M.Sc. – II (Mathematics) Fourth Semester

MSC2497 – Algebraic Topology – II (Optional) Paper – VII

P. Pages : 3 **GUG/W/18/2394**

Time : Three Hours Max. Marks : 100

Notes : 1. Solve all **five** questions.
2. All questions carry equal marks.

UNIT – I

1. a) Prove that **10**

If u is an open subset of R^2 and $f : u \to R^2$ is continuous and injective, then $f(u)$ is open in R^2 and the inverse function $f^{-1} : f(u) \to u$ is continuous.

b) Let C be a simple closed curve in s^2 ; let p and q lie in different components of $s^2 - c$. Then prove that inclusion mapping $j : c \to s^2 - p - q$ induces an isomorphism of fundamental groups. **10**

OR

c) Prove that **10**

Let C be a simple closed curve in s^2. Then C separates s^2 into precisely two components W_1 and W_2. Each of the sets W_1 and W_2 has C as its boundary : that is, $C = \overline{W}_i - W_i$ for $i = 1, 2$.

d) Prove that **10**

Let X be a theta space that is a subspace of S^2 ; let A, B and C be the arcs whose union is X. Then X separates S^2 into three components, whose boundaries are $A \cup B$, $B \cup C$ and $A \cup C$, respectively. The component having $A \cup B$ as its boundary equals one of the components of $S^2 - A \cup B$.

UNIT – II

2. a) Let $G = G_1 * G_2$. Let Ni be a normal subgroup of Gi, for $i = 1, 2$. If N is the least normal subgroup of G that contains N_1 and N_2, then prove that $G/N \cong \left(G_1/N_1\right) * \left(G_2/N_2\right)$. **10**

b) Prove that **10**

Given G, the subgroup [G, G] is a normal subgroup of G and the quotient group $G/[G, G]$ is abelian. If $h : G \to H$ is any homomorphism from G to an abelian group H, then the Kernel of h contains [G, G], so h induces a homomorphism $k : G/[G, G] \to H$.

OR

c) Prove that **10**

The fundamental group of the n – fold dunce cap is a cyclic group of order n.

d) Let X be the wedge of the circles $S_1, ..., S_n$; let p be the common point of these circles. Then show that $\pi_1(X, p)$ is a free group. If f_i is a loop in Si that represents a generator of $\pi_1(Si, p)$ then show that the loops $f_1, ..., f_n$ represent a system of free generators for $\pi_1(X, p)$. **10**

UNIT – III

3. a) Prove that **10**

Let X be the space obtained from a finite collection of polygonal regions by pasting edges together according to some labelling scheme. Then X is a compact Housdorff Space.

b) Prove that **10**

Let w be a proper scheme of the form

$$w = [y_0]a[y_1]a[y_2],$$

where some of the y_i may be empty. Then one has the equivalence

$$w \sim aa\left[y_0\, y_1^{-1}\, y_2\right]$$

where y_1^{-1} denotes the formal inverse of y_1.

OR

c) If w is a scheme of projective type, then prove that w is equivalent to a scheme of the same length having the form **10**

$$(a_1\, a_1)(a_2\, a_2).......(a_k\, a_k)w_1,$$

where $k \geq 1$ and w_1 is either empty or of torus type.

d) If X is the m – fold connected sum of projective planes, then prove that torsion subgroup $T(X)$ of $H_1(X)$ has order 2, and $H_1(X)/T(X)$ is a free abelian group of rank (m – 1). **10**

UNIT – IV

4. a) Prove that **10**

Let B be path connected, locally path connected, and semi locally simply connected. Let $b_0 \in B$. Given a subgroup H of $\pi_1(B, b_0)$, there exists a covering map $p : E \to B$ and point $e_0 \in p^{-1}(b_0)$ such that $P_*(\pi_1(E, e_0)) = H$.

b) Prove that **10**

Let $P : E \to B$ and $P' : E' \to B$ be covering maps; let $P(e_0) = P'(e_0') = b_0$.

There is an equivalence $h : E \to E'$ such that $h(e_0) = e_0'$ if and only if the group

$H_0 = P_*(\pi_1(E, e_0))$ and $H_0' = P_*'(\pi_1(E', e_0'))$

are equal. If h exists, it is unique.

OR

c) Prove that **10**

Let F be a free group with n + 1 free generators; let H be a subgroup of F. If H has index k in F, then H has k_{n+1} free generators.

d) If X is a linear graph, then prove that X is locally path connected and semi locally simply connected. **10**

5. a) Let X be the utilities graph. Then prove that X cannot be imbedded in the plane. **5**

b) Prove that **5**

Given an index set J, there exists a space X that is wedge of circles S_{α} for $\alpha \in J$.

c) Let W be a proper scheme of the form $W = W_0(cc)(aba^{-1}b^{-1})W_1$. Then prove that W is equivalent to the scheme $W' W_0(aabbcc)W_1$. **5**

d) Prove that **5**

A graph X is connected if and only if every pair of vertices of X can be joined by an edge path in X.

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