effective rates are used to compare different rates of interest and facilitate to find the most beneficial.

Example 16:A bank *X* offers 8% interest compounded semi-annually while another bank offers 8.5% interest compounded monthly. Which bank gives more interest at the end of the year?

Answer: The effective rate for bank *X* is:

$$\left(1 + \frac{8}{2 \text{ x } 100}\right)^2 - 1 = (1.04)^2 - 1 = 0.0816 = 8.16\% \qquad \dots (1)$$

While, the effective rate for bank *Y* is:

$$\left(1 + \frac{8.5}{12 \text{ x } 100}\right)^{12} - 1 = (1.0071)^{12} - 1 = 0.0886 = 8.86\% \qquad \dots (2)$$

From (1) and (2), we conclude that the bank *Y* gives more return at the end of the year.

Example 17: What is the effective rate equivalent to the normal rate of 10% compounded monthly?

Answer: Given $r = 10$ and $p = 12$

$$\therefore R_e = \left(1 + \frac{r}{p \text{ x } 100}\right)^p - 1 = \left(1 + \frac{10}{1200}\right)^{12} - 1 = 1.1047 - 1 = 0.1047$$

$\therefore$ the effective rate is 10.47%

1.5 LET US SUM UP

In this chapter we have learn:

- To calculate simple interest.
- To calculate compound interest.
- To calculate nominal and effective rate of interest.
- Difference between compound interest.

1.6 UNIT END EXERCISE

1. Find the SI and amount for the following data giving principal, rate of interest and number of years:

(*i*) 1800, 6%, 4 years. (*ii*) 4500, 8%, 5 years
(*iii*) 7650, 5.5%, 3 years. (*iv*) 6000, 7.5%, 6 years
(*v*) 25000, 8%, 5 years (*vi*) 20000, 9.5%, 10 years.

2. Find the S.I. and the total amount for a principal of Rs. 6000 for 3 years at 6% rate of interest.

3. Find the S.I. and the total amount for a principal of Rs. 11000 for 5 years at 4% rate of interest.

4. Find the S.I. and the total amount for a principal of Rs. 3300 for 6 years at 3½ % rate of interest.

5. Find the S.I. and the total amount for a principal of Rs. 10550 for 2 years at 10¼ % rate of interest.

6. Find the S.I. and the total amount for a principal of Rs. 4360 for 4 years at 7.5% rate of interest.

7. Find the rate of interest if a person invests Rs. 1000 for 3 years and receives a S.I. of Rs. 150.

8. Find the rate of interest if a person invests Rs. 1200 for 2 years and receives a S.I. of Rs. 168.

9. A person invests Rs. 4050 in a bank which pays 7% S.I. What is the balance of amount of his savings after (*i*) six months, (*ii*) one year?

10. A person invests Rs. 5500 in a bank which pays 10.5% S.I. What is the balance of amount of his savings after 5 years?

11. A person invests Rs. 3000 in a bank which offers 9% S.I. After how many years will his balance of amount will be Rs. 3135?

12. Find the principal for which the SI for 4 years at 8% is 585 more than the SI for 3½ years at 11%.

13. Find the principal for which the SI for 5 years at 7% is 250 more than the SI for 4 years at 10%.

14. Find the principal for which the SI for 8 years at 7.5% is 825 more than the SI for 6½ years at 10.5%.

15. Find the principal for which the SI for 3 years at 6% is 230 less than the SI for 3½ years at 5%.

16. After what period of investment would a principal of Rs. 12,350 amount to Rs. 17,043 at 9.5% rate of interest?

17. A person lent Rs. 4000 to Mr. X and Rs. 6000 to Mr. Y for a period of 10 years and received total of Rs. 3500 as S.I. Find (*i*) rate of interest, (*ii*) S.I. from Mr. X, Mr. Y.

18. Miss PankajKansra lent Rs. 2560 to Mr. Abhishek and Rs. 3650 to Mr. Ashwin at 6% rate of interest. After how many years should he receive a total interest of Rs. 3726?

19. If the rate of S.I. on a certain principal is same as that of the period of investment yields same interest as that of the principal, find the rate of interest.

20. If the rate of S.I. on a certain principal is same as that of the period of investment yields interest equal to one-ninth of the principal, find the rate of interest.

21. Find the principal and rate of interest if a certain principal amounts to Rs. 2250 in 3 years and to Rs. 3750 in 5 years.

22. Find the principal and rate of interest if a certain principal amounts to Rs. 3340 in 4 years and to Rs. 4175 in 5 years.

23. If Rs. 6400 amount to Rs. 7552 in 3 years at a certain rate of interest, what will Rs. 8600 amount to in 4 years at the same rate of interest?

24. If Rs. 2700 amount Rs. 3078 in 2 years at a certain rate of interest, what will Rs. 7200 amount to in 4 years at the same rate on interest?

25. At what rate on interest will certain sum of money amount to three times the principal in 20 years?

26. Mr. Chintan earns as interest Rs. 1020 after 3 years by lending Rs. 3000 to Mr. Bhavesh at a certain rate on interest and Rs. 2000 to Mr. Pratik at a rate on interest 2% more than that of Mr. Bhavesh. Find the rates on interest.

27. Miss Bhagyashree lends at a certain rate on interest Rs. 2400 to Mr. Shardul and Rs. 4200 to MrVishwas at rate of interest 3% less than that of Mr. Shardul. If after 5 years she receives Rs. 1890 as total S.I. find the rates of interest.

28. Mr. Chaitanya invested a certain principal for 3 years at 8% and received an interest of Rs. 2640. Mr. Chihar also invested the same amount for 6 years at 6%. Find the principal of Mr. Chaitanya and the interest received by Mr. Chihar after 6 years.

29. Mr. Ashfaque Khan invested some amount in a bank giving 8.5% rate of interest for 5 years and some amount in another bank at 9% for 4 years. Find the amounts invested in both the banks if his total investment was Rs. 75,000 and his total interest was Rs. 29,925.

30. Mrs. Prabhu lent a total of Rs. 48,000 to Mr. Diwakar at 9.5% for 5 years and to Mr. Ratnakar at 9% for 7 years. If she receives a total interest of Rs. 25,590 find the amount she lent to both.

31. Compute the compound amount and compound interest of Rs. 5000 if invested at 11% for 3 years and the interest compounded *i*) annually, (*ii*) semi annually, (*iii*) quarterly and (*iv*) monthly.

32. Compute the compound amount and compound interest of Rs. 1200 if invested at 9% for 2 years and the interest compounded *i*) annually, (*ii*) semi annually, (*iii*) quarterly and (*iv*) monthly.

33. Miss Daizy invested Rs. 25,000 for 5 years at 7.5% with the interest compounded semi-annually. Find the compound interest at the end of 5 years.

34. Mr. Dayanand borrowed a sum of Rs. 6500 from his friend at 9% interest compounded quarterly. Find the interest he has to pay at the end of 4 years?

35. Mr. Deepak borrowed a sum of Rs. 8000 from his friend at 8% interest compounded annually. Find the interest he has to pay at the end of 3 years?

36. Mr. Deshraj borrowed Rs. 1,25,000 for his business for 3 years at 25% interest compounded half yearly. Find the compound amount and interest after 3 years.

37. Find the principal which will amount to Rs. 13468.55 in 5 years at 6% interest compounded quarterly. [$(1.015)^{20} = 1.346855$]

38. Find the principal which will amount to Rs. 30626.075 in 3 years at 7% interest compounded yearly.

39. Find after how many years will Rs. 4000 amount to Rs. 4494.40 at 6% rate of interest compounded yearly.

40. Find after how many years Rs. 10,000 amount to Rs. 12,155 at 10% rate of interest compounded half-yearly.

41. Mrs. ManishaLokhande deposited Rs. 20,000 in a bank for 5 years. If she received Rs.3112.50 as interest at the end of 2 years, find the rate of interest p.a. compounded annually.

42. On a certain principal for 3 years the compound interest compounded annually is Rs. 1103.375 while the simple interest is Rs. 1050, find the principal and the rate of interest.

43. On a certain principal for 4 years the compound interest compounded annually is Rs. 13923 while the simple interest is Rs. 12000, find the principal and the rate of interest.

44. A bank *X* announces a super fixed deposit scheme for its customers offering 10% interest compounded half yearly for 6 years. Another bank offers 12% simple interest for the same period. Which bank's scheme is more beneficial for the customers?

45. *ABC* bank offers 9% interest compounded yearly while *XYZ* bank offers 7% interest compounded quarterly. If Mr. Arunachalam wants to invest Rs. 18000 for 5 years, which bank should he choose?

Multiple Choice Questions:

1. How long it will take a sum of money invested at 5% p.a. simple interest to increase its value by 30%?

a) 5 years b) 6 years c) 7 years d) 8 years

2. The difference between the compound and simple interests on an amount P at

r % p.a. for one year is Rs. ________.

a) $P \times \left(\frac{r}{100} - 1\right)$ b) 0 c) 100 d) $r - 1$

3. The compound interests earned for two consecutive years, calculated annually, are Rs. 800 and Rs.864 respectively. Therefore, the rate of interest is

a) 10% b) 8% c) 6% d) 9%

4. A sum of money doubles itself at compound interest in 15 years. It will become 4 times in

a) 30 years b) 60 years c) 45 years d) 75 years

5. Rs 1,200 is lent out at 9% p.a. simple interest for 3 years than the amount after 3 years is __.

a) Rs. 1,416 b) Rs. 1614 c) Rs. 1,564 d) Rs. 1,524

6. The difference in simple interest at 13% and 12% p.a. of a sum in one year is Rs. 110. Then the sum is

a) Rs.11,000 b) Rs.15,000 c) Rs.12,000 d) Rs.14,000

7. The sum of money which increases 1/10 of itself every year and amounts to Rs. 450 in 5 years at Simple interest than the sum is ___.

a) Rs.400 b) Rs.250 c) Rs.200 d) Rs.300

8. At what rate percent per annum will a sum of Rs 2000 amount to Rs 2205 in 2 years, compounded annually?

a) 7% b) 7.5% c) 5% d) 9%

9. Simple interest on a certain sum for 4 years at 7% p.a. is more than simple interest on the same sum for 2.5 years at the same rate by Rs. 840 then the principal amount is

a) Rs. 10,000 b) Rs. 8,000 c) Rs. 12,000 d) Rs. 9,000

10. A sum of Rs. 1600 lent at simple interest at 12.5% per annum will become double in

a) 7years b) 8years c) 7.5years d) 10years

1.7 LIST OF REFERENCES

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2

ANNUITY

Unit Structure

2.0 OBJECTIVE

After going through this chapter you will able to know:

- The meaning and different terminologies of annuity
- Different types of annuity
- Derivation of formulas for different annuities
- Calculation of present value of an annuity.
- Calculation of future value and sinking fund.
- EMI and method of calculation of EMI base on different types of interest.

2.1 INTRODUCTION

In our day-to-day life we observe lots of money transactions. In many transactions payment is made in single transaction or in equal installments over a certain period of time. The amounts of these installments are determined in such a way that they compensate for their waiting time. In other cases, in order to meet future planned expenses, a regular saving may be done. Many people have had the experience of making a series of fixed payments over a course of time - such as rent, premium or vehicle payments - or obtaining a series of payments for a course of time, such as the certificate of deposit (CD) or interest from a bond or lending money. These ongoing or recurring payments are technically called "annuities".

2.2 ANNUITY

A series of equal periodic payments is called *annuity*. The payments are of *equal size* and *at equal time interval*.

The common examples of annuity are: monthly recurring deposit schemes, premiums of insurance policies, loan installments etc.

Types of Annuities:

Types of Annuities - There are three types of annuities.

- **Annuity Certain:** An annuity certain is an investment that provides a series of payments for a set period of time to a person or to the person's beneficiary. It is an investment in retirement income offered by insurance companies. The annuity may also be taken as a lump sum.

- **Contingent Annuity:** Contingent annuity is a form of annuity contract that provides payments at the time when the named contingency occurs. For instance, upon death of one spouse, the surviving spouse will begin to receive monthly payments. In a contingent annuity policy the payment will not be made to the annuitant or the beneficiary until a certain stated event occurs.

- **Perpetual Annuity or Perpetuity**: A perpetual annuity, also called a perpetuity promises to pay a certain amount of money to its owner forever.

Classification of Annuities: Annuities, in this sense of the word, are divided into 2 basic types: ordinary annuities and annuities due.

1. **Ordinary Annuities (Immediate annuity):**An ordinary annuity makes (or needs) payments at the termination of each period. For example, bonds usually pay interest at the termination of every 6 months.

2. **Annuities Due:** With an annuity due, payments, on the contrary come at the start of each time period. Rent, which landlords typically need at the initiation of each month, is one of the common annuity examples.

Note:

1. We consider only uniform and certain annuities.

2. If the type of an annuity is not mentioned, we assume that the annuityis immediate annuity.

3. If there is no mention of the type ofinterest, then it is assumed that the interest is compounded per annum.

There are various ways to measure the annuity rate changes or the cost of making such payments or what they're ultimately worth. However, it is first better to know about calculating the present value of the annuity or the future value of the annuity.

2.3 PRESENT VALUE

The sum of all periodic payments of an annuity is called its *present value.*

In simple words, it is that sum which if paid *now* will give the same amount which the periodic payments would have given at the end of the decided period. It is the one time payment of an annuity.

The formula to find the present value (PV) is as follows:

$$PV = \frac{P}{\left(\frac{r}{p \times 100}\right)}\left[1-\frac{1}{\left(1+\frac{r}{p \times 100}\right)^{np}}\right]$$

Where
P: periodic equal payment

r: rate of interest p.a.

p: period of annuity

Let $i = \frac{r}{p \times 100}$, then the above formula can be rewritten as follows:

$$PV = \frac{P}{i}\left[1-\frac{1}{(1+i)^{np}}\right]$$

Example 1:*Find the present value of an ordinaryannuity of Rs.70,000 p.a. for 4 years at*

14% p.a. compounded annually.

Answer:*Given that C = Rs. 70,000, n= 4 years, r= 14%p.a.*

- $i = \frac{r}{100} = \frac{14}{100} = 0.14$

the present value of an annuity immediate is given by

- $PV = \frac{C}{i}[1-(1+i)^{-n}]$
- $PV = \frac{70{,}000}{0.14}[1-(1+0.14)^{-3}]$
- $PV = 5{,}00{,}000[1-(1.14)^{-3}]$
- $PV = 5{,}00{,}000[1-0.5921]$
- $PV = 5{,}00{,}000[0.4079]$
- $PV = \text{Rs. } 2{,}03{,}950$

***Example* 2:** Mr. Ashok Rane borrowed Rs. 20,000 at 4% p.a. compounded annually for 10 years. Find the periodic payment he has to make.

Answer: Given PV = Rs. 20,000; $n = 10$; $p = 1$ and $r = 4 \Rightarrow i = 0.04$

Now to find the periodic payments P we use the following formula:

$$PV = \frac{P}{i}\left[1 - \frac{1}{(1+i)^{np}}\right]$$

$$\therefore 20000 = \frac{P}{0.04}\left[1 - \frac{1}{(1+0.04)^{10}}\right] = \frac{P}{0.04} \text{ x } 0.3244$$

$$\therefore P = \frac{20000 \text{ x } 0.04}{0.3244} = 2466.09$$

Thus, the periodic payments are of Rs. 2466.09

2.4 FUTURE VALUE

The sum of all periodic payments along with the interest is called the *future value* (*accumulated amount*) of the annuity.

The formula to find the future value (A) of an immediate annuity is as follows:

$$A = P\left[\frac{\left(1 + \frac{r}{p \text{ x } 100}\right)^{np} - 1}{\frac{r}{p \text{ x } 100}}\right] = P\left[\frac{(1+i)^{np} - 1}{i}\right]$$

Here, P: periodic equal payment

r: rate of interest p.a.

p: period of annuity i.e. yearly, half yearly, quarterly or monthly

and $i = \frac{r}{p \text{ x } 100}$

***Example* 3:** Find the future value after 2 years of an immediate annuity of Rs. 5000, the rate of interest being 6% p.a.

Answer: Given $n = 2$, P = Rs. 5000, $r = 6$ and $p = 1 \Rightarrow i = \frac{6}{100} = 0.06$

$$A = P\left[\frac{(1+i)^{n} - 1}{i}\right] = 5000\left[\frac{(1+0.06)^{2} - 1}{0.06}\right] = 5000\left[\frac{1.1236 - 1}{0.06}\right]$$

$\therefore A$ = 5000 x 2.06 = Rs. 10300

Example 4: Find the future value of an immediate annuity after 3 years with the periodic payment of Rs. 1200 at 5% p.a. if the period of payments is (i) yearly, (ii) half-yearly, (iii) quarterly and (iv) monthly.

Answer : Given P = Rs. 1200, n = 3, r = 5

(i) period p = 1 then $i = \frac{5}{100} = 0.05$

$$A = P\left[\frac{(1+i)^n - 1}{i}\right] = 1200\left[\frac{(1+0.05)^3 - 1}{0.05}\right] = 12000\left[\frac{1.1576-1}{0.05}\right]$$

$\therefore A$ = 1200 x 3.1525 = Rs. 3783

(ii) period p = 2 then $i = \frac{5}{2 \text{ x } 100} = 0.025$

$$A = P\left[\frac{(1+i)^{2n} - 1}{i}\right] = 1200\left[\frac{(1+0.025)^6 - 1}{0.025}\right] = 1200\left[\frac{1.1597-1}{0.025}\right]$$

$\therefore A$ = 1200 x 6.388 = Rs. 7665.60

(iii) period p = 4 then $i = \frac{5}{4 \text{ x } 100} = 0.0125$

$$A = P\left[\frac{(1+i)^{4n} - 1}{i}\right] = 1200\left[\frac{(1+0.0125)^{12} - 1}{0.0125}\right] = 1200\left[\frac{1.1608-1}{0.0125}\right]$$

$\therefore A$ = 1200 x 12.864 = Rs. 15436.80

(iv) period p = 12 then $i = \frac{5}{12 \text{ x } 100} = 0.00417$

$$A = P\left[\frac{(1+i)^{12n} - 1}{i}\right] = 1200\left[\frac{(1+0.00417)^{36} - 1}{0.00417}\right] = 1200\left[\frac{1.16147-1}{0.00417}\right]$$

$\therefore A$ = 1200 x 38.722 = Rs. 46466.83

Example 5: Mr. Nagori invested certain principal for 3 years at 8% interest compounded half yearly. If he received Rs.72957.5 at the end of 3rd year, find the periodic payment he made. [assume $(1.04)^6 = 1.2653$]

Answer: Given $n = 3$, $r = 8$, $p = 2 \Rightarrow i = \frac{8}{2 \text{ x } 100} = 0.04$

$$\text{Now, } A = P\left[\frac{(1+i)^{np} - 1}{i}\right]$$

$$\therefore\ 72957.5 = P\left[\frac{(1+0.04)^6 - 1}{0.04}\right] = P\left[\frac{1.263-1}{0.04}\right]$$

$\therefore 72957.5 = P[6.6325]$

$$\therefore P = \frac{72957.5}{6.6325} = 11000$$

Thus, the periodic payment is Rs. 11,000

Example 6: Find the number of years for which anannuity of Rs.500 is paid at the end ofevery year, if the accumulated amountworks out to be Rs. 1,655 when interest iscompounded annually at 10% p.a.

Answer: Given thatC = Rs. 500, A = Rs. 1,655 n= ?

r= 10% p.a.,$i = \frac{r}{100} = \frac{10}{100} = 0.1$

Accumulated value is given by

$$A = \frac{C}{i}[(1+i)^n - 1]$$

$$1{,}655 = \frac{500}{0.1}[(1+0.1)^n - 1]$$

$$\frac{1{,}655 \times 0.1}{500} = [(1.1)^n - 1]$$

$$0.331 = [(1.1)^n - 1]$$

$$1.331 = (1.1)^n$$

$$(1.1)^n = (1.1)^3$$

n = 3 year

2.4.1 SINKING FUND

The fund (money) which is kept aside to accumulate a certain sum in a fixed period through periodic equal payments is called as sinking fund.

We can consider an example of a machine in a factory which needs to be replaced after say 10 years. The amount for buying a new machine 10 years from now may be very large, so a proportionate amount is accumulated every year so that it amounts to the required sum in 10 years. This annual amount is called as *sinking fund*. Another common example is of the *maintenance tax* collected by any Society from its members.

A sinking fund being same as an annuity, the formula to compute the terms is same as that we have learnt in section 1.11.

Example 7:A company sets aside a sum of Rs. 15,000 annually to enable it to pay off a debenture issue of Rs. 1,80,000 at the end of 10 years. Assuming that the sum accumulates at 6% p.a., find the surplus after paying off the debenture stock.

Answer: Given P = Rs. 15000, $n = 10$, $r = 6 \Rightarrow i = 0.06$

$$\therefore A = P\left(\frac{(1+i)^n - 1}{i}\right) = 15000 \text{ x } \frac{(1+0.06)^{10} - 1}{0.06} = 15000 \text{ x } \frac{1.7908 - 1}{0.06}$$

$\therefore A$ = Rs. 1,97,712

Thus, the surplus amount after paying off the debenture stock

= 197712 – 180000 = Rs. 17712.

Example 8: Shriniketan Co-op Hsg. Society has 8 members and collects Rs. 2500 as maintenance charges from every member per year. The rate of compound interest is 8% p.a. If after 4 years the society needs to do a work worth Rs. 100000, are the annual charges enough to bear the cost?

Answer: Since we want to verify whether Rs. 2500 yearly charges are enough or not we assume it to be *P* and find its value using the formula:

$$A = P\left[\frac{(1+i)^n - 1}{i}\right]$$

Here A = Rs. 100000, $n = 4$, $r = 8 \Rightarrow i = 0.08$

$$\therefore P = \frac{A \text{ x } i}{(1+i)^n - 1} = \frac{100000 \text{ x } 0.08}{(1+0.08)^4 - 1} = 22192$$

Thus, the annual payment of all the members i.e. 8 members should be Rs. 22192.

$\therefore$ the annual payment per member = $\frac{22192}{8}$ = Rs. 2774

This payment is less than Rs. 2500 which the society has decided to take presently. Thus, the society should increase the annual sinking fund.

2.5 EQUATED MONTHLY INSTALLMENTS (EMI)

Suppose a person takes a loan from a bank at a certain rate of interest for a fixed period. The equal payments which the person has to make to the bank per month are called as *equated monthly installments* in short EMI.

Thus, EMI is a kind of annuity with period of payment being monthly and the present value being the sum borrowed.

We will now study the method of finding the EMI using reducing balance method.

Let us recall the formula of finding the present value of an annuity.

$$PV = \frac{P}{i}\left[1 - \frac{1}{(1+i)^{np}}\right]$$

The equal periodic payment (P) is our EMI which is denoted it by E.

The present value (PV) is same as the sum (S) borrowed.

Also the period being monthly $p = 12$.

Substituting this in the above formula we have:

$$S = \frac{E}{i}\left[1 - \frac{1}{(1+i)^{12n}}\right]$$

Thus, if S is the sum borrowed for n years with rate of interest r % p.a. then the EMI is calculated by the formula:

$$E = \frac{S \text{ x } i}{1 - \frac{1}{(1+i)^{12n}}}$$

Example 9: Mr. Sudhir Joshi has taken a loan of Rs. 10,00,000 from a bank for 10 years at 11% p.a. Find his EMI using reducing balance method.

Answer: Given S = Rs. 1000000, $n = 10$, $r = 11 \Rightarrow i = 0.11$

$$\text{Now, } E = \frac{S \text{ x } i}{1 - \frac{1}{(1+i)^{12n}}} = \frac{1000000 \text{ x } 0.11}{1 - \frac{1}{(1+0.11)^{120}}} = \frac{110000}{0.9999}$$

$\therefore E$ = Rs. 1,10,000

Example 10: Mr. PrabhakarNaik has borrowed a sum of Rs. 60,000 from a person at 6% p.a. and is due to return it back in 4 monthly installments. Find the EMI he has to pay and also prepare the amortization table of repayment.

Answer : Given S = Rs. 60,000; $n = 4$ months;

$$r = 6\% \text{ p.a.} = \frac{6}{12} = 0.5\% \text{ p.m} \quad \Rightarrow i = 0.005$$

$$\text{Now, } E = \frac{S \text{ x } i}{1 - \frac{1}{(1+i)^{n}}} = \frac{60000 \text{ x } 0.005}{1 - \frac{1}{(1+0.005)^{4}}} = \frac{300}{0.01975}$$

$\therefore$ ***E* = Rs.**

Now, we will prepare the *amortization table* i.e. the table of repayment of the sum borrowed using reducing balance method.

In the beginning of the 1st month the outstanding principal is the sum borrowed i.e. Rs. 60000 and the EMI paid is Rs. 15187.97

The interest on the outstanding principal is 0.005 x 60000 = Rs. 300 … (1)

Thus, the principal repayment is 15187.97 – 300 = Rs. 14887.97 … (2)

The outstanding principal (O/P) in the beginning of the 2^{nd} month is now

60000 – 14887.97 = 45112.03.

Note:

- (1) is called the *interest part* of the EMI and (2) is called as the *principal part* of the EMI.
- As the tenure increases the interest part reduces and the principal part increases.

This calculation can be tabulated as follows:

Month	O/P	EMI	Interest Part	Principal Part
	(*a*)	(*b*)	(*c*) = (*a*) x *i*	(*b*) - (*c*)
1	60000	15187.97	300	14887.97
2	45112.03	15187.97	225.56	14962.45
3	30141.02	15187.97	150.75	15037.22
4	15111.80	15187.97	75.56	15112.41

In the beginning of the 4^{th} month the outstanding principal is Rs. 15111.80 but the actual principal repayment in that month is Rs. 15112.41. This difference is due to rounding off the values to two decimals, which leads the borrower to pay 61 paise more!!

Example 11: Mr. ShyamRane has borrowed a sum of Rs. 100000 from a bank at 12% p.a. and is due to return it back in 5 monthly installments. Find the *EMI* he has to pay and also prepare the amortization table of repayment.

Answer: Given S = Rs. 100000; n = 5 months;

$$r = 12\% \text{ p.a.} = \frac{12}{12} = 1\% \text{ p.m} \quad \Rightarrow i = 0.01$$

Now, $$E = \frac{S \text{ x } i}{1 - \frac{1}{(1+i)^n}} = \frac{100000 \text{ x } 0.01}{1 - \frac{1}{(1+0.01)^5}} = \frac{1000}{0.0485343} = 20603.98$$

The amortization table is as follows:

Month	O/P	EMI	Interest Part	Principal Part
	(*a*)	(*b*)	(*c*) = (*a*) x *i*	(*b*) - (*c*)
1	100000	20603.98	1000	19603.98
2	80396.02	20603.98	803.96	19800.02
3	60596	20603.98	605.96	19998.02
4	40597.98	20603.98	405.98	20198
5	20399.98	20603.98	204	20399.98

Flat Interest Rate (A layman's approach) : In this method the amount is calculated using simple interest for the entire period. This method is hardly using any mathematical concepts except the formula for simple interest, so it is not accurate.

Example 12:A person brought Rs.75,000 at 12% p.a. if he wishes to return the sum within a one year, Find his EMI using Flat interest rate method.

Answer: Given that P = Rs. 75,000, r = 12%, n = 1 year.

$$S.I = \frac{P \times N \times R}{100} = \frac{75000 \times 1 \times 12}{100} = Rs. 9,000$$

Amount =P + S.I. = 75,000 + 9,000 = Rs. 84,000

$$EMI = \frac{Amount}{12N} = \frac{84,000}{12 \times 1} = Rs. 7,000.$$

2.6 LET US SUM UP

In this chapter we have learn:

- Definition of annuity and basic terms related to annuity.
- Different types of annuity.
- Formula and method of calculation of Present value and future value.
- Different method to calculate EMI.

2.7 UNIT END EXERCISE

1. Find the future value of an immediate annuity of Rs. 1200 at 6% p.a. compounded annually for 3 years.

2. Find the future value of an immediate annuity of Rs. 500 at 8% p.a. compounded p.m. for 5 years.

3. Find the accumulated amount of an immediate annuity of Rs. 1000 at 9% p.a. compounded semi-annually for 4 years.

4. Find the future value of an immediate annuity of Rs. 2800 paid at 10% p.a. compounded quarterly for 2 years. Also find the interest earned on the annuity.

5. Find the periodic payment to be made so that Rs. 25000 gets accumulated at the end of 4 years at 6% p.a. compounded annually.

6. Find the periodic payment to be made so that Rs. 30,000 gets accumulated at the end of 5 years at 8% p.a. compounded half yearly.

7. Find the rate of interest if a person depositing Rs. 1000 annually for 2 years receives Rs. 2070.

8. Find the rate of interest compounded p.a. if an immediate annuity of Rs. 50,000 amounts to Rs. 1,03,000 in 2 years.

9. Find the rate of interest compounded p.a. if an immediate annuity of Rs. 5000 amounts to Rs. 10400 in 2 years.

10. What is the value of the annuity at the end of 5 years, if Rs. 1000 p.m. is deposited into an account earning interest 9% p.a. compounded annually? What is the interest paid in this amount?

11. What is the value of the annuity at the end of 3 years, if Rs. 500 p.m. is deposited into an account earning interest 6% p.a. compounded annually? What is the interest paid in this amount?

12. Mr. AshishGokhale borrows Rs. 5000 from a bank at 8% compound interest. If he makes an annual payment of Rs. 1500 for 4 years, what is his remaining loan amount after 4 years?

(Hint: find the amount using compound interest formula for 4 years and then find the accumulated amount of annuity, the difference is the remaining amount.)

13. Find the present value of an immediate annuity of Rs. 10,000 for 3 years at 6% p.a. compounded annually.

14. Find the present value of an immediate annuity of Rs. 100000 for 4 years at 8% p.a. compounded half yearly.

15. Find the present value of an immediate annuity of Rs. 1600 for 2 years at 7% p.a. compounded half yearly.

16. A loan is repaid fully with interest in 5 annual installments of Rs. 15,000 at 8% p.a. Find the present value of the loan.

17. Mr. Suman borrows Rs. 50,000 from Mr. Juman and agreed to pay Rs. 14000 annually for 4 years at 10% p.a. Is this business profitable to Mr. Juman?

(Hint: Find the PV of the annuity and compare with Rs. 50000)

18. Mr. Paradkar is interested in saving a certain sum which will amount to Rs. 3,50,000 in 5 years. If the rate of interest is 12% p.a., how much should he save yearly to achieve his target?

19. Mr. KedarPethkar invests Rs. 10000 per year for his daughter from her first birthday onwards. If he receives an interest of 8.5% p.a., what is the amount accumulated when his daughter turns 18?

20. Dr. Prabhulkar, a dentist has started his own dispensary. He wants to install a machine chair which costs Rs. 3,25,000. The machine chair is also available on monthly rent of Rs. 9000 at 9% p.a. for 3 years. Should Dr. Prabhulkar buy it in cash or rent it?

21. A sum of Rs. 50,000 is required to buy a new machine in a factory. What sinking fund should the factory accumulate at 8% p.a. compounded annually if the machine is to be replaced after 5 years?

22. The present cost of a machine is Rs. 80,000. Find the sinking fund the company has to generate so that it could buy a new machine after 10 years, whose value then would be 25% more than of today's price. The rate of compound interest being 12% p.a. compounded annually.

23. Regency Co-op. Hsg. Society which has 15 members require Rs. 4,60,000 at the end of 3 years from now for the society repairs. If the rate of compound interest is 10% p.a., how much fund the society should collect from every member to meet the necessary sum?

24. Mr. Lalwaney is of 40 years now and wants to create a fund of Rs. 15,00,000 when he is 60. What sum of money should he save annually so that at 13% p.a. he would achieve his target?

25. If a society accumulates Rs. 1000 p.a. from its 200 members for 5 years and receives 12% interest then find the sum accumulated at the end of the fifth year. If the society wants Rs. 13,00,000 for society maintenance after 5 years, is the annual fund of Rs. 1000 per member sufficient?

26. If a society accumulates Rs. 800 p.a. from its 100 members for 3 years and receives 9% interest then find the sum accumulated at the end of the third year. If the society wants Rs. 2,50,000 for society maintenance after 3 years, is the annual fund of Rs. 800 per member sufficient?

27. Mr. Kanishk wants clear his loan of Rs. 10,00,000 taken at 12% p.a. in 240 monthly installments. Find his EMI using reducing balance method.

28. Using the reducing balance method find the EMI for the following:

29. Mr. ArvindKamble has borrowed Rs. 30,000 from his friend for 2 years at 14% p.a. If he is to return this amount in 5 monthly installments, find the installment amount, the interest paid and prepare the amortization table for repayment.

30. Mrs. Chaphekar has taken a loan of Rs. 1,25,000 from a bank at 12% p.a. If the loan has to be returned in 3 years, find the EMI, Mrs. Chaphekar has to pay. Prepare the amortization table of repayment of loan and find the interest she has to pay.

31. A loan of Rs. 75,000 is to be returned with interest in 4 installments at 15% p.a. Find the value of the installements.

32. Find the sum accumulated by paying an EMI of Rs. 11,800 for 2 years at 10% p.a.

33. Find the sum accumulated by paying an EMI of Rs. 1,800 for 2 years at 12% p.a.

Find the sum accumulated by paying an EMI of Rs. 12,000 for 3 years at 9% p.a.

Multiple Choice Questions:

1. The accumulated value of an annuity immediate , of Rs.20,000 p.a. for 3 years at 10% p.a. compounded yearly, is Rs.

a) 66,200 b) 66,221 c) 92,820 d) 60,000

2. An annuity in which each payment is made at the end of the year is called ________.

a) annuity due b) annuity certain
c) Immediate annuity d) uniform annuity

3. A loan of Rs.80,000 is returned in 3 monthly installments at 12% p.a. find the EMI using the flat rate.

a) Rs.25,488.67 b) Rs.27,466.67 c) Rs.28,576.67 d) Rs. 26,567.82

4. The present value is always _______ the future value.

a) More then b) less then c) equal to d) independent of

5. An annuity in which the number of payments is fixed is called

a) Fix annuity b) limited annuity
c) certain annuity d) immediate annuity.

6. An annuity in which all the payments are equal is called
a) Equitable annuity. b) egalitarian annuity.
c) annuity due. d) uniform annuity.

7. The present value of an immediate annuity of Rs.50,000 p.a. for 3 years at 10% p.a. compounded annually is Rs.
a) 1,80,000 b) 1,24,342.60 c) 1,24,234.06 d) 1,50,000

8. The accumulated amount after 3 years of an immediate annuity of Rs.5,000 p.a. with the rate of interest of 6% compounded annually is Rs.
a) 15,000 b) 15,900 c) 15.921.23 d) 15,918

9. If the payment of annuity are made at the beginning of each period, the annuity is called
a) Ordinary annuity b) Annuity due
c) Uniform annuity d) Immediate annuity

10. When the EMI are calculated using present value of the annuity using compound interest, the method is called
a) Flat rate method b) Repayment method
c) Reducing balance method d) Amortization method

2.8 LIST OF REFERENCES

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3

FUNCTIONS

Unit Structure

3.0 OBJECTIVE

After going through this chapter you will able to know:

- Definition of function and types of functions.
- Different types of function used in business and economics.
- Most important concept of business break-even point.
- The concept of equilibrium point.

3.1 INTRODUCTION

A function was the heart of the scientific revolution of the seventeenth century. To understand the general use of function we must study their properties in the general, which is what we do in this chapter.

The reader is no doubt familiar with function of the form $y = f(x)$ for instance, if $f(x) = x^2 - 2$, $x = 2$ and $y = f(2)$ then the value of y is 4.

3.2 FUNCTION

A function can be intuitively understood as a relation between two variables, where one variable which depends on the other is called as dependent variable while the other is called as independent variable. Let y be variable depending on an independent variable say x, then the variable y is said be a function of x which can be denoted as $y = f(x)$ (This is pronounced as y equal to f of x).

We all know that demand of a certain product depends on its price and vice-versa. If D denotes demand of a certain product and P denotes its price then the relation between demand and price can be written as $D = f(P)$ or $P = g(D)$. Here f and g are notations for showing the function (relation).

For *e.g.*, for a certain commodity the relation between demand and price is given as $D = \frac{k}{P}$, which means that, as the price of the commodity increases its demand falls and as the price decreases the demand increases.

3.3 TYPES OF FUNCTION

Constant function:

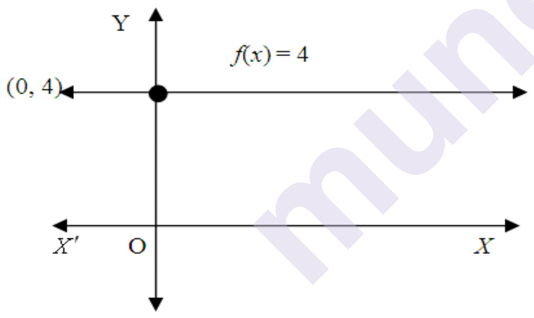
A function which takes a constant value is called as *constant function*.

Example: $f(x) = 4$, $g(x) = a$, where a is any real value.

Graphically, a constant function is a straight line parallel to the X-axis or the Y-axis. In general, for a fixed real value k, a constant function is written as $\boldsymbol{f(x) = k}$.

In both the examples we can see that the function takes a fixed value over all values of the independent variable x.

