Library 13/04/15 (Afternoon)

TYB.Sc (VI) (75:25/60:40)

QP Code : 14624

Topology of Matric Spaces - II

($2\frac{1}{2}$ Hours) [Total Marks: 75]

N.B.: (1) All qustions are compulsory.

(2) Figures to the right indicate marks for respective subquestions.

1. (a) Attempt any one question:

(i) Let f be a continuous real valued periodic function, defined on $[-\pi, \pi]$ and having period 2π. If $\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ is the Fourier series of f on $[-\pi, \pi]$ then prove that : $\sigma_n(x) \longrightarrow f(x)$ as $n \longrightarrow \infty$, where $\sigma_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} S_k(x)$, S_k is the k^{th} partial sum of the Fourier series of f. (8)

(ii) Let f be a continuous real valued periodic function, defined on $[-\pi, \pi]$ and having period 2π. If $f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ is the Fourier series of f on $[-\pi, \pi]$ then prove that : $S_n(x) - f(x) = \frac{2}{\pi}\int_0^{\pi}\left[\frac{f(x+t)+f(x-t)}{2} - f(x)\right] D_n(t)\text{dt}$ where $D_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} S_k(x)$, S_k is the k^{th} partial sum of the Fourier series of f and D_n is the Dirichlet's kernel. (8)

(b) Attempt any three questions:

(i) Define Diriclet's Kernel $D_n(t)$ nad Fejer's Kernel $K_n(t)$. Show that (4)

$$K_n(t) = \frac{\sin^2(\frac{nt}{2})}{2n\sin^2\frac{t}{2}} \quad -\infty < t < \infty,\ t \neq 2k\pi, k \in \mathbb{Z}$$

(ii) Is the series $\sum_{n=1}^{\infty}\left[\frac{\cos nx + \sin nx}{n^{\frac{3}{2}}}\right]$ the Fourier series of a function $f \in C[-\pi, \pi]$? Justfy your answer (4)

(iii) If $f(x) = |x|,\ -\pi \leq x \leq \pi$, and $f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$, compute $\frac{{a_0}^2}{2} + \sum_{n=1}^{\infty}{a_n}^2$. State clearly the result used. (4)

(iv) $f(x) = \cos^3 x + \sin^3 x$ in $[-\pi, \pi]$ and $f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$. Then find the Fourier coefficients a_0, a_1 and b_1 (4)

2. (a) Attempt any one question:

(i) $K \subseteq \mathbb{R}^n$ (distance Euclidean), K is closed and bounded. Show that K is sequentially compact. (8)

(ii) Let $I = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] \subseteq \mathbb{R}^n$ (distance Euclidean). Prove that I is compact. (8)

(b) Attempt any three questions:

(i) Show that the subset $A = \{(x, y) \in \mathbb{R}^2 : |x| \leq 1\}$ of the metric space $(\mathbb{R}^2, d)$, (d Euclidean distance) is not compact. (4)

(ii) Show that the following function (distance in $\mathbb{R}$ is usual) $f : [0,1] \times [0,1] \longrightarrow \mathbb{R}$, $f(x,y) = x + y$ (distance in $\mathbb{R}^2$ Euclidean) is uniformly continuous. (4)

(iii) Let (X, d) be a metric space and $K \subseteq X$ be a compact set. Show that a closed subset F of K is compact. (4)

(iv) Prove or disprove: A closed and bounded subset of a metric space is compact. (4)

3. (a) Attempt any one question:

(i) Show that a metric space (X, d) is connected if and only if every continuous function $f : X \longrightarrow \{1, -1\}$ is constant. (8)

(ii) Let (X, d) be a metric space and A be a connected subset of X. If $A \subset B \subset \bar{A}$ then Show that B is connected. In particular, prove that $\bar{A}$ is connected. Give an example to show that if A, C are connected subsets of X and $A \subset B \subset C$ then B need not be connected. (8)

(b) Attempt any three questions:

(i) Show that the set $A = \{(x, y) \in \mathbb{R}^2 : y^2 = x\}$ is path connected subset of $\mathbb{R}^2$ (distance being Euclidean). (4)

(ii) If (X, d) be a connected metric space and $f : X \longrightarrow \mathbb{Z}$ (distance in $\mathbb{Z}$ being usual distance) is a continuous function then prove that f is a constant function. (4)

(iii) Prove or disprove: The subset $\{(x, y) \in \mathbb{R}^2 : y \neq 0\}$ of $(\mathbb{R}, d)$ (d being Euclidean distance) is connected. (4)

(iv) Let (X, d) be a metric space. If A is a finite subset of X having more than one element, show that A is disconnected. (4)

4. Attempt any three questions:

(a) $f(x) = \dfrac{x^2}{4}$, $-\pi \leq x \leq \pi$. Find the Fourier series of f. Assuming that the Fourier series of f converges to $f(x)$ at $x = 0$, find the sum $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$. (5)

(b) Let $f \in C[-\pi, \pi]$ and f has Fourier sereis $\dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$, show that (5)

$$\sigma_n(t) = \frac{a_0}{2} + \sum_{k=1}^{n-1} \left(1 - \frac{k}{n}\right) (a_k \cos kt + b_k \sin kt)$$

(c) Show that $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is a compact subset of $\mathbb{R}^2$, distance being Euclidean. (5)

(d) Let (X, d) be a compact metric space. If $\{A_n\}$ is a sequence of non-empty closed sets in X such that $A_{n+1} \subseteq A_n$ for each $n \in \mathbb{N}$, then show that $\bigcap_{n \in \mathbb{N}} A_n \neq \emptyset$. (5)

(e) Prove or disprove: $A = \{(x, y) \in \mathbb{R}^2 : xy = 0\}$ is a connected subset of $\mathbb{R}^2$ (distance being Eulcidean) (5)

(f) Prove that a path connected subset of $\mathbb{R}^n$ (distance being Euclidean) is connected. (5)

OM-Con. : 2736-15. ------------------------