Duration: 2 ½ Hrs　　　　　　　　　　　　　　　　　　　　　　Marks: 75

N.B. : (1) All questions are compulsory.
(2) Figures to the right indicate marks.

Notations: The Laplace transform of $f(t) = \mathscr{L}\Big(f(t)\Big)$ and the Fourier transform of $f(t) = \mathscr{F}\Big(f(t)\Big)$.

Q 1 (A) Attempt any One of the following: (8)

(i) If $f(t)$ is a periodic function of period A, and if $\mathscr{L}\Big(f(t)\Big)$ exists, then prove that

$$\mathscr{L}\Big(f(t)\Big) = \frac{1}{1-e^{-As}} \int_0^A e^{-st} f(t)\, dt.$$

Hence find the Laplace transform of $f(t) = \dfrac{2t}{T}$ for $0 < t < T$ and $f(t) = f(t+T)$

(ii) If $\mathscr{L}\Big(f(t)\Big) = F(s)$ then show that $\mathscr{L}\Big(t^n f(t)\Big) = (-1)^n \dfrac{d^n}{ds^n} F(s)$ for all $n \in \mathbb{N}$.

(B) Attempt any Two of the following: (12)

(i) Find $\mathscr{L}\Big(t\ e^{-2t}\ \sinh 3t\Big)$.

(ii) If $\mathscr{L}\Big(f(t)\Big) = F(s)$ then prove that $\mathscr{L}\Big(\dfrac{1}{t}\ f(t)\Big) = \displaystyle\int_s^\infty F(s)\, ds$.

(iii) Find $\mathscr{L}^{-1}\Big(\dfrac{1}{s(s-3)^2}\Big)$.

Q 2 (A) Attempt any One of the following: (8)

(i) If $F(s) = \mathscr{F}\Big(f(t)\Big) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt$ then prove the following.

(I) $\mathscr{F}\Big(e^{ibt}\ f(t)\Big) = F(s+b)$　　(II) $\mathscr{F}\Big(f(at)\Big) = \dfrac{1}{|a|}\ F\Big(\dfrac{s}{a}\Big)\quad a \neq 0.$

(ii) If $F(s) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} f(t)\ e^{ist}\, dt$ is the Fourier transform of $f(t)$ then prove that

(I) $\mathscr{F}\Big(f(-t)\Big) = F(-s)$　　(II) $\mathscr{F}\Big(\overline{f(-t)}\Big) = \overline{F(s)}$.

(B) Attempt any Two of the following: (12)

(i) Obtain the Fourier integral representation of the function $f(t)$ defined as follows.

$$f(t) = \begin{cases} -1 & \text{if } -1 < t < 0, \\ 1 & \text{if } 0 < t < 1, \\ 0. & \text{otherwise.} \end{cases}$$

(ii) Find the Fourier transform of $e^{-a|t|}$, $a > 0$. Hence show that

$$\mathscr{F}\Big(\frac{1}{t^2+a^2}\Big) = \frac{1}{a}\ \sqrt{\frac{\pi}{2}} e^{-a|s|}.$$

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(iii) Express the function $f(x)$ as Fourier sine integral where $f(x) = \begin{cases} \sin x & \text{if } 0 \le x \le \pi, \\ 0 & \text{if } x > \pi. \end{cases}$

and evaluate $\displaystyle\int_0^\infty \frac{\sin sx * \sin \pi s}{1 - s^2}\, ds$.

Q 3 (A) Attempt any One of the following: (8)

(i) (I) Write a short note on one dimensional wave equation.

(II) Find a bounded solution of $\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial t^2} = xt, u(0,t) = 0, u(x,0) = 0 = u_t(x,0)$.

(ii) Let $g(t)$ be a function defined for all $t \ge 0$. If $f(t)$ is a function defined by

$f(t) = \begin{cases} e^{-xt} g(t) & \text{if } t \ge 0, \\ 0 & \text{if } t < 0. \end{cases}$

and $\mathscr{F}\Big(f(t)\Big) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} e^{ist}\, f(t)\, dt$, then

(I) state and prove the relation between $\mathscr{L}\Big(g(t)\Big)$ and $\mathscr{F}\Big(f(t)\Big)$.

(II) verify the above relation for $g(t) = 1$ and $x = s + is$.

(B) Attempt any Two of the following: (12)

(i) Find $\displaystyle\int_0^\infty \frac{e^{-2t} - e^{-3t}}{t}\, dt$ using Laplace transforms.

(ii) Solve the initial value problem $y'' + 2y' + 2y = 2, y(0) = 0, y'(0) = 1$ using Laplace transforms.

(iii) Solve using Laplace transforms $R\dfrac{dy}{dt} + \dfrac{1}{C}\, y = V, y(0) = 0$ where R, C, V are constants.

Q 4 Attempt any Three of the following: (15)

(a) If $\mathscr{L}\Big(f(t)\Big) = F(s) = \dfrac{1}{s(s^2 + 2s + 2)}$, find $\displaystyle\lim_{t \longrightarrow \infty} f(t)$, using Final value theorem.

(b) Find $\mathscr{L}\Big((2 + te^{-3t})^2\Big)$.

(c) Let $f(t) = H(1 - |t|)$ where $H(1 - |t|)$ is defined as $H(1 - |t|) = \begin{cases} 1, & \text{if } 1 > |t| \\ 0, & \text{if } 1 < |t|. \end{cases}$

Find $\mathscr{F}\Big(f(t)\Big)$ and hence find $\mathscr{F}^{-1}\Big(\dfrac{\sin s}{s}\Big)$.

(d) Let $f(t)$ be a real valued even function and let $F(s) = \mathscr{F}\Big(f(t)\Big)$. Prove that $F(s)$ is a real and even function.

(e) For what values of the constant c, $u = \sin t\, \sin x$ is a solution of $\dfrac{\partial^2 u}{\partial t^2} = c^2\, \dfrac{\partial^2 u}{\partial x^2}, t \ge 0, x \ge 0$.

(f) Evaluate $\displaystyle\int_0^\infty e^{-2t}\Big(\int_0^t e^{-4u} \sin 3u\, du\Big)\, dt$, using $\mathscr{L}\Big(\displaystyle\int_0^t e^{-4u} \sin 3u\, du\Big)$.

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