The graph of $f(x) = 4$ is as follows:

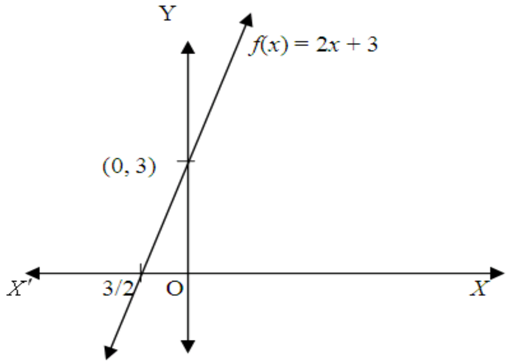


Linear function

A function whose graph is a straight line is called as a *linear function*.

In general, for fixed real values m and c, the function $f(x) = mx + c$ is a linear function. Here m is the slope of the line and c is its y-intercept.

Example: $f(x) = 2x + 3$. The graph of this function is as shown below:



Step function

A function which takes different constant values at different intervals of the independent variable is called a *step function*. The name step function is being given because of the graphical behavior of the function.

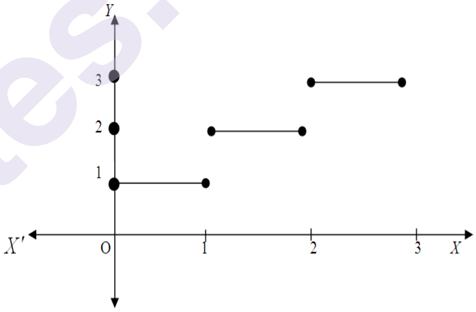
Example: (*i*) Consider the

function

$f(x) = 1 \quad \text{for } 0 < x \leq 1$

$= 2 \quad \text{for } 1 < x \leq 2$

$= 3 \quad \text{for } 2 < x \leq 3$

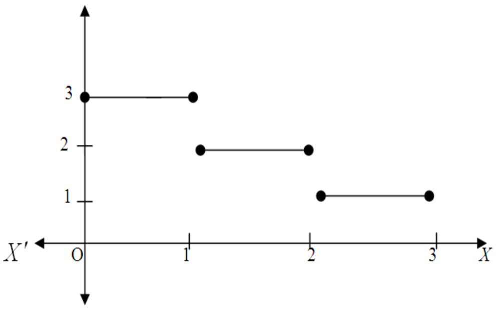


ii) Consider the function

$g(x) = 3 \qquad 0 < x \leq 1$

$= 2 \qquad 1 < x \leq 2$

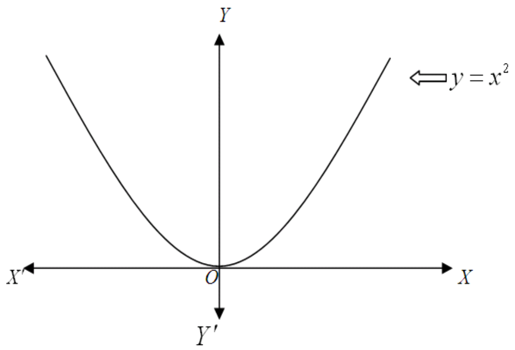
$= 1 \qquad 2 < x \leq 3$



Power function

A function which has the dependent variable as a power of the independent variable is called as a *power function*. Here the variable (x) is the base and a natural number (n) as the index (power).

Example: $f(x) = x^2$, $g(x) = 3x^3$

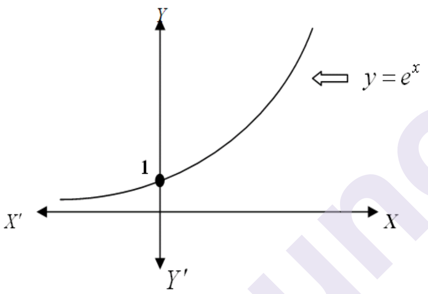


Exponential function

A function which has a constant base and variable index is called as an *exponential function*. If a is any positive integer then $f(x) = a^x$ is an *exponential function*.

A very common example in calculus is when the number e whose approximate value is 2.718 is taken as the base. *i.e.* $f(x) = e^x$.

The graph of e^x is given below:



Logarithmic function

Logarithmic function is an inverse function of exponential function.

- If $x = a^y$ where $a > 0$ and $a \neq 1$, then $y = \log_a x$ is the logarithmic function. Here y is said to be the logarithm of x to the base a.
- If $x = e^y$, then $y = \log_e x = \log x$ is the logarithmic function. Here y is said to be the logarithm of x to the base e. Usually, when e is taken as the base it is not written.

If m and n are positive numbers, then

1. $\log_a (mn) = \log_a m + \log_a n$

2. $\log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n$

3. $\log_a (m^n) = n \log_a m$

4. $\log_a 1 = 0$

5. $\log_a a = 1$

6. $\log_b x = \dfrac{\log_a x}{\log_a b}$

Example.1: If $f(x) = x^2 + x + 1$, find the value of f at x = -1, 0, 1, 2

Answer: At x = -1, $f(-1) = (-1)^2 + (-1) + 1 = 1 - 1 + 1 = 1$

$\therefore f(-1) = 1$

At $x = 0$, $f(0) = 0^2 + 0 + 1 = 1$ $\therefore f(0) = 1$

At $x = 1$, $f(1) = 1^2 + 1 + 1 = 3$ $\therefore f(1) = 3$

At $x = 2$, $f(2) = 2^2 + 2 + 1 = 7$ $\therefore f(2) = 7$

Example.2. If $f(x) = \sqrt{x^2 + 2}$, find $f(x + 1)$ and $f(x - 1)$

Answer: Since $f(x) = \sqrt{x^2 + 2}$

$\therefore f(x+1) = \sqrt{(x+1)^2 + 2}$

$\therefore f(x+1) = \sqrt{x^2 + 2x + 1 + 2} = \sqrt{x^2 + 2x + 3}$

Similarly,

$f(x-1) = \sqrt{(x-1)^2 + 2}$

$\therefore f(x-1) = \sqrt{x^2 - 2x + 1 + 2} = \sqrt{x^2 - 2x + 3}$

Example.3. Find the value of x, if $f(3x) = 3f(x)$ where $f(x) = x^2 + x + 3$.

Answer: Given $f(x) = x^2 + x + 3 \Rightarrow 3f(x) = 3x^2 + 3x + 9$

Also, $f(3x) = (3x)^2 + (3x) + 3 = 9x^2 + 3x + 3$

Also given that, $f(3x) = 3f(x)$

$\therefore 9x^2 + 3x + 3 = 3x^2 + 3x + 9$

$\Rightarrow 6x^2 - 6 = 0 \Rightarrow x^2 - 1 = 0$

$\therefore x^2 = 1$

$\Rightarrow x = \pm 1$

Example 4. If $f(x) = \log x$, show that $f(xy) = f(x) + f(y)$

Answer: $\because f(x) = \log x$

$\therefore f(y) = \log y$

Also, $f(xy) = \log(xy)$ … (1)

Using the properties of logarithms, we have

$\log(xy) = \log x + \log y$ … (2)

Thus, $f(xy) = \log x + \log y$ … from (1) and (2)

$\therefore f(xy) = f(x) + f(y)$. Hence proved.

3.4 THE FUNCTION USED IN BUSINESS AND ECONOMICS

Demand function

Let p be the price of a certain commodity and Dbe its demand. As per the economic convention price is considered as a function of demand and hence we can denote this relation as $p = f(D)$ or $p = f(Q_d)$. (Q_d is the quantity of demand). This is called as the *demand function*.

The graph of demand function called as demand curve, is drawn by taking the price (p) on the Y-axis and the demand (D) on the X-axis.

The graph of demand function always has a negative slope, *i.e.* demand curve falls downwards from left to right indicating that as the demand increases the price of that commodity decreases.

Examples: $p = 3 + 7D$, $p = \frac{5}{D}$, $p = 3D^2 - 4D + 11$, $p = \frac{4}{D^2 - 2}$ or

$Q_d = 5 + 3p$, $Q_d = \frac{10}{p+1}$, $Q_d = 2p^2 - p + 4$ etc.

Supply function

If p is the price of a certain commodity and S is its supply, then the *supply function* is given by $p = f(S)$ or $p = f(Q_S)$, where Q_S is the quantity of supply.

The graph of supply function called as supply curve, is drawn by taking the price(p) on the Y-axis and the supply (S) on the X-axis.

The graph of the supply curve has a positive slope, *i.e.* the supply curve move upwards from left to right.

Examples: $p = 10 - 4S$, $p = 3 + \sqrt{S^2 + 4}$, or $Q_S = 2p - 3$, $Q_S = 3p^2 - p$ etc.

Total Cost function

If x is the number of units produced of a certain commodity and C is the total cost incurred then the cost function is given as $C = f(x)$

Thus, cost depends on the quantity produced.

The cost is divided into two parts:

a) **Fixed cost (*FC*)**: This is independent of the quantity produced. *e.g.* labor cost, maintenance, rent etc.
b) **Variable cost (*V*(*x*))**: This depends on the quantity produced.

Thus, $C = FC + V(x)$.

Examples: $C = 7x + 6$, $C = x^2 + 2x + 5$.

Average Cost function

If x is the number of units produced of a certain commodity and C is the total cost, then the cost per unit of the commodity is called as the *average cost* (AC).

Average cost = $AC = \frac{C(x)}{x}$

Total Revenue

If p is the price per unit and D is the demand of a certain product, then the *total revenue function* (TR) is given by $TR = pD$

Average Revenue

Average revenue (AR) is the revenue per unit demand of the product. In other words, AR is the price per unit of the product.

$$AR = \frac{R}{D} = \frac{pD}{D} = p$$

$$\therefore AR = p$$

***Example* 5**. The demand function for a product is given by $p = 6 - 4D^2$, where p is the price per unit of the product and D is the demand. Find the total revenue, average revenue for the product.

Ans: Let R be the total revenue function.

w.k.t. $R = pD$

$$\therefore R = (6 - 4D^2)D$$

$$\therefore R = 6D - 4D^3$$

And average revenue is given by, $AR = p$

$\therefore AR = 6 - 4D^2$

***Example*.6.**If the demand function of a product is given by $p = D^2 + 6D + 11$, find the total revenue when $D = 5$.

Ans: Given $p = D^2 + 6D + 11$

$\therefore R = pD = (D^2 + 6D + 11)D$

$\therefore R = D^3 + 6D^2 + 11D$

When $D = 5$,

$[R]_{D=5} = 5^3 + 6(5)^2 + 11(5)$

$\therefore [R]_{D=5} = 125 + 150 + 55 = 330$.

3.5 BREAK-EVEN POINT

Profit function

If R is the total revenue and C is the total cost of a product, then the *profit function* denote as π, is given by $\pi = R - C$.

- If $\pi > 0$ then there is a profit.
- If $\pi < 0$ then there is a loss.
- If $\pi = 0$ then there is a situation wherein there is no loss or no profit. Such a point is called as

Break Even Point (BEP). In other words, at the **BEP** the total revenue is same as the total cost.

Now we know that, $\pi = R - C$.

If $\pi = 0$, $R - C = 0 \quad \Rightarrow R = C$.

***Example* 7.**The total cost C(in lakhs of Rs.) in manufacturing x units of a cell phone is given by $C = 0.055x + 219$. Each cell phone is sold at price of Rs. 8,500. Find the profit function and the *break even point*.(BEP)

Ans: The revenue function is given by $R = 8500 \text{ x } \dfrac{x}{100000}$

$\therefore R = 0.085x$

The profit function is given by $\pi = R - C$

$\therefore \pi = 0.085x - (0.055x + 219)$

$\therefore \pi = 0.085x - 0.055x - 219 = 0.03x - 219$

$\therefore \pi = 0.03x - 219$.

The BEP is achieved when π = 0

Thus, $0.03x - 219 = 0$

$\Rightarrow 0.03x = 219$

$\Rightarrow x = 7300$

Hence the BEP is achieved when 7,300 cell phones are manufactured.

***Example* 8.** The demand function for a product is given by $p = ax + b$. If the prices for producing 1unit and 6 units of the product are 10 and 30 respectively, find the demand function explicitly. Also find the total revenue when $x = 10$.

Ans: When $x = 1$, the demand function is $10 = a + b$ … (1)

When $x = 6$, the demand function is $30 = 6a + b$ … (2)

Solving the eqns (1) and (2), we get

$a = 4$ and $b = 6$.

Hence the demand function is $p = 4x + 6$

The total revenue is $R = px = (4x + 6)x = 4x^2 + 6x$

When $x = 10$, $[R]_{x=10} = 4(10)^2 + 6(10) = 400 + 60 = 460$

***Example* 9** The fixed cost for manufacturing a product is Rs. 45,000 and the variable cost is Rs. 500 per unit of the product. The revenue function is given by $R = 100x^2 - 3600x$. Find the profit function, BEP and the values of x for which there will be profit.

Ans: The total cost function is given by:

$C = FC + V(x)$

From the given data we have,

$C = 45000 + 500x$ … (1)

The revenue function is given as:

$R = 100x^2 - 3600x$ … (2)

Thus, the profit function $\pi = R - C$ from (1) and (2) is,

$\pi = 100x^2 - 3600x - (45000 + 500x)$

$\therefore \pi = 100x^2 - 4100x - 45000$ … (3)

The BEP is achieved when $\pi = 0$. Equating eqn (3) to 0, we get

$$100x^2 - 5900x - 45000 = 0$$

Dividing by 100 on both sides, we get

$$\therefore x^2 - 41x - 450 = 0$$

$$\therefore (x+9)(x-50) = 0 \qquad \ldots (4)$$

$$x = -9 \text{ or } x = 50$$

Discarding $x = -9$, as number of units produced cannot be negative, the BEP is at $x = 50$.

We know that, the manufacturer will earn profit when the sales are above the BEP values.

Thus, profit is achieved for any value of x greater than 50.

3.6 EQUILIBRIUM POINT

As we have seen above that price is a function of demand as well as supply. If both the curves are drawn for the same commodity, we see that the curves intersect at a point. This point, where the supply and demand is equal is called as the equilibrium price denoted as p_e.

***Example* 10**. The demand function for a certain product is given by $p = 7 - 2D$ and the supply function is given by $p = 4S - 5$, where p is the price per unit, D is the demand and S is the supply. Find the equilibrium price and quantity for the product.

Ans:We will first write the demand and supply functions as functions of price.

$$p = 7 - 2D \qquad \Rightarrow D = \frac{7-p}{2} \qquad \ldots(1)$$

And $p = 4S - 5 \qquad \Rightarrow S = \dfrac{p+5}{4} \qquad \ldots(2)$

Now, w.k.t., equilibrium price is that price where the demand and supply coincides. So we equate both the above equations,

$$i.e. \frac{7-p}{2} = \frac{p+5}{4}$$

$$\therefore 4(7-p) = 2(p+5)$$

$$\therefore 28 - 4p = 2p + 10$$

$$\therefore 6p = 18 \qquad \Rightarrow p = 3$$

Thus the equilibrium price is $p_e = 3$.

To find the equilibrium quantity, we substitute the equilibrium price in any of the eqns (1) and (2);

Substituting $p = 3$ in eqn (1), we get

$$Q_D = D = \frac{7-p}{2} = \frac{7-3}{4} = 1.$$

Thus the equilibrium quantity is $Q_D = 1$.

3.7 LET US SUM UP

In this chapter we have learn:

- Function and its types.
- Different business and economics functions
- To solve different Problem to using function.
- The Concept of Break-even point and equilibrium point.

3.8 UNIT END EXERCISE

1. Define dependent variable and independent variable.

2. Explain the demand and supply function with proper examples from economics.

3. Write a short note on *Break even Point*.

4. Classify the following functions as constant function, linear function, power function, exponential function, step function and logarithmic function:

(i) $f(x) = \log 10$, (ii) $f(x) = 10x - 15$ (iii) $f(x) = 3^x$ (iv) $f(x) = e^x$

$$(v)\quad \begin{aligned} f(x) &= 4 \quad 0 < x \le 2 \\ &= 8 \quad 2 < x \le 4 \\ &= 12 \quad 4 < x \le 6 \end{aligned}$$

5. Given that $f(x) = x^2 + \sqrt{2x-3}$, find $f(x+2)$.

6. If $f(x) = 3x^2 - 6$, show that $f(2x) - 4f(x) - 18 = 0$.

7. Find x, if $f(2x) = 4f(x)$, where $f(x) = 3x^2 + x - 6$.

8. If $f(x) = \frac{2x-3}{x+1}$, find $f\left(\frac{2x-3}{x+1}\right)$.

9. If $f(x)=\log\left(\frac{1+x}{1-x}\right)$, show that $f(x)+f(y)=f\left(\frac{x+y}{1+xy}\right)$

10. If $f(x)=\log\left(\frac{1-x}{1+x}\right)$, show that $f(x)+f(y)=f\left(\frac{x+y}{1+xy}\right)$

11. If $f(x)=\log\left(\frac{1+x}{1-x}\right)$, show that $f(x)-f(y)=f\left(\frac{x-y}{1-xy}\right)$

12. If the demand function is $Q_D=p-15$ and the supply function is $Q_S=10-\frac{p}{3}$, find the equilibrium price and equilibrium quantity.

13. If the demand function is $Q_D=2p-12$ and the supply function is $Q_S=3p-18$, find the equilibrium price and equilibrium quantity.

14. If the demand function is $Q_D=12p-110$ and the supply function is $Q_S=15-\frac{p}{2}$, find the equilibrium price and equilibrium quantity.

15. Find the equilibrium price if $Q_D=\frac{8p}{p-2}$ and $Q_S=p^2$.

16. If the demand function is $p=D+10$ and the supply function is $p=20-2S$, find the equilibrium price and equilibrium quantity.

17. If the demand function is $p=3D+16$ and the supply function is $p=20-S$, find the equilibrium price and equilibrium quantity.

18. The demand functions for two products A and B are as follows: $D_A=10-p_B$ and $D_B=12-p_A+p_B$. The corresponding supply functions are $S_A=2+p_A+p_B$ and $S_B=2+p_A-p_B$. Find the equilibrium prices and equilibrium quantities.

19. Mr. BhaveshShah who owns a Tea factory, finds that variable cost for processing of1 gm of tea is Re.1 and his fixed costs are Rs. 3000. Write down the cost function, hence find the cost for processing 1kg of Tea.

20. Mr. Gunderia's glass factory has 35 workers. The cost of producing 1 sheet of glass is Rs. 34.50. The fixed cost per worker is Rs. 25. If the glass sheet is sold at Rs.52, find the number of sheets to be produced so that there is no loss. If a discount of Rs.2 is given per sheet what is the BEP? At this price what is the profit if 200 sheets are produced?

21. A firm has the cost function $C=x(x^2-2)$ and the revenue function $R=14x-x^2$. Find the average cost, average revenue and profit function. Hence find the profit at the BEP.

22. The price of a chocolate was Rs. 5 when its demand was 10,000. Its demand came down to 8,000 at the price of Rs.8 per chocolate. If the price is a linear function of the demand, find the explicit relation. Estimate the demand when the price will be Rs. 10 per chocolate. Which of these three prices gives more profit for the manufacturer?

23. The total cost and the total revenue of a firm is given by $C = 5x + 350$ and $R = 50x - x^2$. Find the break even points.

24. Mr. VinodMenon manufactures wall clocks. The cost of setting up the factory is Rs. 5 lakhs. The cost of production per wall clock is Rs. 100. Determine the cost function. If each wall clock is sold at Rs. 250, write down the revenue and profit function. If 2000 walls clocks are produced and sold does Mr. Vinod earn any profit? If so, what is his profit? Also find the BEP.

25. Mr. Roy manufactures pens, whose total cost is given by $C = 4x + 25200$. If each pen is sold at Rs. 12, find the minimum number of pens Mr. Roy should produce so that there is no loss. If a discount of Rs. 2 is given per pen, what would be the BEP? If 5000 pens are sold everyday at the discounted cost, does Mr. Roy earn any profit? If yes, find the profit.

Multiple Choice Questions:

i) For the demand function $D=20-5p+p^2$ then the demand when p=3 is___
a) 16 b) 15 c) 14 d)10

ii) The cost function is $C=x^2+x+100$ then the average cost of producing 10 items is ___
a) 20 b) 21 c) 210 d) 200

iii) If f(x)=3x+5\x then the value of f(x) at x=0 is______
a) 0 b) 1 c) 3 d) function does not exist

iv) If the total cost function is $C=4x^2+x+8$ then the average cost when x=2 is____
a) 13.75 b) 13 c) 26 d) 24

v) For the function $f(x) = 2x + 3$ is
a) constant function b) linear function
c) quadratic function d) identity function

3.9 LIST OF REFERENCES

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4

PERMUTATION AND COMBINATION

Unit Structure

4.0 OBJECTIVE

After going through this chapter you will able to know:

- Factorial notation.
- Definition of permutation and its formula.
- Arrangements of *n* different objects may be all are same or distinct.
- Definition of Combination and its formula.
- Using combination we can calculate different ways of selection.

4.1 INTRODUCTION

We often hear people saying that 'probably it will rain today'; 'it is likely that India will win the match against Australia on this pitch'; 'the chance of passing the CET is 10% only' and so on. The words 'probably', 'by chance' or 'likely' are statistical terms but are very commonly used by all of us. It is very natural that people are interested to know about the possibility that something happens. People interested in sports are eager to know, the possibility of their team to win a game; political activists want to be sure their chances of winning an election, meteorological department would like to know about the weather, an economists may want to know the chance that sales will increase if the price of a commodity is decreased etc. Also it becomes necessary sometimes to know in how many different ways a particular event may happen.

All these calculations can be broadly classified into two types namely *permutations* and *combinations*.

4.2 PREREQUISITES AND TERMINOLOGY

Before going into the detailed study of these two methods, we shall discuss some of the prerequisites which are useful in understanding the concepts and formulae related to them.

Factorial:

The factorial of a natural number n is defined as the product of all numbers from 1to n. It is denoted as $n!$

For example:

Factorial of 3 *i.e.* $3! = 3 \times 2 \times 1 = 6$ (*This is read as 3 factorial equal to 6*)

Similarly, $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

In general, $n! = n(n-1)(n-2)(n-3)\ldots\ldots 3 \times 2 \times 1$

This formula gives a recursive relation: $\boldsymbol{n! = n(n-1)!}$

For example:

$6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = \mathbf{6 \times 5!}$

<u>Remark</u>: We define $\mathbf{0! = 1}$

Fundamental Principal of Counting:

If there are m ways doing one thing and n ways of doing another thing then the total number of ways of doing both the things together is mn.

4.3 PERMUTATION

A *permutation* of n objects is an arrangement of some of these (or all) objects in a definite order. The order in which the arrangement is done is important in permutations.

***Example* 1:** In how many different ways can three friends Mitesh, Ritesh and Paresh stand for a group photograph?

Ans: Let us denote these friends by their first alphabet M, R and P. The three friends can be arranged as shown below:

M R P M P R R M P R P M P M R P RM

The number of ways is 6.

One should not be satisfied with this answer. The question that should come to our mind is how did we arrange them so? Well, if we observe the arrangement again, it can be seen that first M's place was fixed and the remaining two were arranged. This step was repeated again for R and P. In terms of permutation what we did was arranging 3 objects amongst themselves. The next question should be what if there are 10 friends? Can you write down their different arrangements explicitly as above?

4.4 FORMULA TO COMPUTE PERMUTATIONS OF *NDIFFERENT* OBJECTS

^nP_r: The number of ways of arranging r objects out of n objects is denoted by ^nP_r and is calculated by the formula: $^nP_r = \frac{n!}{(n-r)!}$

For e.g: $^4P_2 = \frac{4!}{(4-2)!} = \frac{4 \text{ x } 3 \text{ x } 2 \text{ x } 1}{2 \text{ x } 1} = 12$

Remark: The number of ways of arranging all n objects is thus $^nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$

Now if we go back to the first example, where we had to arrange 3 friends, then the number of ways of their arrangement using the above formula is $3! = 6$, which is the same answer what we had got by arranging them explicitly.

***Example* 2:** In how many ways can 6 people be photographed, if only 4 can be seated at a time?

Ans: Here $n = 6$ and $r = 4$.

$\therefore$ no. of ways of arranging 4 out of 6 people = $^6P_4 = \frac{6!}{(6-4)!} =$

$\frac{6 \text{ x } 5 \text{ x } 4 \text{ x } 3 \text{ x } 2 \text{ x } 1}{2 \text{ x } 1} = 720$

***Example* 3:** If $^nP_5 = 42\,^nP_3$ then find the value of n.

Ans: $^nP_5 = \frac{n!}{(n-5)!}$ and $^nP_3 = \frac{n!}{(n-3)!}$

Since $^nP_5 = 42\,^nP_3$

$\therefore \frac{n!}{(n-5)!} = 42 \frac{n!}{(n-3)!}$

$\therefore \frac{(n-3)!}{(n-5)!} = 42$

i.e. $\frac{(n-3)(n-4)(n-5)!}{(n-5)!} = 42$

$\therefore (n-3)(n-4) = 42$

$\therefore n^2 - 7n + 12 = 42$

$\therefore n^2 - 7n - 30 = 0$

$\therefore\ (n - 10)(n + 3) = 0$

$\Rightarrow n = 10,\ -3$

As $n \neq -3 \quad \therefore \boldsymbol{n = 10}$

***Example* 4:** Show that ${}^{n}P_r = n.{}^{n-1}P_{r-1}$

Ans: Consider RH.S. = $n.{}^{n-1}P_{r-1} = n.\ \dfrac{(n-1)!}{[(n-1)-(r-1)]!}$

$$= \frac{n(n-1)!}{(n-1-r+1)!} = \frac{n!}{(n-r)!} = {}^{n}P_r = \text{L.H.S.}$$

***Example* 5:** In how many different ways can a 4 digit number be formed from the numbers 1, 2, 3, …, 9, with no digit being repeated?

Ans: Since no repetition is allowed, the number of ways of forming a four digit number from the given 9 digits is

$${}^{9}P_4 = \frac{9!}{(9-4)!} = \frac{9 \text{ x } 8 \text{ x } 7 \text{ x } 6 \text{ x } 5!}{5!} = 3024.$$

***Example* 6**: In how many different ways can the letters of the word "MATHS" be arranged if no letter is to be repeated in the same word?

Ans: The word "MATHS" consists of four letter M, A, T, H and S.

The required number of ways is arranging 4 objects all at a time.

Thus, the no. of different ways = 4! = 24.

***Example* 7:** Eight candidates are to appear for an interview in a company. In how many ways can the HR manager schedule the candidates for their interview? What if two of the eight candidates are not be interviewed?

Ans: The number of ways of arranging 8 candidates is 8! = 40320

If two out the eight are not be arranged, it means to arrange the remaining six out of 8.

This can be done in ${}^{8}P_6$ ways.

$\therefore$ the number of ways of scheduling the candidates now = $\dfrac{8!}{(8-6)!}$

$$= \frac{8!}{2!} = \frac{40320}{2} = 20160$$

***Example* 8:** Find the number of ways in which 3 books on Economics, 4 books on Mathematics and 2 books on Law can be arranged on a book shelf so that books of the same subject are together. Find the number of arrangements if books are to be arranged at random.

Ans: We can consider this problem of arranging three blocks **E** (books on Economics), **M** (books on Mathematics) and **L** (books on Law), which can be done in 3! ways.

Now,

3 books in Economics can be arranged amongst themselves in 3! ways,

4 books in Mathematics can be arranged amongst themselves in 4! ways,

and 2 books in Law can be arranged amongst themselves in 2! Ways

Thus, by the fundamental principle of counting,

the total number of arrangements = 3! x 3! x 4! x 2! = 1728 ways.

If there is no condition on the position of any book of any subject we may consider this of arranging 3 + 4 + 2 = 9 books, which can be done in 9! Ways

i.e. 362880 ways.

***Example* 9:** In how many ways can 3 boys and 4 girls be seated for a group photograph if (i) no two boys sit together, (ii) no two girls sit together, (iii)all boys sit together

Ans: (i) *no two boys sit together*

4 girls can be arranged in **4! = 24** number of ways

In each such arrangement there are five places (marked as X) where boys can be seated so that no two boys sit together as shown below:

X **G** X **G** X G X **G** X

Now, 3 boys can be seated in these 5 places in $^5P_3$ ways = **60 ways**

Thus, the total number of arrangements = 24 x 60 = 1440.

(ii) *no two girls sit together*:

3 boys can be arranged in **3!**number of ways

In each such arrangement there are four places (marked as X) where girls can be seated so that no two girls sit together as shown below:

X **B** X **B** X **B** X

Now, 4 girls can be seated in these 4 places in $^4P_4$ ways = **4! ways**

Thus, the total number of arrangements = 3! x 4! =144.

(iii) *all boys sit together*

All boys can be considered as one block and the remaining 4 girls as four blocks. The required arrangement is of these 1 + 4 = 5 blocks, which can be done in **5! ways**.

The 3 boys can be arranged amongst themselves in **3! ways**.

Thus the total number of ways in which 3 boys and 4 girls can be seated for a group photograph such that all boys are seated together is = 5! x 3! = **720**.

***Example* 10:** In how many different ways can the letters of the word "JOGESHWARI" be arranged such that (i) there is no restriction, (ii) the word starts with 'A', (iii) the word ends with 'W', (iv) the word begins with 'A' and ends with 'W', (v) the vowels are together, (vi) the letters W, A, R are never together.

Ans: The given word "JOGESHWARI" consists of 10 distinct letters, of which 4 are vowels (A, E, I, O) and 6 are consonants (G, H, J, R, S, W)

(i) If no restriction is there then the total number of arrangements = 8! = 40320

(ii) *The word starts with* 'A'
Since the first place is fixed for the 10 letter word, it remains to arrange the remaining 9 letters.

Thus, the total number of ways = 9! = 362880
(iii) *The word ends with* 'W'

This is similar to the above problem
(iv) *the word begins with* 'A' *and ends with* 'W'

Since two letters *A* and *W* are fixed, it remains to arrange the remaining 8 letters which can be done in 8! = 40320 ways.

(v) *the vowels are together*

4 vowels can be considered as one block and the remaining 6 consonants as 6 blocks.

Thus, the problem now reduces to arranging these 7 blocks, which can be done in 7! = 5040 ways.

The 4 vowels can be arranged amongst themselves in 4! = 24 ways.

Thus by fundamental principle of counting, the total number of ways = 5040 x 24 = 120960.

(vi) *the letters* W, A, R *are never together*

The required number of ways = total number of all arrangements – number of ways of arrangements where W, A, R are together.

Now, the total number of ways of arranging 10 letters = 10!

No of ways of arranging the letters W, A, R together = 8! x 3! (from the above case (v))

Thus the required number of ways = 10! – (8! x 3!)= 3628800 – 241920

= 3386880.

4.5 PERMUTATIONS OF *N* OBJECTS NOT ALL DIFFERENT

Consider a permutation of 2 white and 4 black hats of the same type. Then the permutation $W_1W_2B_1B_2B_3B_4$ is same as $W_2W_1B_1B_2B_3B_4$ as we cannot differentiate between the two white hats. Thus, in permutations where certain numbers of objects are similar, we have to remove the duplications of same permutations.

Let n_1 objects be of one kind, n_2 objects be of second kind,…, n_k objects be of k^{th} kind then the number of distinct permutations of all objects taken together i.e. $n = n_1 + n_2 + \ldots + n_k$ is given by: $\frac{n!}{n_1!n_2!n_3!\ldots\ldots n_k!}$

***Example* 11:** A College Librarian Mrs. Akalpita orders 15 books of which 5 books are of Business Law, 5 books of Business Mathematics, 3 books on Principles of Management and 2 books on F.H.S. In how many different ways can she arrange them on the shelf?

Ans: Given $n = 15$, $n_1 = 5$, $n_2 = 5$, $n_3 = 3$ and $n_4 = 2$

The number of ways of arranging the 20 books on the shelf

$$= \frac{15!}{5!5!3!2!} = 7567560$$

***Example* 12:** In how many different ways can the letters of the word "MALAYALAM" be arranged horizontally?

Ans: In the given word "MALAYALAM" has $n = 9$ letters of which M, A and L are repeating.

Here 'M' is repeated 2 times, $\therefore n_1 = 2$

'A' is repeated 4 times, $n_2 = 4$

'L' is repeated 2 times, $n_3 = 2$

$\therefore$ the number of permutations = $\frac{9!}{2!4!2!} = 3780.$

***Example* 13:** Find the number of permutations of the letters of the word "TOMMORROW" such that (i) no two M's are together, (ii) all the O's are not together.

Ans: The given word "TOMMORROW" has 9 letters of which 'O', 'M' and 'R' are repeated.

Now, 'O' is repeated 3 times, $n_1 = 3$

'M' is repeated 2 times, $n_2 = 2$

'R' is repeated 2 times, $n_3 = 2$

$\therefore$ number of permutations of letters of the word "TOMMORROW"

$$= \frac{9!}{3!2!2!} = 15120 \qquad \ldots (1)$$

(i) *no two M*'s *are together*

We assume the two *M*'s as one block and the remaining 7 letters as remaining 7 blocks.

The number of ways of arranging these 7+1 = 8 blocks is 8!, in which the letters *R* and *O* are repeated 2 and 3 times respectively.

Hence, the number of permutations in which both the *M*'s are together is

$$= \frac{8!}{3!2!} = 3360. \qquad \ldots (2)$$

From (1) and (2), the number of distinct permutations in which no two *M*'s are together = 15120 – 3360 = 11760.

(ii) *all the O's are not together*

We consider the three *O*'s as one block and the remaining 6 letters as one block.

The number of ways of arranging the 6 + 1 = 7 blocks is 7!, in which the letters *M* and *R* are repeated 2 times each.

Hence, the number of permutations in which all the *O*'s are together is

$$= \frac{7!}{2!2!} = 1260 \qquad \ldots (3)$$

From (1) and (3), the number of distinct permutations in which all three *O*'s are not together = 15120 – 1260 = 13860

***Example* 14:** Find the number of permutations of the letters of the word "VOWEL" such that the vowels occupy the odd places in the arrangement.

Ans: The given word "VOWEL" has 2 vowels 'E', 'O' and 3 consonants.

There are two odd places viz. 1^{st}, 3^{rd} and 4^{th}.

1^{st} ___ 3^{rd} ___ 5^{th}

↑ ↑ ↑

Two vowels can be placed in three places in ${}^{3}P_{2}$ ways. For each such way the 3 consonants can be placed in the remaining three places in 3! ways. By Fundamental principal of counting, the total number of arrangements is given by ${}^{3}P_{2}$ x 3! = 6 x 6 = 36 ways.

4.6 COMBINATION

A combination of n objects is an arrangement of some of these (or all) objects where the order of arrangement is not considered.

For e.g.: A combination considers the arrangements '*ab*' and '*ba*', of two letters '*a*' and '*b*' as the same.

$\boldsymbol{{}^{n}C_{r}}$: The selection (or rejection) of r objects out of n objects is denoted by ${}^{n}C_{r}$ and is calculated by the formula: ${}^{n}C_{r} = \dfrac{n!}{r!(n-r)!}$

For example: $${}^{5}C_{2} = \frac{5!}{2!(5-2)!} = \frac{5!}{2!3!} = \frac{5 \text{ x } 4 \text{ x } 3 \text{ x } 2 \text{ x } 1}{2 \text{ x } 3 \text{ x } 2 \text{ x } 1} = 10$$

We know that the factorial formula can be written recursively. Hence we can write the numerator $n!$ as $n(n-1)(n-2)\ldots.(n-r+1)(n-r)!$ Due to this recursive formula, the calculation of ${}^{n}C_{r}$ becomes easier as show below:

$$\text{Now, } {}^{n}C_{r} = \frac{n(n-1)(n-2)....(n-r+1)(n-r)!}{r!(n-r)!} = \frac{n(n-1)(n-2)....(n-r+1)}{r!}$$

For example: ${}^{10}C_{4} = \dfrac{10 \text{ x } 9 \text{ x } 8 \text{ x } 7}{4 \text{ x } 3 \text{ x } 2 \text{ x } 1} = 210$. This method simplifies and speeds up the calculations

<u>Results (without proof)</u>:

1. ${}^{n}C_{0} = {}^{n}C_{n} = 1$
2. ${}^{n}C_{r} = {}^{n}C_{n-r}$
3. ${}^{n}C_{r} + {}^{n}C_{r-1} = {}^{n+1}C_{r}$

***Example* 15:** In how many ways can 2 students be selected for a student's committee out of 7 students?

Ans: Here n= 7 and r = 2

$\therefore$ the no. of ways of selecting 5 out of 7 students = ${}^{7}C_{2} = \dfrac{7 \text{ x } 6}{2 \text{ x } 1} = 21$.

***Example* 16:** A question paper has 8 questions and only 5 questions are to be attempted. I how many ways a student can select any 5 questions?

Ans: Here $n = 8$, $r = 5$

$\therefore$ the required number of ways = ${}^8C_5 = {}^8C_3 = \dfrac{8 \times 7 \times 6}{3 \times 2 \times 1} = 56$

***Example* 17:** In how many ways can 2 boys and 2 girls be selected from a group of 6 boys and 5 girls?

Ans: Here there are two calculations to be made which are interdependent.

(*i*) 2 boys out of 6 boys can be selected in ${}^6C_2$ ways and

(*ii*) 2 girls out of 5 girls can be selected in ${}^5C_2$ ways.

The total number of ways of such a selection = ${}^6C_2 \times {}^5C_2 = \dfrac{6 \times 5}{2} \times \dfrac{5 \times 4}{2}$ =150.

***Example* 18:** A book shelf has 10 books of which 6 are of Accounts and remaining of Management. In how many ways can a person select 3 books on Accounts and 1 book on Management?

Ans: There are 6 books of Accounts and 10 – 6 = 4 books on Management.

3 books out of 6 books on Accounts can be selected in ${}^6C_3 = 20$ ways.

1 book out of 4 books on Management can be selected in ${}^4C_1 = 4$ ways.

By fundamental principal of counting, the total number of ways is 20 x 4 = **80.**

***Example* 19:** A Committee of 6 people is to be formed from a Staff of 4 Managers, 6 Officers and 2 peons. Find the number of distinct committees in which there are: (*i*) 2 persons from each category; (*ii*) no peons; (*iii*) exactly 2 Managers and (*iv*) atleast 2 Managers.

Ans:

(*i*) 2 persons from each category can be selected in

${}^4C_2 \times {}^6C_2 \times {}^2C_2 = 90$ ways.

(*ii*) No peons to be selected means the selection of 6 people is to be done from remaining 4 + 6 = 10 people.

This can be done in ${}^{10}C_6 = {}^{10}C_4 = \dfrac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} = 210$ ways.

(*iii*) Exactly 2 Managers out of 4 can be selected in ${}^4C_2 = 6$ ways.

The remaining 4 persons in the committee are to be selected from the remaining 6 + 2 = 8 persons, which can be done in $^8C_4 = \frac{8 \times 7 \times 6 \times 5}{4 \times 3 \times 2 \times 1} = 70$ ways.

The total number of ways = 6 x 70 = **420**

(*iv*) Atleast two managers can be selected in the following ways:

	Managers	**Others**
Selection I	2 out of 4	4 out of 8
Selection II	3 out of 4	3 out of 8
Selection III	4 out of 4	2 out of 8

Selection I can be done in $^4C_2 \times {}^8C_4 = 420$ ways

Selection II can be done in $^4C_3 \times {}^8C_3 = 192$ ways

Selection II can be done in $^4C_4 \times {}^8C_2 = 28$ ways

Thus, the total number of ways = 420 + 192 + 28 = **640**.

4.7 LET US SUM UP

In this chapter we have learn:

- Basic prerequisites and terminology like factorial notation, fundamental theorem addition and multiplication.
- Definition and formula of permutation and some small problems of real life.
- Definition and formula of combination and some small problems of real life.

4.8 UNIT END EXERCISE

1. Find the number of ways in which 4 boys can be seated for a group photograph.
2. In how many ways can 3 boys and 2 girls be seated for a photograph.
3. In how many ways can 7 books be arranged on a book shelf?
4. Find the number of ways of making 3 people sit on 3 chairs?
5. Find the number of ways of making 5 people sit on 3 chairs?
6. In how many ways can 3 boys and 2 girls be seated for a photograph in two lines with the first line of boys and second line of girls?
7. 20 students of a class are seated in three lines for a group photograph. The first line has 8 chairs; second line has 3 chairs and third 5 chairs. In how many ways can this be done?

8. In how many ways can 6 people be selected for 3 posts in a company?

9. In how many ways can 10 students be selected for Student's Council which consists of only 4 members?

10. Find the number of ways in which a 5 digit number be formed from the numbers 1, 2, 3, ……, 9 if (*i*) no digit is repeated, (*ii*) repetition is allowed.

11. Find the number of ways of forming a 4 digit even number from the digits 1, 2, 3, ……, 9 if no digit is to be repeated?

12. In how many ways can 3-digit odd number be formed from the digits 1, 2, 3, ……, 9 if (*i*) no digit is repeated, (*ii*) repetition is allowed.

13. Find the number of ways of arranging the letters of the word (*i*) MASK,
(*ii*)MOTHER, (*iii*) RATION, (*iv*) YES, (*v*) BHARTI, (*vi*) GREATFUL.

14. Find the value of n from the following:

i. ${}^nP_5 = 6\,{}^nP_3$

ii. ${}^nP_6 = 56\,{}^nP_4$

iii. ${}^nP_7 = 12\,{}^nP_5$

15. Show that ${}^{n+1}P_{r+1} = (n+1)\,{}^nP_r$.

16. Find the number of ways in which 2 books on Mathematics, 3 books on Law and 2 books on Economics can be arranged on a shelf so that books of the same subject are together. Also find the number of arrangements if books are to be arranged at random.

17. Find the number of ways in which 4 books on Physics, 3 books on Chemistry and 2 books on Biology can be arranged on a shelf so that (*i*) books of the same subject are together; (*ii*) No two books on Biology are together; (*iii*)No two books on Chemistry are together and (*iv*) No two books on Physics are together.

18. In how many ways can 4 boys and two girls be seated if (*i*) no two girls sit together, (*ii*) no two boys sit together and (*iii*) both the girls sit together?

19. In how many ways can 3 Africans and 3 Americans be seated so that (*i*) atleast two Americans always sit together and (*ii*) exactly two Africans sit together?

20. In how many distinct ways can the letters of the word "CHEMISTRY" be arranged such that (*i*) there is no restriction; (*ii*) the word begins with a vowel; (*ii*) the word begins and ends with a vowel; (*iii*) the vowels are together and (*iv*)the letters T, R and Y are never together.

21. In how many distinct ways can the letters of the word "ANDHERI" be arranged such that (*i*) the word begins with *A*; (*ii*) the word begins with *A* and ends with *R*; (*iii*) the word begins with *vowel* and ends with *vowel* and (*iv*) the letters *N*, *R* and *I* are never together?

22. A College Librarian Mrs. Parita orders 12 books of which 4 are of Maths, 3 are of English Literature, 3 books on Sociology and 4 books on Philosophy. In how many different ways can the books be arranged?

23. In how many distinct ways can the letters of the word "DISMISS" ?

24. In how many distinct ways can the letters of the word "STATISTICS" such that (*i*) no two *I*'s are together; (*ii*) no two *T*'s are together ?

25. Find the number of permutations of the letters of the word "VIRAR" such that the vowels occupy the even places in the arrangement.

26. In how many ways can 6 books be selected out of 10 books?

27. In how many ways can 4 boys be selected out of 7 boys?

28. A question paper contains two sections. Section I consists of 4 questions of which two are to be attempted and Section II consists of 5 questions of which 3 are to be attempted. Find the number of ways of attempting the questions in the paper.

29. There are 6 bulbs of which 3 are defective are to be put in two sockets in a room. Find the number of ways in which the room is lighted.

30. There are 8 books on History and 4 books on Geography. In how many ways can 4 books on History and 3 books on Geography be selected?

31. A box contains 4 white and 5 black balls. How many selections of 2 balls can be made so that (*i*) one ball of each color is selected; (*ii*) both balls are white; (*iii*) no white ball is selected.

32. A box contains 6 blue, 4 green and 2 white balls. How many selections of three balls can be made so that (*i*) one ball of each color is selected; (*ii*) atleast one white ball is selected and (*iii*) no white ball is selected.

33. 10 candidates appear for an interview. The selection committee has time to interview only 6 candidates. Find the number of ways of selecting the candidates.

34. 10 candidates appear for an interview. 2 candidates are disqualified as they did not bring necessary documents and the selection panel has time to interview only 5 people. In how many ways can this be done?

35. A committee consisting of 4 men and 3 women is to be formed from 7 men and 6 women. Find the number of selections in which (*i*) a particular man is selected; (*ii*) a particular women is not selected; (*iii*) a particular man is selected and a particular is rejected.

36. A College has 8 professors and 5 lecturers. In how many ways can a committee of 5 teachers be formed such that it consists of (*i*) two professors, (*ii*)two lecturers, (*iii*) atleast two professors, (*iv*) atleast two lecturers, (*v*)atleast two professors and lecturers and (*vi*) no lecturer.

37. A Cultural Committee of 8 persons from 6 men and 5 women is to be constituted in a Society. Find the number of distinct committees if it should consist of (*i*) atleast 3 women, (*ii*) exactly 2 women and (*iii*) atmost 4 men.

38. A question paper has three sections with each consisting of 4 questions. A student has to attempt a total of 5 questions with atleast one from each section. Find the number of ways in which this can be done.

39. A case is under discussion in front of 5 judges. Find the number of ways in which the judgment is given with a majority.

40. The Mumbai Royal Cricket Club has 22 players of which 3 are wicket keepers, 6 are fast bowlers, 2 are spinners and 3 all rounder's. In how many ways can a team be formed if it should include (*i*) one wicket keeper, 2 fast bowlers and one all rounder; (*ii*) 2 fast bowlers, 1 spinner and 1 wicket keeper; (*iii*) atleast one spinner, (*iv*) atleast one spinner and one all rounder.

Multiple Choice Questions:

1) If one task can be done in m ways and other task can be done in *n* ways then both tasks can be done together in how many ways?

a) 2^{m+n} b) 2^{mn} c) m + n d) m × n

2) How many words can be formed using all the letters of the word 'GREAT'?

a) 5 b) 24 c) 120 d) 720

3) How many 3 digit numbers can be formed using the digits 6, 7, 8, 9 if no digit can be repeated?

a) 4 b) 24 c) 48 d) 64

4) $^6C_4$ equals

a) 15 b) 30 c) 120 d)720

5) Number of 3 letter words that can be formed using the letters of the word 'DRINK', exactly once are

a) 60 b) 120 c) 240 d) 480

4.9 LIST OF REFERENCES

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5

MATRICES

Unit Structure

5.0 OBJECTIVE

After going through this chapter you will able to know:

- Define matrix and its order.
- The different types of matrices.
- The condition of equality of two matrices.
- The operations on matrices.
- That to convert real life example in to matrices.

5.1 INTRODUCTION

The theory of matrices was introduced by the English Mathematician Arthur Gayley in 1858. Since then, a matrix has become a powerful tool to solve the system of linear equations. In most economic problems we see the economic variables as linear functions, although non-linear relations do exist in practice. ‘Matrix algebra’ enables us to:

(i) write a system of equations in compact form

(ii) test the existence of a unique solution to a system

(iii) solve for equilibrium prices and quantities etc.

Computer analysis of problems involving system of equations applies ‘matrix algebra’ methods for finding solution. With growing use of computer technology in this field it has become must to study and understand the basics of matrices and its operations.

5.2 DEFINITION OF A MATRIX

A matrix is a rectangular arrangement of numbers (or variables or parameters) in horizontal rows and vertical columns enclosed in square bracket [ ] or parenthesis (). As a convention, we represent matrices with capital letters A, B, …etc and numbers in it called as "elements" by small case letters.

Example 1: $A = \begin{bmatrix} 1 & 2 & 3 \\ -4 & 0 & 1 \end{bmatrix}$ $\boldsymbol{R_2}$

$\boldsymbol{C_1 C_2 C_3}$

The matrix A has 2 rows and 3 columns and hence 2 × 3 = 6 elements viz. 1, 2, 3, -4, 0 and 1.

***Example* 2:** $B = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

The matrix B has 3 rows and 1 column and hence 3 × 1 = 3 elements.

***Example* 3:** $C = \begin{bmatrix} x & y \end{bmatrix}$

Similarly C has 1 row and 2 columns and 1 × 2 = 2 elements.

In general, a matrix with m rows and n columns has m x n number of elements.

5.3 ORDER OF A MATRIX

Order of a matrix gives the number of rows and columns in a given matrix. A matrix is said to be of order $m \times n$ (read as m by n) if it has m rows and n columns.

Referring to the above ***examples***, the order of matrix A is 2 x 3, that of matrix B is 3 × 1 and that of matrix C is 1 × 2.

Symbolic Representation

A matrix A of order $m \times n$ is represented generally as $A = \begin{bmatrix} a_{ij} \end{bmatrix} \begin{matrix} 1 \le i \le m \\ 1 \le i \le n \end{matrix}$

Here $\boldsymbol{a_{ij}}$ represents element in the $\boldsymbol{i}^{th}$ row and $\boldsymbol{j}^{th}$ column.

Let us understand with an example:

***Example* 4:** Let $A = \begin{bmatrix} 7 & -4 & 6 \\ 5 & 2 & 1 \end{bmatrix}$

The order of matrix $A$is 3 × 2 and there are 6 elements.

7 is in the 1st row 1st column, so $a_{11} = 7$

–4 is in the 1st row 2nd column, so $a_{12} = –4$

6 is in the 1st row 3rd column, so $a_{13} = 6$

5 is in the 2nd row 1st column, so $a_{21} = 5$

2 is in the 2nd row 2nd column, so $a_{22} = 2$

–1 is in the 2nd row 3rd column, so $a_{231} = –1$

$$\text{Thus } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} 7 & -4 & 6 \\ 5 & 2 & -1 \end{bmatrix}$$

A matrix B of order 4 × 3 would be written symbolically as follows:

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \\ b_{41} & b_{42} & b_{43} \end{bmatrix} = \left[b_{ij}\right]_{4 \times 3}, \quad \begin{matrix} 1 \le i \le 4 \\ 1 \le j \le 3 \end{matrix}$$

Note that in the notation $\boldsymbol{b_{ij}}$ 'i' corresponds to the row number and 'j' corresponds to the column number.

5.4 TYPES OF MATRICES

1. Null (Zero) Matrix

A matrix with all elements zero is called as null matrix.

e.g. $[0]_{1 \times 1}$, $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}_{1 \times 3}$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2 \times 2}$

2. Row Matrix

A matrix which has only one row is called as a row matrix. Order of a row matrix is 1 × n.

e.g. $[-1]_{1 \times 1}$, $\begin{bmatrix} 1 & -1 & 2 \end{bmatrix}_{1 \times 3}$, $\begin{bmatrix} 7 & 2 & 5 & 3 \end{bmatrix}_{1 \times 4}$

3. Column Matrix

A matrix which has only one column is called as a column matrix. Order of a column matrix is m × 1.

e.g. $[10]_{1 \times 1}$, $\begin{bmatrix} 3 \\ 5 \end{bmatrix}_{2 \times 1}$, $\begin{bmatrix} 1 \\ 4 \\ 9 \\ 16 \end{bmatrix}_{4 \times 1}$

4. Square Matrix

A matrix which has equal number of rows and columns is called a square matrix. Order of a square matrix is $n \times n$.

e.g. $[4]_{1 \times 1}$, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2 \times 2}$, $\begin{bmatrix} 1 & 4 & 3 \\ 3 & 1 & 4 \\ 4 & 3 & 1 \end{bmatrix}_{3 \times 3}$

5. Triangular Matrix

(i) Upper triangular matrix: A square matrix whose elements below the diagonal are zero is called as upper triangular matrix.

e.g. $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}_{2 \times 2}$, $\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}_{3 \times 3}$

In above examples; elements {1, 3,} and {a, b, f} are the diagonal elements.

(ii) Lower triangular matrix: A square matrix whose elements above the diagonal are zero is called as lower triangular matrix.

e.g. $\begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}_{2 \times 2}$, $\begin{bmatrix} -1 & 0 & 0 \\ 2 & 6 & 0 \\ 7 & 5 & 8 \end{bmatrix}_{3 \times 3}$

6. Diagonal Matrix

A square matrix whose all elements other than the diagonal are zero is called as a diagonal matrix.

e.g. $\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$, $\begin{bmatrix} 13 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 5 & 17 \end{bmatrix}$

7. Scalar Matrix

A diagonal matrix whose all diagonal elements are same is called as scalar matrix.

e.g. $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, $\begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$

8. Identity Matrix

A scalar matrix whose all entries are 1 is called identity matrix and is denoted by ***I***.

e.g. $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

9. Transpose a Matrix

The matrix obtained by interchanging row and column in a given matrix is called its transpose.

Let $\boldsymbol{A = (a_{ij})_{m \times n}}$ be a matrix of order $m \times n$, then its transpose denoted by A^T is given by $\boldsymbol{A^T = (a_{ji})_{n \times m}}$

Order of transpose is $n \times m$, which is reverse of $m \times n$ as the rows and columns are interchanged.

e.g. $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}_{1 \times 3}$ then $A^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{3 \times 1}$

$B = \begin{bmatrix} 7 & 8 & -1 \\ 11 & 13 & -2 \end{bmatrix}_{2 \times 3}$ then $B^T = \begin{bmatrix} 7 & 11 \\ 8 & 13 \\ -1 & -2 \end{bmatrix}_{3 \times 2}$

$C = \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}_{2 \times 2}$ then $C^T = \begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}$

$D = \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix}$ then $D^T = \begin{bmatrix} d_{11} & d_{21} & d_{31} \\ d_{12} & d_{22} & d_{32} \\ d_{13} & d_{23} & d_{33} \end{bmatrix}$

10. Symmetric Matrix

A square matrix which is same as its transpose is called as symmetric matrix. Any matrix A is said to be symmetric if $\boldsymbol{A = A^T}$

e.g. $A = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 1 & 7 \\ 1 & 6 & 3 \\ 7 & 3 & 8 \end{bmatrix}$

We can easily check $A = A^T$ and $B = B^T$.

11. Skew-symmetric Matrix

A square matrix A is said to be skew-symmetric if $\boldsymbol{A = -A^T}$.

e.g. $A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$ then $A^T = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} = -\boldsymbol{A^T}$

also $B = \begin{bmatrix} 0 & -3 & -4 \\ 3 & 0 & 5 \\ 4 & -5 & 0 \end{bmatrix}$ then $B^T = \begin{bmatrix} 0 & 3 & 4 \\ -3 & 0 & -5 \\ -4 & 5 & 0 \end{bmatrix} = -B$

5.4.1 Equality of two matrices:

Two matrices are said to be equal if and only if (i) their order is same and (ii) corresponding elements are same. For equality of matrices both conditions should be satisfied.

If $A = \left[a_{ij}\right]_{m \times n}$ and $B = \left[b_{ij}\right]_{m \times n}$ and $a_{ij} = b_{ij}$ for all i and j, then $A = B$.

***Example* 5:** Find a, b, x, y if $A = B$ where $A = \begin{bmatrix} 5 & 7 \\ x & y \end{bmatrix}$ and $B = \begin{bmatrix} a & b \\ 2 & 3 \end{bmatrix}$

Ans: Since $A = B$, using the equality of matrices conditions we have,

$a = 5, b = 7, x = 2$ and $y = 3$.

***Example* 6:** $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Though the elements may appear to be same, the first condition that the order of the matrices should be same is not satisfied. Hence the matrices are not equal.

5.5 ALGEBRA OF MATRICES

1) Addition to Matrices

Two matrices can be added if their order is same and the addition is done element wise.

If $A = \left[a_{ij}\right]_{m \times n}$, $B = \left[b_{ij}\right]_{m \times n}$ are matrices of same order m x n then the sum $A + B = \left[a_{ij} + b_{ij}\right]$ is also a matrix of order m x n.

***Example* 7:** $P = \begin{bmatrix} 1 & -1 & 2 \end{bmatrix}_{1 \times 3}, Q = \begin{bmatrix} -1 & 1 & -1 \end{bmatrix}_{1 \times 3}$

$$P + Q = \begin{bmatrix} (1 - 1) & (-1+1) & (2-1) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}_{1 \times 3}$$

***Example* 8:** $X = \begin{bmatrix} 9 \\ 5 \\ 2 \end{bmatrix}_{3 \times 1}, Y = \begin{bmatrix} -3 \\ 1 \\ 4 \end{bmatrix}_{3 \times 1}$

$$\therefore X + Y = \begin{bmatrix} 9 - 3 \\ 5 + 1 \\ 2 + 4 \end{bmatrix}_{3 \times 1} = \begin{bmatrix} 6 \\ 6 \\ 6 \end{bmatrix}_{3 \times 1}$$

***Example* 9:** $A = \begin{bmatrix} 11 & -4 & 3 \\ 6 & 9 & 2 \end{bmatrix}_{2 \times 3}, B = \begin{bmatrix} -2 & 6 & 9 \\ 3 & -3 & 7 \end{bmatrix}_{2 \times 3}$

$$\therefore A + B = \begin{bmatrix} 11 - 2 & -4 + 6 & 3 + 9 \\ 6 + 3 & 9 - 3 & 2 + 7 \end{bmatrix} = \begin{bmatrix} 9 & 2 & 12 \\ 9 & 6 & 9 \end{bmatrix}_{2 \times 3}$$

Properties of Addition of Matrices

For any matrices A, B, C of order m x n,

(a) $A + B = B + A$, matrix addition is commutative.

(b) $(A + B) + C = A + (B + C)$, matrix addition is associative.

(c) For any matrix, $A + O = O + A = A$, where O is the null matrix of same order as that of A.

2) Subtraction of Matrices

Two matrices can be subtracted if their order is same and the subtraction is done element wise.

If $A = \left[a_{ij}\right]_{m \times n}$, $B = \left[b_{ij}\right]_{m \times n}$ are matrices of order m x n then $A - B = \left[a_{ij} - b_{ij}\right]_{m \times n}$

***Example* 10:** $A = \begin{bmatrix} 4 & 6 \\ 3 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 2 \\ -1 & 1 \end{bmatrix}$

$$A - B = \begin{bmatrix} 4 - 0 & 6 - 2 \\ 3 - (-1) & 1 - 1 \end{bmatrix} \begin{bmatrix} 4 & 4 \\ 4 & 0 \end{bmatrix}$$

3) Scalar Multiplication of a Matrix

A scalar is any constant (number). Let $A = \left[a_{ij}\right]_{m \times n}$ and α be any scalar

Then $\alpha A = \left[\alpha a_{ij}\right]_{m \times n}$

When a matrix is multiplied by a scalar, every element of the matrix gets multiplied by that scalar.

***Example* 11:** $C = \begin{bmatrix} 7 & 4 \end{bmatrix}$ then $2C = \begin{bmatrix} 14 & 8 \end{bmatrix}$

***Example* 12:** $A = \begin{bmatrix} 5 & -3 & 2 \end{bmatrix}$ then $-4A = \begin{bmatrix} -20 & 12 & -8 \end{bmatrix}$

***Example* 13:** $B = \begin{bmatrix} 4 & 6 \\ 8 & 2 \end{bmatrix}$ then $\frac{1}{2}B = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$

Properties of scalar multiplication

Let A, B matrices of same order and α, β be scalars:

(a) $\alpha(A + B) = \alpha A + \alpha B$

(b) $(\alpha + \beta) A = \alpha A + \beta A$

(c) $\alpha(\beta A) = \beta(\alpha A) = (\alpha\beta)A$

Before we proceed to next section, let us understand the above properties and with simple examples.

***Example* 14:** Write down the order of each of the following matrix. State which pair of matrices can be added and write the resultant matrix.

$$A = \begin{bmatrix} 6 & 8 \end{bmatrix}, B = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, C = \begin{bmatrix} 4 \end{bmatrix}, D = \begin{bmatrix} -2 & 4 \end{bmatrix}, E = \begin{bmatrix} 6 & 2 \\ 5 & 4 \end{bmatrix}, F = \begin{bmatrix} 2 \end{bmatrix},$$

$$G = \begin{bmatrix} 5 & 4 & 6 \\ 2 & 3 & -4 \end{bmatrix}, H = \begin{bmatrix} -1 & 2 \\ 4 & 3 \end{bmatrix}, J = \begin{bmatrix} 11 \\ -8 \end{bmatrix}, K = \begin{bmatrix} 1 & 4 & 4 \\ 4 & 7 & 1 \\ 7 & 1 & 7 \end{bmatrix},$$

Ans: The order of a matrix is $m \times n$, where m = no. of rows and n = no. of columns. Thus, the order of given matrices are:

Matrix	**Order**	**Matrix**	**Order**
A	1×2	E	2×2
B	2×1	F	1×1
C	1×1	G	2×3
D	1×2	H	2×2
J	2×1	K	3×3

Now, we know that two matrices can be added only if their order is same. Thus, from the above table, we can conclude that the following matrices can be added:

A*and*D*, *B*and*I*, *C*and*F*, *E*and*H and their sums are:

$$A + D = \begin{bmatrix} 6 & 8 \end{bmatrix} + \begin{bmatrix} -2 & 4 \end{bmatrix} = \begin{bmatrix} 6-2 & 8+4 \end{bmatrix} = \begin{bmatrix} 4 & 12 \end{bmatrix}$$

$$B + J = \begin{bmatrix} 2 \\ 5 \end{bmatrix} + \begin{bmatrix} 11 \\ -8 \end{bmatrix} = \begin{bmatrix} 2+11 \\ 5-8 \end{bmatrix} = \begin{bmatrix} 13 \\ -3 \end{bmatrix}$$

$$C + F = \begin{bmatrix} 2 \end{bmatrix} + \begin{bmatrix} 2 \end{bmatrix} = \begin{bmatrix} 4+2 \end{bmatrix} = \begin{bmatrix} 6 \end{bmatrix}$$

$$E + H = \begin{bmatrix} 6 & 2 \\ 5 & 4 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 6-1 & 2+2 \\ 5+4 & 4+3 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 9 & 7 \end{bmatrix}$$

***Example* 15:** Verify the commutative and associative laws of addition for

$$A = \begin{bmatrix} 3 & -1 \\ 4 & 2 \end{bmatrix}, B = \begin{bmatrix} -3 & 2 \\ 4 & 1 \end{bmatrix}, C = \begin{bmatrix} 4 & -3 \\ 2 & 1 \end{bmatrix}$$

Ans: To verify:

(i) Commutative law: $\begin{matrix} A + B = B + A \\ B + C = C + B \\ A + C = C + A \end{matrix}$

(ii) Associative law: $A + (B + C) = (A + B) + C$

(i) We will verify here A + B = B + A. The other two being similar is left for readers as exercise!

$$A+B = \begin{bmatrix} 3 & -1 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} -3 & 2 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 3-3 & -1+2 \\ 4+4 & 2+1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 8 & 3 \end{bmatrix} \quad \ldots (1)$$

$$B+A = \begin{bmatrix} -3 & 2 \\ 4 & 1 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} -3+3 & 2-1 \\ 4+4 & 1+2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 8 & 3 \end{bmatrix} \quad \ldots (2)$$

From (1) and (2), A + B = B + A. Hence commutative law holds true.

(ii) $B+C = \begin{bmatrix} -3 & 2 \\ 4 & 1 \end{bmatrix} + \begin{bmatrix} 4 & -3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -3+4 & 2-3 \\ 4+2 & 1+1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 6 & 2 \end{bmatrix}$

$$\therefore A+(B+C) = \begin{bmatrix} 3 & -1 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 6 & 2 \end{bmatrix} = \begin{bmatrix} 3+1 & -1-1 \\ 4+6 & 2+2 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 10 & 4 \end{bmatrix} \ldots (3)$$

Now, $A+B = \begin{bmatrix} 3 & -1 \\ 4 & 2 \end{bmatrix} + \begin{bmatrix} -3 & 2 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 3-3 & -1+2 \\ 4+4 & 2+1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 8 & 3 \end{bmatrix}$

$$(A+B)+C = \begin{bmatrix} 0 & 1 \\ 8 & 3 \end{bmatrix} + \begin{bmatrix} 4 & -3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 0+4 & 1-3 \\ 8+2 & 3+1 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 10 & 4 \end{bmatrix} \ldots (4)$$

From (3) and (4), A + (B + C) = (A+ B) + C

Thus, Associative law holds true.

***Example* 16:** Write down elements of a matrix A of order 2 × 3, if $a_{ij} = (i + j)^2$.

Ans: Given matrix $A = (a_{ij})$ of order 2 × 3 i.e. containing 6 elements.

$\therefore$ Symbolically, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ … (*) Also

given is the formula to calculate each a_{ij} i.e. $a_{ij} = (i + j)^2$

$\therefore a_{11} = (1 + 1)^2 = 2^2 = 4$
$a_{12} = (1 + 2)^2 = 3^2 = 9$
$a_{13} = (1 + 3)^2 = 4^2 = 19$
$a_{21} = (2 + 1)^2 = 3^2 = 9$
$a_{22} = (2 + 2)^2 = 4^2 = 16$
$a_{23} = (2 + 3)^2 = 5^2 = 25$

Substituting these values in (*), we have A = $\begin{bmatrix} 4 & 9 & 16 \\ 9 & 16 & 25 \end{bmatrix}$

***Example* 17:** Find a matrix B such that $A + B =$ O, where $A = \begin{bmatrix} 4 & -6 & 8 \\ -3 & 2 & -7 \end{bmatrix}$

Ans: Let us first recollect that two matrices can be added if their order is same and the resultant matrix is also of the same order.

Given matrix A has order 2 × 3.

∴ B and 0 also must be order 2 × 3. As we know 0 is the Null Matrix, we have

A + B = 0

$$\therefore B = 0 - A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 4 & -6 & 8 \\ -3 & 2 & -7 \end{bmatrix}$$

$$\therefore B = \begin{bmatrix} -4 & 6 & -8 \\ 3 & -2 & 7 \end{bmatrix}$$

***Example* 18:** Find a matrix such that 2A + 3B = O where A = $\begin{bmatrix} 3 & -12 \\ -9/2 & -18 \end{bmatrix}$.

Ans: As we have done in previous example, we know that B and O are matrices of order 2 × 2.

Given   2A + 2B = O

$$\therefore 3B = O - 2A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - 2\begin{bmatrix} 3 & -12 \\ -9/2 & -18 \end{bmatrix}$$

$$3B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 6 & -24 \\ -9 & 36 \end{bmatrix} = \begin{bmatrix} -6 & 24 \\ 9 & -36 \end{bmatrix}$$

$$\therefore B = \frac{1}{3}\begin{bmatrix} -6 & 24 \\ 9 & -36 \end{bmatrix} = \begin{bmatrix} -6/3 & 24/3 \\ 9/3 & -36/3 \end{bmatrix}$$

$$\therefore B = \begin{bmatrix} -2 & 8 \\ 3 & -12 \end{bmatrix}$$

***Example* 19:** Find all 2 × 2 matrices A such that $A + A^T = 0$ (in other words, A is a skew-symmetric matrix)

Ans: Since A is a 2 × 2 matrix, writing A symbolically,

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$$

Given $A + A^T = 0$

$$\therefore \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} a_{11} + a_{11} & a_{12} + a_{21} \\ a_{21} + a_{12} & a_{22} + a_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 2a_{11} & a_{12} + a_{21} \\ a_{21} + a_{12} & 2a_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\therefore 2a_{11} = 0 \Rightarrow a_{11} = 0$$

$$a_{12} + a_{21} = 0 \Rightarrow a_{12} = -a_{21}$$

and $2a_{22} = 0 \Rightarrow a_{22} = 0$

$$\therefore A = \begin{bmatrix} 0 & -a_{21} \\ a_{12} & 0 \end{bmatrix}$$

Thus, only 2 × 2 skew-symmetric matrix is of the form: $\begin{bmatrix} 0 & -x \\ x & 0 \end{bmatrix}$

WORD PROBLEMS

***Example* 20:** Three friends Amit, Sumit and Vinit go to a retail shop and purchase following items :Amit purchases one mango, 2 kg potatoes and 1 kg rice. Sumit purchases half a dozen lemons, 1 kg potato and 1 kg sugar. Vinit purchases one dozen lemon, 3 kg rice and 1 kg sugar. Construct a proper matrix giving various purchases of the three friends.

Ans: There are three friends which can be taken as three rows. And there are 5 items which can be taken as five columns. Thus we gave the matrix of purchase as:

	Mangoes	Potatoes	Rice	Lemon	Sugar
Amit	12	2	1	0	0
Sumit	0	1	0	6	1
Vinit	0	0	3	12	1

$$\therefore A = \begin{bmatrix} 12 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 6 & 1 \\ 0 & 0 & 3 & 12 & 1 \end{bmatrix}_{3 \times 5}$$

***Example* 21:** A multinational bank has its offices in Mumbai and Delhi. In both the cities it has 3 branches. Every branch has 1 manager, 4 officers and 3 peons. In one of Mumbai branch they have 2 executives, while in all the Delhi branches there are 5 typists. Use matrix notation and find (*i*) total number of posts of each type in all branches together (*ii*) total number of employees of each type is both offices taken together.

Ans. Consider the following rows matrices:

$$M_1 = \begin{pmatrix} 1 & 4 & 3 & 5 & 0 \end{pmatrix},\ M_2 = \begin{pmatrix} 1 & 4 & 3 & 0 & 0 \end{pmatrix},\ M_3 = \begin{pmatrix} 1 & 4 & 3 & 0 & 0 \end{pmatrix}$$

$$D_1 = \begin{pmatrix} 1 & 4 & 3 & 0 & 5 \end{pmatrix},\ D_2 = \begin{pmatrix} 1 & 4 & 3 & 0 & 5 \end{pmatrix},\ D_3 = \begin{pmatrix} 1 & 4 & 3 & 0 & 5 \end{pmatrix}$$

M_1, M_2, M_3 represents branches in Mumbai, elements represent number of manager, officers, peons, executives and typists in the order of appearance. Similarly D_1, D_2, D_3 represent the three branches in Delhi

(i) Total number of posts of each type

(a) in Mumbai : M_1, M_2, M_3

$$= \begin{pmatrix} 1 & 4 & 3 & 2 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 4 & 3 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 4 & 3 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 12 & 9 & 2 & 0 \end{pmatrix}$$

(b) in Delhi : D_1, D_2, D_3

$$= \begin{pmatrix} 1 & 4 & 3 & 0 & 5 \end{pmatrix} + \begin{pmatrix} 1 & 4 & 3 & 0 & 5 \end{pmatrix} + \begin{pmatrix} 1 & 4 & 3 & 0 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 12 & 9 & 0 & 15 \end{pmatrix}$$

(ii) Total number of employees in both offices together is

$$(M_1, M_2, M_3) + (D_1, D_2, D_3)$$

$$= \begin{pmatrix} 3 & 12 & 9 & 2 & 0 \end{pmatrix} + \begin{pmatrix} 3 & 12 & 9 & 0 & 15 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 24 & 18 & 2 & 15 \end{pmatrix}$$

***Example* 22:** An automobile company sells two models of a car AC and Non AC, made available in three colors silver, black and red. In the first month of launching the sales for AC version were 120, 180, 205 for silver, black and red color respectively, while for the Non AC version the corresponding numbers were 220, 350, 310 respectively. In the second month the sales for AC and Non AC Models were: 360, 275, 327, and 530, 320, 382 respectively. Represent the above data in matrix from and find the total sales for each model and color for both the months together.

Ans. Les us represent the given data in matrix form:

	1st Month		2nd Month	
	AC	**Non AC**	**AC**	**Non AC**
Silver	120	220	360	530
Black	180	350	275	320
Red	205	310	327	382

Let $A = \begin{bmatrix} 120 & 220 \\ 180 & 350 \\ 205 & 310 \end{bmatrix}$ and $B = \begin{bmatrix} 360 & 530 \\ 275 & 320 \\ 327 & 382 \end{bmatrix}$ be matrix representing sales in the two months respectively.

The total sales can be obtained by adding A and B.

$$A + B = \begin{bmatrix} 120 & 220 \\ 180 & 350 \\ 205 & 310 \end{bmatrix} + \begin{bmatrix} 360 & 530 \\ 275 & 320 \\ 327 & 382 \end{bmatrix} = \begin{bmatrix} 480 & 750 \\ 455 & 670 \\ 532 & 692 \end{bmatrix}$$

4) Multiplication of Two matrices:

Two matrices A and B are said to conformable or in other words they can be multiplied, if **number of columns of A is same as number of rows of B**. The product *AB* makes sense only if the above condition is satisfied. Let A = $[a_{ij}]$ be a $m \times n$ matrix and B = $[b_{jk}]$ be a $n \times p$ matrix. Then the product AB is defined as:

AB = $[c_{ik}]$, where $c_{jk} = a_{i1}b_{1k} + a_{i2}b_{2k} + ... + a_{in}b_{nk}$ and order of AB is $m \times p$.

Note:

1. Observe that product AB has number of rows that of A and number of columns that of B.

$$[A]_{m \times n} \text{ x } [B]_{n \times p} = [AB]_{m \times p}$$

2. The multiplication of matrices is not element wise as in addition or subtraction of matrices.

As mentioned the above matrix AB = $[c_{ik}]$ is of order $m \times p$.

Here $c_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + ... + a_{in}b_{nk}$is the ik^{th}element of AB, obtained by multiplying the row of A with column of B.

Let us understand this with examples of square matrices order 2×2 and 3×3:

***Example* 23:** Let A = $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, B = $\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ be matrices of order 2 × 2.

Thus product AB also is a matrix of order 2 × 2.

Let AB = $\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$

The four elements c_{11} , c_{12} , c_{21} and c_{22} are calculated as follows:

1. c_{11} is obtained by multiplying 1st row of A with 1st column of B i.e. $\begin{bmatrix} a_{11} & a_{12} \end{bmatrix}$ with $\begin{bmatrix} b_{11} \\ b_{12} \end{bmatrix}$

$\therefore c_{12} = a_{11}b_{11} + a_{12}b_{21}$

2. c_{12} is obtained by multiplying 1st row of A with 2nd column of B i.e. $\begin{bmatrix} a_{11} & a_{12} \end{bmatrix}$ with $\begin{bmatrix} b_{12} \\ b_{21} \end{bmatrix}$

$\therefore c_{12} = a_{11}b_{12} + a_{12}b_{22}$

3. c_{21} is obtained by multiplying the 2nd row of *A* with 1st row of *B*.

$\therefore c_{21} = a_{21}b_{11} + a_{22}b_{21}$

4. Similarly, $c_{22} = a_{21}b_{12} + a_{22}b_{22}$

Thus, $AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$

***Example* 24:**Let A = $\begin{bmatrix} 1 & -4 \\ 3 & 2 \end{bmatrix}$, B = $\begin{bmatrix} 5 & 2 \\ -3 & 1 \end{bmatrix}$

$$C_{11} = 1 \times 5 + (-4) \times (-3) = 5 + 12 = 17$$

$$C_{12} = 1 \times 2 + (-4) \times 1 = 2 - 4 = -2$$
$$C_{21} = 3 \times 5 + 2 \times (-3) = 15 - 6 = 9$$
$$C_{22} = 3 \times 2 + 2 \times 1 = 6 + 2 = 8$$

AB = $\begin{bmatrix} 17 & -2 \\ 9 & 8 \end{bmatrix}$

It will take a few problems of practice to get accustomed to multiplication of matrices. Les us revise again the method of how did we get the elements of first row in AB:

It is by multiplying 1st row of A with 1st column of B and 1st row of A by 2nd Column B. While the elements in 2nd row of AB, again, by multiplying 2nd row of A with 1st column of B and 2nd row of A with 2nd column of B.

***Example* 25:** For $A = \begin{bmatrix} 7 & 3 \\ 1 & 4 \\ 2 & 5 \end{bmatrix}_{3 \times 2}$ $B = \begin{bmatrix} 6 & -3 & 0 \\ -4 & 2 & -1 \end{bmatrix}_{2 \times 3}$ find AB

Ans: A and B are conformable as no. of columns of A = no. of rows of B = 2

and AB has order 3 × 3.

Let $AB = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$

C_{11} is obtained by adding the products of elements in 1st row of A and 1st column of B.

$c_{11} = 7 \text{ x } 6 + 3\,(-4) = 42 - 12 = 30$

$c_{12} = 7\,(-3) + 3 \text{ x } 2 = -21 + 6 = -15$

$c_{13} = 7 \text{ x } 0 + 3\,(-1) = -3$

$c_{21} = 1 \text{ x } 6 + 4(-4) = 6 - 16 = -10$

$c_{22} = 1(-3) + 4 \text{ x } 2 = 5$

$c_{23} = 1 \text{ x } 0 + 4(-1) = -4$

$c_{31} = 2 \text{ x } 6 + 5(-4) = -8$

$c_{32} = 2(-3) + 5 \text{ x } 2 = 4$

$c_{33} = 2 \text{ x } 0 + 5(-1) = -5$

Thus, $AB = \begin{bmatrix} 30 & -15 & -3 \\ -10 & 5 & -4 \\ -8 & 4 & -5 \end{bmatrix}$

***Example* 26:** $A = \begin{bmatrix} 3 & 1 \\ -2 & 2 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 4 \\ 0 & 6 \end{bmatrix}$, find AB and BA.

Ans. Firstly A and B are matrices of order 2 × 2. Hence AB and BA both products are defined.

$$BA = \begin{bmatrix} -2 & 4 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} (-2) \times 3 + 4 \times (-2) & (-2) \times 1 + 4 \times 2 \\ 0 \times 3 + 6 \times (-2) & 0 \times 1 + 6 \times 2 \end{bmatrix}$$

$$\therefore BA = \begin{bmatrix} -14 & 6 \\ -12 & 12 \end{bmatrix} \quad \ldots (1)$$

$$\text{And } AB = \begin{bmatrix} 3 & 1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} -2 & 4 \\ 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \times (-2) + 1 \times 0 & 3 \times 4 + 1 \times 6 \\ (-2) \times (-2) + 2 \times 0 & (-2) \times 4 + 2 \times 6 \end{bmatrix}$$

$$\therefore BA = \begin{bmatrix} -6 & 18 \\ 4 & 4 \end{bmatrix} \quad \ldots (2)$$

From (1) and (2) it is very clear that AB ≠ BA. This is very important fact regarding product of matrices. Matrix multiplication in general is not commutative. *i.e.* AB ≠ BA for any matrices A and B.

There are some examples, as we see further, where AB = BA, but as a rule, it is not true always.

***Example* 27:** For $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ show that AB = BA.

Ans: $AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ ...1

$BA = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ ...2

From …1 and …2 AB = BA

***Example* 28:** Let $A = \begin{bmatrix} 11 & -8 \end{bmatrix}$, $B = \begin{bmatrix} 3 \\ -6 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$, $D = \begin{bmatrix} 1 & -1 & 6 \\ 7 & -8 & 2 \end{bmatrix}$

Can you compute AB, BA, AC, OB, DA, CD, AD? If yes ,find it or else justify your answer.

Ans: Given

Matrix	Order
A	1 × 2
B	2 × 1
C	2 × 2
D	2 × 3

We know that two are matrices conformable if number of columns of first matrix is same as number of rows of second matrix. From the above table, we can say that A and B, B and A, A and C, C and D, A and D are conformable. Matrices D and B, D and A are not conformable.

∴ DB and DA cannot be computed.

(i) $\therefore AB = \begin{bmatrix} 11 & -8 \end{bmatrix}_{1 \times 2} \begin{bmatrix} 3 \\ -6 \end{bmatrix}_{2 \times 1} = \begin{bmatrix} 11 \times 3 + (-8)(-6) \end{bmatrix}_{1 \times 1} = \begin{bmatrix} 33 + 48 \end{bmatrix}$

∴ AB = [81]

$$BA = \begin{bmatrix} 3 \\ -6 \end{bmatrix}_{2\times 1} \begin{bmatrix} 11 & -8 \end{bmatrix}_{2\times 2}$$

$$BA = \begin{bmatrix} 3\times 11 & 3\times(-8) \\ -6\times 11 & (-6)\times(-8) \end{bmatrix}_{2\times 2} = \begin{bmatrix} 33 & -24 \\ -66 & 48 \end{bmatrix}$$

(ii) $AC = \begin{bmatrix} 11 & -8 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}_{2\times 2}$

$$= \begin{bmatrix} 11\times 0 + (-8)(-1) & 11\times 1 + (-8)\times 2 \end{bmatrix}_{1\times 2}$$

$AC = \begin{bmatrix} 8 & -5 \end{bmatrix}$

(iii) $CD = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}_{2\times 2} \begin{bmatrix} 1 & -1 & 6 \\ 7 & -8 & 2 \end{bmatrix}_{2\times 3} =$

$$\begin{bmatrix} 0\times 1 + 1\times 7 & 0\times(-1) + 1\times(-8) & 0\times 6 + 1\times 2 \\ -1\times 1 + 2\times 7 & (-1)\times(-1) + 2\times(-8) & (-1)\times 6 + 2\times 2 \end{bmatrix}_{2\times 3}$$

$\therefore CD = \begin{bmatrix} 7 & -8 & 2 \\ 13 & -15 & -2 \end{bmatrix}$

(iv) $AD = \begin{bmatrix} 11 & -8 \end{bmatrix}_{1\times 2} \begin{bmatrix} 1 & -1 & 6 \\ 7 & -8 & 2 \end{bmatrix}_{2\times 3}$

$$= \begin{bmatrix} 11\times 1 + (-8)\times 7 & 11\times(-1) + (-8)(-8) & 11\times 6 + (-8)x2 \end{bmatrix}_{1\times 3}$$

$AD = \begin{bmatrix} -45 & 53 & 50 \end{bmatrix}_{1\times 3}$

Properties of Matrix Multiplication:

(1) Law of Commutativity does not hold true: For any two conformable matrices A, B; $AB \neq BA$

(2) Law of Associativity: A(BC) = (AB) C

(3) Existence of Multiplicative Identity: $AI = IA = A$, where I is the identity matrix.

(4) Law of cancellation does not hold true: If $AB = AC$, it does not $B = C$.

(5) If AB = 0, it does not imply $A = 0$ or $B = 0$ (Refer to Ex. 5)

(6) Positive integer powers of a matrix:

$A^2 = A.A.$; $A^3 = A \times A \times A$

In general $A^n = A \times A \times \ldots.. \times A$ n times.

(7) Law of Distributivity: For conformable matrices A, B, C;

A(B + C) = AB + AC and (A + B) C = AC + BC

(8) In view of property (1), we have some more results:

(i) $(A + B)^2 = A^2 + AB + BA + B^2$ (and not $A^2 + 2AB + B^2$)

(ii) $(A - B)^2 = A^2 - AB - BA + B^2$

(iii) $(A + B)(A - B) = A^2 - AB + BA - B^2$

***Example* 29:** If $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 4 & -1 \end{bmatrix}$ then show that

$$(A + B)^2 = A^2 + B^2$$

Ans: We know that $(A + B)^2 = A^2 + AB + BA + B^2$ … (1)

To prove $(A + B)^2 = A^2 + B^2$ … (2)

$\therefore$ it is enough to prove $AB + BA = 0$ … from (1) & (2)

$$AB = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} 1 \times 1 + (-1) \times 4 & 1 \times 1 + (-1) \times (-1) \\ 2 \times 1 + (-1) \times 4 & 2 \times 1 + (-1) \times (-1) \end{bmatrix}$$

$$\therefore AB = \begin{bmatrix} -3 & 2 \\ -2 & 3 \end{bmatrix} \quad \ldots (3)$$

$$BA = \begin{bmatrix} 1 & 1 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 \times 1 + 1 \times 2 & 1 \times (-1) + 1 \times (-1) \\ 4 \times 1 + (-1) \times 2 & 4 \times (-1) + (-1) \times (-1) \end{bmatrix}$$

$$BA = \begin{bmatrix} 3 & -2 \\ 2 & -3 \end{bmatrix} \quad \ldots (4)$$

Adding (3) & (4), we have

$$AB + BA = \begin{bmatrix} -3 & 2 \\ -2 & 3 \end{bmatrix} + \begin{bmatrix} 3 & -2 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

Thus $(A + B)^2 = A^2 + B^2$

***Example* 30:** If $A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$, compute $A^2 - 8A + 6I$

Ans: $A^2 = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 16 + 2 & 8 + 6 \\ 4 + 3 & 2 + 9 \end{bmatrix} = \begin{bmatrix} 18 & 14 \\ 7 & 11 \end{bmatrix}$ ...1

$8A = 8\begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 32 & 16 \\ 8 & 24 \end{bmatrix}$ ... 2

$6I = 6\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$ ... 3

from (1), (2) and (3), we have

$$A^2 - 8A + 6I = \begin{bmatrix} 18 & 14 \\ 7 & 11 \end{bmatrix} - \begin{bmatrix} 32 & 16 \\ 8 & 24 \end{bmatrix} + \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 18 - 32 + 6 & 14 - 16 + 0 \\ 7 - 8 + 0 & 11 - 24 + 6 \end{bmatrix} = \begin{bmatrix} -8 & -2 \\ -1 & -7 \end{bmatrix}$$

Word Problems

***Example* 31:** Mr. Narayan buys 3 kg potatoes, 1 kg onion, 5 kg rice, 1 kg wheat and 2 kg oil from a shop. If the cost per kg of above items is Rs. 10/-, Rs. 12/-, Rs. 18/-, Rs. 15/- and Rs. 50/- respectively, find using matrix algebra the total amount spent by him.

Ans. Let $A = \begin{bmatrix} 3 & 1 & 5 & 1 & 2 \end{bmatrix}$ represent the items bought by Mr. Narayan and $B = \begin{bmatrix} 10 & 12 & 18 & 15 & 50 \end{bmatrix}^T$ represent the cost per kg of items.

Then the total expenditure = $AB = \begin{bmatrix} 3 & 1 & 5 & 1 & 2 \end{bmatrix} \begin{bmatrix} 10 \\ 12 \\ 18 \\ 15 \\ 50 \end{bmatrix}$

$\therefore AB = \begin{bmatrix} 3 \times 10 + 1 \times 12 + 5 \times 18 + 1 \times 15 + 2 \times 50 \end{bmatrix}$

$\therefore AB = \begin{bmatrix} 30 + 12 + 90 + 15 + 100 \end{bmatrix} = \begin{bmatrix} 247 \end{bmatrix}$

$\therefore$ Total amount spent = Rs. 247/-.

***Example* 32:** A company produces three types of products say A, B, C. Let the matrix X represent the sales of these products in two cities as follows: $X = \begin{bmatrix} 10 & 40 & 30 \\ 20 & 30 & 40 \end{bmatrix}$ If manufacturing cost of the products is Rs. 100/-, Rs. 150/- and Rs. 250/- respectively and is sold at Rs. 120/-, Rs. 140/- and Rs. 300/- respectively, find the total profit using matrix algebra.

Ans: Profit = Selling Price – Cost Price

Let $C = \begin{bmatrix} 100 & 150 & 250 \end{bmatrix}$ and $S = \begin{bmatrix} 120 & 140 & 300 \end{bmatrix}$

Represent the cost price and selling price per product of type A, B, C.

Total manufacturing cost:

Consider $CX^T = \begin{bmatrix} 100 & 150 & 250 \end{bmatrix} \begin{bmatrix} 10 & 20 \\ 40 & 30 \\ 30 & 40 \end{bmatrix}$

$\therefore CX^T = \begin{bmatrix} 1000 + 6000 + 7500 & 2000 + 4500 + 10000 \end{bmatrix}$

$\therefore CX^T = \begin{bmatrix} 14500 & 16500 \end{bmatrix}$

$\therefore$ Total manufacturing cost = 14500 + 16500 = Rs. 31,000/- …1

Total Selling price

$$\therefore \text{Consider } SX^{T} = \begin{bmatrix} 120 & 140 & 300 \end{bmatrix} \begin{bmatrix} 10 & 20 \\ 40 & 30 \\ 30 & 40 \end{bmatrix}$$

$$\therefore SX^{T} = \begin{bmatrix} 1200 + 5600 + 9000 & 2400 + 4200 + 12000 \end{bmatrix}$$

$$\therefore SX^{T} = \begin{bmatrix} 15800 & 18600 \end{bmatrix}$$

$\therefore$ Total money after sales = 15800 + 18600 = Rs. 34,400/- …2

From (1) and (2)

Profit = 34400 – 310000 = Rs. 3400/-

***Example* 33:** Mr. Mittal owns two factories A and B which produces two products P and Q. Factory A produces 12 units of P and 6 units of Q and factory B produces 14 units of P and 5 units of Q, represented by the matrix:

$X = \begin{bmatrix} 12 & 14 \\ 6 & 5 \end{bmatrix}$. If $Y = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ represents number of days the two factories operate, what does XY represent ?

Ans. $Y = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ means, factory A operates for 4 days a week and factory B 5 days a week.

$$\therefore XY = \begin{bmatrix} 12 & 14 \\ 6 & 5 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 12 \times 4 + 14 \times 5 \\ 6 \times 4 + 5 \times 5 \end{bmatrix} = \begin{bmatrix} 48 + 70 \\ 24 + 25 \end{bmatrix}$$

$\therefore XY = \begin{bmatrix} 118 \\ 49 \end{bmatrix}$ represents number of units of P produced in both the factories in a week (ie 118) and number of units of Q produced in both the factories in a week (ie 49).

5.6 LET US SUM UP

In this chapter we have learn:

- Definition and order of matrix.
- Types of matrices.
- Equality of two matrices.
- Operations on matrices.

5.7 UNIT END EXERCISE

1) Write down the order of the following matrices:

(a) $\begin{bmatrix} 24 \\ 36 \\ 8 \end{bmatrix}$, (b) $\begin{bmatrix} 4 & 7 & 9 \\ 5 & 8 & 6 \end{bmatrix}$, (c) $\begin{bmatrix} x^2 & xy & y^2 \\ y^2 & yz & z^2 \\ z^2 & xz & x^2 \end{bmatrix}$, (d) $\begin{bmatrix} a & b & c \end{bmatrix}$, (e) [100]

2) $A = \begin{bmatrix} 11 & 4 & 16 \\ 4 & 22 & 9 \\ 16 & 9 & 33 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 2 & 4 \\ 2 & 10 & 16 \\ 4 & 16 & 15 \end{bmatrix}$, $C = \begin{bmatrix} 1 & -9 & -16 \\ -9 & 2 & -11 \\ -16 & -22 & 3 \end{bmatrix}$

Verify the following matrices:

(a) A + B = B + A, (b) A + (B + C) = (A + B) + C, (c) A + C = C + A,

(d) A – (B + C) = A – B – C, (e) A – (B – C) = A – B + C

3) For $A = \begin{bmatrix} -6 & 7 \\ -4 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -3 \\ 7 & 6 \end{bmatrix}$

Verify:

(i) $(A + B)^T = A^T + B^T$

(ii) $(A^T)^T = A$

4) If $A = [a_{ij}]_{3 \times 3}$ is such that $a_{ij} = i - j$. Find all elements of A and write down the matrix A.

5) If $P = \begin{bmatrix} 3 & -4 \\ 7 & 11 \end{bmatrix}$, $Q = \begin{bmatrix} -4 & 6 \\ 12 & -10 \end{bmatrix}$. Find 3P – 2Q.

6) (i) Find the matrix B if 4A + 5B = 3I, where $A = \begin{bmatrix} -2 & 1 \\ -1 & 2 \end{bmatrix}$ and I is the identity matrix.

(ii) If $A = \begin{bmatrix} 4 & -1 \\ 2 & -3 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 4 \\ 0 & -1 \end{bmatrix}$, $C = \begin{bmatrix} 8 & 11 \\ -6 & 5 \end{bmatrix}$. Find the matrix X such that 2A – 3B + 4C = 5X.

7) If $\begin{bmatrix} x - y & x - z \\ z + y & x + y \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 8 & 6 \end{bmatrix}$, find x, y, z.

8) If $A = \begin{bmatrix} 1 & -1 & 4 \\ x & -5 & z \\ y & 7 & 6 \end{bmatrix}$ is symmetric matrix, find x, y, z.

9) If $A = \begin{bmatrix} 0 & 5 & a \\ b & 0 & 1 \\ -6 & c & 0 \end{bmatrix}$ is a skew-symmetric matrix, find a, b, c.

10) Find the values of x, y, z if $\begin{bmatrix} x & 1 & 7 \\ -6 & y & 5 \\ 11 & 3 & z \end{bmatrix} + \begin{bmatrix} 3 & 1 & 2 \\ 4 & 3 & -2 \\ 6 & 5 & 4 \end{bmatrix} = \begin{bmatrix} 2 & x & 9 \\ -z & 1 & 3 \\ 17 & 4y & 8 \end{bmatrix}$

11) If, $\begin{bmatrix} x+y & 6 \\ 7 & xy \end{bmatrix} = \begin{bmatrix} 8 & 6 \\ -7 & 12 \end{bmatrix}$ find x, y.

12) Find a, b, c and d, if

(i) $\begin{bmatrix} 2a-3b & c-a & 3 \\ 1 & a+4b & 3c+4d \end{bmatrix} = \begin{bmatrix} -1 & 1 & 3 \\ 1 & 16 & 10 \end{bmatrix}$

(ii) $\begin{bmatrix} a+5b & c+a & a+d \\ 4 & a+3c & 7 \end{bmatrix} = \begin{bmatrix} 4 & 1 & 3 \\ 4 & 1 & 7 \end{bmatrix}$

13) Solve the following to find matrices P and Q.

(i) $3P - Q = \begin{pmatrix} 4 & 4 \\ 3 & -1 \end{pmatrix}$, $P - 3Q = \begin{pmatrix} 4 & -4 \\ 1 & 21 \end{pmatrix}$

(ii) $P + 2Q = \begin{pmatrix} 4 & 1 & 5 \\ -1 & 4 & -4 \end{pmatrix}$, $2P - Q = \begin{pmatrix} 3 & -3 & 0 \\ 3 & 3 & 2 \end{pmatrix}$

(iii) $2P + Q = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$, $P - 3Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

14)

(i) If $A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$, find a matrix B such that A + B is a diagonal matrix.

(ii) If $A = \begin{bmatrix} 1 & 0 & 2 \\ 4 & 3 & 1 \\ 2 & 1 & 4 \end{bmatrix}$, find a matrix B such that A – B is scalar matrix.

(iii) If $A = \begin{bmatrix} 1 & -2 \\ -3 & 100 \end{bmatrix}$, find a matrix B, such that A + B is a Null matrix.

(iv) If $A = \begin{bmatrix} 1 & -2 & 3 \\ -7 & 6 & -4 \\ 8 & -9 & 5 \end{bmatrix}$, find a matrix B such that A + B is an upper triangular matrix.

(v) If $A = \begin{bmatrix} -1 & 6 & -7 \\ 2 & -8 & 9 \\ -3 & 4 & -5 \end{bmatrix}$, find a matrix B such that A + B is a lower triangular matrix.

15) From a grocery store Ram purchased 3 kilo potatoes, 1 kilo onions, and 2 kilo rice. Shyam also purchased 2 kilo potatoes, 1 kilo rice and half a kilo wheat. Represent the above data in matrix form.

16) The number of students in Arts, Science and Commerce faculty studying in F.Y., S.Y. and T.Y. of a certain college of as follows: In Arts there are 120, 60 and 25, in Science there are 440, 280 and 120 and in Commerce there are 720, 500 and 300 students in F.Y., S.Y. and T.Y. respectively. Represent the above data in matrix form.

17) In an entrance examination, 20 students from college A, students from college B and 35 from college C appeared. Only 12 students from A, B and 10 students from C passed the examination. Out of them 5 students from A, 7 students from B and 8 from C secured 100% marks. Represents the above data in matrix from.

18) Let $A = \begin{bmatrix} 12 & 14 & 15 \\ 13 & 15 & 17 \\ 14 & 16 & 15 \end{bmatrix}$, where the three rows represent number of three items and the columns represent three shops respectively. If $B = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & 3 & 0 \end{bmatrix}$, represent the sales per day, find the stock left after the sales done in five days using matrix notation.

19) In a city there are 12 colleges and 52 schools. Each institution has 6 peons, 4 clerks, and 1 cashier. Each college in addition to this as 1 head clerk and 1 accountant. Using matrix method, find the total number of posts in school and colleges taken together.

20) The number of students enrolled for a personality contest from three colleges A, B and C were 15, 24 and 8 respectively. Of these those who enter the semi-final were 8, 16 and 4 respectively. Only 4, 9 and 1 students from the three colleges respectively qualified for the finals. Represent the above data in matrix form.

21) The price of one kilo of rice, wheat and oil in a retail shop is Rs. 16/-, Rs. 12/- and Rs. 65/- respectively. The same items costs Rs. 15/-, Rs. 10/- and Rs. 50/- in a wholesale shop respectively. Ramesh and Suresh purchased 5 kilos of each items from the retail and whole sale shop respectively. Using matrix method, find the total cost incurred by each of them.

22) A firm manufacturers three items; A, B and C. The sales in first quarter are 1.18, 2.25 and 3.02 (in million tons) respectively, while that in the second quarter are 1.06, 2.40, 3.01. Find the half yearly sales for the three items.

23) Which of the following matrices are conformable?

(i) $A = \begin{bmatrix} 100 & -90 \\ 75 & 32 \end{bmatrix}$, (ii) $B = \begin{bmatrix} 1001 & 1101 & 1201 \\ 11 & 101 & 1001 \end{bmatrix}$, (iii) $C = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$,

(iv) $D = \begin{bmatrix} -6 & 2 & 11 \\ 4 & -1 & 4 \\ 1 & 13 & 6 \end{bmatrix}$, (v) $E = \begin{bmatrix} 14 & -10 & 9 \end{bmatrix}$

24) If $A = \begin{bmatrix} 4 & -6 & 3 \\ 1 & 2 & 9 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 1 \\ -6 & 2 \\ 3 & 9 \end{bmatrix}$, compute AB, BA. Is AB = BA justify your answer.

25) (i) Let $P = \begin{bmatrix} 4 & 0 \\ -1 & 2 \end{bmatrix}$, $Q = \begin{bmatrix} 0 & 2 \\ 3 & 1 \end{bmatrix}$, $R = \begin{bmatrix} -1 & 1 \\ 4 & 2 \end{bmatrix}$. Compute PQ, PR, QR, P(QR), (PQ)R. What can you say about P(QR) and (PQ)R ?

(ii) For above matrices, P, Q, R,; compute (P + Q) R, (P + Q) R. Hence verify P(Q + R) = PQ + PR, (P + Q) R = PR + QR.

26) If $A = \begin{bmatrix} 1 & 1 & 3 \\ 3 & 3 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 \\ 5 & 2 \\ -2 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 4 & 0 \\ -3 & 5 \\ 5 & 2 \end{bmatrix}$. Verify that AB = AC but $B \neq C$.

27) If $f(x) = x^2 - 3x + 6$, find $f(A)$ where $A = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$.

28) Verify that AB = 0, where $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ -1 & 4 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 4 & 9 \end{bmatrix}$

29) If $A \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} -3 & -3 & -3 \\ 4 & 5 & 6 \end{bmatrix}$ find A.

(Hint : Assume A to be a matrix of order 2 × 2 and solve.)

30) If $A = \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}$, find A + 5A – 10I.

31) Verify that $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ satisfies $A^2 - 4A + I = 0$

32) Find matrix x such that $\begin{bmatrix} 3 & -1 \\ 2 & 3 \end{bmatrix} X = \begin{bmatrix} 0 & 11 & 5 \\ 11 & 11 & 18 \end{bmatrix}$

33) If A = $\begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$, find the value of x such that $A^2 - xA + 2I = 0$

34) If A = $\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$, B = $\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$, show that AB = BA.

Prove that, a diagonal matrices commute under multiplication.

35) For matrices P = $\begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$, Q = $\begin{bmatrix} 1 & -2 & 3 \end{bmatrix}$, verify that $(PQ)^T = Q^T P^T$.

36) If $X = \begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix}$, show that $(X - 2I)(X - 3I) = 0$

37) If A = $\begin{bmatrix} 2 & -1 & 1 \\ -2 & 3 & -2 \\ 4 & 4 & -3 \end{bmatrix}$ show that A (A –

38) A soft drink company manufactures two types of soft drinks A and B. The sales of these products in two cities are represented by the matrix X as follows:

$$X = \begin{pmatrix} 1000 & 1200 \\ 1500 & 2500 \end{pmatrix} \begin{matrix} \leftarrow \text{City 1} \\ \leftarrow \text{City 2} \end{matrix}$$

(columns: A, B)

If Y = $\begin{bmatrix} 12 \\ 10 \end{bmatrix}$ represents the cost of each product A and B respectively, what does XY represent.

39) A manufacturer produces three products A, B and C and sells it in two cities the total sales are shown as follows:

$$\begin{bmatrix} \text{A} & \text{B} & \text{C} \\ 13500 & 11000 & 4500 \\ 8200 & 15000 & 7800 \end{bmatrix} \begin{matrix} \\ \leftarrow \text{City 1} \\ \leftarrow \text{City 2} \end{matrix}$$

(i) If the selling price per unit is Rs. 4/-, Rs. 6/- and Rs. 3/- for the product A, B and C respectively, find the total revenue using matrix algebra.

(ii) If the manufacturing cost per unit is Rs. 2/-, Rs. 3/- and Rs. 0.50/- for the product A, B and C respectively, find the overall profit using matrix algebra.

40) The sales of three items of a company from Monday to Saturday is represented by the matrix X as follows:

$$X = \begin{bmatrix} \text{Mon} & \text{Tue} & \text{Wed} & \text{Thu} & \text{Fri} & \text{Sat} \\ 2 & 5 & 3 & 1 & 4 & 6 \\ 4 & 3 & 1 & 2 & 5 & 6 \\ 6 & 5 & 2 & 3 & 1 & 4 \end{bmatrix} \begin{matrix} \\ \leftarrow \text{Item 1} \\ \leftarrow \text{Item 2} \\ \leftarrow \text{Item 3} \end{matrix}$$

Let $P = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ be the matrix representing the price of each item respectively.

(i) What does $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} X^T$ represent ?

(ii) What does $X^T \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ represent?

(iii) Find the total amount obtained at the end of week.

41) The sales of three brands of Tea A, B, C during a week from Monday to Saturday are shown below by matrix X.

$$X = \begin{bmatrix} \text{Mon} & \text{Tue} & \text{Wed} & \text{Thu} & \text{Fri} & \text{Sat} \\ 10 & 15 & 12 & 11 & 10 & 20 \\ 20 & 20 & 30 & 30 & 25 & 30 \\ 5 & 10 & 8 & 7 & 5 & 10 \end{bmatrix} \begin{matrix} \\ \leftarrow \text{A} \\ \leftarrow \text{B} \\ \leftarrow \text{C} \end{matrix}$$

Let $C = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$ be the cost per packet of Tea brand respectively.

(i) Find total sales of the three brands during a week.

(ii) Find the total earnings in a week.

(iii) Find total sales in a week of all brands together per day.

(iv) Find total sales during a week of all brands taken together.

42) 'Nirman Constructions' has taken a order to construct 10 houses of type A, 20 houses of type B and 5 houses of type C. The following matrix represents the raw material required in each type of house.

$$X = \begin{bmatrix} 105 & 50 & 70 & 20 & 100 \\ 150 & 70 & 50 & 15 & 120 \\ 75 & 35 & 20 & 10 & 50 \\ \text{Cement} & \text{Steel} & \text{Wood} & \text{Glass} & \text{Paint} \end{bmatrix} \begin{matrix} \leftarrow \text{A} \\ \leftarrow \text{B} \\ \leftarrow \text{C} \\ \\ \end{matrix}$$

The cost per unit raw material is given by Y = [10 18 6 15 9]

(i) Find how much raw material the company should order.

(ii) Find cost of all materials for a house of each type.

(iii)Find total cost of raw materials for all houses taken together.

5.8 LIST OF REFERENCES

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6

DETERMINANT

Unit Structure

6.0 OBJECTIVE

After going through this chapter you will able to know:

- Determinant of square matrices of order upto 3×3.
- Properties of determinants.
- Using determinant able to solve system of equations.

6.1 INTRODUCTION

Determinant is a value associated with a square matrix. Let A be a square matrix then determinant of A is denoted as $|A|$.

As we know now that determinant is defined for square matrices only, let us learn how to compute determinants of square matrices of order 2x2 and 3x3.

6.2 DETERMINANT OF MATRIX OF ORDER 2X2

Let $A = [a_{ij}]$ be a matrix of order 2x2. Explicitly we can write $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$.

Then,

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}\, a_{22} - a_{12}\, a_{21} \qquad (*)$$

Note: The minus sign between the two terms is the part of the formula.

***Example* 1.**If $A=\begin{pmatrix}2 & 4\\3 & 7\end{pmatrix}$, then $|A|=\begin{vmatrix}2 & 4\\3 & 7\end{vmatrix}$ = 2 x 7 – 4 x 3 = 14 – 12 = 2

***Example* 2.** Let $A=\begin{pmatrix}3 & -1\\6 & -2\end{pmatrix}$

Then, $|A|=\begin{vmatrix}3 & -1\\6 & -2\end{vmatrix}$ = 3 x (–2) – (–1) x 6 = –6 + 6 = 0

***Example* 3.**If $\begin{vmatrix}11 & -2\\3 & x\end{vmatrix}$ = 50, then find the value of x .

Ans: Evaluating the determinant on the LHS, we have,

$11x-(-2)\times 3=50$

i.e. $11x+6=50$

$\therefore 11x=50-6=44$

$\therefore x=4$

6.3 DETERMINANT OF MATRIX OF ORDER 3X3

Let A = [a_{ij}] be a matrix of order 3x3. Explicitly we can write A as: $A=\begin{pmatrix}a_{11} & a_{12} & a_{13}\\a_{21} & a_{22} & a_{23}\\a_{31} & a_{32} & a_{33}\end{pmatrix}$

$$|A|=\begin{vmatrix}a_{11} & a_{12} & a_{13}\\a_{21} & a_{22} & a_{23}\\a_{31} & a_{32} & a_{33}\end{vmatrix}=a_{11}\begin{vmatrix}a_{22} & a_{23}\\a_{32} & a_{33}\end{vmatrix}-a_{12}\begin{vmatrix}a_{21} & a_{23}\\a_{31} & a_{33}\end{vmatrix}+a_{13}\begin{vmatrix}a_{21} & a_{22}\\a_{31} & a_{32}\end{vmatrix}$$

Note: There are three terms on the RHS of the formula

1. The first element a_{11} is multiplied with the minor determinant of order 2x2 obtained by deleting the row and column containing a_{11}.

2. The second element a_{12} taken with a minus sign is multiplied with its minor determinant of order 2x2 obtained by deleting the row and column containing a_{12}

3. The third element a_{13} is multiplied with its minor determinant as described above for the first two terms.

Initially readers are advised to use the above formula to evaluate determinants of order 3x3. After gaining confidence one can write the same directly as follows:

$$|A|=a_{11}(a_{22}a_{33}-a_{23}a_{32})-a_{12}(a_{21}a_{33}-a_{23}a_{31})+a_{13}(a_{21}a_{32}-a_{22}a_{31})$$

***Example* 4.** Evaluate the determinant $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 1 & 4 \end{vmatrix}$

Ans: $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 1 & 4 \end{vmatrix} = 1\begin{vmatrix} 4 & 5 \\ 1 & 4 \end{vmatrix} - 2\begin{vmatrix} 2 & 5 \\ 3 & 4 \end{vmatrix} + 3\begin{vmatrix} 2 & 4 \\ 3 & 1 \end{vmatrix}$

= 1 (4 x 4 – 5 x 1) – 2(2 x 4 – 5 x 3) + 3(2 x 1 – 4 x 3)

= 11 – 2(– 7) + 3(– 10) = 11 + 14 – 30

= – 5

6.4 PROPERTIES OF DETERMINANTS

Determinants satisfy certain important properties which makes it more useful in solving complex problems. Here we will be discussing some elementary properties of determinant.

1. The value of determinant remains unchanged if its rows and columns are interchanged. In other words the determinant of a matrix is same as that of its transpose, $|A| = |A^T|$. For example,

$\begin{vmatrix} 4 & 5 \\ 1 & 2 \end{vmatrix} = 8 - 5 = 3$

And $\begin{vmatrix} 4 & 1 \\ 5 & 2 \end{vmatrix} = 8 - 5 = 3$

2. If any two rows (or columns) are interchanged then the value of determinant changes only by sign. For example,

$\begin{vmatrix} 4 & 1 \\ 5 & 2 \end{vmatrix} = 8 - 5 = 3$. Now if we interchange the first and second row,

i.e. $R_1 \leftrightarrow R_2$ gives $\begin{vmatrix} 5 & 2 \\ 4 & 1 \end{vmatrix} = 5 - 8 = -3$

Or if we interchange the columns of the original determinant i.e $C_1 \leftrightarrow C_2$

$\begin{vmatrix} 5 & 4 \\ 2 & 1 \end{vmatrix} = 5 - 8 = -3$

3. If any two rows (or columns) are identical or if one row (column) is a multiple of another row (column) then the value of determinant is zero.

For example, $\begin{vmatrix} 4 & 2 \\ 2 & 1 \end{vmatrix} = 4 - 4 = 0$ … (as $R_1 = 2R_2$.)

Also, $\begin{vmatrix} 4 & 4 \\ 2 & 2 \end{vmatrix} = 8 - 8 = 0$ … (as $C_1 = C_2$)

4. If any row (or column) is multiplied by a scalar, say α then the value of determinant is also α times the original value i.e. $|\alpha A| = \alpha |A|$.

For example, $\begin{vmatrix} 3 & -1 \\ 2 & 4 \end{vmatrix} = 12 + 2 = 14$ and if we multiply first row by 2, then the determinant i.e $2R_1$ gives $\begin{vmatrix} 6 & -2 \\ 2 & 4 \end{vmatrix} = 24 + 4 = 28 = 2(14)$

Similarly, $\begin{vmatrix} 2 & 3 & 0 \\ 3 & 0 & 4 \\ 0 & 4 & 5 \end{vmatrix} = 2(0-16) - 3(15-0) + 0 = -32 - 45 = -77$

Then $3C_3$ gives, $\begin{vmatrix} 2 & 3 & 0 \\ 3 & 0 & 12 \\ 0 & 4 & 15 \end{vmatrix} = 2(0-48) - 3(45-0) + 0 = -96 - 135 = -231 = 3(-77)$

5. Any operation of the type $R_i + kR_j$ (or $C_i + kC_j$) does not change the value of the determinant. The meaning of the row operation is that to R_1 we add k times R_2. For example,

$$\begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} = 2 + 3 = 5$$

Now to R_1 if we add $2R_2$ i.e. $R_1 + 2R_2$ then the determinant becomes $\begin{vmatrix} 2+2(3) & -1+2(1) \\ 3 & 1 \end{vmatrix} = \begin{vmatrix} 8 & 1 \\ 3 & 1 \end{vmatrix} = 8 - 3 = 5$ (same !!)

This is the most powerful property of determinants which is extensively used in solving determinants and inverses of matrices.

Let us solve some problems using the above properties of determinants.

***Example* 1.** Evaluate the determinant without actually expanding it $\begin{vmatrix} 1 & 11 & 121 \\ 1 & 12 & 144 \\ 1 & 13 & 169 \end{vmatrix}$.

Ans: Consider $\begin{vmatrix} 1 & 11 & 121 \\ 1 & 12 & 144 \\ 1 & 13 & 169 \end{vmatrix} = \begin{vmatrix} 1 & 11 & 121 \\ 1 & 12 & 144 \\ 0 & 1 & 25 \end{vmatrix}$ (by the operation $R_3 - R_2$)

$= \begin{vmatrix} 1 & 11 & 121 \\ 0 & 1 & 23 \\ 0 & 1 & 25 \end{vmatrix}$ (by the operation $R_2 - R_1$)

$= \begin{vmatrix} 1 & 11 & 121 \\ 0 & 1 & 23 \\ 0 & 0 & 2 \end{vmatrix}$ (by the operation $R_3 - R_2$)

$$= \begin{vmatrix} 1 & 0 & 0 \\ 11 & 1 & 0 \\ 121 & 23 & 2 \end{vmatrix} \qquad \text{(by interchanging rows to columns)}$$

$= 1(2 - 0) - 0 + 0$

$= 2$

***Example* 2.** Without expanding the determinant show that $\begin{vmatrix} 12 & 6 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} -4 & 3 \\ 12 & 6 \end{vmatrix} = 0$

Ans: Now (by the property 2) we know that if any two rows (or columns) are interchanged then the determinant changes by a sign,

$$\text{L.H.S.} = \begin{vmatrix} 12 & 6 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} -4 & 3 \\ 12 & 6 \end{vmatrix} = -\begin{vmatrix} -4 & 3 \\ 12 & 6 \end{vmatrix} + \begin{vmatrix} -4 & 3 \\ 12 & 6 \end{vmatrix} \qquad \text{(by the operation } R_1 \leftrightarrow R_2\text{)}$$

$= 0 \quad = \text{R.H.S.}$

***Example* 3.** Without expanding the determinant show that $\begin{vmatrix} x & x+1 & x+2 \\ x+4 & x+5 & x+6 \\ x+7 & x+8 & x+9 \end{vmatrix} = 0$

Ans: L.H.S. $= \begin{vmatrix} x & x+1 & x+2 \\ x+4 & x+5 & x+6 \\ x+7 & x+8 & x+9 \end{vmatrix}$

$$= \begin{vmatrix} x & x+1 & 1 \\ x+4 & x+5 & 1 \\ x+7 & x+8 & 1 \end{vmatrix} \qquad \text{(by the operation } C_3 - C_2\text{)}$$

$$= \begin{vmatrix} x & 1 & 1 \\ x+4 & 1 & 1 \\ x+7 & 1 & 1 \end{vmatrix} \qquad \text{(by the operation } C_2 - C_1\text{)}$$

$= 0$

$\because$ if any two rows (or columns) are identical then the determinant is zero. In this problem $C_3 = C_2$.

***Example* 4.** Without expanding show that $\begin{vmatrix} a & b & c \\ b & a & c \\ b & c & a \end{vmatrix} = (a-b)(a-c)(a+b+c)$.

Ans: Consider

$$\text{L.H.S.}=\begin{vmatrix} a & b & c \\ b & a & c \\ b & c & a \end{vmatrix} = \begin{vmatrix} a-b & b-a & c-c \\ b & a & c \\ b & c & a \end{vmatrix} = \begin{vmatrix} a-b & -(a-b) & 0 \\ b & a & c \\ b & c & a \end{vmatrix}$$

(by the operation R_1 - R_2)

$$=(a-b)\begin{vmatrix} 1 & -1 & 0 \\ b-b & a-c & c-a \\ b & c & a \end{vmatrix} = (a-b)\begin{vmatrix} 1 & -1 & 0 \\ 0 & a-c & -(a-c) \\ b & c & a \end{vmatrix}$$

(by the operation $R_2 - R_3$)

$$=(a-b)(a-c)\begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ b & c & a \end{vmatrix} = (a-b)(a-c)\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ b & c+b & a \end{vmatrix}$$

(by the operation $C_1 + C_2$)

$= (a - b)(a - c)[1(a + b + c) - 0 + 0] = (a - b)(a - c)(a + b + c) =$ R.H.S.

6.5 CRAMER'S RULE

System of Linear Equations :

In this section we shall discuss the method to solve a system of linear equations using Cramer's rule.

Consider a system of two linear equations in *two variables* :

$$\begin{aligned} a_1x + b_1y &= c_1 \\ a_2x + b_2y &= c_2 \end{aligned}$$; the solution of these equations is given by

$$x = \frac{D_x}{D}, y = \frac{D_y}{D} \text{ where D}=\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \text{ and } D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

This is called as Cramer's rule.

- Consider the three equations in *three variables* :

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_2z &= d_3 \end{aligned}$$; the solution of these equations is given by

$$x = \frac{D_x}{D}, y = \frac{D_y}{D} \text{ and } z = \frac{D_z}{D},$$

$$\text{where D } = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}, D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \text{ and } D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

The Cramer's rule is applicable only when $D \neq 0$.

Let us solve some system of linear equations using Cramer's rule

***Example* 1.** $2x+3y=5$ and $x+2y=2$

Ans: The determinant of the coefficients is $D=\begin{vmatrix}2 & 3\\ 1 & 2\end{vmatrix}=4-3=1$

Also, $D_x=\begin{vmatrix}5 & 3\\ 2 & 2\end{vmatrix}=10-6=4$ and $D_y=\begin{vmatrix}2 & 5\\ 1 & 2\end{vmatrix}=4-5=-1$

By Cramer's rule we have,

$$x=\frac{D_x}{D},\ y=\frac{D_y}{D}$$

$$\therefore x=\frac{4}{1}=4 \text{ and } y=\frac{-1}{1}=-1$$

Hence the solution of the given system of equations is

$x=4$ and $y=-1$

***Example* 2.** $2x+3y+z=9$, $x+2y+z=6$ and $3x+y+2z=8$

Ans: We have $D=\begin{vmatrix}2 & 3 & 1\\ 1 & 2 & 1\\ 3 & 1 & 2\end{vmatrix}=2(4-1)-3(2-3)+1(1-6)=6+3-5=4$

$$D_x=\begin{vmatrix}9 & 3 & 1\\ 6 & 2 & 1\\ 8 & 1 & 2\end{vmatrix}=9(4-1)-3(12-8)+1(6-16)=27-12-10=5$$

$$D_y=\begin{vmatrix}2 & 9 & 1\\ 1 & 6 & 1\\ 3 & 8 & 2\end{vmatrix}=2(12-8)-9(2-3)+1(8-18)=8+9-10=7$$

$$D_z=\begin{vmatrix}2 & 3 & 9\\ 1 & 2 & 6\\ 3 & 1 & 8\end{vmatrix}=2(16-6)-3(8-18)+9(1-6)=20+30=45=5$$

$$\therefore D=4,\quad D_x=5,\quad D_y=7,\quad D_z=5$$

By Cramer's rule we have $x=\dfrac{D_x}{D}, y=\dfrac{D_y}{D}$ and $z=\dfrac{D_z}{D}$,

$$\therefore x=\frac{5}{4},\ y=\frac{7}{4} \quad \text{and } z=\frac{5}{4}.$$

***Example* 3.** In a factory two products are produced, A and B using two machines

M_1 and M_2.The time taken by product A to be processed through M_1 and M_2 is 2 and 1 hour respectively. The time taken by product B to be processed through M_1 and M_2 is 1 and 2 hours respectively. If the time available for both the machines is 30 hours, how many units of the two products can be produced?

Ans: Let x units of product A and y units of product B be produced. Then from the given information we have the following system of equations:

$$2x + y = 30 \qquad \ldots. (1)$$

$$x + 2y = 30 \qquad \ldots. (2)$$

We solve the above linear equations using Cramer's rule.

$$D = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4 - 1 = 3 \qquad \therefore D = 3 .$$

$$D_x = \begin{vmatrix} 30 & 1 \\ 30 & 2 \end{vmatrix} = 60 - 30 = 30 \qquad \therefore \; D_x = 30$$

$$D_y = \begin{vmatrix} 2 & 30 \\ 1 & 30 \end{vmatrix} = 60 - 30 = 30 \qquad \therefore D_y = 30$$

Now we know that $x = \frac{D_x}{D}, y = \frac{D_y}{D}$.

Thus, $x = \frac{30}{3} = 10$ and $y = \frac{30}{3} = 10$.

i.e 10 units of each product can be produced.

***Example* 2.** A pharmaceutical company manufactures three types of medicines; M_1, M_2 and M_3. Each type of medicine contains three drugs D_1, D_2 and D_3 as given in the following table:

Medicine	Drug (in mg)		
	D_1	D_2	D_3
M_1	1	3	2
M_2	2	1	3
M_3	3	2	1

What amount of the three medicines should be taken to consume 14 mg of D_1, 11 mg of D_2 and 11 mg of D_3?

Ans: Let x units of M_1, y units of M_2 and z units of M_3 be consumed. Then according to the given information we have the following system of linear equations:

$$x + 2y + 3z = 14$$
$$3x + y + 2z = 11$$
$$2x + 3y + z = 11$$

$$\text{Now },D = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{vmatrix} = 1(1-6) - 2(3-4) + 3(9-2) = -5 + 2 + 21 = 18$$

$$D_x = \begin{vmatrix} 14 & 2 & 3 \\ 11 & 1 & 2 \\ 11 & 3 & 1 \end{vmatrix} = 14(1-6) - 2(11-22) + 3(33-11) = -70 + 22 + 66 = 18$$

$$D_y = \begin{vmatrix} 1 & 14 & 3 \\ 3 & 11 & 2 \\ 2 & 11 & 1 \end{vmatrix} = 1(11-22) - 14(3-4) + 3(33-22) = -11 + 14 + 33 = 36$$

$$D_z = \begin{vmatrix} 1 & 2 & 14 \\ 3 & 1 & 11 \\ 2 & 3 & 11 \end{vmatrix} = 1(11-33) - 2(33-22) + 14(9-2) = -22 - 22 + 98 = 54$$

By Cramer's rule we have $x = \frac{D_x}{D}, y = \frac{D_y}{D}$ and $z = \frac{D_z}{D}$,

$\therefore x = \frac{18}{18} = 1, y = \frac{36}{18} = 2$ and $z = \frac{54}{18} = 3$.

1 unit of M_1, 2 units of M_2 and 3 units of M_3.

6.6 LET US SUM UP

In this chapter we have learn:

- Determinant of square matrices.
- Properties of determinant.
- Use of Cramer's rule for solving system of equations.

6.7 UNIT END EXERCISE

1) Evaluate the following determinants :

(a) $\begin{vmatrix} 10 & 5 \\ -6 & -3 \end{vmatrix}$ (b) $\begin{vmatrix} 100 & 20 \\ 13 & -3 \end{vmatrix}$ (c) $\begin{vmatrix} 4 & 1 & 7 \\ 2 & 6 & 3 \\ 1 & 2 & 1 \end{vmatrix}$ (d) $\begin{vmatrix} 10 & -1 & 4 \\ 1 & 3 & -2 \\ 4 & 0 & 1 \end{vmatrix}$

(e) $\begin{vmatrix} x+y & x-y \\ x-y & x+y \end{vmatrix}$ (f) $\begin{vmatrix} a & b & c \\ -b & 0 & -a \\ -c & a & b \end{vmatrix}$

2) Solve the following to find the value of x.

(a) $\begin{vmatrix} x+2 & 3 \\ 1 & -1 \end{vmatrix} = 5$ (b) $\begin{vmatrix} x+2 & 3 \\ 1 & x-2 \end{vmatrix} = 2$ (c) $\begin{vmatrix} 3x & 1 \\ x & 2 \end{vmatrix} = 50$

(d) $\begin{vmatrix} 1 & 0 & 3 \\ -1 & x & -2 \\ 0 & 2 & x+1 \end{vmatrix} = 0$ (e) $\begin{vmatrix} 2 & 1 & -1 \\ x & 4 & x \\ 0 & x-1 & 1 \end{vmatrix} = -2$

3) Evaluate the following determinants without expanding actually.

1. $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{vmatrix}$ 2. $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & 3 \end{vmatrix}$ 3. $\begin{vmatrix} y^2z^2 & yz & y+z \\ z^2x^2 & zx & z+x \\ x^2y^2 & xy & x+y \end{vmatrix}$

4) Without expanding actually prove the following :

1. $\begin{vmatrix} 3 & 6 & 11 \\ 2 & 7 & 11 \\ 2 & 6 & 12 \end{vmatrix} = 20$ **2.** $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$

3. $\begin{vmatrix} a+b & b & a \\ b & b+c & c \\ a & c & a+c \end{vmatrix} = 4abc$ **4.** $\begin{vmatrix} x & x-y & x-z \\ z & x+y & z-x \\ y & y-x & z+x \end{vmatrix} = 4xyz$

5. $\begin{vmatrix} 1 & xy & x+y \\ 1 & yz & y+z \\ 1 & xz & z+x \end{vmatrix} = (x-y)(y-z)(z-x)$ **6.** $\begin{vmatrix} 1 & a & b+c \\ 1 & b & a+c \\ 1 & c & a+b \end{vmatrix} = 0$

5) Solve the following system of linear equations using Cramer's rule.

1. $2x+5y=1$, $3x+8y=2$. **2.** $x+3y=6$, $2x+5y=12$

3. $x+y=2$, $2x-3y=14$ **4.** $x+2y=1$, $3x+y=2$

5. $3x-2y=5$, $x-3y=-3$ **6.** $7x-y=10$, $x+3y=14$.

7. $7x+y+z=13$, $3x-y+2z=6$, $4x+y-2z=1$.

8. $x-y+5z=2$, $3x+y+2z=3$, $4x-2y+3z=3$

9. $x-3y-8z=-10$, $3x+y-4z=0$, $2x+5y+6z=13$

10. $2x+y+z=6$, $x-y-z=-3$, $3x+y+2z=7$

11. $0.2x+0.3y-0.1z=0.9$, $0.1x+0.1y+0.1z=0.9$, $0.3x-0.1y-0.1z=-0.1$

12. $x - 3y - 8z = -10$, $3x + y - 4z = 0$, $2x + 5y + 6z = 13$

13. $2x + 3y + 4z = 29$, $x - 2y + 3z = 8$, $3x + y + 2z = 17$

14. $5x + 2z = 7$, $x - 2y + 3z = 2$, $3x + y + 2z = 6$

15. $6x + y + 4z = 1$, $3x - 2y + 3z = -1$, $x + y - 5z = 2$

Word Problems:

6) Mr. Manish invests amount of Rs. 20,000 in three schemes at the rate of 8%, 10% and 12 % p.a. He had a combined income of Rs. 2,060. The total income from the second and third scheme was Rs. 1,660. Find the amount Mr. Manish invested in each scheme using Cramer's rule.

7)Three students Shahnawaz, Shoaib and Sakina invested their savings of Rs. 750, Rs. 1200 and Rs. 1545 in 3 different shares with prices x, y and z respectively as follows. Find the price of each share using Cramer's rule.

Student	**share 1** (x)	**share 2** (y)	**share 3** (z)
Shahnawaz	6	20	30
Shoaib	30	50	20
Sakina	42	40	43

8) Dr. Ghate 's earnings (E) from his clinic is a linear function of a fixed expense of maintenance (M) plus the number of patients(P), i.e.E = xP – M. If his first half yearly earning is Rs.60, 000 with P = 1440 and the second half yearly earning is Rs. 80,000 with P = 1840, find the fixed maintenance per month and the average charge per patient using Cramer's rule.

9) The prices per unit of three commodities X, Y and Z are Rs. x, Rs. y and Rs. z respectively. A person '*A*' purchases 3 units of X, 5 units of Y and sells 4 units of Z. '*B*' purchases 2 units of X, 1 unit of Z and sells 3 units of Y. '*C*' purchases 4 units of Y, 6 units of Z and sells 1 unit of X. In this transaction *A, B* and *C* earn Rs. 6000, Rs. 5000 and Rs. 13000 respectively. Find the prices per unit of the three commodities.

10) Three friends Kanishka, Neha and Darshana invested their pocket money of Rs. 1450, Rs. 1800 and Rs. 950 in buying three items A, B and C from the market. Kanishka buys 7 units of A, 9 units of B and 10 units of C. Neha buys 6 units of A, 12 units of B and 20 units of C. Darshana buys 4 units of A 5 units of B and 10 units of C. Formulate this problem into system of linear equations and solve using Cramer's rule to give the price per unit of each item A, B and C.

11) A mixture is to be made of three foods X, Y and Z. Each type consists of nutrients A, B and C as shown below :

mg per gram of

Food	Nutrient A	Nutrient B	Nutrient C
X	3	3	2
Y	2	1	2
Z	4	2	0

How to form a mixture containing 8 mg of A, 5 mg of B and 6 mg of C?

12) The quarterly sales of a salesman for three products P_1, P_2 and P_3 of a company are given below with the total commission at different rates:

Quarter	P_1	P_2	P_3	Commission
I	100	80	40	600
II	120	100	80	860
III	50	80	100	740

Find, using Cramer's rule, the rate of commission on sale per unit of the products P_1, P_2 and P_3.

13) For the following two related market models (a) $Q_s = 5 + 2P_a - P_b$ and $Q_d = 4 - P_a + P_b$ (b) $Q_s = 5 + 3P_a + P_b$ and $Q_d = 14 + 2P_a - 3P_b$; find the equilibrium prices for each market model by Cramer's rule.

(Hint: Equilibrium conditions are : $Q_s = Q_d$ for both models.)

14) For the following two related market models (a) $Q_s = 15 + 2P_a + P_b$ and $Q_d = 38 - 2P_b$ (b) $Q_s = 5 - 3P_a + 2P_b$ and $Q_d = 11 - 7P_a + 4P_b$; find the equilibrium prices for each market model by Cramer's rule.

15) The equilibrium conditions for three market models is given below:

$5P_1 - 6P_2 + 4P_3 = 15$, $7P_1 + 4P_2 - 3P_3 = 19$ and $2P_1 + P_2 + 6P_3 = 46$. Find the equilibrium price for each market.

6.8 LIST OF REFERENCES:

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7

INVERSE OF MATRIX AND APPLICATION

Unit Structure

7.0 OBJECTIVE:

After going through this chapter you will able to know:

- Define inverse of matrix.
- Elementary transformation method to find inverse of matrix.
- Adjoint method to find inverse of matrix.
- Application of inverse of matrix.

7.1 INTRODUCTION:

In the previous chapter we have seen how to add, subtract and multiply matrices. Though there is no division of two matrices as such, for a square matrix A, 1/A is written as A^{-1} and is called as inverse of A. The concept of inverse of a matrix is useful in solving system of linear equations, input-output analysis etc.

A very important fact regarding this is that, not every matrix has an inverse. Only those matrices whose determinant is not zero have a inverse.

7.2 INVERSE OF A MATRIX:

This leads us to define two more types of matrices :

Singular Matrices: A square matrix X is said to be singular matrix if its determinant is zero, i.e. $|X| = 0$.

Non-Singular Matrices : A square matrix X is said to be non-singular matrix if

its determinant is non-zero, i.e. $|X| \neq 0$.

A matrix is said to be invertible if it has an inverse.

Before moving ahead let us remember the following properties.

- Every square matrix is not invertible.
- A square matrix is invertible iff it is non-singular.
- Inverse of a matrix is unique.
- For a square matrix A of order n, if A^{-1} is its inverse then $A.A^{-1} = I = A^{-1}.A$, where I is the identity matrix of order n.
- The inverse of the inverse of an invertible matrix is itself, *i.e.* $(A^{-1})^{-1} = A$
- The inverse of transpose of a matrix is transpose of its inverse, *i.e.* $(A^T)^{-1} = (A^{-1})^T$.
- If A, B are two invertible matrices of same order then, $(AB)^{-1} = B^{-1}.A^{-1}$

Note that the RHS is $B^{-1}.A^{-1}$ and not $A^{-1}B^{-1}$, as we know that matrix multiplication is not Commutative.

There are many methods to find inverse of an invertible square matrix. In

this section we are going to learn to find inverses using elementary row(or column) transformations.

7.3 INVERSE OF A MATRIX USING ELEMENTARY TRANSFORMATION:

Elementary Operations: An elementary operation is either an operation on row or on column of any of the following three types:

Type 1: Interchanging of two rows (or columns). e.g. $R_1 \leftrightarrow R_3$, $C_2 \leftrightarrow C_3$

Type 2: Scalar multiplication of any row (or column). e.g. $3R_2$, $\frac{1}{4}C_1$

Type 3: The addition of multiple of any row(or column) to another row (or

column). e.g. $R_1 + 2R_2$, $C_1 - 3C_2$.

To find inverse of a matrix, we follow the following steps :

STEP I: We first check that the given matrix is non-singular, *i.e.* its

determinant is non-zero.

STEP II: Then a partitioned matrix is created by putting an identity matrix next to the given matrix separated by a vertical line.

STEP III: Now using the row operations the left of the partition is made an identity matrix *proceeding columnwise*. Thus what we get on the right of the partition is the required inverse of the given matrix.

Let us now find inverse of some matrices using the above steps.

Find the inverse of the following matrices, if they exists:

(*i*) $\begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}$ (*ii*) $\begin{pmatrix} 5 & -1 \\ 1 & 3 \end{pmatrix}$ (*iii*) $\begin{pmatrix} -4 & -9 & 5 \\ -4 & -7 & 4 \\ 1 & 2 & -1 \end{pmatrix}$ (*iv*) $\begin{pmatrix} -7 & 1 & 4 \\ -3 & 3 & 0 \\ 8 & -2 & -2 \end{pmatrix}$

Ans : (*i*) Let A = $\begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}$

$\Rightarrow |A| = \begin{vmatrix} 3 & 1 \\ 6 & 2 \end{vmatrix} = 6 - 6 = 0$

Since |A|=0, the given matrix is singular and hence not invertible.

i.e. A^{-1} does not exists.

(*ii*) Let B = $\begin{pmatrix} 5 & -1 \\ 1 & 3 \end{pmatrix}$ $\therefore |B| = \begin{vmatrix} 5 & -1 \\ 1 & 3 \end{vmatrix} = 15 + 1 = 16 \neq 0$

B is non-singular, B^{-1} exists.

Now consider the partitioned matrix $\left(\begin{array}{cc|cc} 5 & -1 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right)$

$R_1 \leftrightarrow R_2$ gives, $\left(\begin{array}{cc|cc} 1 & 3 & 0 & 1 \\ 5 & -1 & 1 & 0 \end{array}\right)$ ..(1)

$R_2 - 5R_1$ gives $\left(\begin{array}{cc|cc} 1 & 3 & 0 & 1 \\ 0 & -16 & 1 & -5 \end{array}\right)$ ..(2)

$\left(\frac{-1}{16}\right)R_2$ gives $\left(\begin{array}{ccc|c} 1 & 3 & 0 & 1 \\ 0 & 1 & \frac{-1}{16} & \frac{5}{16} \end{array}\right)$ ..(3)

$R_1 - 3R_2$ gives $\left(\begin{array}{ccc|c} 1 & 0 & \frac{3}{16} & \frac{1}{16} \\ 0 & 1 & \frac{-1}{16} & \frac{5}{16} \end{array}\right)$ ..(4)

Thus the inverse of the given matrix is $B^{-1} = \begin{pmatrix} \frac{3}{16} & \frac{1}{16} \\ \frac{-1}{16} & \frac{5}{16} \end{pmatrix}$

or $B^{-1}=\frac{1}{16}\begin{pmatrix}3 & 1\\ -1 & 5\end{pmatrix}$.

Let us understand the solution stepwise:

- In the first step we checked the determinant is non-zero and hence proceeded ahead by creating a partitioned matrix.

- Then we transform left side of the partition to an identity matrix by making the first column as $\begin{pmatrix}1\\0\end{pmatrix}$. To make the first entry as 1 we use the first type of elementary operation of interchanging two rows (as the given matrix already had 1 in the second row), as shown in (1).

- Then we use R_2-5R_1 to make the second entry in the first column as 0.

- Now we make the second column as $\begin{pmatrix}0\\1\end{pmatrix}$. To make the second entry 1 we use second type of elementary operation $\left(\frac{-1}{16}\right)R_2$.

- Finally to make the first entry of second column as 0 we use the operation R_1-3R_2.

- Simultaneously same operations are performed on the right side of the partition. The right side of the partitioned matrix in the last step is our required inverse.

- When we are using row operations the left side matrix should be transformed to identity matrix column-wise as described above.

(*iii*) Let C

$$=\begin{pmatrix}-4 & -9 & 5\\ -4 & -7 & 4\\ 1 & 2 & -1\end{pmatrix}$$

$$|C|=\begin{vmatrix}-4 & -9 & 5\\ -4 & -7 & 4\\ 1 & 2 & -1\end{vmatrix}=-4(7-8)-(-9)(4-4)+5(-8+7)=4+0-5=-1$$

$\because |C|\neq 0$, C^{-1} exists.

Now consider the partitioned matrix $\left(\begin{array}{ccc|ccc}-4 & -9 & 5 & 1 & 0 & 0\\ -4 & -7 & 4 & 0 & 1 & 0\\ 1 & 2 & -1 & 0 & 0 & 1\end{array}\right)$

$R_1\leftrightarrow R_3$ gives $\left(\begin{array}{ccc|ccc}1 & 2 & -1 & 0 & 0 & 1\\ -4 & -7 & 4 & 0 & 1 & 0\\ -4 & -9 & 5 & 1 & 0 & 0\end{array}\right)$

R_2+4R_1 and R_3+4R_1 gives $\left(\begin{array}{ccc|ccc}1 & 2 & -1 & 0 & 0 & 1\\ 0 & 1 & 0 & 0 & 1 & 4\\ 0 & -1 & 1 & 1 & 0 & 4\end{array}\right)$

R_1-2R_2 and R_3+R_2 gives $\left(\begin{array}{ccc|ccc}1 & 0 & -1 & 0 & -2 & -7\\ 0 & 1 & 0 & 0 & 1 & 4\\ 0 & 0 & 1 & 1 & 1 & 8\end{array}\right)$

R_1+R_3 gives $\left(\begin{array}{ccc|ccc}1 & 0 & 0 & 1 & -1 & 1\\ 0 & 1 & 0 & 0 & 1 & 4\\ 0 & 0 & 1 & 1 & 1 & 8\end{array}\right)$

$\therefore C^{-1}=\begin{pmatrix}1 & -1 & 1\\ 0 & 1 & 4\\ 1 & 1 & 8\end{pmatrix}$.

(*iv*) Let D = $\begin{pmatrix}-7 & 1 & 4\\ -3 & 3 & 0\\ 8 & -2 & -2\end{pmatrix}$

$|D|=\begin{vmatrix}-7 & 1 & 4\\ -3 & 3 & 0\\ 8 & -2 & -2\end{vmatrix}=-7(-6)-1(6)+4(6-24)=42-6-72=-36\neq 0$

$\because |D|\neq 0 \Rightarrow D^{-1}$ exists.

Consider $\left(\begin{array}{ccc|ccc}-7 & 1 & 4 & 1 & 0 & 0\\ -3 & 3 & 0 & 0 & 1 & 0\\ 8 & -2 & -2 & 0 & 0 & 1\end{array}\right)$

R_1+R_3 gives $\left(\begin{array}{ccc|ccc}1 & -1 & 2 & 1 & 0 & 1\\ -3 & 3 & 0 & 0 & 1 & 0\\ 8 & -2 & -2 & 0 & 0 & 1\end{array}\right)$

R_2+3R_1 and R_3-8R_1 gives $\left(\begin{array}{ccc|ccc}1 & -1 & 2 & 1 & 0 & 1\\ 0 & 0 & 6 & 3 & 1 & 3\\ 0 & 6 & -18 & -8 & 0 & -7\end{array}\right)$

$\left(\frac{1}{6}\right)R_3$ and $\left(\frac{1}{6}\right)R_2$ gives $\left(\begin{array}{ccc|ccc}1 & -1 & 2 & 1 & 0 & 1\\ 0 & 0 & 1 & 1/2 & 1/6 & 1/2\\ 0 & 1 & -3 & -4/3 & 0 & -7/6\end{array}\right)$

$R_2\leftrightarrow R_3$ gives $\left(\begin{array}{ccc|ccc}1 & -1 & 2 & 1 & 0 & 1\\ 0 & 1 & -3 & -4/3 & 0 & -7/6\\ 0 & 0 & 1 & 1/2 & 1/6 & 1/2\end{array}\right)$

$$R_1 + R_2 \text{ gives} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & -1/3 & 0 & -1/6 \\ 0 & 1 & -3 & -4/3 & 0 & -7/6 \\ 0 & 0 & 1 & 1/2 & 1/6 & 1/2 \end{array}\right)$$

$$R_1 + R_3 \text{ and } R_2 + 3R_3 \text{ gives} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/6 & 1/6 & 1/3 \\ 0 & 1 & 0 & 1/6 & 1/2 & 1/3 \\ 0 & 0 & 1 & 1/2 & 1/6 & 1/2 \end{array}\right)$$

$$D^{-1} = \begin{pmatrix} 1/6 & 1/6 & 1/3 \\ 1/6 & 1/2 & 1/3 \\ 1/2 & 1/6 & 1/2 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 1 & 1 & 2 \\ 1 & 3 & 2 \\ 3 & 1 & 3 \end{pmatrix}$$

7.4 INVERSE OF A MATRIX USING ADJOINT METHOD:

The inverse of a matrix A is calculated by the formula:

$$A^{-1} = \frac{1}{|A|} \; adj\,(A)$$

5.6.1 Adjoint of a 2 x 2 matrix:

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then **Adjoint of A = adj(A) =** $\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$

And

$$A^{-1} = \frac{1}{|A|}\begin{pmatrix} d & -c \\ \end{pmatrix}$$

***Example* 18:** Find the inverse of $\begin{pmatrix} 2 & -3 \\ 7 & 1 \end{pmatrix}$ using adjoint method.

Ans: $|A| = \begin{vmatrix} 2 & -3 \\ 7 & 1 \end{vmatrix} = 2 + 21 = 23$

Now, adj(A) = $\begin{pmatrix} 1 & 3 \\ -7 & 2 \end{pmatrix}$

Hence $A^{-1} = \frac{1}{23}\begin{pmatrix} 1 & 3 \\ -7 & 2 \end{pmatrix} = \begin{pmatrix} 1/23 & 3/23 \\ -7/23 & 2/23 \end{pmatrix}$

5.6.2 Adjoint of a 3 x 3 matrix:

Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ then

Matrix of Co-factors: $C = \begin{pmatrix} A_{11} & -A_{12} & A_{13} \\ -A_{21} & A_{22} & -A_{23} \\ A_{31} & -A_{32} & A_{33} \end{pmatrix}$

Where each $\boldsymbol{A_{ij}}$: determinant of the minor matrix of A deleting the i^{th} row and j^{th} column.

Observe the signs of the cofactors in side the matrix. They are alternately +and −.

$\mathbf{Adj}(A) = C^T$

***Example* 19:** Find the inverse of $A = \begin{pmatrix} 1 & 2 & -1 \\ 4 & -1 & 8 \\ 6 & 3 & 5 \end{pmatrix}$ using adjoint method.

Ans: $|A| = \begin{vmatrix} 1 & 2 & -1 \\ 4 & -1 & 8 \\ 6 & 3 & 5 \end{vmatrix} = 1(-29) - 2(-28) - 1(18) = \mathbf{9}$

Now we find all the nine cofactors of the matrix A as follows:

A_{11} is obtained by deleting the 1st row and 1st column of A

$A_{11} = \begin{vmatrix} -1 & 8 \\ 3 & 5 \end{vmatrix} = -29$

A_{12} is obtained by deleting the 1st row and 2nd column of A

$A_{12} = \begin{vmatrix} 4 & 8 \\ 6 & 5 \end{vmatrix} = -28$

A_{13} is obtained by deleting the 1st row and 3rd column of A

$A_{13} = \begin{vmatrix} 4 & -1 \\ 6 & 3 \end{vmatrix} = 18$

A_{21} is obtained by deleting the 2nd row and 1st column of A

$A_{21} = \begin{vmatrix} 2 & -1 \\ 3 & 5 \end{vmatrix} = 13$

A_{22} is obtained by deleting the 2nd row and 2nd column of A

$A_{22} = \begin{vmatrix} 1 & -1 \\ 6 & 5 \end{vmatrix} = 11$

A_{23} is obtained by deleting the 2nd row and 3rd column of A

$$A_{23} = \begin{vmatrix} 1 & 2 \\ 6 & 3 \end{vmatrix} = -9$$

A_{31} is obtained by deleting the 3rd row and 1st column of A

$$A_{31} = \begin{vmatrix} 2 & -1 \\ -1 & 8 \end{vmatrix} = 15$$

A_{32} is obtained by deleting the 3rd row and 2nd column of A

$$A_{32} = \begin{vmatrix} 1 & -1 \\ 4 & 8 \end{vmatrix} = 12$$

A_{33} is obtained by deleting the 3rd row and 3rd column of A

$$A_{33} = \begin{vmatrix} 1 & 2 \\ 4 & -1 \end{vmatrix} = -9$$

Thus, the matrix of cofactors: $C = \begin{pmatrix} A_{11} & -A_{12} & A_{13} \\ -A_{21} & A_{22} & -A_{23} \\ A_{31} & -A_{32} & A_{33} \end{pmatrix} = \begin{bmatrix} -29 & 28 & 18 \\ -13 & 11 & 9 \\ 15 & -12 & -9 \end{bmatrix}$

$$\text{Adj}(A) = C^T = \begin{bmatrix} -29 & -13 & 15 \\ 28 & 11 & -12 \\ 18 & 9 & -9 \end{bmatrix}$$

Hence $A^{-1} = \frac{1}{|A|}\text{adj}(A) = \frac{1}{9}\begin{bmatrix} -29 & -13 & 15 \\ 28 & 11 & -12 \\ 18 & 9 & -9 \end{bmatrix}$

Inversion method:

Example: Solve the following equations using Matrix inversion method

$2x - 3y + 6 = 0, \quad 6x + y + 8 = 0$

Solution: Given equations can be written as

$2x - 3y = -6$

$6x + y = -8$

The corresponding matrix equation AX=B form is

$$\begin{bmatrix} 2 & -3 \\ 6 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -6 \\ -8 \end{bmatrix}$$

Where A= $\begin{bmatrix} 2 & -3 \\ 6 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $B = \begin{bmatrix} -6 \\ -8 \end{bmatrix}$

Let $AX = B$

Multiple by A^{-1} on both sides we get

$A^{-1}AX = A^{-1}B$

$IX = A^{-1}B \qquad \because AA^{-1} = I$

$X = A^{-1}B$----I

Now

A= $\begin{bmatrix} 2 & -3 \\ 6 & 1 \end{bmatrix} \Rightarrow |A| = 2 + 18 = 20$

Hence A^{-1} is exists.

Adjoint of $A = \begin{bmatrix} 1 & 3 \\ -6 & 2 \end{bmatrix}$

$A^{-1} = \frac{1}{|A|}\, Adjoint\ A$

$A^{-1} = \frac{1}{20}\begin{bmatrix} 1 & 3 \\ -6 & 2 \end{bmatrix}$

Using equation I we get

$X = A^{-1}B$

$X = \frac{1}{20}\begin{bmatrix} 1 & 3 \\ -6 & 2 \end{bmatrix}\begin{bmatrix} -6 \\ -8 \end{bmatrix} = \frac{1}{20}\begin{bmatrix} -30 \\ 20 \end{bmatrix} = \begin{bmatrix} -3/2 \\ 1 \end{bmatrix}$

Hence, $x = -3/2, \ \ y = 1.$

Example:Solve the following equations using Matrix inversion method

$x + y + z = 6, \qquad x - 2y + z = 0, \qquad 2x + y - z = 1$

Solution: Given equations can be written as

$x + y + z = 6, \qquad x - 2y + z = 0, \qquad 2x + y - z = 1$

The corresponding matrix equation AX=B form is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ 2 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 1 \end{bmatrix}$$

Where A= $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ 2 & 1 & -1 \end{bmatrix}, \ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \ B = \begin{bmatrix} 6 \\ 0 \\ 1 \end{bmatrix}$

Let $AX = B$

Multiple by A^{-1} on both sides we get

$A^{-1}AX = A^{-1}B$

$IX = A^{-1}B \qquad \because AA^{-1} = I$

$X = A^{-1}B$----I

Now

A= $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ 2 & 1 & -1 \end{bmatrix} \Rightarrow |A| = 1\begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} + 1\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix}$

$= 1(2-1) - 1(-1-2) + 1(1+4) = 1 + 3 + 5 = 9$

Hence A^{-1} is exists.

A= $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ 2 & 1 & -1 \end{bmatrix}$

First find cofactor of A:

$a_{ij} = (-1)^{i+j}M_{ij}$

$a_{11} = (-1)^{1+1}\begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix} = 1(2-1) = 1$

$a_{12} = (-1)^{1+2}\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = -1(-1-2) = 3$

$a_{13} = (-1)^{1+3}\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} = 1(1+4) = 5$

$a_{21} = (-1)^{2+1}\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -1(-1-1) = 2$

$a_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = 1(-1-2) = -3$

$a_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = -1(1-2) = 1$

$a_{31} = (-1)^{3+1}\begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} = 1(1+2) = 3$

$a_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = -1(1-1) = 0$

$a_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = 1(-2-1) = -3$

Cofactor of A = $\begin{bmatrix} 1 & 3 & 5 \\ 2 & -3 & 1 \\ 3 & 0 & -3 \end{bmatrix}$

Adjoint of $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -3 & 0 \\ 5 & 1 & -3 \end{bmatrix}$

$$A^{-1} = \frac{1}{|A|} \, Adjoint\ A$$

$$A^{-1} = \frac{1}{9}\begin{bmatrix} 1 & 2 & 3 \\ 3 & -3 & 0 \\ 5 & 1 & -3 \end{bmatrix}$$

Using equation I we get

$$X = A^{-1}B$$

$$X = \frac{1}{9}\begin{bmatrix} 1 & 2 & 3 \\ 3 & -3 & 0 \\ 5 & 1 & -3 \end{bmatrix}\begin{bmatrix} 6 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 9 \\ 18 \\ 27 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Hence, $x = 1$, $y = 2$, $z = 3$.

7.5 APPLICATION OF MATRIX INVERSION TO INPUT-OUTPUT ANALYSIS

Input-Output analysis is a technique which was put forward by Wassily W. Leontief. It is a mathematical method which uses matrix technology to study the interdependencies of different sectors on each other in a economy. The output of one industry may form the input of another industry and vice versa. The effect of changes in the input/output of various sectors is calculated using matrix inversion method.

Let us now first go through the terminology and concepts of input-output analysis which is done using the input-output table:

Input: The material demanded by a producer for his production is called as input. It is the expenditure occurred to the producer.

Output: The outcome of a production process is called as output. It is the income gained by the producer.

Total cost: The total money value of all the inputs is called as the total cost of the producer.

Total revenue: The total money value of all the outputs is called as the total revenue of the producer.

Input – Output table: This is a matrix consisting of various rows and columns representing the distribution of the outputs and inputs of different sectors in an economy.

Now, let us see an example of an input-output table consisting of three interdependent sectors.

Producer sector (output)	Consumer sector (input)			Final demand	Total output
	I	II	III		
I	a_{11}	a_{12}	a_{13}	d_1	x_1
II	a_{21}	a_{22}	a_{23}	d_2	x_2
III	a_{13}	a_{32}	a_{33}	d_3	x_3

Table 5.1

Recall that in the expression a_{ij}, the first suffix letter 'i' represents the row and the second letter 'j' represents the corresponding column.

a_{11} is the money value of the output sector I which is used by the input sector I. d_1 is the money value of the demand of the output and x_1 represents the money value of the total output of the sector I. Similarly, a_{23} is the money value of the output sector II which is used by the input sector III.d_2 is the money value of the demand of the output and x_2 represents the money value of the total output of the sector II.

Thus, from the table **5.**1 we can write the following equations;

$$\left.\begin{array}{l} x_1 = a_{11} + a_{12} + a_{13} + d_1, \quad x_2 = a_{21} + a_{22} + a_{23} + d_2 \text{ and} \\ x_3 = a_{31} + a_{32} + a_{33} + d_3. \end{array}\right\} \dots (1)$$

Technical Coefficients: We know that the output of one sector forms the input of another sector. The number of units of output of a sector required to produce one unit output of another sector are called as the technical coefficients. These coefficients are the ratio of the money value of the output of one sector required by another sector to the money value of the total output of the supplying sector.

From the table **5.**1 and eqn (1) we can write the technical coefficients as follows:

$$\left.\begin{array}{l} b_{11} = \dfrac{a_{11}}{x_1}, b_{12} = \dfrac{a_{12}}{x_2}, b_{13} = \dfrac{a_{13}}{x_3} \\ b_{21} = \dfrac{a_{21}}{x_1}, b_{22} = \dfrac{a_{22}}{x_2}, b_{23} = \dfrac{a_{23}}{x_3} \\ b_{31} = \dfrac{a_{31}}{x_1}, b_{12} = \dfrac{a_{32}}{x_2}, b_{13} = \dfrac{a_{33}}{x_3} \end{array}\right\} \dots (2)$$

Technology matrix: The matrix of all the technical coefficients is called as the technology matrix. For the above example, we have the technology matrix as: $B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$

From (1) and (2), we have the following equations:

$$\left.\begin{aligned} x_1 &= a_{11} + a_{12} + a_{13} + d_1 = b_{11}x_1 + b_{12}x_2 + b_{13}x_3 + d_1 \\ x_2 &= a_{21} + a_{122} + a_{23} + d_2 = b_{21}x_1 + b_{22}x_2 + b_{23}x_3 + d_2 \\ x_3 &= a_{31} + a_{32} + a_{33} + d_3 = b_{31}x_1 + b_{32}x_2 + b_{33}x_3 + d_3 \end{aligned}\right\} \quad \ldots (3)$$

If we denote $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, D = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$ then the above eqn (3) becomes:

$X = BX + D$

$\Rightarrow X - BX = D \quad \Rightarrow (I - B)X = D$

Thus, $\boldsymbol{X = (I - B)^{-1}D}$

The importance of the above matrix equation is that if the final demand change then the corresponding change in the total output can be computed easily as the matrix $(I - B)^{-1}$ is already known and is independent of the final demands hence is constant. The matrix $(I - B)$ is called the **Leontief's matrix**.

Before we proceed in solving problems related to input-output tables let us go through the assumptions of input-output analysis:

- No joint product by different sectors, i.e. every sector produces only one product.
- The ratio of input and output is fixed.
- Constant returns of scale.
- No external economies or diseconomies.
- Prices remain unchanged.

***Example* 20:** From the following input-output table find the technical coefficients, technology matrix and the total output of the two industries.

Industry	X	Y	Final demand
X	100	60	240
Y	150	80	70

Ans: The total output for sector X is 100 + 60 + 240 = 400
The total output for sector Y is 150 + 80 +70 = 300.
The total input for sector X is 100 + 150 = 250
The total input for sector Y is 60 + 80 = 140

The above data can be put in the given table as follows:

Producer sector (output)	Consumer sector (input)		Final demand	Total output
	X	Y		
X	100	60	240	400
Y	150	80	70	300
Total input	250	140	310	700

Now, the four technical coefficients are:

$b_{11} = \frac{a_{11}}{x_1} = \frac{100}{400} = 0.25,\qquad b_{12} = \frac{a_{12}}{x_2} = \frac{60}{400} = 0.15$

$b_{21} = \frac{a_{21}}{x_1} = \frac{150}{300} = 0.2,\qquad b_{22} = \frac{a_{22}}{x_2} = \frac{80}{300} = 0.27$

$\therefore$ the technology matrix is $B = \begin{pmatrix} 0.25 & 0.15 \\ 0.2 & 0.27 \end{pmatrix}$

***Example* 21:** From the following input-output table find the technical coefficients, technology matrix and the total output of the two industries. If the final demand changes to 200 and 100 for the sectors X and Y respectively, find the total output.

Industry	X	Y	Final demand
X	20	50	130
Y	30	70	50

Ans:
The total output for sector X is 20 + 50 + 130 = 200
The total output for sector Y is 30 + 70 +50 = 150.
The total input for sector X is 20 + 30 = 50
The total input for sector Y is 50 + 70 = 120

Producer sector (output)	Consumer sector (input)		Final demand	Total output
	X	Y		
X	20	50	130	200
Y	30	70	50	150
Total input	50	120	180	350

The above data can be put in the given table as follows:

Now, the four technical coefficients are:

$b_{11} = \frac{a_{11}}{x_1} = \frac{20}{200} = 0.1,$ $\qquad b_{12} = \frac{a_{12}}{x_2} = \frac{50}{200} = 0.25$

$b_{21} = \frac{a_{21}}{x_1} = \frac{30}{150} = 0.2,$ $\qquad b_{22} = \frac{a_{22}}{x_2} = \frac{70}{150} = 0.47$

$\therefore$ the technology matrix is $B = \begin{pmatrix} 0.1 & 0.25 \\ 0.2 & 0.47 \end{pmatrix}$

To find the total output (X) we will use the formula: $X = (I - B)^{-1}D$

Now, $B = \begin{pmatrix} 0.1 & 0.25 \\ 0.2 & 0.47 \end{pmatrix}$, $D = \begin{pmatrix} 200 \\ 100 \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\therefore (I - B) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.1 & 0.25 \\ 0.2 & 0.47 \end{pmatrix} = \begin{pmatrix} 0.9 & -0.25 \\ -0.2 & 0.53 \end{pmatrix}$$

Using the formula to find inverse of a 2 x 2 matrix by adjoint method from section **5.6.1**, we have

$$\text{Adj}(I - B) = \begin{pmatrix} 0.53 & 0.25 \\ 0.2 & 0.9 \end{pmatrix} \text{ and } |I - B| = \begin{vmatrix} 0.9 & -0.25 \\ -0.2 & 0.53 \end{vmatrix} = 0.427$$

$$\therefore (I - B)^{-1} = \frac{1}{0.427}\begin{pmatrix} 0.53 & 0.25 \\ 0.2 & 0.9 \end{pmatrix} = \begin{pmatrix} 1.24 & 0.59 \\ 0.47 & 2.11 \end{pmatrix}$$

$\therefore$ The total output is given by $X = (I - B)^{-1}D = \begin{pmatrix} 1.24 & 0.59 \\ 0.47 & 2.11 \end{pmatrix}\begin{pmatrix} 200 \\ 100 \end{pmatrix}$

$$\therefore X = \begin{pmatrix} 1.24 \text{ x } 200 + 0.59 \text{ x } 100 \\ 0.47 \text{ x } 200 + 2.11 \text{ x } 100 \end{pmatrix} = \begin{pmatrix} 307 \\ 305 \end{pmatrix}$$

$\therefore$ If the final demand changes to 200 and 100 the corresponding total output of the two sectors X and Y are:

Total output of sector X is $x_1 = 307$

The total output of sector Y is $x_2 = 305$

***Example* 22:** The following table gives the technology matrix of three sectors of an economy. If the final demands change to 100, 150 and 50 find the total output using Leontif's matrix.

Sector	*P*	*Q*	*R*
P	0.2	0.34	0.15
Q	0.18	0.67	0.5
R	0.45	0.28	0.75

Ans: Let the technology matrix be $B = \begin{pmatrix} 0.2 & 0.34 & 0.15 \\ 0.18 & 0.67 & 0.5 \\ 0.45 & 0.28 & 0.75 \end{pmatrix}$

$$(I-B) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.2 & 0.34 & 0.15 \\ 0.18 & 0.67 & 0.5 \\ 0.45 & 0.28 & 0.75 \end{pmatrix} = \begin{pmatrix} 0.8 & -0.34 & -0.15 \\ -0.18 & 0.33 & -0.5 \\ -0.45 & -0.28 & 0.25 \end{pmatrix}$$

$$|I-B| = \begin{vmatrix} 0.8 & -0.34 & -0.15 \\ -0.18 & 0.33 & -0.5 \\ -0.45 & -0.28 & 0.25 \end{vmatrix} = 0.8(-0.0575) + 0.34(-0.27) - 0.15(0.1989)$$

$$\therefore |I-B| = -0.11$$

The matrix of cofactors $= C = \begin{pmatrix} -0.0575 & -0.27 & 0.1989 \\ -0.043. & 0.1325 & -0.377 \\ 0.2195 & -0.427 & 0.2028 \end{pmatrix}$

$$\text{Adj}(I-B) = C^T = \begin{pmatrix} -0.0575 & -0.043 & 0.2195 \\ -0.27 & 0.1325 & -0.427 \\ 0.1989 & -0.377 & 0.2028 \end{pmatrix}$$

$$\therefore (I-B)^{-1} = \frac{1}{|I-B|}\,\text{adj}(I-B) = \frac{-1}{0.11}\begin{pmatrix} -0.0575 & -0.043 & 0.2195 \\ -0.27 & 0.1325 & -0.427 \\ 0.1989 & -0.377 & 0.2028 \end{pmatrix}$$

$$\therefore (I-B)^{-1} = \begin{pmatrix} 0.52 & 0.39 & -2 \\ 2.45 & -1.2 & 3.88 \\ -1.81 & 3.43 & -1.84 \end{pmatrix}$$

Now $D = \begin{bmatrix} 100 \\ 150 \\ 50 \end{bmatrix}$, thus, the change in total output is given by the formula

$$X = (I-B)^{-1}D = \begin{pmatrix} 0.52 & 0.39 & -2 \\ 2.45 & -1.2 & 3.88 \\ -1.81 & 3.43 & -1.84 \end{pmatrix}\begin{bmatrix} 100 \\ 150 \\ 50 \end{bmatrix} = \begin{bmatrix} 10.5 \\ 259 \\ 241.5 \end{bmatrix}$$

Thus, the changes in final demand to 100, 150 and 50 lead to the following total output of the three sectors:

For sector P: $x_1 = 10.5$, for sector Q: $x_2 = 259$ and for sector R: $x_3 = 241.5$

7.6 LET US SUM UP:

In this chapter we have learn:

- Definition inverse of matrix. And its properties.
- Different method of calculation of inverse of matrix.
- Different application of inverse of matrices.

7.7 UNIT END EXERCISE:

Q.1 Find the inverse of the following matrices using elementary row operations and also by adjoint method:

1. $\begin{pmatrix} 3 & 4 \\ 1 & -1 \end{pmatrix}$ **2.** $\begin{pmatrix} 6 & 2 \\ 5 & 3 \end{pmatrix}$ **3.** $\begin{pmatrix} 7 & -1 \\ 2 & 10 \end{pmatrix}$ **4.** $\begin{pmatrix} 3 & 6 \\ -1 & 2 \end{pmatrix}$

5. $\begin{pmatrix} 1 & 1 & 2 \\ 1 & 3 & 2 \\ 3 & 1 & 3 \end{pmatrix}$ **6.** $\begin{pmatrix} 1 & 1 & 1 \\ 3 & 2 & -4 \\ 2 & -3 & 5 \end{pmatrix}$ **7.** $\begin{pmatrix} 1 & 1 & 3 \\ 1 & 1 & 1 \\ 1 & 2 & -1 \end{pmatrix}$ **8.** $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$

9. $\begin{pmatrix} 1 & 2 & 4 \\ 4 & 3 & -2 \\ 1 & 0 & 3 \end{pmatrix}$ 10. $\begin{pmatrix} 3 & 2 & 6 \\ 1 & -3 & 1 \\ 2 & -1 & 0 \end{pmatrix}$

Q.2 Solve the following system of equations by inversion method.

i) $3x - 2y = 1, 5x + 3y = 3.$

ii) $3x - y = 8,\ x + y = 1.$

iii) $4x + 2y - z = 3, x - 2y + z = -8, 2x - y + z = -7.$

iv) $x + y + z = 3, 3x - 2y + 3z = 4, 5x + 4y + 3z = 12.$

v) $2x + y - z = -2,\ 3x - z = 5, 4y + 3z = 9$

Q.3From the following input-output table find the technical coefficients, technology matrix and the total output of the two industries, find the total output with respect change in demand.

i)

Industry	*X*	*Y*	Final demand	Demand change
X	40	50	120	250
Y	50	70	70	100

ii)

Industry	X	Y	Final demand	Demand change
X	20	40	80	120
Y	30	50	100	150

iii)

Industry	X	Y	Final demand	Demand change
X	40	60	150	300
Y	50	80	100	200

iv)

Industry	X	Y	Final demand	Demand change
X	30	70	180	280
Y	60	90	130	220

v)

Industry	X	Y	Final demand	Demand change
X	20	35	150	180
Y	25	60	130	160

7.8 LIST OF REFERENCES

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8

DERIVATIVES

Unit Structure

8.0 OBJECTIVE

After going through this chapter you will able to know:

- Definition of Derivatives.
- Standard formula of Derivatives.
- Rule of Derivatives.

8.1 INTRODUCTION

In business and economics many decisions are made on the basis of marginal analysis, like marginal cost, marginal revenue or marginal productivity etc.

The branch of mathematics which deals with such type of marginal analysis, or in other words, the process wherein rate of change in one variable with reference to a change in other variable is to be analyzed is called **Calculus**. The study of Calculus is divided onto two major aspects: *Differential Calculus* and *Integral Calculus*. In this chapter we will be studying the former i.e. differential calculus.

Before we begin with differentiation, let us first understand **limit of a function**.

Limit of Functions: Before introducing the concept of *derivative of a function* it is important to understand the concept of limit of a function. We would not define limit of a function formally, as it is not our area of interest. But an overview of this concept is given below.

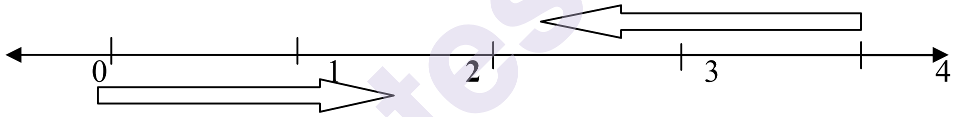
Let y be a function of x defined as: $y = \frac{x^2 - 4}{x - 2}$

If we want to know the value of y when $x = 2$, one would simply put $x = 2$ in above equation.

This would give us, $y = \frac{2^2 - 4}{2 - 2} = \frac{0}{0}$ which is a value not defined in mathematics!

To solve this problem we have to understand, what are the values of y when x approaches the value 2. The following tables give a clearer picture:

Table 1		**Table 2**	
X	***y***	***x***	***y***
0	2	4	6
1	3	3	5
1.5	3.5	2.5	4.5
1.8	3.8	2.1	4.1
1.9	3.9	2.01	4.01
1.99	**3.99**	**2.001**	**4.001**



The Table 1 gives the value of y as x approaches the value 2 from left side of the number line i..e. from 0 to 2, while the Table 2 gives the value of y as x approaches the value 2 from right side of the number line i.e. from 4 to 2. As we can see from both the tables that as $x$nears the value 2, the value of y becomes approximately 4. This can be said as follows: the limit of y as x tends to (approaches to) 2 is 4, which is represented symbolically as $\lim_{x \to 2} y = 4$.

In general if $y = f(x)$, then we say that the limit of y (if it exists) is l as x tends to a, if y approaches the value l as x approaches a, from both sides of the number line. Symbolically this is written as $\lim_{x \to a} y = \lim_{x \to a} f(x) = l$.

This is an informal definition of limit of a function. A more mathematical definition can be obtained by curious students from any book on Calculus.

8.2 DERIVATIVESAS A MEASURE OF RATE

Differentiation is the process of finding the derivative of a function. As mentioned in the introductory part above, economics and problems in business deal with marginal analysis of variables. So we are interested in, what and how much is the change in y with reference to a change in x, where $y = f(x)$ is a continuous function of x.

Let δx be the small change in x and δy be the corresponding change in y.

Then we have $y + \delta y = f(x + \delta x)$

$\therefore \delta y = f(x + \delta x) - y$

Dividing both sides by δx, we get

$$\frac{\delta y}{\delta x} = \frac{f(x+\delta x) - f(x)}{\delta x}$$

Taking limit as $\delta x \to 0$, on both sides, we get

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

The limit on the LHS if exists, is called the derivative of *y* **w**ith **r**espect **t**o (hereafter written as w.r.t.) *x* and is denoted as $\frac{dy}{dx}$.

Thus, $\frac{dy}{dx}$ is the rate of change of y w.r.t. the change in x and is called as the derivative of y w.r.t. x

Remark:

1. The expression $\frac{dy}{dx}$, **should not be misinterpreted** as two values *dy* divided by *dx*

2. $\frac{d}{dx}$ is the differential operator, which when operated on functions gives its derivative. For example, consider our demand function $D = f(P)$. Here P is the independent variable, so the derivative of *D* w.r.t. *P* is $\frac{d}{dP}(D)$ or in short $\frac{dD}{dP}$

3. One more important thing to remember is that the variables (both dependent and independent) may keep changing in the expression of the derivative from problem to problem, *i.e.* $\frac{dy}{dx}$ is not the only notation for derivative it can be $\frac{dP}{dD}$ or $\frac{dQ}{dP}$ or $\frac{dC}{dx}$ as the function may be.

4. If $\frac{dy}{dx}$ exists for a function $y = f(x)$ then we say that *y* is a differentiable function of *x*.

Let us now study derivatives of some standard functions.

1. Derivative of a constant function

Let $y = k$ be a constant function, where k is any real value then $\frac{dy}{dx} = \frac{d}{dx}(k) = 0$

Thus, Derivative of a constant function is zero

All real values like 11, -304, 5/17, -4.234, log3, e, $\sqrt{2}$ etc are examples of constant functions.

Solved examples:

Find the derivatives of the following functions:

(i) 12, (ii) -10, (iii) 1/13, (iv) -7/16, (v) 0, (vi) log 4,

(vii) $\sqrt{2}$

Ans : Since all the functions above are constant functions, their derivative is zero using the above formula .

$$i.e.\ \frac{d}{dx}(12) = \frac{d}{dx}(-10) = \frac{d}{dx}(1/13) = \frac{d}{dx}(-7/16) = \frac{d}{dx}(0) = \frac{d}{dx}(\log 4) = \frac{d}{dx}(\sqrt{2}) = 0$$

2. Derivative of a power function

If $y = x^n$ is a power function of x, where n is any real number then, $\frac{dy}{dx} = n.x^{n-1}$

Solved examples :

Find the derivatives of the following functions:

(i) x^{12}, (ii) x^{-3}, (iii) $x^{2/3}$, (iv) $x^{-6/7}$, (v) $x^{1/2}$, (vi) $\sqrt{x}$, (vii) $\frac{1}{\sqrt{x}}$

(i) $\frac{d}{dx}\left(x^{12}\right) = 12.x^{12-1} = 12x^{11}$

(ii) $\frac{d}{dx}\left(x^{-3}\right) = (-3).x^{-3-1} = -3x^{-4}$

(iii) $\frac{d}{dx}\left(x^{\frac{2}{3}}\right) = \frac{2}{3}.x^{\frac{2}{3}-1} = \frac{2}{3}.x^{\frac{2-3}{3}} = \frac{2}{3}.x^{\frac{-1}{3}}$

(iv) $\frac{d}{dx}\left(x^{\frac{-6}{7}}\right) = \frac{-6}{7}.x^{\frac{-6}{7}-1} = \frac{-6}{7}.x^{\frac{-6-7}{7}} = \frac{-6}{7}.x^{\frac{-13}{7}}$

(v) $\frac{d}{dx}\left(x^{\frac{1}{2}}\right) = \frac{1}{2}.x^{\frac{1}{2}-1} = \frac{1}{2}.x^{\frac{1-2}{2}} = \frac{1}{2}.x^{\frac{-1}{2}} = \frac{1}{2x^{1/2}} = \frac{1}{2\sqrt{x}}$

$\because x^{1/2} = \sqrt{x}$, from above problem, $\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$

(*vi*) $\frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right) = \frac{d}{dx}\left(\frac{1}{x^{1/2}}\right) = \frac{d}{dx}(x^{-1/2})$

$$= \left(\frac{-1}{2}\right)x^{\left(\frac{-1}{2}-1\right)} = \left(\frac{-1}{2}\right)x^{\frac{-1-2}{2}} = \left(\frac{-1}{2}\right)x^{\frac{-3}{2}}$$

$$\therefore \frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right) = \frac{-x^{-3/2}}{2}$$

3. Derivative of an exponential function

If $\boldsymbol{y = a^x}$ is an exponential function of x, where $a > 0$, then $\boldsymbol{\frac{dy}{dx} = a^x \log a}$

Solved examples :

Find the derivatives of the following functions: 2^x, 9^x

(*i*) Let $y = 2^x$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(2^x) = 2^x \log 2$$

(*ii*) Let $y = 9^x$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}(9^x) = 9^x \log 9$$

4. If $\boldsymbol{y = e^x}$, then by the above formula:

$\frac{dy}{dx} = \frac{d}{dx}(e^x) = e^x \log e = e^x$ (since $\log e = 1$)

Thus, we have $\boldsymbol{\frac{d}{dx}(e^x) = e^x}$

5. Derivative of a logarithmic function

If $y = \log x$, (where $x \neq 0$) then $\frac{dy}{dx} = \frac{d}{dx}(\log x) = \frac{1}{x}$

Hence, $\boldsymbol{\frac{d}{dx}(\log x) = \frac{1}{x}}$

Let us sum up all the formulae of differentiation in tabular form:

Function : y	**Derivative :** $\frac{dy}{dx}$
1. k, constant function	$\frac{d}{dx}(k)=0$
2. x^n	$\frac{d}{dx}(x^n)=n.x^{n-1}$
3. a^x, (where $a>0$)	$\frac{d}{dx}(a^x)=a^x \log a$
4. e^x	$\frac{d}{dx}(e^x)=e^x$
5. $\log x$ (where $x \neq 0$)	$\frac{d}{dx}(\log x)=\frac{1}{x}$

Here are some problems for you to try now !

8.3 RULES OF DERIVATIVES

Now when we are through with the basic formulas of differentiation, one would be interested in knowing the derivatives of $5x^4$, $7\log x$, $\frac{\log x}{\log 6}$, $2e^x+3x^e-4\log x$ or $2^x.3^x$ etc. For that we require certain rules of differentiation. In this section we will study these *rule of differentiation*.

Rule I **:** Derivative of scalar multiple of a funcDion :

I $f(x)$ is a function of x, then

$$\frac{d}{dx}\left[kf(x)\right]=k\frac{d}{dx}[f(x)],$$ where k is a scalar (*i.e.* constant)

➢ **Solved problems based on Rule I :**

Differentiate the following functions W.r.t. x :

(i) $4x^3$, (ii) $\frac{x^{-1/3}}{3}$, (iii) $2\sqrt{x}$, (iv) 3.4^x, (v) $\frac{3^x}{\log 3}$, (fi) $\frac{2\log x}{3}$

Ans:

(i) Let $y=4x^3$

Differentiating y w.r.t.x, we have,

$$\frac{d}{dx}(y)=\frac{d}{dx}(4x^3)=4\frac{d}{dx}(x^3)=4(3.x^{3-1})=12x^2$$

$\therefore \frac{dy}{dx} = 12x^2$.

(ii) Let $y = \frac{x^{-1/3}}{3}$

Differentiating y w.r.t.x, we have,

$$\frac{d}{dx}(y) = \frac{d}{dx}\left(\frac{x^{-1/3}}{3}\right) = \frac{1}{3}\frac{d}{dx}(x^{-1/3}) = \frac{1}{3}.\left(-\frac{1}{3}\right).x^{\frac{-1}{3}-1} \quad \text{(Here the scalar is } \frac{1}{3}\text{)}$$

$$\therefore \frac{dy}{dx} = -\frac{1}{9}x^{-4/3}$$

(iii)Let $y = 2\sqrt{x}$.

Differentiating y w.r.t.119x, we Have,

$$\frac{d}{dx}(y) = \frac{d}{dx}\left(2\sqrt{x}\right) = 2\frac{d}{dx}(\sqrt{x}) = 2.\left(\frac{1}{2\sqrt{x}}\right)$$

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{x}}.$$

$(iv$ Let $y = 3.4^x$

Differentiating y w.r.t.x, we have,

$$\frac{d}{dx}(y) = \frac{d}{dx}\left(3.4^x\right) = 3\frac{d}{dx}(4^x)$$

$$\therefore \frac{dy}{dx} = 3(4^x \log 4)$$

(v) Let $y = \frac{3^x}{\log 3}$

Differentiating y w.r.t.x, we have,

$$\frac{d}{dx}(y) = \frac{d}{dx}\left(\frac{3^x}{\log 3}\right) = \frac{1}{\log 3}\frac{d}{dx}(3^x) \qquad \text{(Here the scalar is } \frac{1}{\log 3}\text{)}$$

$$\therefore \frac{dy}{dx} = \frac{3^x \log 3}{\log 3} = 3^x$$

(vi) Let $y = \frac{2\log x}{3}$

Differentiating y w.r.t.x, we have,

$$\frac{d}{dx}(y)=\frac{d}{dx}\left(\frac{2\log x}{3}\right)=\frac{2}{3}\frac{d}{dx}(\log x) \qquad \text{(Here the scalar is } \frac{2}{3}\text{)}$$

$$\therefore \frac{dy}{dx}=\frac{2}{3}.\frac{1}{x}=\frac{2}{3x}$$

Rule II : Derivative of sum or difference of two (or more) functions

If $f(x)$ and $g(x)$ are functions of x, then the derivative of $f(x)+g(x)$ is given by: $\boldsymbol{\frac{d}{dx}[f(x)+g(x)]=\frac{d}{dx}[f(x)]+\frac{d}{dx}[g(x)]}$ **(Sum Rule)**

And the derivative of $f(x)-g(x)$ is given by

$$\boldsymbol{\frac{d}{dx}[f(x)-g(x)]=\frac{d}{dx}[f(x)]-\frac{d}{dx}[g(x)]} \qquad \textbf{(Difference Rule)}$$

This rule can also be extended to more than two functions as given below:

$$\frac{d}{dx}[f(x)\pm g(x)\pm h(x)]=\frac{d}{dx}[f(x)]\pm\frac{d}{dx}[g(x)]\pm\frac{d}{dx}[h(x)]$$

- **Solved problems based on Rule II :**

Differentiate the following functions w.r.t x :

(i) $e^x+\log 5$, (ii) $2^x-\log x$, (iii) x^3+3^x,(iv) $\log x-7^x$

Ans:

(i) Let $y=e^x+\log 5$

Differentiating y w.r.t.x, we have,

$$\frac{d}{dx}(y)=\frac{d}{dx}(e^x+\log 5)=\frac{d}{dx}(e^x)+\frac{d}{dx}(\log 5)=e^x+0$$

$$\therefore \frac{d}{dx}(e^x+\log 5)=e^x.$$

(ii) Let $y=2^x-\log x$

Differentiating y w.r.t.x, we have,

$$\frac{d}{dx}(y)=\frac{d}{dx}(2^x-\log x)=\frac{d}{dx}(2^x)-\frac{d}{dx}(\log x)$$

$$\therefore \frac{d}{dx}(2^x-\log x)=2^x\log 2-\frac{1}{x}.$$

(iii) Let $y = x^3 + 3^x$

Differentiating y w.r.t.x, we have,

$$\frac{d}{dx}(y) = \frac{d}{dx}(x^3 + 3^x) = \frac{d}{dx}(x^3) + \frac{d}{dx}(3^x) = 3x^{3-1} + 3^x \log 3$$

$$\therefore \frac{d}{dx}(x^3 + 3^x) = 3x^2 + 3^x \log 3 .$$

(iv) Let $y = \log x - 7^x$

Differentiating y w.r.t.x, we have,

$$\frac{d}{dx}(y) = \frac{d}{dx}(\log x - 7^x) = \frac{d}{dx}(\log x) - \frac{d}{dx}(7^x)$$

$$\therefore \frac{d}{dx}(x^3 + 3^x) = \frac{1}{x} - 7^x \log 7 .$$

Now get ready for some problems for practice!

Rule III : Derivative of product of two functions:

If $f(x)$ and $g(x)$ are functions of x, then the derivative of product of $f(x)$ and $g(x)$ *i.e.* $f(x).g(x)$ is given by:

$$\frac{d}{dx}\left[f(x).g(x)\right] = g(x)\frac{d}{dx}[f(x)] + f(x)\frac{d}{dx}[g(x)] \qquad \textbf{(Product Rule)}$$

(*This can be remembered as* 2^{nd} *into derivative of* 1^{st} + 1^{st} *into derivative of* 2^{nd}!)

➢ **Solved problems based on Rule III :**

Differentiate the following functions w.r.t x :

(i) $2^x.3^x$, (ii) $x^2 \log x$,

Ans:

(i) Let $y = 2^x.3^x$

Differentiating y w.r.t.x, we have,

$$\frac{d}{dx}(y) = \frac{d}{dx}(2^x.3^x) = 3^x\frac{d}{dx}(2^x) + 2^x\frac{d}{dx}(3^x) = 3^x(2^x \log 2) + 2^x(3^x \log 3)$$

$$\therefore \frac{d}{dx}(2^x.3^x) = 2^x.3^x(\log 2 + \log 3) .$$

(ii) Let $y = x^2 \log x$

Differentiating y w.r.t.x, we have,

$$\frac{d}{dx}(y) = \frac{d}{dx}(x^2 \log x) = \log x \frac{d}{dx}(x^2) + x^2 \frac{d}{dx}(\log x) = \log x (2x^{2-1}) + x^2 \left(\frac{1}{x}\right)$$

$$\therefore \frac{d}{dx}(x^2 \log x) = 2x \log x + x$$

Rule IV **:** Derivative of quotient of two functions**:**

If $f(x)$ and $g(x)$ are functions of x, then the derivative of quotient of $f(x)$ and $g(x)$, *i.e.* $\frac{f(x)}{g(x)}$ is given by

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2} \qquad \textbf{(Quotient Rule)}$$

- **Solved problems based on Rule IV :**

Differentiate the following functions w.r.t x :

(i) $\frac{4^x}{3^x}$ $\qquad$ (ii) $\frac{x^4}{\log x}$

Ans:

(i) Let $y = \frac{4^x}{3^x}$

Differentiating y w.r.t.x, we have,

$$\frac{d}{dx}(y) = \frac{d}{dx}\left(\frac{4^x}{3^x}\right) = \frac{3^x \frac{d}{dx}(4^x) - 4^x \frac{d}{dx}(3^x)}{(3^x)^2} = \frac{3^x 4^x \log 4 - 4^x 3^x \log 3}{(3^x)^2}$$

$$\therefore \frac{dy}{dx} = \frac{3^x 4^x (\log 4 - \log 3)}{(3^x)^2} = \frac{4^x (\log 4 - \log 3)}{3^x}$$

(ii) Let $y = \frac{x^4}{\log x}$

Differentiating y w.r.t.x, we have,

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^4}{\log x}\right) = \frac{\log x \frac{d}{dx}(x^4) - x^4 \frac{d}{dx}(\log x)}{(\log x)^2} = \frac{(\log x) 4x^3 - x^4 (1/x)}{(\log x)^2}$$

$$\therefore \frac{dy}{dx} = \frac{x^3 (4 \log x - 1)}{(\log x)^2} = \frac{4^x (\log 4 - \log 3)}{3^x}$$

The actual problems of differentiation are not segregated like those solved using rule I, or rule II or so, but would require more than one rule to be used. In such cases the method to differentiate the given functions is to use the above rules one by one and simplify the problem to basic functions whose derivatives we know. In this section we shall learn how to solve problems using all the rules and then we shall have some problems for practice.

***Solved problems using all the four rules and formulae* :**

Differentiate the following functions w.r.t. x.

$(i)\ 3x^2 - 6\log x \qquad (ii)\ \frac{3^x.e^x}{\log 3} \quad (iii)\ \frac{4x^{1/4}}{\log x} \quad (iv)\ 2xe^x + \frac{x^e}{e}$

$(v)\ \frac{3x^{1/3} - 4.2^x}{2\sqrt{x} + 5\log x}$

Ans:

(i) Let $y = 3x^2 - 6\log x$

Differentiating w.r.t.x, we have,

$$\frac{d}{dx}(y) = \frac{d}{dx}(3x^2 - 6\log x) = \frac{d}{dx}(3x^2) - \frac{d}{dx}(6\log x) \quad (\textit{usingRule II})$$

$$\therefore \frac{dy}{dx} = 3\frac{d}{dx}(x^2) - 6\frac{d}{dx}(\log x) \quad (\textit{using Rule I})$$

$$\therefore \frac{dy}{dx} = 3(2x^{2-1}) - 6\left(\frac{1}{x}\right) = 6x - \frac{6}{x} = 6(x - 1/x). \quad (\textit{using formulae})$$

(ii) Let $y = \frac{3^x.e^x}{\log 3}$

Differentiating w.r.t.x, we have,

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{3^x.e^x}{\log 3}\right) = \frac{1}{\log 3}\frac{d}{dx}(3^x.e^x) \quad (\textit{using Rule I})$$

$$\therefore \frac{dy}{dx} = \frac{1}{\log 3}\left[e^x\frac{d}{dx}(3^x) + 3^x\frac{d}{dx}(e^x)\right] \quad (\textit{using Rule III})$$

$$\therefore \frac{dy}{dx} = \frac{1}{\log 3}\left[e^x(3^x\log 3) + 3^x(e^x)\right] \quad (\textit{using formulae})$$

$$\therefore \frac{dy}{dx} = \frac{3^x e^x(\log 3 + 1)}{\log 3}$$

(iii)Let $y = \frac{4x^{1/4}}{\log x}$

Differentiating w.r.t.x, we have,

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{4x^{1/4}}{\log x}\right)=4\frac{d}{dx}\left(\frac{x^{1/4}}{\log x}\right) \qquad (\textit{using Rule I})$$

$$\therefore \frac{dy}{dx}=4\left[\frac{(\log x)\frac{d}{dx}(x^{1/4})-x^{1/4}\frac{d}{dx}(\log x)}{(\log x)^2}\right] \qquad (\textit{using Rule IV})$$

$$\therefore \frac{dy}{dx}=\frac{4}{(\log x)^2}\left[(\log x)\frac{1}{4}(x^{1/4-1})-x^{1/4}.\frac{1}{x}\right] \qquad (\textit{using formulae})$$

$$\frac{dy}{dx}=\frac{4}{(\log x)^2}\left[\frac{(x^{-3/4})(\log x)}{4}-x^{1/4-1}\right]=\frac{4}{(\log x)^2}\left[\frac{(x^{-3/4})(\log x)}{4}-x^{-3/4}\right]$$

$$\therefore \frac{dy}{dx}=\frac{x^{-3/4}}{\log x}\left[1-\frac{4}{\log x}\right].$$

(iv) Let $y=2xe^x+\frac{x^e}{e}$

Differentiating w.r.t.x, we have,

$$\frac{dy}{dx}=\frac{d}{dx}\left(2xe^x+\frac{x^e}{e}\right)=\frac{d}{dx}(2xe^x)+\frac{d}{dx}\left(\frac{x^e}{e}\right) \qquad (\textit{using Rule II})$$

$$\therefore \frac{dy}{dx}=2\frac{d}{dx}(xe^x)+\frac{1}{e}.\frac{d}{dx}(x^e) \qquad (\textit{using Rule I})$$

$$\therefore \frac{dy}{dx}=2\left[e^x.\frac{d}{dx}(x)+x\frac{d}{dx}(e^x)\right]+\frac{1}{e}.(ex^{e-1}) \qquad (\textit{using Rule III})$$

$$\therefore \frac{dy}{dx}=2\left[e^x.(1x^{1-1})+x(e^x)\right]+x^{e-1}=2(e^x+xe^x)+x^{e-1}$$

$(\textit{using formulae}$ and $\because x^0=1)$

(v) Let $y=\frac{3x^{1/3}-4.2^x}{2\sqrt{x}+6\log x}$

$$\frac{dy}{dx}=\frac{d}{dx}\left[\frac{3x^{1/3}-4.2^x}{2\sqrt{x}+6\log x}\right]=\frac{(2\sqrt{x}+6\log x)\frac{d}{dx}(3x^{1/3}-4.2^x)-(3x^{1/3}-4.2^x)\frac{d}{dx}(2\sqrt{x}+6\log x)}{(2\sqrt{x}+6\log x)^2}$$

$(\textit{using Rule IV})$

$$\therefore \frac{dy}{dx} = \frac{(2\sqrt{x}+6\log x)\left[\frac{d}{dx}(3x^{1/3}) - \frac{d}{dx}(4.2^x)\right] - (3x^{1/3} - 4.2^x)\left[\frac{d}{dx}(2\sqrt{x}) + \frac{d}{dx}(6\log x)\right]}{(2\sqrt{x}+6\log x)^2}$$

(*using Rule II*)

$$\therefore \frac{dy}{dx} = \frac{(2\sqrt{x}+6\log x)\left[3.\frac{1}{3}(x^{1/3-1}) - 4(2^x \log 2)\right] - (3x^{1/3} - 4.2^x)\left[\frac{2}{2\sqrt{x}} + \frac{6}{x}\right]}{(2\sqrt{x}+6\log x)^2}$$

(*using Rule I* and *formulae*)

$$\therefore \frac{dy}{dx} = \frac{(2\sqrt{x}+6\log x)\left[(x^{-2/3}) - 4(2^x \log 2)\right] - (3x^{1/3} - 4.2^x)\left[\frac{1}{\sqrt{x}} + \frac{6}{x}\right]}{(2\sqrt{x}+6\log x)^2}.$$

8.4 SECOND ORDER DERIVATIVES

Let $y = f(x)$ be a differentiable function of x. $\frac{dy}{dx}$ is called the *first derivative* of $f(x)$. $\frac{dy}{dx}$ is itself a function of x and can be differentiated again w.r.t. x.

If the derivative of $\frac{dy}{dx}$ exists it is called the *second derivative* of $f(x)$ and is denoted as $\frac{d^2y}{dx^2}$. In this way we can proceed to get further derivatives of higher order.

Thus, for a function $y = f(x)$

$\frac{dy}{dx}$ is called the first derivative of $f(x)$

$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$ is called the second derivative of f(x)

Before we solve problems related to second derivative of functions, readers should get themselves thorough with all the formulas and rules of differentiation.

Solved Examples

1. If $y = 3x^2 - 6x + 9$, find $\frac{d^2y}{dx^2}$.

Ans: Differentiating y w.r.t. x, we have: $\frac{dy}{dx} = 6x - 6$

Differentiating $\frac{dy}{dx}$ w.r.t. x, we get, $\frac{d^2y}{dx^2} = 6.$

2. If $y = \frac{x^4}{4} - 6x^{7/6} + 12\sqrt{x} - 10$, find $\frac{d^2y}{dx^2}$

Ans: Differentiating y w.r.t. x, we have: $\frac{dy}{dx} = x^3 - 7x^{1/6} + \frac{6}{\sqrt{x}}$

Differentiating $\frac{dy}{dx}$ w.r.t. x, we get, $\frac{d^2y}{dx^2} = 3x^2 - \frac{7}{6}x^{-5/6} - 3x^{-3/2}$.

3. If $y = x^2e^x$, prove that $\frac{d^2y}{dx^2} = 2e^x(1 + 2x + x^2)$

Ans: Differentiating y w.r.t. x, we have:

$$\frac{dy}{dx} = e^x(2x) + x^2e^x = 2xe^x + x^2e^x$$

Differentiating $\frac{dy}{dx}$ w.r.t. x, we get

$$\therefore \frac{d^2y}{dx^2} = 2(e^x.1 + xe^x) + 2xe^x + x^2e^x$$

$$= 2e^x + 2xe^x + 2xe^x + x^2e^x$$

$$\therefore \frac{d^2y}{dx^2} = 2e^x(1 + 2x + x^2)$$

4. If $y = \frac{x\log x}{2^x}$, find $\frac{d^2y}{dx^2}$

Ans: Differentiating y w.r.t. x, we have

$$\frac{dy}{dx} = \frac{2^x\left[\log x.1 + x.\frac{1}{x}\right] - x\log x(2^x\log 2)}{(2^x)^2}$$

$$\therefore \frac{dy}{dx} = \frac{2^x\left[(\log x + 1) - (x\log x)(\log 2)\right]}{(2^x)^2} = \frac{(\log x + 1) - (x\log x)(\log 2)}{2^x}$$

Differentiating $\frac{dy}{dx}$ w.r.t. x, we get

$$\therefore \frac{d^2y}{dx^2} = \frac{2^x\left[\frac{1}{x} + 0 - (\log 2)(\log x.1 + x.\frac{1}{x})\right] - [(\log x + 1) - (x\log x)(\log 2)]2^x\log 2}{(2^x)^2}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{\frac{1}{x} - (\log 2)(\log x + 1) - [(\log x + 1) - (x \log x)(\log 2)] \log 2}{2^x}$$

8.5 LET US SUM UP

In this chapter we have learn:

- Formulas of differentiation
- Rule of differentiation and its basic problem

8.6 UNIT END EXERCISE

Q.1 Find the derivatives of the following functions :

(*i*) -100, (*ii*) 21/12, (*iii*) 3^2, (*iv*) log 16, (*v*) x^4 (*vi*) x^{-10},

(*vii*) $x^{\frac{4}{5}}$, (*viii*) $x^{\frac{-1}{6}}$, (*ix*) $\sqrt{x}$, (*x*) 5^x

Q.2 Differentiate the following w.r.t. x.

(*i*) $\frac{x^6}{6}$, (*ii*) $\frac{x^{-4}}{2}$, (*iii*) $5x^{2/5}$, (*iv*) $\frac{7}{x^{-1/14}}$, (*v*) $4\sqrt{x}$

(*vi*) 14.8^x, (*vii*) $\frac{9^x}{\log 9}$, (*viii*) $5^x.\log 5$, (*ix*) $2e^x$, (*x*) $\frac{5\log x}{4}$

Q.3 Differentiate the following functions w.r.t x :

(*i*) $e^x + \log x$, (*ii*) $e^x + x^9$, (*iii*) $e^x + 2^x$, (*iv*) $x^{1/3} - x^{1/2}$, (*v*) $x^6 + 6^x$
(*vi*) $x^{1/3} - x^{1/2} + x^{1/6}$, (*vii*) $x^6 - \log x$,(*viii*) $7^x - x^{4/5} + \log 2$,(*ix*) $10^x - x^{10}$

(*x*) $x^2 + 2^x - 2^2$.

Q.4 Differentiate the following functions w.r.t x :

(*i*) $\sqrt{x}.e^x$ (*ii*) $x^5 \log x$ (*iii*) $x^5 5^x$ (*iv*) $x^{3/2} 2^x$ (*v*) $e^x \log x$

(*vi*) $e^x x^e$ (*vii*) $a^x x^e$ (*viii*) $3^x 9^x$ (*ix*) $\frac{3^x}{\sqrt{x}}$ (*x*) $\frac{4^x}{\sqrt{x}}$

Q.5 Differentiate the following functions w.r.t x :

(*i*) $\frac{\sqrt{x}}{\log x}$ (*ii*) $\frac{x^6}{6^x}$ (*iii*) $\frac{3^x}{6^x}$ (*iv*) $\frac{e^x}{2^x}$ (*v*) $\frac{\log x}{7^x}$

Q.6 Differentiate the following w.r.t. x :

(*i*) $(4x-5)^2$ (*ii*) $x^4 - 3x^3 + 10x - 1000$ (*iii*) $5^x.6^x + 2e^x - 6\sqrt{x}$ (*iv*) $\frac{1}{x}$

$(v)\ \frac{9e^x + \frac{5}{\sqrt{x}}}{e^x}$ $(vi)\ \frac{(x+2)^2}{5x^5 5^x}$ $(vii)\ \frac{(x-6)\log x}{7+8x^{1/8}}$ $(viii)\ \sqrt{x^3}$ $(ix)\ \frac{10}{x\log x}$

$(x)\ x\log x - \frac{x^2}{2}$ $(xi)\ \frac{10x-4x^{5/4}}{e^x}$ $(xii)\ \frac{x^3-12x+9}{\sqrt{x}}$ $(xiii)\ \frac{7x-3}{x^2-4}$

$(xiv)\ \left(x+\frac{2}{x}\right)^2$ $(xv)\ 14x^{9/7} - \frac{14^x}{\log 216} + e^x\log x + \frac{8x}{11-2e^x}$ $(xvi)\ (4x-7)e^x$ $(xvii)$ $\frac{3x^{4/3}-4x^{3/4}+2\log x}{100-16x^{9/8}}$ $(xviii)\ x^3\log x + \frac{2^x}{\log 2}$ $(xix)\ \frac{x+1}{x-2}$

$(xx)\ (2x+1)(x-2)$

Q.7 Find the second derivative for the following functions:

$(i)\ (4x^2+3)(3x^3-4)$ $(ii)\ 7x^4-4x^3-8x^2+11$ $(iii)\ x^3-3x^2+9x$

$(iv)\ 3^x e^x$ $(v)\ (3-x^2)(x^2+3)$ $(vi)\ x^2\log x$ $(vii)\ e^x\log x$ $(viii)\ \frac{9^x\log 3}{3^x\log 9}$

$(ix)\ \frac{3x^3-9)}{3\log x}$ $(x)\ x+\log x$

8.7 LIST OF REFERENCES

- Business Mathematics by QaziZameeruddin (Author), Vijay K. Khanna (Author), S.K. Bhambri (Author)
- A Textbook Of Business Mathematics by HazarikaPadmalochan
- Business Mathematics by S.R. Arora, Taxmann

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9

APPLICATION OF DERIVATIVES

Unit Structure :

9.0 OBJECTIVES

After going through this chapter you will able to know:

- Application of economic function and its used.
- Elasticity of demand and supply.
- Maxima and minima of single variable or increasing and decreasing function.
- Extreme value of function using second order derivative test.

9.1 INTRODUCTION

We know that the process of differentiation is very useful in solving problems related to marginal analysis in business and economics. One more important area of application of derivatives in economics is related to optimization problems like, profit maximization or cost minimization. The significant points where the profit for a certain product is maximum or the cost per unit production is minimum can be found using differentiation. In this chapter we are going to learn these different applications of derivatives to Economics and Business.

9.2 APPLICATION OF ECONOMIC FUNCTIONS

Marginal Cost :

If the total cost function for producing x units of a product is given by $C(x)$ then the **marginal cost** is the rate of change of total cost w.r.t x. In other words, marginal cost is approximate cost of production per extra unit of the product.

Marginal Cost is denoted by MC and is calculated by the formula:

$$MC = \frac{d}{dx}[C(x)]$$

Marginal Average Cost

We have already defined average cost (AC) in the previous chapter. The **marginal average cost** is the rate of change of average cost (AC).

It is denoted by MAC and is given by: $MAC = \frac{d}{dx}(AC)$

Marginal revenue

If the total revenue function is given by $R(x)$, then the marginal revenue is the rate of change of total revenue w.r.t. the quantity demanded.

Marginal revenue is denoted by MR and is calculated by the formula:

$$MR = \frac{d}{dx}[R(x)]$$

Relation between marginal cost (MC) and average cost (AC)

We know that $AC = \frac{C}{x}$, differentiating w.r.t. x, we get

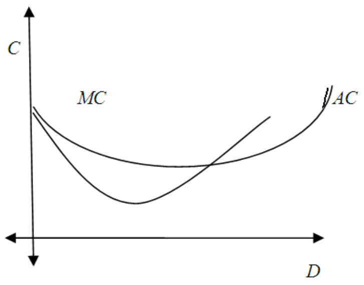
$$\frac{d}{dx}(AC) = \frac{d}{dx}\left(\frac{C}{x}\right) = \frac{x\frac{d}{dx}(C) - C}{x^2}$$

$$\therefore MAC = \frac{x}{x^2}.\frac{d}{dx}(C) - \frac{C}{x^2} = \frac{1}{x}\left(\frac{dC}{dx} - \frac{C}{x}\right) \qquad \left(\because MAC = \frac{d}{dx}(AC)\right)$$

$$\therefore MAC = \frac{1}{x}\left(\frac{dC}{dx} - \frac{C}{x}\right) \qquad \Rightarrow MAC = \frac{1}{x}(MC - AC)$$

Thus, $MAC = \frac{1}{x}(MC - AC)$ or $\frac{d}{dx}(AC) = \frac{1}{x}(MC - AC)$

An analysis of the above relation indicates that if MC = ACthen $\frac{d}{dx}(AC) = 0$.



Which means the slope of AC is zerowhen MC and AC cut each other at the minima of AC as shown in the adjacent figure.

Ex.1. If the total cost function of a producing a product is given by

$C = 0.002x^3 - 0.05x^2 + 25x + 1000$, find (*a*) average cost (*AC*), (*b*) marginal cost (*MC*) and (*c*) marginal average cost (*MAC*).

Ans: Given $C = 0.002x^3 - 0.05x^2 + 25x + 1000$

Now, $AC = \frac{C}{x} = \frac{0.002x^3 - 0.05x^2 + 25x + 1000}{x}$

$\therefore AC = 0.002x^2 - 0.05x + 25 + \frac{1000}{x}$ … (I)

And $MC = \frac{dC}{dx}$. Differentiating *C* w.r.t. *x*, we get,

$\therefore MC = \frac{d}{dx}\left(0.002x^3 - 0.05x^2 + 25x + 1000\right)$

$\therefore MC = 0.006x^2 - 0.1x + 25$ … (II)

Also, $MAC = \frac{d}{dx}(AC)$. Differentiating (I) w.r.t *x*, we get,

$\therefore MAC = \frac{d}{dx}\left(0.002x^2 - 0.05x + 25 + \frac{1000}{x}\right)$

$\therefore MAC = 0.004x - 0.05 - \frac{1000}{x^2}$ … (III)

Ex.2. If $C = 2x^3 - 5x^2 + 15x + 100$ is the total cost function, find (*a*) *AC*, (*b*) *MC*, (*c*) *MC* when *x* = 10 and (*d*) actual cost of producing 11^{th} unit of the product.

Ans:

(*a*) **Average Cost**

$\therefore AC = \frac{C}{x} = \frac{2x^3 - 5x^2 + 15x + 100}{x} = 2x^2 - 5x + 15 + \frac{100}{x}$ … (I)

(*b*) **Marginal Cost**

$\therefore MC = \frac{dC}{dx} = \frac{d}{dx}(2x^3 - 5x^2 + 15x + 100)$

$\therefore MC = 6x^2 - 10x + 15$ … (II)

(*c*) **when** $x = 10$**,**

$\therefore [MC]_{x=10} = 6(10)^2 - 10(10) + 15 = 600 - 100 + 15$

$\therefore [MC]_{x=10} = \text{Rs. } 515$

(*d*) Actual cost of producing an extra unit *i.e.* the 11^{th} unit is calculated as:

$$[C]_{x=11}-[C]_{x=10}=[2(11)^3-5(11)^2+15(11)+100]-[2(10)^3-5(10)^2+15(10)+100]$$

$$=[2662-605+165+100]-[2000-500+150+100]$$

$$=2322-1750=572$$

Thus, the actual cost of producing the 11th unit is Rs. 572.

Ex.3. Verify that $\frac{d}{dx}(AC)=\frac{1}{x}(MC-AC)$, for the cost function given as

$C=1000+120x-10x^2+2x^3$.

Ans: $AC=\frac{C}{x}=\frac{1000+120x-10x^2+2x^3}{x}=\frac{1000}{x}+120-10x+2x^2$ … (I)

$$\frac{d}{dx}(AC)=\frac{d}{dx}\left(\frac{1000}{x}+120-10x+2x^2\right)=-\frac{1000}{x^2}-10+4x \quad \text{… (II)}$$

$$MC=\frac{dC}{dx}=\frac{d}{dx}\left(1000+120x-10x^2+2x^3\right)=120-20x+6x^2 \quad \text{… (III)}$$

Subtracting (II) from (III), we get,

$$MC-AC=\left(120-20x+6x^2\right)-\left(\frac{1000}{x}+120-10x+2x^2\right)$$

$$i.e.\ MC-AC=-\frac{1000}{x}-10x+4x^2$$

Thus, $\frac{1}{x}\left(MC-AC\right)=\frac{1}{x}\left(-\frac{1000}{x}-10x+4x^2\right)=-\frac{1000}{x^2}-10+4x=\frac{d}{dx}(AC)$

$$\therefore \frac{d}{dx}(AC)=\frac{1}{x}(MC-AC).$$

Ex.4. The total cost function of a product is $C=10000+100x-10x^2+\frac{x^3}{3}$. Find the (*a*) *AC*, (*b*) *MC*, (*c*) *VC*, (*d*) *AVC*, (*e*) $\frac{d}{dx}(MC)$ and (*f*) no. of units when *MC* = *AVC*.

Ans:

(*a*) **Average Cost**

$$AC=\frac{C}{x}=\frac{10000+100x-10x^2+x^3/3}{x}=\frac{10000}{x}+100-10x+\frac{x^2}{3}$$

(b) **Marginal Cost**

$$MC = \frac{dC}{dx} = \frac{d}{dx}\left(10000 + 100x - 10x^2 + \frac{x^3}{3}\right) = 100 - 20x + x^2$$

(c) **Variable cost**

$$\because C = FC + VC(x)$$

$$\therefore VC = 100x - 10x^2 + \frac{x^3}{3}$$

(d) **Average variable cost**

$$AVC = \frac{VC}{x} = \frac{100x - 10x^2 + x^3/3}{x} = 100 - 10x + \frac{x^2}{3}$$

(e) $\frac{d}{dx}(MC) = \frac{d}{dx}\left(100 - 20x + x^2\right) = -20 + 2x$

(f) when $MC = AVC$, from (b) and (d) we have,

$$100 - 20x + x^2 = 100 - 10x + \frac{x^2}{3}$$

$$\therefore x^2 - \frac{x^2}{3} - 10x = 0 \quad \Rightarrow 2x^2 - 30x = 0$$

$$\Rightarrow x(2x - 30) = 0 \qquad \Rightarrow x = 0 \text{ or } x = 15$$

Discarding $x = 0$, we have $MC = AVC$ when $x = 15$.

Ex.5. Find AR and MR if the revenue function is given by $R = 100 + \frac{x^2}{10}$. Find the MR when $x = 2000$? Interpret your answer.

Ans: (a) $AR = \frac{R}{x} = \frac{100}{x} + \frac{x}{10}$

(b) $MR = \frac{dR}{dx} = \frac{d}{dx}\left(100 + \frac{x^2}{10}\right) = \frac{x}{5}$

(c) when $x = 2000$;

$$[MR]_{x=2000} = \frac{2000}{5} = 400$$

When an additional unit above 2000 units is sold then the revenue increases by Rs. 400.

Ex.6. The total revenue function for a product is given by $R = 102x^2 + 240x + 1000$ Find (a) AR, (b) MR, (c) MR when $x = 15$, interpret your answer and (d) what is the actual revenue from the 16th unit produced.

Ans: (a) **Average Revenue**

$$AR = \frac{R}{x} = \frac{102x^2 + 240x + 1000}{x} = 102x + 240 + \frac{1000}{x}$$

(b) **Marginal Revenue**

$$MR = \frac{dR}{dx} = \frac{d}{dx}\left(102x^2 + 240x + 1000\right) = 204x + 240$$

(c) When $x = 15$;

$$\left[MR\right]_{x=15} = 204(15) + 240 = 3060 + 240 = 3300$$

When an additional unit above 15 units is sold then the revenue increases by Rs. 3300.

(d) The actual revenue from the 16th unit is calculated as follows;

$$\left[R\right]_{x=16} - \left[R\right]_{x=15} = [102(16)^2 + 240(16) + 1000] - [102(15)^2 + 240(15) + 1000]$$

$$= [26112 + 3840 + 1000] - [22950 + 3600 + 1000]$$

$$= 30952 - 27550 = 3402$$

Thus, the actual revenue from the extra unit sold is Rs. 3402.

9.3 ELASTICITY

Elasticity of a function $y = f(x)$ is defined as the rate of proportional change in y per unit proportional change in x.

Price Elasticity of demand

The price elasticity of demand is the proportionate change in quantity demanded to proportionate change in price. If $D = f(p)$ or $p = g(D)$ is the price /demand function then the price elasticity of demand is denoted by η_d (Greek alphabet eta) and is given by:

$$\eta_d = \frac{-p}{D}\frac{dD}{dp}$$

Since the demand curve has a negative slope, η_d is negative. Usually we take the absolute value of η_d. Depending upon the values of η_d, we have different types of elasticity's as follows:

(i) If $|\eta_d| > 1$, the demand is **elastic**. This means that the demand increases heavily with a small fall in price and decreases heavily with a small increase in price.

If $|\eta_d| < 1$, the demand is **inelastic**. This means that the demand increases in a small proportion with a high fall in price and vice versa.

(ii) If $|\eta_d|$= 1, the demand is **unitary elastic**. This means that the proportionate change in quantity demanded is equal to the proportionate change in price.

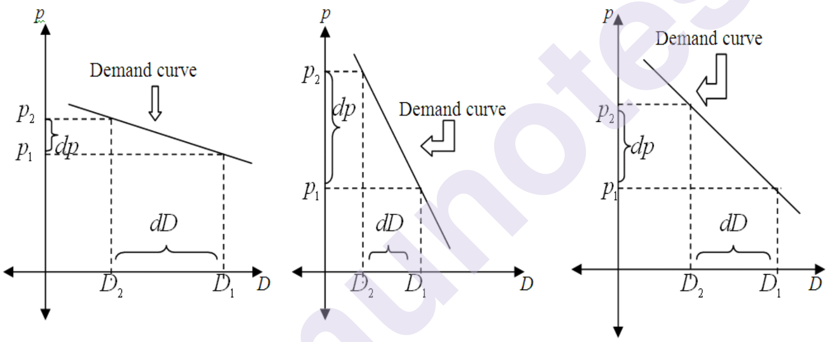
(iii) If, $|\eta_d|$ = 0 the demand is **perfectly inelastic**. This means a change in price does not make any change in quantity demanded.

(iv) If, $|\eta_d|$ B∞ the demand is **perfectly elastic**. This means the demand increases infinitely with a small fall in price.

The value of $|\eta_d|$ indicates the percentage increase or decrease in demand due to 1% increase or decrease in price of a commodity.

The price elasticity of demand is useful to any firm in analyzing the sensitivity of sales of a certain commodity, to a change in its price. Necessary commodities exhibit inelastic demand, *e.g.* salt, water, electricity *etc*. The luxury commodities have a high price sensitivity *i.e.* they exhibit elastic demand, *e.g.* cars, diamonds, gold *etc*.

The following are examples of graphs of linear demand functions representing the three types of elasticity's of demand:



$\eta_d > 1 \; \eta_d < 1 \; \eta_d = 1$

8.5.2 Price Elasticity of Supply

The price elasticity of demand is the proportionate change in supply of a commodity to proportionate change in its price. If $S = f(p)$ or $p = g(S)$ is the price /demand function then the price elasticity of demand is denoted by η_s (Greek alphabet eta) and is given by:

$$\eta_s = \frac{p}{S}\frac{dS}{dp}$$

Since the supply curve has a positive slope, η_S is positive.

Relation between Marginal Revenue (MR) and Average Revenue (AR)

Let $R = pD$ be the total revenue function, where p is the price and D is the quantity of demand of a commodity. We know that, $AR = p$and $MR = \frac{dR}{dD}$.

Differentiating R w.r.t. D, we have,

$$\frac{dR}{dD} = \frac{d}{dD}(pD) \qquad \therefore MR = p + D\frac{dp}{dD} = p\left(1 + \frac{D}{p}.\frac{dp}{dD}\right)$$

Now, $\eta_d = \frac{-p}{D}\frac{dD}{dp}$ $\qquad \therefore MR = p\left(1 - \frac{1}{\eta_d}\right) = AR\left(1 - \frac{1}{\eta_d}\right)$.

Thus, $MR = AR\left(1 - \frac{1}{\eta_d}\right)$

The above relationship can also be expressed as follows: $\eta_d = \frac{AR}{AR - MR}$

An analysis from the above equation indicates that (i) if the commodity is perfectly elastic, *i.e.* $\eta_d = \infty$, then $\frac{1}{\eta_d} = 0 \Rightarrow MR = AR$. ($ii$) If the commodity is unitary elastic, *i.e.* $\eta_d = 1$, then $MR = 0$ which means that TR will remain constant irrespective of increase or decrease in the quantity demanded.

Ex.7. If the demand function is given by $p = 20D - D^2$, find the elasticity of demand when $D = 5$ and interpret your answer.

Ans: Given $p = 20D - D^2$. Differentiating w.r.t. D, we get,

$$\frac{dp}{dD} = \frac{d}{dD}\left(20D - D^2\right) = 20 - 2Da$$

Now, $\eta_d = \frac{-p}{D}\frac{dD}{dp} = \frac{-(20D - D^2)}{D}\left(\frac{1}{20 - 2D}\right)$

$$\therefore \eta_d = \frac{D - 20}{20 - 2D}$$

When $D = 5$, $\left[\eta_d\right]_{D=5} = \frac{5 - 20}{20 - 10} = -1.5$

Since $\left|\eta_d\right| > 1$, the demand function is elastic.

Ex.8. If $AR = 28$ and $\eta = 1.6$, find MR.

Ans: We know that $MR = AR\left(1-\frac{1}{\eta}\right)$

Substituting the given values, we get

$$MR = 28\left(1-\frac{1}{1.6}\right) = 28(1-0.625)$$

$\Rightarrow MR = 10.5$

Ex.9. If the demand function for a certain commodity is given by $p = 20 + 5D^2$, find the demand $D$when $|\eta| = 1$.

Ans: Given $p = 20 + 5D^2$. Differentiating w.r.t. D, we get

$$\frac{dp}{dD} = \frac{d}{dD}\left(20+5D^2\right) = 10D$$

Now, $\eta = \frac{-p}{D}\frac{dD}{dp} = \frac{-(20+5D^2)}{D}\left(\frac{1}{10D}\right)$ $\qquad \therefore \eta = -\frac{20+5D^2}{10D^2}$

If $|\eta| = 1$, we have $\frac{20+5D^2}{10D^2} = 1$

$\Rightarrow 20 + 5D^2 = 10D^2$

$\Rightarrow 5D^2 = 20$ $\qquad \Rightarrow D^2 = 4$

$\therefore D = 2$ $\qquad$ (since $D \neq -2$)

Ex.10 Mr. Bharat Shah manufactures memory cards. The total revenue function is given as $R = -0.4x^2 + 1400x$. Find η_d and interpret it when x is 400, 482 and 560.

Ans: Given $R = -0.4x^2 + 1400x = (-0.4x + 1400)x$

$\therefore p = -0.4x + 1400$. Differentiating w.r.t. x, we have

$\frac{dp}{dx} = -0.4$. Now, $\eta_d = \frac{-p}{x}\frac{dx}{dp}$.

$$\therefore \eta_d = \frac{-(-0.4x+1400)}{x}(-0.4) = \frac{-0.16x+560}{x}$$

When (*i*) $x = 400$ $\qquad$ (*ii*) $x = 482$ $\qquad$ (*iii*) $x = 560$

$\eta_d = 1.24$ $\qquad \eta_d \approx 0.001$ $\qquad \eta_d = 0.84$

Interpretation:

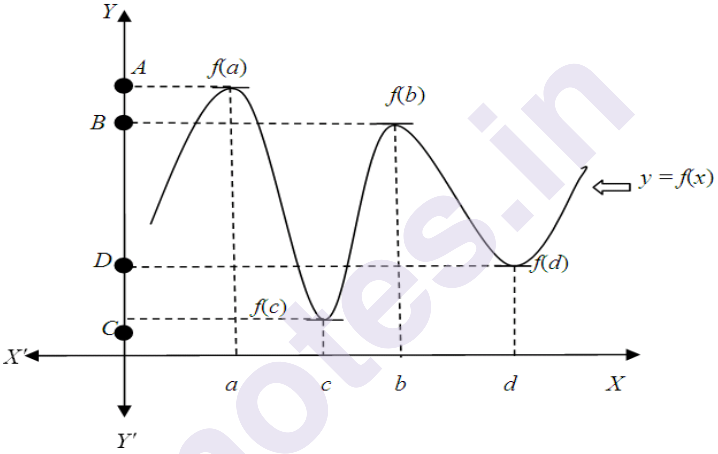
(i) When $x = 400$, $|\eta_d| > 1$. Thus, demand function is inelastic.

(ii) When $x = 482$, $|\eta_d|$ B1. Thus, demand function is unitary elastic.

(iii) When $x = 560$, $|\eta_d| < 1$. Thus, demand function is elastic.

9.4 MAXIMA AND MINIMA FOR A FUNCTION OF SINGLE VARIABLE

Before formally defining maxima and minima for a function, let us observe the graph of a function $f(x)$ given below:



In the above graph of a function $y = f(x)$we observe that at the points A, B, C and D the function attains extreme values. At A and B, the function is reaching a peak while at B and C the function reaches a bottom. A and$B$ are said to be the points where $f(x)$ has a **maxima**, while C and D are the points where $f(x)$ has a **minima**.

From the graph we make the following observations:

- The function $f(x)$ isincreasing (has an upward slope) till the point A and starts decreasing after it reaches A.
- The function $f(x)$is decreasing (has a downward slope) till the point C and starts increasing after it reaches C.

The same is true about the points B and D.

The values a, b, c, and d, at which $f(x)$ takes extreme values are called as **extreme points** or **stationary points.** These points are called stationary as the derivative of the function, evaluated at these points is zero. Graphically, if a tangent is drawn at these points then it is parallel to the X-axis. In other words, the slope of the tangent at these points is zero.

It is important to remember here, that a function may or may not have any extreme points. For example,

- A constant function has no maxima or minima.
- A linear function, which is increasing (positive slope) or decreasing (negative slope) throughout the domain, also does not attain maxima or minima.

Point of inflexion: The points where the $f(x)$ has neither a maxima nor a minima is called as **point of inflexion**.

Maxima

A function $f(x)$ is said to attain a maxima or a maximum value at point $x = a$ if it stops to increase and begins to decrease at $x = a$.

Minima

A function $f(x)$ is said to attain a minima or a minimum value at a point $x = a$ if it stops to decrease and begins to increase at $x = a$.

There are different methods to find the extrema of a function:

First Derivative test for maxima and minima

Let us write down the steps involved in finding the extreme values using the first derivative test:

Step I: Find the first derivative $f'(x)$ of $f(x)$.

Step II: Equate the first derivative to zero. *i.e.* put $f'(x)= 0$ and solve to get the stationary points, say $x = x_1, x_2$ etc.

Step III: Find the sign of $f'(x)$ at a point just less than and just greater than each of the stationary points.

Step IV: If the sign of $f'(x)$ changes from positive to negative then $f(x)$ has local maxima at that point. If the sign of $f'(x)$ changes from negative to positive then $f(x)$ has a local minima at that point. If there is no change in sign of $f'(x)$ then such a point is called as point of inflexion.

Solved Examples

Find all the points of maxima and minima of the function using first derivative test.

1. $f(x) = x^3 - 2x^2 - 4x - 12$

Ans: Given $f(x) = x^3 - 2x^2 - 4x - 12$

Differentiating w.r.t. x,

$f'(x) = 3x^2 - 4x^2 - 4$

Equating $f'(x) = 0$, we get

$3x^2 - 4x - 4 = 0$

i.e. $3x^2 - 6x + 2x - 4 = 0$

$\therefore (x-2)(3x+2) = 0$

$\Rightarrow x = 2$ and $x = -2/3$

Thus, $x = -2/3$ and $x = 2$ are two stationary points.

Consider the point $x = -2/3$

We evaluate $f'(x)$ at $x = -1$ and $x = 0$ $(\because -1 < -2/3 < 0)$

$\therefore f'(-1) = 3(-1)^2 - 4(-1) - 4 = 3 + 4 - 4 = 3$ $\qquad \therefore f'(-1) > 0$

and $f'(0) = 3(0)^2 - 4(0) - 4 = -4$ $\qquad \therefore f'(0) < 0$

Since $f'(x)$ changes sign from positive to negative about the point $x = -2/3$, $f(x)$ has a local maxima at $x = -2/3$.

Now consider the point $x = 2$

We evaluate $f'(x)$ at $x = 1$ and $x = 3$ $(\because 1 < 2 < 3)$

$\therefore f'(1) = 3(1)^2 - 4(1) - 4 = 3 - 4 - 4 = -5$ $\qquad \therefore f'(1) < 0$

and $f'(3) = 3(3)^2 - 4(3) - 4 = 27 - 12 - 4 = 11$ $\quad \therefore f'(3) > 0$

Since $f'(x)$ changes sign from negative to positive about the point $x = 2$, $f(x)$ has a local minima at $x = 2$.

2. $f(x) = 2x^3 - 6x^2$

Ans: Differentiating w.r.t. x,

$f'(x) = 6x^2 - 12x$

Equating $f'(x) = 0$, we get $6x^2 - 12x = 0$

$\therefore 6x(x-2) = 0$

$x = 0$ and $x = 2$

Thus, $x = 0$ and $x = 2$ are two stationary points.

Consider the point $x = 0$

We evaluate $f'(x)$ at $x = -1$ and $x = 1$ $(\because -1 < 0 < 1)$

$\therefore f'(-1) = 6(-1)^2 - 12(-1) = 6 + 12 = 18$

$\Rightarrow f'(-1) > 0$

and $f'(1) = 6(1)^2 - 12(1) = 6 - 12 = -6$

$\Rightarrow f'(1) < 0$

Since $f'(x)$ changes sign from positive to negative about the point $x = 0$, $f(x)$ has a local maxima at $x = 0$.

Now consider the point $x = 2$

We evaluate $f'(x)$ at $x = 1$ and $x = 3$ ($\because 1 < 2 < 3$)

$f'(1) = 6(1)^2 - 12(1) = 6 - 12 = -6 \qquad \Rightarrow f'(1) < 0$

and $f'(3) = 6(3)^2 - 12(3) = 54 - 36 = 18 \Rightarrow f'(3) > 0$

Since $f'(x)$ changes sign from negative to positive about the point $x = 2$, $f(x)$ has a local minima at $x = 2$.

9.5 SECOND DERIVATIVE TEST FOR MAXIMA AND MINIMA

Let $y = f(x)$ be the given function. The steps involved in finding the extreme points are as follows:

STEP I: Find the first derivative of $f(x)$.*i.e.* $\frac{dy}{dx}$

STEP II: Equate the first derivative to zero i.e. put $\frac{dy}{dx} = 0$ and solve to find the stationary points say x_1, x_2 etc.

Step III: Find the second derivative of $f(x)$. *i.e.* $\frac{d^2y}{dx^2}$.

Step IV : Evaluate $\frac{d^2y}{dx^2}$ at the stationary points.

Step V : (*i*) If at $x = x_i$, $\frac{d^2y}{dx^2} > 0$ then the function $f(x)$ has a minima at $x = x_i$ and the minimum value of $f(x)$ is $f(x_i)$.

(*ii*) If at $x = x_i$, $\frac{d^2y}{dx^2} < 0$ then the function $f(x)$ has a maxima at $x = x_i$ and the maximum value of $f(x)$ is $f(x_i)$.

Solved Examples

Find all the extreme values of the following functions:

1. $y = x^3 - 12x - 10$

Ans:(*i*) Differentiating *y* w.r.t. *x*,

$$\frac{dy}{dx} = 3x^2 - 12$$

(*ii*) put $\frac{dy}{dx} = 0$: $\therefore 3x^2 - 12 = 0$

$$\therefore x^2 = 4 \qquad \Rightarrow x = \pm 2$$

$x = 2$ and $x = -2$ are the stationary points.

(*iii*) Differentiating $\frac{dy}{dx}$ w.r.t *x*, we have,

$$\frac{d^2y}{dx^2} = 6x$$

(*iv*) At $x = 2$: $\frac{d^2y}{dx^2} = 6(2) = 12 > 0$

Thus, $y = x^3 - 12x - 10$ has a **minima** at $x = 2$ and the minimum value of the function is $y_{\min} = (2)^3 - 12(2) - 10 = 8 - 24 - 10 = -26$.

At $x = -2$: $\frac{d^2y}{dx^2} = 6(-2) = -12 < 0$

Thus, $y = x^3 - 12x - 10$ has a **maxima** at $x = -2$ and the maximum value of the function is $y_{\max} = (-2)^3 - 12(-2) - 10 = -8 + 24 - 10 = 6$

2. $y = x^4 - 4x^3 + 4x^2 + 24$

Ans: (*i*) Differentiating *y* w.r.t. *x*,

$$\frac{dy}{dx} = 4x^3 - 12x^2 + 8x$$

(*ii*) put $\frac{dy}{dx} = 0$: $4x^3 - 12x^2 + 8x = 0$

$$\therefore x(4x^2 - 12x + 8) = 0$$

$$\therefore x(4x^2 - 4x - 8x + 8) = 0$$

$$\therefore x[4x(x-1) - 8(x-1)] = 0$$

$$\therefore x(x-1)(4x-8) = 0$$

$x = 0$, $x = 1$ and $x = 2$ are the stationary points.

(iii)Differentiating $\frac{dy}{dx}$ w.r.t x,

$$\frac{d^2y}{dx^2}=12x^2-24x+8$$

(iv) At $x=0:\frac{d^2y}{dx^2}=12(0)-24(0)+8=8>0$

Also at x = 2: $\frac{d^2y}{dx^2}=12(2^2)-24(2)+8=48-48+8=8>0$

Thus, at x = 0 and x = 2, the function has a minima and the minimum value is $y_{\min}=(0)^4-4(0)^3+4(0)^2+24=24$

At x = 1: $\frac{d^2y}{dx^2}=12(1^2)-24(1)+8=12-24+8=-4<0$

Thus, at x = 1 the function has a maxima and the maximum value is $y_{\max}=(1)^4-4(1)^3+4(1)^2+24=1+24=25$.

3. Find two positive numbers such that their sum is 100 and product is maximum.

Ans: Let the two numbers be x and $100-x$.

Since we want their product to be maximum,

consider the product function $y=x(100-x)=100x-x^2$.

Differentiating w.r.t. x,

$$\therefore\frac{dy}{dx}=100-2x$$

put $\frac{dy}{dx}=0$: $i.e.$ $100-2x=0$

$\therefore 2x=100 \quad \Rightarrow x=50$.

Now, $\frac{dy}{dx}=100-2x \qquad \Rightarrow\frac{d^2y}{dx^2}=-2<0$

At x = 50: $\frac{d^2y}{dx^2}<0 \qquad \Rightarrow$ there is a maxima at x = 50.

Hence the required two numbers are 50 and 100 – 50 = 50.

4. If the total revenue function and the cost function for a product are given as $R=30x-2x^2$ and $C=-6000-6x+x^2$, find the profit function. At what output is the profit maximum? What is the maximum profit?

Ans: Given $R=30x-2x^2$ and $C=-6000-6x+x^2$

The profit function is calculated as $\pi=R-C$

$$\therefore\pi=\left(30x-2x^2\right)-\left(-6000-6x+x^2\right)$$

$$\therefore\pi=6000+36x-3x^3 .$$

Since we want to maximize π , we follow the steps as mentioned above:

Differentiating π w.r.t. x we get,

$$\frac{d\pi}{dx} = \frac{d}{dx}\left(6000 + 36x - 3x^3\right)$$

$$\therefore \frac{d\pi}{dx} = 36 - 9x^2$$

Equating $\frac{d\pi}{dx}$ = 0, we have: $36 - 9x^2 = 0$

Thus, $x^2 = 4$ $\quad \Rightarrow x = \pm 2$.

Discarding $x = -2$, we have $x = 2$ as the stationary point.

Now, $\frac{d^2\pi}{dx^2} = \frac{d}{dx}\left(\frac{d\pi}{dx}\right) = \frac{d}{dx}\left(36 - 9x^2\right) = -18x$

At $x = 2$, $\frac{d^2\pi}{dx^2} = -18(2) = -36 < 0$.

Thus, the profit is maximum when 2 units of the product are produced.

The maximum profit is $\pi_{\max} = 6000 + 36(2) - 3(2)^3 = 6000 + 72 - 24$

$$\therefore \pi_{\max} = \text{Rs. } 6048.$$

5. If the total cost function of a firm is $C = x^3/3 - 18x^2 + 160x$, find how many items be produced to have the average cost, marginal cost minimum. Also show that the average and marginal cost are equal at the minimum average cost.

Ans: Given $C = x^3/3 - 18x^2 + 160x$.

Average cost:

$$AC = \frac{C}{x} = \frac{x^3/3 - 18x + 160x}{x} = x^2/3 - 18x + 160 \qquad \dots \text{(I)}$$

Differentiating w.r.t x, we get

$$\frac{d}{dx}\left(AC\right) = \frac{d}{dx}\left(x^2/3 - 18x + 160\right) = \frac{2}{3}x - 18 \qquad \dots \text{(II)}$$

Equating this to zero, we have,

$$\frac{2}{3}x - 18 = 0 \Rightarrow x = 27.$$

Differentiating (II), w.r.t. x, we get $\frac{d^2}{dx^2}\left(AC\right) = \frac{2}{3} > 0$

$\therefore$ at $x = 27$, AC is minimum and the minimum average cost is

$$\left[AC\right]_{x=27} = (27)^2/3 - 18(27) + 160 = 243 - 486 + 160 = -83 \quad \dots \text{(III)}$$

Marginal Cost

$$MC = \frac{dC}{dx} = \frac{d}{dx}\left(x^3/3 - 18x^2 + 160x\right) = x^2 - 36x + 160 \qquad \ldots \text{(IV)}$$

Differentiating (IV) w.r.t. x, we have,

$$\frac{d}{dx}(MC) = \frac{d}{dx}\left(x^2 - 36x + 160\right) = 2x - 36 \qquad \ldots \text{(V)}$$

Equating this to zero, we have $2x - 36 = 0 \Rightarrow x = 18$.

Differentiating (V) w.r.t. x, we get $\frac{d^2}{dx^2}(MC) = \frac{d}{dx}(2x - 36) = 2 > 0$

$\therefore$ at $x = 18$, MC is minimum.

To show $AC = MC$ at minimum AC

At $x = 27$, the MC is:

$$[MC]_{x=27} = (27)^2 - 36(27) + 160 = 729 - 972 + 160 = -83 \qquad \ldots \text{(VI)}$$

From (III) and (VI), we conclude that at the point of minima for AC,

$AC = MC$.

9.6 LET US SUM UP

In this chapter we have learn:

- Marginal function of economics.
- Elasticity of demand and supply.
- Maxima and Minima of single variable function.
- Maxima and Minima using second order derivative test.

9.7 UNIT END EXERCISE

1. Define elasticity of demand, elasticity of supply. Interpret η at different values graphically.

2. Write a short not on elasticity.

3. Find TR, AR, MR and η for the following demand functions at the specified point. Determine whether the demand is elastic, inelastic or unitary.

a) $p = 250 - 3D$ at $D = 10$

b) $p = 100 - 2D + 0.2D^2$ at $D = 5$

c) $p = 6 - 0.03x$ at $x = 100$

d) $p = 30 - 0.05x$ at $x = 1000$

e) $p = 50 - 0.008D$ at $D = 500$

f) $p = 120 - 4x$ at $x = 25$

g) $p = \alpha - \beta x$ at $x = a/2b$

h) $p = 10 - 0.02D$ at $D = 150$

i) $D = p^2 - 4p + 200$ at $p = 20$

k) $D = \sqrt{1000 - 2p}$ at $p = 400$

4. If $R = 300 + x^2/4$, find *AR*, *MR*. What is the *MR* at $x = 20$? What is the actual revenue from the 21st unit?

5. If $R = D^3 - 28D^2 + 100D + 1150$, find (a) *AR*, (b) *MR*, (c) *MR* if $D = 50$ and (d) actual revenue from the 51st unit.

6. The total revenue function for a product is given as $R = -0.7x^2 + 1400x$. Find η at (*a*) $x = 500$, (*b*) $x = 1000$ and (*c*) $x = 1500$. Interpret your answer.

7. The demand function for a commodity is $p = 25 - D^2$ and the supply function is $p = 4S^2 + 10$. Find η_d and η_s.

8. (a) If *AR* = 280, η = 2.01, find *MR*. (b) If *MR* = 104 and η = 1.2, find *MR*. (c) If *AR* = 16, *MR* = 10.5 find η.

9. It is known that the demand function for a certain commodity is a linear function. When *D* =1200, *p* = 6 and when *D* = 1600, *p* =2. Find (a) demand function, (b) total revenue and (c) marginal revenue.

10. Find the *AC*, *MC*, *MAC* for the following cost functions:

a) $C = 10x^3 + 12x^2 - 4x + 18$ b) $C = \frac{x^3}{3} + 2x^2 + 10x + 100$

c) $C = 200x - 12x^2 + 145$ d) $C = \frac{15}{x} + 8 + 0.7x$

e) $C = 0.006x^3 - 0.4x^2 - 20x + 1000$ f) $C = 3x^3 + 18x^2 + 42x + 1500$

11. If $AC = 0.009x^2 - 0.04x + 6 + \frac{1000}{x}$, find (a) cost function, (b) *MC*. Show that $C = x^2(MAC) - x(MC)$.

12. If $AC = 200 - 5x + 0.05x^2$, find (a) C(*x*), (b) *MC*, (c) *x*, when *AC* = *MC*.

13. If $AC = \frac{25}{x} - 2 + 6x + x^2$, find (a) C(*x*), (b) *MC*, (c) *MC* when x = 5. What is the actual cost of production for the 6th unit produced?

14. Find all the points of maxima/ minima of the following functions using the first derivative test.

(i) $f(x) = 3x^3 - 12x^2 + 45x - 75$
(ii) $f(x) = 6x^3 - 33x^2 + 168x + 110$
(iii) $f(x) = 4x^3 + 15x^2 - 72x + 12$
(iv) $f(x) = x^3 - 4x$
(v) $f(x) = 4x^4 - 12x^3 + 46x^2 - 60x + 50$

15. Examine for maxima and minima the following functions:

1. $5x^2 - 30x + 10$ **2.** $x^3 - 3x^2 - 9x + 24$ **3.** $x^3 - 48x + 18$

4. $x^3 - 3x^2 - 9x - 36$ **5.** $x^3 + 5x^2 + 8x + 15$ **6.** $x^3 - 3x^2 - 9$

7. $x^3 - 24x^2 + 50$ **8.** $(x-1)(x+2)^2$ **9.** $x^3 - 9x^2 + 15x + 9$

10. $x + \frac{1}{x+1}$ **11.** $\frac{x}{(x+p)(x+q)}$ **12.** $\frac{(x+3)^2}{x^2+1}$

13. $x^3 + 5x^2 + 8x + 16$ **14.** $2x^3 + 3x^2 - 36x + 15$ **15.** $5x^3 - 3x^2 - \frac{8x}{3}$

16. $x^4 - 2x^2 + 4$ **17.** $x^5 - 5x^4 + 5x^2 - 20$ **18.** $x^4 + 2x^2 - 3x^2 - 4x + 16$

19. $3x^4 - 10x^3 + 6x^2 + 50$ **20.** $x^5 - 5x^4 + 5x^3 + 10$

16. Divide 50 into two positive numbers such that their product is maximum.

17. Find two positive numbers whose product is 16 and sum is minimum.

18. Divide 100 into two positive numbers such that sum of their squares is minimum.

19. Find two positive numbers whose product is 100 and sum is minimum.

20. Divide 200 into two positive numbers such that their product is maximum.

21. The total cost function of a firm for producing x unit of output is given by $C = x^3 - 28x^2 + 300x + 1400$. Find the output at which the AC and MC is minimum.

22. If the total cost function is $C = a + bx + cx^2$:

(*i*) Find (a) *AC*, (b) *MC*. How many units are to be produced to minimize the AC and what is the minimum *AC*? Show that at minimum *AC*, the $MC = AC$.

(*ii*) Verify that $\frac{d}{dx}(AC) = \frac{MC - AC}{x}$.

23. The profit function of a factory producing pens is given by $\pi = -0.03x^2 + 900x - 600$. How many pens must be produced to achieve maximum profit? What is the profit per pen?

24. Mr. Diwakar Mishra manufactures electric switches. Based on the product and sale of x units of switches per day, the profit function is $\pi = -x^2 + 80x - 400$. How may switches should Mr. Mishra manufacture and sell so as to earn maximum profit? What is the maximum profit of the factory per day? What is the profit per switch?

25. Mr.AmitPandey manufactures transistors. His initial cost is Rs. 1000 and variable cost per transistor is $0.7x - 20$, where x is no. of transistors produced. The total revenue function is found to be $R = x^2 + 100x$. Find

(i) The total cost function,

(ii) Profit function,

(iii) no. of units to be produced to gain maximum profit,

(iv) maximum profit earned Mr. Pandey and

(v) profit per transistor.

9.8 LIST OF REFERENCES

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10

NUMERICAL ANALYSIS [INTERPOLATION]

Unit Structure :

10.0 OBJECTIVE

After going through this chapter you will able to know:

- Finite difference method
- Forward and backward difference using finite difference.
- Newton's interpolation formula and its used.

10.1 INTRODUCTION

There are many mathematical problems which either cannot be solved by the exiting analytical methods or even if they can be solved, their solutions are in complex form and may not give the desired information. We used numerical methods to deal with such problems.

The branch of mathematics which deals with this aspect is called Numerical analysis. In this chapter we are going to learn forward and backward difference operator, relation between forward difference and backward difference. We find polynomial and approximate value using Newton's forward or backward difference interpolation.

10.2 FINITE DIFFERENCES

Suppose the function $y = f(x)$ takes discrete data as given below:

x	x_0	x_1	……	…..	x_n
y	y_0	y_1			y_n

The process of finding the value of y corresponding to any value of x_i lies between x_0 and x_n is called interpolation. Thus interpolation is the technique of estimating the value of a function for any intermediate value of the independent variable. We can deal with such a situation using the method of finite differences. We begin by deriving two important interpolation formulae by means of forward and backward difference of a function. We begin with some basic concepts.

10.3 FORWARD DIFFERENCE OPERATOR

Let $y = f(x)$ be a function. Where x is the independent variable and y is dependent variable.

In the method of finite differences is that the values of the variable x are equally spaced. Suppose x assumes values such that the difference between any two consecutive values is a constant, as shown below.

x: x_0, $x_1 = x_0 + h$, $x_2 = x_0 + 2h$, $x_3 = x_0 + 3h$, … ….

Where h is called the interval difference.

The corresponding values of y is given by

y: $f(x_0)$, $f(x_0 + h)$, $f(x_0 + 2h)$, … … … … ….

The difference of the value of the function $f(x_0 + h) - f(x_0)$ these difference are called the first forward difference of the function $y = f(x)$.It is denoted by $\Delta f(x_0)$ (read as delta *f* of *x* zero).

$$\Delta f(x) = f(x + h) - f(x)$$

Where Δ is called forward difference operator and h is called interval difference.

Similarly we find second forward difference of the function$y = f(x)$,

$$\Delta^2 f(x) = \Delta[\Delta f(x)]$$

$$= \Delta[f(x + h) - f(x)]$$

$$= \Delta f(x + h) - \Delta f(x)$$

$$= [f(x + 2h) - f(x + h)] - [f(x + h) - f(x)]$$

$$= f(x + 2h) - 2f(x + h) + f(x)$$

x	$f(x)$	$\Delta f(x)\Delta^2 f(x)\Delta^3 f(x)$		
x_0	$f(x_0)$			
		$\Delta f(x_0)$		
$x_0 + h$	$f(x_0 + h)$			$\Delta^2 f(x_0)$
$\Delta f(x_0 + h)$			$\Delta^3 f(x_0)$	
$x_0 + 2h$	$f(x_0 + 2h)$			$\Delta^2 f(x_0 + h)$
$\Delta f(x_0 + 2h)$				
$x_0 + 3h$	$f(x_0 + 3h)$			

The above table is called forward difference table. Similarly we can find other differences.

The Shift operator *E*:

Let $y = f(x)$ be a function of x. Let $x, x + h, x + 2h, \ldots\ldots$ be the consecutive values of x. Then we define the shift operator E as

$Ef(x) = f(x + h)$

Similarly, $E^2 f(x) = f(x + 2h), \ldots\ldots\ldots\ldots\ldots E^n f(x) = f(x + nh)$

Relation between the operator $E$and Δ:

We have learn , $\Delta f(x) = f(x + h) - f(x)$ and $Ef(x) = f(x + h)$

$\Delta f(x) = Ef(x) - f(x)$

$\Delta f(x) = f(x)(E - 1)$

$\Delta = (E - 1)$

$\therefore$ Hence we can write $E = \Delta + 1$ which gives the relation between the operators $E$and Δ.

Note: Using the relation between the operators $E$and Δ, we can express

$\Delta^2 f(x_0) = (E - 1)^2 f(x_0) = (E^2 - 2E + 1) f(x_0)$

$\Delta^2 f(x_0) = E^2 f(x_0) - 2Ef(x_0) + f(x_0)$

$\Delta^2 f(x_0) = f(x_0 + 2h) - 2f(x_0 + h) + f(x_0)$

$\Delta^2 f(x_0) = f(x_2) - 2f(x_1) + f(x_0)$

Similarly,

$$\Delta^3 f(x_0) = f(x_3) - 3f(x_2) + 3f(x_1) - f(x_0)$$

$$\Delta^4 f(x_0) = f(x_4) - 4f(x_3) + 6f(x_2) - 4f(x_1) + f(x_0)$$

Example 1: Construct a forward difference table for the following data:

x	0	1	2	3	4
y	3	2	7	24	59

Solution : Prepare the forward difference table :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	3			
		-1		
1	2		6	
		5		6
2	7		12	
		17		6
3	24		18	
		35		
4	59			

Example 2 : If $f(x) = 2x^3 - x^2 + 5$, construct a forward difference table by taking $x = 0(1)5$.

Solution: given function $f(x) = 2x^3 - x^2 + 5$ by taking $x = 0(1)5$ i.e. $x = 0,1,2,3,4,5$

$$f(0) = 2(0)^3 - (0)^2 + 5 = 5$$

$$f(1) = 2(1)^3 - (1)^2 + 5 = 6$$

$$f(2) = 2(2)^3 - (2)^2 + 5 = 17$$

$$f(3) = 2(3)^3 - (3)^2 + 5 = 50$$

$$f(4) = 2(4)^3 - (4)^2 + 5 = 117$$

$$f(5) = 2(5)^3 - (5)^2 + 5 = 230$$

Prepare the forward difference table :

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	5			
1				
1	6		10	
		11		12
2	17		22	
		33		12
3	50		34	
		67		12
4	117		46	
		113		
5	230			

Example 3: Estimate the missing value in the following table:

x	0	1	2	3	4
y	7	10	13	-	43

Solution: Since four values of $f(x)$ are given, we assume that $f(x)$ is a polynomial of degree 3.

$\therefore \Delta^4 f(x) = 0$

$\therefore \Delta^4 f(0) = 0$

$\therefore (E-1)^4 f(0) = 0$

$\therefore E^4 f(0) - 4E^3 f(0) + 6E^2 f(0) - 4Ef(0) + f(0) = 0$

$\therefore f(4) - 4f(3) + 6f(2) - 4f(1) + f(0) = 0$

$\therefore 43 - 4f(3) + 6(13) - 4(10) + 7 = 0$

$\therefore 88 - 4f(3) = 0$

$\therefore f(3) = 22$

Therefore, the missing value is 22.

10.4 NEWTON'S FORWARD DIFFERENCE INTERPOLATION FORMULA

Let the function $y = f(x)$, the values of the variable x are equally spaced. Suppose x assumes values such that the difference between any two consecutive values is a constant, as shown below.

$x: \quad x_0, \quad x_1 = x_0 + h, \quad x_2 = x_0 + 2h, \quad x_3 = x_0 + 3h, \dots \dots$

Where h is called the interval difference.

The corresponding values of y is given by

$y: f(x_0), \quad f(x_0 + h), \quad f(x_0 + 2h), \dots \dots \dots \dots \dots$

Suppose that we want to estimate $f(x)$ for $x = x_0 + ph$.

We have $p = \frac{x-x_0}{h}$

Now, $f(x) = f(x_0 + ph) = E^p f(x_0) = (\Delta + 1)^p f(x_0)$

$$f(x) = \left[1 + p\Delta + \frac{p(p-1)}{2!}\Delta^2 + \cdots \ldots \ldots \ldots \ldots + \frac{p(p-1)(p-2)\ldots..(p-n+1)}{n!}\Delta^n\right] f(x_0)$$

$$= f(x_0) + p\Delta f(x_0) + \frac{p(p-1)}{2!}\Delta^2 f(x_0) + \cdots \ldots \ldots \ldots \ldots + \frac{p(p-1)(p-2)\ldots..(p-n+1)}{n!}\Delta^n f(x_0)$$

This is known as Newton's forward difference interpolation formula.
This formula can also be expressed in the form:

$$y_x = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \cdots \ldots \ldots \ldots \ldots + \frac{p(p-1)(p-2)\ldots..(p-n+1)}{n!}\Delta^n y_0$$

Example 4: Using Newton's forward difference interpolation formula, estimate $f(3)$.

x	2	4	6	8
$f(x)$	4	7	11	18

Solution: First we have to prepare forward difference table:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
2	4			
		3		
4	7		1	
4		2		
6	11		3	
7				
8	18			

Here, $x_0 = 2$, $h = 4 - 2 = 2$, $x = 3$

$$p = \frac{x - x_0}{h} = \frac{3-2}{2} = \frac{1}{2} = 0.5$$

Also, $f(2) = 4, \quad \Delta f(2) = 3, \quad \Delta^2 f(x) = 1, \quad \Delta^3 f(x) = 2$

Now, by Newton's forward difference interpolation formula,

$$f(x) = f(x_0) + p\Delta f(x_0) + \frac{p(p-1)}{2!}\Delta^2 f(x_0) + \cdots \dots \dots \dots \dots$$
$$+ \frac{p(p-1)(p-2)\dots\dots(p-n+1)}{n!}\Delta^n f(x_0)$$

$$f(3) = f(2) + (0.5)\Delta f(2) + \frac{0.5(0.5-1)}{2!}\Delta^2 f(2)$$
$$+ \frac{0.5(0.5-1)(0.5-2)}{3!}\Delta^3 f(2)$$

$$f(3) = 4 + 0.5(3) + \frac{0.5(-0.5)}{2} \times 1 + \frac{0.5(-0.5)(-1.5)}{6} \times 2$$

$$f(3) = 4 + 1.5 - 0.125 + 0.125$$

$$f(3) = 5.5$$

Example 5: From the following table, find the number of students who have obtained less than 45 marks, using Newton's forward difference interpolation.

Marks	30-40	40-50	50-60	60-70	70-80
No. of students	31	42	51	35	31

Solution: Here we have to prepare cumulative frequency table.

x	Below 40	Below 50	Below 60	Below 70	Below 80
cf(y)	31	73	124	159	190

Now construct forward difference table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	31				
		42			
50	73		9		
		51		-25	
60	124		-16		37
		35		12	
70	159		-4		
		31			
80	190				

Here $x_0 = 40, \quad x = 45, \quad h = 10$

$$\therefore p = \frac{x - x_0}{h} = \frac{45-40}{10} = \frac{1}{2} = 0.5$$

Also

$y_0 = 31, \quad \Delta y_0 = 42, \quad \Delta^2 y_0 = 9, \quad \Delta^3 y_0 = -25, \quad \Delta^4 y_0 = 37$

Using Newton's forward difference interpolation formula,

$$y_x = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \cdots \ldots \ldots \ldots \ldots + \frac{p(p-1)(p-2)\ldots..(p-n+1)}{n!}\Delta^n y_0$$

$$y(45) = 31 + (0.5 \times 42) + \frac{0.5(0.5-1)}{2!} \times 9 + \frac{0.5(0.5-1)(0.5-2)}{3!} \times (-25) \quad + \frac{0.5(0.5-1)(0.5-2)(0.5-3)}{4!} \times 37$$

$$y(45) = 31 + 21 - 1.125 - 1.5625 - 1.4453$$

$$y(45) = 47.86 \approx 48$$

Therefore, the number of students who have obtained marks less than 45 is 48.

Example 6: Find second degree polynomial passing through the points (0, 5), (1, 4), (2, 5)
and (3, 8) using Newton's forward difference interpolation.
Solution: First prepare forward difference table.

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$
0	5		
		-1	
1	4		2
		1	
2	5		2
		3	
3	8		

Here $x_0 = 0, \quad h = 1.$

$$\therefore\ n = \frac{x - x_0}{h} = \frac{x-0}{1} = x$$

Also, $f(0) = 5, \quad \Delta f(0) = -1,\ \Delta^2 f(0) = 2.$

By Newton's forward difference interpolation formula

$$f(x) = f(x_0) + p\Delta f(x_0) + \frac{p(p-1)}{2!}\Delta^2 f(x_0) + \cdots \ldots \ldots \ldots \ldots + \frac{p(p-1)(p-2)\ldots..(p-n+1)}{n!}\Delta^n f(x_0)$$

$$f(x) = f(0) + x\Delta f(0) + \frac{x(x-1)}{2!}\Delta^2 f(0)$$

$$f(x) = 5 + x(-1) + \frac{x^2 - x}{2} \times 2$$

$$f(x) = 5 - x + x^2 - x$$

$$f(x) = x^2 - 2x + 5$$

Therefore, the required polynomial is $x^2 - 2x + 5$.

10.5 BACKWARD DIFFERENCE OPERATOR

Let $y = f(x)$ be a function. Where x is the independent variable and y is dependent variable.

In the method of finite differences is that the values of the variable x are equally spaced. Suppose x assumes values such that the difference between any two consecutive values is a constant, as shown below.

x: x_0, $x_1 = x_0 + h$, $x_2 = x_0 + 2h$, $x_3 = x_0 + 3h$, ...

Where h is called the interval difference.

The corresponding values of y is given by

y: $f(x_0)$, $f(x_0 + h)$, $f(x_0 + 2h)$, ...

The difference of the value of the function $f(x_0) - f(x_0 - h)$ these difference are called the first forward difference of the function $y = f(x)$. It is denoted by $\nabla f(x_0)$ (read as del f of x zero).

$$\nabla f(x) = f(x) - f(x - h)$$

Where ∇ is called backward difference operator and h is called interval difference.

Similarly we find second forward difference of the function $y = f(x)$,

$$\nabla^2 f(x) = \nabla[\nabla f(x)]$$

$$= \nabla[f(x) - f(x - h)]$$

$$= \Delta f(x) - \Delta f(x - h)$$

$$= [f(x) - f(x - h)] - [f(x - h) - f(x - 2h)]$$

$$= f(x) - 2f(x - h) + f(x - 2h)$$

x	$f(x)\nabla f(x)\nabla^2 f(x)\nabla^3 f(x)$
x_0	$f(x_0)$
$\nabla f(x_0 + h)$	
$x_0 + h$	$f(x_0 + h)\nabla^2 f(x_0 + 2h)$
$\nabla f(x_0 + 2h)\nabla^3 f(x_0 + 3h)$	
$x_0 + 2h$	$f(x_0 + 2h)\nabla^2 f(x_0 + 3h)$
$\nabla f(x_0 + 3h)$	
$x_0 + 3h$	$f(x_0 + 3h)$

The above table is called forward difference table. Similarly we can find other differences.

The Shift operator E^{-1}:

Let $y = f(x)$ be a function of x. Let $x, x+h, x+2h, \ldots\ldots$ be the consecutive values of x. Then we define the shift operator E as

$$E^{-1}f(x) = f(x-h)$$

Similarly, $E^{-2}f(x) = f(x-2h), \ldots\ldots\ldots\ldots E^{-n}f(x) = f(x-nh)$

Relation between the operator E^{-1} and ∇:

We have learn , $\nabla f(x) = f(x) - f(x-h)$ and $E^{-1}f(x) = f(x-h)$

$$\nabla f(x) = f(x) - E^{-1}f(x)$$

$$\nabla f(x) = f(x)(1 - E^{-1})$$

$$\nabla = (1 - E^{-1})$$

$\therefore$ Hence we can write $E^{-1} = 1 - \nabla$ which gives the relation between the operators E and ∇.

Example 7: Construct a backward difference table for the following data:

x	0	1	2	3	4
y	3	6	11	18	27

Solution: Prepare the backward difference table.

x	y	∇y	$\nabla^2 y$
0	3		
		3	
1	6		2
		5	
2	11		2
		7	
3	18		2
		9	
4	27		

Example 8:Prepare backward difference table for $f(x) = x^3 - 2x^2 - x + 1$ with $x = -1(1)7$.

Solution: Given function $f(x) = x^3 - 2x^2 - x + 1$ with $x = -1(1)7$.

i.e. $x = -1, 0, 1, 2, 3, 4, 5, 6, 7$

Prepare the backward difference table.

x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$
-1	-1			
			2	
0	1			-4
-2 6				
1	-1			2
0 6				
2	-1			8
8 6				
3	7			14
		22		
4	29			20
			42	
5	71			26
			68	
6	139			32
			100	
7	239			

10.6 NEWTON'S BACKWARD INTERPOLATION FORMULA

Let the function $y = f(x)$, the values of the variable x are equally spaced. Suppose x assumes values such that the difference between any two consecutive values is a constant, as shown below.

x: x_0, $x_1 = x_0 + h$, $x_2 = x_0 + 2h$, $x_3 = x_0 + 3h$, ...

Where h is called the interval difference.

The corresponding values of y is given by

y: $f(x_0)$, $f(x_0 + h)$, $f(x_0 + 2h)$, ...

Suppose that we want to estimate $f(x)$ for $x = x_n + ph$.

We have $p = \frac{x - x_n}{h}$

$$f(x) = \left[1 + p\nabla + \frac{p(p+1)}{2!}\nabla^2 + \cdots \dots \dots \dots \dots + \frac{p(p+1)(p+2)\dots..(p-n+1)}{n!}\nabla^n\right] f(x_n)$$

$$= f(x_n) + p\nabla f(x_n) + \frac{p(p+1)}{2!}\nabla^2 f(x_n) + \cdots \dots \dots \dots \dots + \frac{p(p-1)(p-2)\dots..(p-n+1)}{n!}\nabla^n f(x_n)$$

This is known as Newton's backward difference interpolation formula.

This formula can also be expressed in the form:

$$y_x = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \cdots \ldots \ldots \ldots \ldots$$
$$+ \frac{p(p-1)(p-2)\ldots..(p-n+1)}{n!}\nabla^n y_n$$

Example 9: Using the Newton's backward difference interpolation formula, find $f(3.5)$ for the following data:

x	0	1	2	3	4
$f(x)$	2	8	17	29	42

Solution: Prepare the backward difference table.

x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$
0	2		
		6	
1	8		3
		9	
2	17		3
12			
3	29		3
		15	
4	44		

Here $x_n = 4, \quad x = 3.5, \quad h = 1.$

$$p = \frac{x - x_n}{h} = \frac{3.5 - 4}{1} = -0.5$$

Also, $f(x_n) = 44, \ \nabla f(x_n) = 15, \quad \nabla^2 f(x_n) = 3$

Using Newton's backward difference interpolation formula,

$$f(x) = f(x_n) + p\nabla f(x_n) + \frac{p(p+1)}{2!}\nabla^2 f(x_n) \ldots \ldots . \ldots \ldots$$
$$+ \frac{p(p-1)(p-2)\ldots..(p-n+1)}{n!}\nabla^n f(x_n)$$

$$f(3.5) = 44 + (-0.5) \times 15 + \frac{-0.5(-0.5+1)}{2!} \times 3$$

$$f(3.5) = 44 - 7.5 - 0.375$$

$$f(3.5) = 36.125$$

Example 10: By using Newton's backward difference interpolation formula, estimate the profit for the year 2007, from the following data.

$x(year)$	1990	1995	2000	2005	2010
Profit (inCrs.)	32	45	62	85	107

Solution: Prepare the backward difference table.

x	y	$\nabla y\nabla^2 y$	$\nabla^3 y$
1990	3 2		
13			
1995	454		
17 2			
2000	626		
23 2			
2005	85		8
	31		
2010	116		

Here $x_n = 2010, \quad x = 2007, \quad h = 5.$

$$p = \frac{x - x_n}{h} = \frac{2007 - 2010}{5} = -0.6$$

Also, $y_n = 116, \ \nabla y_n = 31, \quad \nabla^2 y_n = 8, \quad \nabla^3 y_n = 2$

Using Newton's backward difference interpolation formula,

$$y_x = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \cdots \ldots \ldots \ldots \ldots$$
$$+ \frac{p(p-1)(p-2)\ldots..(p-n+1)}{n!}\nabla^n y_n$$

$$f(2007) = 116 + (-0.6) \times 31 + \frac{-0.6(-0.6+1)}{2!} \times 8$$
$$+ \frac{-0.6(-0.6+1)(-0.6+2)}{3!} \times 2$$

$$f(2007) = 116 - 18.6 - 0.96 - 0.112$$
$$f(2007) = 96.328$$

Example 11: Use Newton's backward difference interpolation formula to find $f(x)$ given that $f(0) = 1, f(1) = 6, f(2) = 19, f(3) = 46.$

Solution : Prepare the backward difference table.

x	$f(x)\nabla f(x)\nabla^2 f(x)\nabla^3 f(x)$		
0	1		
		5	
1	6		8
		13	
2	19		14
		27	
3	46		

Here $x_n = 3, \quad h = 1.$

$$p = \frac{x - x_n}{h} = \frac{x-3}{1} = (x-3)$$

Also, $f(x_n) = 46,\ \nabla f(x_n) = 27,\quad \nabla^2 f(x_n) = 14,\quad \nabla^2 f(x_n) = 6$

Using Newton's backward difference interpolation formula,

$$f(x) = f(x_n) + p\nabla f(x_n) + \frac{p(p+1)}{2!}\nabla^2 f(x_n) \ldots \ldots\ldots \ldots$$
$$+ \frac{p(p-1)(p-2)\ldots..(p-n+1)}{n!}\nabla^n f(x_n)$$

$$f(x) = 46 + (x-3)\times 27 + \frac{(x-3)(x-3+1)}{2!}\times 14$$
$$+ \frac{(x-3)(x-3+1)(x-3+2)}{3!}\times 6$$

$$f(x) = 46 + 27x - 81 + (x-3)(x-2)\times 7 + (x-3)(x-2)(x-1)$$

$$f(x) = 46 + 27x - 81 + 7x^2 - 35x + 42 + x^3 - 6x^2 + 11x - 6$$

$$f(x) = x^3 + x^2 + 3x + 1$$

Therefore, the required polynomial is $x^3 + x^2 + 3x + 1.$

10.7 LET US SUM UP

In this chapter we have learn:

- Forward difference and backward difference using finite difference.
- Relation between shift operator and Delta.
- Relation between shift inverse operator and Del.
- Newton's forward and backward difference interpolation formula.

10.8 UNIT END EXERCISE

1. Construct the forward difference table for the following points:

   a) (1,5), (2, 12), (3, 23), (4, 53)

   b) $(0,3), (2,8), (4,15), (6,23), (10,37)$

2. Construct the backward difference table for the following points:

   a)(3, 12), (5, 27), (7, 45), (9, 69)

   b)(1, -3), (2, 6), (3, 18), (4, 31), (5, 56), (6, 98)

3. Construct the forward difference table for the following function:

   a) for $f(x) = 2x^2 - 5x + 6 \quad with \quad x = 0(1)5$

   b) for $f(x) = x^3 - x^2 - 2x + 1 \quad with \quad x = 0(2)10$

4. Construct the backward difference table for the following function:

a) for $f(x) = x^2 - 3x - 1 \quad with \quad x = 0(1)6$

b) for $f(x) = 2x^3 - 3x^2 - x + 10 \quad with \quad x = 0(1)5$

5. Estimate the missing value in the following tables by forward / backward difference.

a)

x	1	3	5	7
y	1	27	-	343

b)

x	0	1	2	3	4
y	5	7	-	23	37

6. Estimate $f(1.5)$ and $f(3.5)$ from the following data using Newton's forward/ backward difference interpolation formula.

x	0	1	2	3	4
$f(x)$	0	1	8	27	64

7. The population of a town is given below:

Year	1980	1990	2000	2010	2020
Population (in Thousand)	10	34	69	98	124

Estimate the population for the year 1983 using Newton's forward difference interpolation formula.

8. Find the number of persons getting wages less than Rs. 25 from the following table by Newton's forward difference interpolation.

Wages(Rs.)	0-20	20-40	40-60	60-80	80-100
No. of Persons:	11	30	26	23	10

9. By Newton's backward interpolation formula , estimate $f(4.2)$ for the following data.

x	1	2	3	4	5
$f(x)$	3	9	21	43	69

10. By Newton's backward interpolation formula , estimate $f(9)$ for the following data.

x	2	4	6	8	10
$f(x)$	7	18	32	56	88

11. From the following data estimate $f(18)$ by Newton's backward difference interpolation.

x	5	10	15	20
$f(x)$	50	70	100	145

12.Use Newton's forward difference formula to find $f(x)$ given that

$f(0) = 1, f(1) = 6, f(2) = 19, f(3) = 43$.

13. Find $f(x)$ for the following table by Newton's forward/ backward formula:

a)

x	0	1	2	3	4
$f(x)$	3	6	11	18	27

b)

x	0	1	2	3
$f(x)$	1	1	7	13

c)

x	0	1	2	3
$f(x)$	-1	1	1	-2

d)

x	1	2	3	4	5
$f(x)$	-3	0	5	12	21

14. Multiple choice questions:

i) Let h be the finite difference, then forward difference operator is defined as

a) $\Delta f(x) = f(x+h)$
b) $\Delta f(x) = f(x+h) + f(x)$
c) $\Delta f(x) = f(x+h) - f(x)$
d) $\Delta f(x) = f(x-h)$

ii) Let h be the finite difference, then which of the following is true for sift operate?

a) $E^n f(x) = f(x)$
b) $E^n f(x) = f(x+nh)$
c) $E^n f(x) = f(x-nh)$
d) $E^n f(x) = nf(x+nh) - h$

iii) Using Newton's forward interpolation formula for give data $f(2) = 4, f(4) = 7, f(6) = 11$ then value of $f(5)$ is

a) 8.257
b) 8.575
c) 8.875
d) 8.375

iv) Using Newton's backward interpolation formula for give data $f(0) = 0, f(1) = 1,$

$f(2) = 8, f(3) = 27$ then value of $f(2.5)$ is

a) 16.625
b) 15.625
c) 16.675
d) 15.525

v) Which of the following is true?

a) $\nabla f(x) = f(x) - E^{-1}(x)$
b) $\nabla f(x) = f(x) + E^{-1}(x)$
c) $\nabla f(x) = f(x+h) - E^{-1}(x)$
d) $\nabla f(x) = f(x+h) + E^{-1}(x)$

10.9 LIST OF REFERENCES

- Introduction method of Numerical Analysis by S.S. Sastry
- Numerical Method by Jain and Iyengar

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